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Zero-temperature dielectric response of the charged Bose gas in a uniform magnetic field

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A closed-form expression for the dielectric function of a zero-temperature charged Bose gas in a uniform magnetic field is obtained in the random-phase approximation. This dielectric function is used to explore the dispersion relation for the frequencies of longitudinal oscillation of the gas as well as the form of the electrostatic potential about a test charge in the gas.

I. INTRODUCTION

In an earlier paper,[1] hereafter known as I, we studied the dielectric response of the charged Bose gas (CBG) in the random-phase approximation (RPA). In that paper we obtained forms for the dispersion relation for the frequencies of allowed longitudinal vibrations of the gas, and also for the long-ranged electrostatic potential about a test charge in the gas. In I we stated that an interesting extension of the work presented there would be to study the CBG in an external magnetic field.

This is what we have done in this paper. Taking the case of a uniform magnetic field, we investigate the dielectric response of the CBG in the RPA. As in I, we specifically study the dispersion relation for the frequencies of longitudinal vibration of the gas, as well as the potential about a test charge in the gas. We do this explicitly at zero temperature only, but set up the machinery enabling the study of the response of the gas to be done at all other temperatures.

We note that analogous studies have been made of the charged Fermi gas (CFG), specifically an electron gas, in an external magnetic field.[2] The CFG in a magnetic field exhibits the interesting de Haas— van Alphen effect at low temperatures, and is also important in the study of transport phenomena in metals. Analogously, the CBG is important because it exhibits the Meissner-Ochsenfeld effect at low temperatures. This has been shown in the case of the ideal CBG by Schafroth,[3] and for the interacting CBG by Fetter.[4] As was pointed out by Schafroth, the CBG is therefore of importance as a model for a superconductor.

Furthermore, as well as its intrinsic interest as an unsolved many-body problem, the CBG is also important because of its applicability to the phenomena of pion condensation in neutron stars.[5,6] The studies referred to in Refs. 5 and 6 are all zero-magnetic-field treatments, but it is believed that magnetic fields of significant strength exist in neutron stars. Hence this paper, which studies the CBG in a magnetic field beyond the Meissner-Ochsenfeld effect, that is, in a magnetic field strong enough to penetrate the gas, is of physical interest.

The CBG in a uniform magnetic field has never been studied at zero temperature before, although the high-temperature classical-Boltzmann-region behavior has been investigated.[7,8] Das[9] has made an attempt to extend the study of the gas into the quantum-mechanical region, but his treatment suffers from the following limitations. Das studied the electrostatic potential about a test charge, but his work is based on a semiclassical theory, and he eventually takes a high-temperature limit to obtain an asymptotic expansion for the screening length. Therefore, all he is doing, in effect, is obtaining quantum corrections to the classical result. Furthermore, Das has linearized his equations to obtain a spherically symmetric form of the electrostatic potential about the test charge. This cannot be correct because the introduction of a magnetic field into the gas imposes a necessary spatial anisotropy on the system. This is clearly demonstrated in Sec. V of this paper.

In this paper we study the dielectric response of the CBG in a uniform magnetic field when the gas is in the total quantum region (zero temperature). This is the first time the quantum effects, which dominate at low temperatures, have been studied for the CBG in a magnetic field.

In Sec. II of the paper, we set up the form of the dielectric tensor in the RPA for the gas, and in Sec. III we explicitly give the zero-temperature dielectric function. In Sec. IV we investigate the dispersion relation for the frequencies of longitudinal oscillation of the gas, and in Sec. V we look at the electrostatic potential about a test charge. In Sec. VI we discuss the results.

II. DIELECTRIC TENSOR

Consider a gas of N identical spinless bosons with mass m and charge e in a box of volume Ω, together with a background of stationary particles of opposite charge to preserve charge neutrality, and assume the usual periodic boundary conditions. Consider the gas to be in a steady-state electromagnetic field specified by the potentials $\vec{A}_0(\vec{x})$ and $\phi_0(\vec{x})$, the vector and scalar potentials, respectively. By considering small perturbations on these potentials, Harris[10] is able to use a second quantized formalism in the RPA to obtain an expression for the Fourier-transformed conductivity tensor $\tilde{\sigma}(\vec{q}, \vec{q}\,', \omega)$ of the gas. We note, as we did in I, that the RPA is valid in the high-density limit. Defining $\tilde{\sigma}(\vec{q}, \vec{q}\,', \omega)$ by the equation

$$\langle \vec{J}_1(\vec{q}, \omega)\rangle = \sum_{\vec{q}} \tilde{\sigma}(\vec{q}, \vec{q}\,', \omega)\cdot \vec{E}_1(\vec{q}\,', \omega),$$

where $\vec{E}_1$ is the perturbation of the electric field and $\langle \vec{J}_1\rangle$ is the ensemble average of the current density operator, he finds

$$\tilde{\sigma}(\vec{q}, \vec{q}\,', \omega) = -\sum_b \sum_{b'} \frac{ie^2}{2\Omega} \frac{F_0(b) - F_0(b')}{h\omega - (E_b - E_{b'})} \times\langle b'|\, \vec{v} e^{-i\vec{q}\cdot\vec{x}} + e^{-i\vec{q}\cdot\vec{x}}\vec{v}\,| b\rangle \times\left\{\frac{\vec{q}\,'}{(q')^2}\langle b|\, e^{i\vec{q}\,'\cdot\vec{x}}| b'\rangle + \frac{1}{\omega}\langle b|\, \vec{v} e^{i\vec{q}\,'\cdot\vec{x}}| b'\rangle \times\left(\tilde{I} - \frac{\vec{q}\,'\vec{q}\,'}{(q')^2}\right)\right\}, \tag{1}$$

where $|b\rangle$ represents an eigenstate of a boson in the unperturbed gas, $F_0(b)$ is the frequency distribution of bosons in the eigenstate $|b\rangle$, and E_b is the energy eigenvalue of a boson in the eigenstate $|b\rangle$. $\tilde{I}$ is the second-order identity tensor, ω is the frequency of a small oscillation of the gas about equilibrium, $\vec{q}$ is a Fourier-transform parameter which represents the wave number of the frequency ω, and $\vec{v} = (1/m)[\vec{p} - (e/c)\vec{A}_0]$ is the velocity operator.

We note that, as found in I, to eliminate singularities in the conductivity tensor the Landau prescription[11] needs to be taken, and ω in the denominator of Eq. (1) replaced by $\omega + i\gamma$ where $\gamma \to 0+$. However, we will eventually be working only at zero temperature ($T = 0$), and as the singularity does not occur at this temperature we will work without introducing $i\gamma$.

We are interested in the special case where the gas is in a uniform magnetic field. Thus we set $\phi_0 = 0$, and choose $\vec{A}_0(\vec{x}) = (-By, 0, 0)$, which means the gas is in a uniform magnetic field of magnitude B and direction parallel to the z axis. We note that this choice of vector potential satisfies the Coulomb gauge condition, $\nabla\cdot\vec{A} = 0$, which is inherent in Harris's derivation.

The eigenfunctions and energy eigenvalues are of course well known for this system, and we have,[12]

$$\langle \vec{x}| b\rangle \equiv \langle \vec{x}| n, k_x, k_z\rangle \equiv \chi_{n,k_x,k_z}(\vec{x}) = \frac{1}{\Omega^{1/3}}\left(\frac{1}{2^n n!}\right)^{1/2}\left(\frac{m\omega_B}{\pi\hbar}\right)^{1/4} e^{i(xk_x+zk_z)}\exp\left[-\frac{m\omega_B}{2\hbar}\left(y+\frac{\hbar k_x}{m\alpha_B}\right)^2\right] H_n\left[\left(\frac{m\omega_B}{\hbar}\right)^{1/2}\left(y+\frac{\hbar k_x}{m\alpha_B}\right)\right], \tag{2}$$

$$E_b = E_{n,k_x,k_z} = \frac{\hbar^2 k_z^2}{2m} + \hbar\omega_B(n+\tfrac{1}{2}), \tag{3}$$

where $\omega_B = |e|B/mc$ is the cyclotron frequency, $\alpha_B = eB/mc = \omega_B\,\mathrm{sgn}(e)$, H_n is a Hermite polynomial,[13] $\hbar k_x$ and $\hbar k_z$ are the eigenvalues of the operators p_x, and p_z and $n = 0, 1, 2, 3, \ldots$.

We can now evaluate the matrix elements appearing in Eq. (1). This is done in Appendix A. It is clear that when the results of Eqs. (A8)–(A10) are used in Eq. (1), we will obtain the following structure for $\tilde{\sigma}(\vec{q}, \vec{q}\,', \omega)$:

$$\tilde{\sigma}(\vec{q}, \vec{q}\,', \omega) = \sum_{n,n'}\sum_{k_x,k_x'}\sum_{k_z,k_z'} \tilde{G}(n, n', k_x, k_x', k_z, k_z', q_x, q_y, q_z, q_x', q_y', q_z')\,\delta_{k_x,k_x'+q_x}\delta_{k_z,k_z'+q_z}\delta_{k_x,k_x'+q_x'}\delta_{k_z,k_z'+q_z'}.$$

We may use the properties of the Kronecker δ function to rearrange thus,

$$\tilde{\sigma}(\vec{q}, \vec{q}\,', \omega) = \sum_{n,n'}\sum_{k_x}\sum_{k_z} \tilde{G}(n, n', k_x, k_x - q_x, k_z, k_z - q_z, q_x, q_y, q_z, q_x, q_y', q_z)\delta_{q_x,q_x'}\delta_{q_z,q_z'},$$

$$\equiv \sum_{n,n'}\sum_{k_x}\sum_{k_z} \tilde{H}(n, n', k_x, k_z, q_x, q_z, q_y, q_y')\delta_{q_x,q_x'}\delta_{q_z,q_z'}.$$

If the plasma is uniform in space, then $F_0(n, k_x, k_z) = F_0(n, k_z)$, and since E_{n,k_x,k_z} is also independent of k_x, it is clear that the only k_x dependence in $\tilde{H}$ is that occurring in the matrix elements, and by inspection of Eqs. (A8)–(A10) we can see that

$$\tilde{H}(n, n', k_x, k_z, q_x, q_z, q_y, q_y') = \exp\left(-i\,\frac{\hbar k_x}{m\alpha_B}\,(q_y' - q_y)\right)\tilde{K}(n, n', q_x, q_y, q_z, q_y').$$

The sum on k_x may now be done, and using the fact that k_x is quantized by periodic boundary conditions, together with the prescription valid in the limit $\Omega \to \infty$,

$$\sum_{k_x} \to \frac{\Omega^{1/3}}{2\pi}\int_{-\infty}^{\infty} dk_x\,,$$

it is a simple matter to show,

$$\sum_{k_x} \exp\left(-i\,\frac{\hbar k_x}{m\alpha_B}\,(q_y' - q_y)\right) = \frac{m\omega_B\Omega^{2/3}}{2\pi\hbar}\,\delta_{q_y, q_y'}\,.$$

The obtaining of this final δ function enables us to write,

$$\tilde{\sigma}(\vec{q}, \vec{q}', \omega) = \tilde{\sigma}(\vec{q}, \omega)\delta_{\vec{q}, \vec{q}'}\,,$$

where, after collecting all the terms and making use of Eq. (A7) we have, expressing the result in matrix form,

$$\tilde{\sigma}(\vec{q}, \omega) = \frac{im\omega_B e^2}{4\pi\hbar\omega\Omega^{1/3}}\exp\left(-\frac{\hbar}{2m\omega_B}(q_x^2 + q_y^2)\right)\sum_{n,n'}\sum_{k_z}\frac{F_0(n', k_z - q_z) - F_0(n, k_z)}{\hbar\omega + \hbar\omega_B(n' - n) + (\hbar^2 q_z/2m)(q_z - 2k_z)}\,M_1 M_2 \qquad (4)$$

where

$$M_1 = \begin{bmatrix} \frac{\hbar q_x}{m}\,{}^nF^*_{n'} + 2\,\mathrm{sgn}(e)\left(\frac{\hbar\omega_B}{m}\right)^{1/2}\left[\left(\frac{n'+1}{2}\right)^{1/2}{}^nF^*_{n'+1} + \left(\frac{n'}{2}\right)^{1/2}{}^nF^*_{n'-1}\right] \\ \frac{\hbar q_y}{m}\,{}^nF^*_{n'} - 2i\left(\frac{\hbar\omega_B}{m}\right)^{1/2}\left[\left(\frac{n'+1}{2}\right)^{1/2}{}^nF^*_{n'+1} - \left(\frac{n'}{2}\right)^{1/2}{}^nF^*_{n'-1}\right] \\ \frac{\hbar}{m}(2k_z - q_z)\,{}^nF^*_{n'} \end{bmatrix},$$

$$M_2^T = \begin{bmatrix} \frac{q_x}{q^2}\left(\omega + \omega_B(n'-n) + \frac{\hbar}{2m}(q^2 + q_z^2 - 2k_z q_z)\right){}^nF_{n'} + \mathrm{sgn}(e)\left(\frac{\hbar\omega_B}{m}\right)^{1/2}\left[\left(\frac{n'+1}{2}\right)^{1/2}{}^nF_{n'+1} + \left(\frac{n'}{2}\right)^{1/2}{}^nF_{n'-1}\right] \\ \frac{q_y}{q^2}\left(\omega + \omega_B(n'-n) + \frac{\hbar}{2m}(q^2 + q_z^2 - 2k_z q_z)\right){}^nF_{n'} + i\left(\frac{\hbar\omega_B}{m}\right)^{1/2}\left[\left(\frac{n'+1}{2}\right)^{1/2}{}^nF_{n'+1} - \left(\frac{n'}{2}\right)^{1/2}{}^nF_{n'-1}\right] \\ \frac{1}{q^2}\left([\omega + \omega_B(n'-n)]q_z + \frac{\hbar}{2m}(q_x^2 + q_y^2)(2k_z - q_z)\right){}^nF_{n'} \end{bmatrix},$$

and ${}^nF_{n'}$, is defined by Eq. (A6) in Appendix A. M_2^T is the transpose of M_2.

Equation (4) is the result in the RPA for the conductivity tensor where all that remains to be determined is the distribution function $F_0(n, k_z)$. As was done in I, $F_0(n, k_z)$ may be approximated by the distribution function of an ideal gas of charged bosons in a magentic field. This procedure is consistent with the RPA and we in fact make this substitution in Sec. III.

We note that in Eq. (4) we not only have obtained the conductivity tensor but also the dielectric tensor $\tilde{\epsilon}(\vec{q}, \omega)$, since $\tilde{\epsilon}(\vec{q}, \omega)$ and $\tilde{\sigma}(\vec{q}, \omega)$ are connected by the equation,

$$\tilde{\epsilon}(\vec{q}, \omega) = \tilde{I} + (4\pi i/\omega)\tilde{\sigma}(\vec{q}, \omega). \qquad (5)$$

III. DIELECTRIC FUNCTION

As in the earlier paper we wish to look at the longitudinal properties of the Bose gas. To do this we investigate the dielectric function $\epsilon(\vec{q}, \omega)$ (also called the dielectric constant or dielectric-response function) which is given in terms of the dielectric tensor by

$$\epsilon(\vec{q}, \omega) = (1/q^2)\vec{q}\cdot\tilde{\epsilon}(\vec{q}, \omega)\cdot\vec{q}. \qquad (6)$$

By contracting the tensor in this way, the complexity inherent in Eq. (4) is vastly reduced. Substituting

Eqs. (4) and (5) into Eq. (6), and making use of Eq. (A7) yields the following, fairly manageable result for $\epsilon(\vec{q},\omega)$,

$$\epsilon(\vec{q},\omega)=1+\frac{2me^2\omega_B}{\hbar^2q^2\Omega^{1/3}\omega}\exp\left(-\frac{\hbar}{2m\omega_B}(q_x^2+q_y^2)\right)\sum_{n,n'}\sum_{k_z}\left(\frac{\hbar\omega_B(n'-n)+(\hbar^2q_z/2m)(q_z-2k_z)}{\hbar\omega+\hbar\omega_B(n'-n)+(\hbar^2q_z/2m)(q_z-2k_z)}\right)$$
$$\times[F_0(n',k_z-q_z)-F_0(n,k_z)]^nF_{n'}{}^nF^*_{n'}. \quad (7)$$

Breaking the double sum on n and n' into three sums, namely $n=n'$, $n>n'$, and $n<n'$ yields

$$\epsilon(\vec{q},\omega)=1+\frac{2me^2\omega_B}{\hbar^2q^2\Omega^{1/3}\omega}e^{-x}\sum_{k_z}\Bigg[\sum_{m=0}^{\infty}\frac{(\hbar^2q_z/2m)(q_z-2k_z)}{\hbar\omega+(\hbar^2q_z/2m)(q_z-2k_z)}[F_0(m,k_z-q_z)-F_0(m,k_z)][L_m^0(x)]^2$$
$$+\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(l+1)\hbar\omega_B+(\hbar^2q_z/2m)(q_z-2k_z)}{\hbar\omega+(l+1)\hbar\omega_B+(\hbar^2q_z/2m)(q_z-2k_z)}\frac{m!}{(m+l+1)!}$$
$$\times x^{l+1}[F_0(m+l+1,k_z-q_z)-F_0(m,k_z)][L_m^{l+1}(x)]^2$$
$$+\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{-(l+1)\hbar\omega_B+(\hbar^2q_z/2m)(q_z-2k_z)}{\hbar\omega-(l+1)\hbar\omega_B+(\hbar^2q_z/2m)(q_z-2k_z)}\frac{m!}{(m+l+1)!}$$
$$\times x^{l+1}[F_0(m,k_z-q_z)-F_0(m+l+1,k_z)][L_m^{l+1}(x)]^2\Bigg], \quad (8)$$

where

$$x=\frac{\hbar}{2m\omega_B}(q_x^2+q_y^2).$$

As mentioned in Sec. II, we now take for $F_0(n,k_z)$ the distribution function of ideal gas of charged bosons in a magnetic field. This means[14]

$$F_0(n,k_z)=\left\{z^{-1}\exp\left[\frac{1}{kT}\left(\hbar\omega_B(n+\tfrac{1}{2})+\frac{\hbar^2k_z^2}{2m}\right)\right]-1\right\}^{-1},$$

where z is the fugacity of the gas. As was done in I, $\epsilon(\vec{q},\omega)$ could be investigated at all temperatures, using a knowledge of z at all temperatures. However, in this paper we will concern ourselves only with the behavior of the gas at $T=0$. At $T=0$ all the bosons in a gas of ideal bosons will be in the lowest energy level. That is, at $T=0$,

$$F_0(n,k_z)=N(2\pi\hbar/m\omega_B\Omega^{2/3})\delta_{n,0}\delta_{k_z,0}, \quad (9)$$

where the factor $2\pi\hbar/m\omega_B\Omega^{2/3}$ arises from the degeneracy in k_x of the energy levels of the gas. Substituting Eq. (9) into Eq. (8) yields

$$\epsilon(\vec{q},\omega,T=0)=1-\frac{m\omega_p^2}{\hbar q^2\omega}e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\left(\frac{\hbar^2q_z^2}{2m}+l\hbar\omega_B\right)\left(\frac{1}{\hbar\omega+l\hbar\omega_B+\hbar^2q_z^2/2m}+\frac{1}{\hbar\omega-l\hbar\omega_B-\hbar^2q_z^2/2m}\right), \quad (10)$$

where $\omega_p^2=4\pi e^2N/m\Omega$ is the plasma frequency of the gas. Equation (10) is a closed-form expression in the RPA, for the $T=0$ dielectric function of the CBG in an external magnetic field B. To the best of the authors's knowledge, this is a new result that has not before been displayed. Equation (10) is to be compared with Eq. (7) of I, which gives the result for $\epsilon(\vec{q},\omega,T=0)$ for $B=0$. Namely,

$$\epsilon(\vec{q},\omega,T=0,B=0)=1-\frac{\omega_p^2}{\omega^2-\hbar^2q^4/4m^2}. \quad (11)$$

It is not at all clear that Eq. (10) will reduce to Eq. (11) in the limit $B=0$. In fact, Eq. (10) gives $\epsilon(\vec{q},\omega,T=0)$ in a form which is not at all suitable for looking at the case of weak B, in which we are interested. In Appendix B we develop an expansion which enables us to recast Eq. (10) into a more useful form. Using the results of Eqs. (B2) and (B3) in Eq. (10) yields

$$\epsilon(\vec{q},\omega,T=0)=1-\frac{\omega_p^2}{\omega^2-\hbar^2q^4/4m^2}-\frac{m\omega_p^2}{\hbar q^2}\sum_{p=0}^{\infty}\left(\frac{\hbar}{2m}(q_x^2+q_y^2)\right)^p\Bigg[\frac{1}{\prod_{n=0}^{p}(\omega-\hbar q_z^2/2m-n\omega_B)}-\frac{1}{(\omega-\hbar q_z^2/2m)^{p+1}}$$
$$+\frac{(-1)^{p+1}}{\prod_{n=0}^{p}(\omega+\hbar q_z^2/2m+n\omega_B)}-\frac{(-1)^{p+1}}{(\omega+\hbar q_z^2/2m)^{p+1}}\Bigg]. \quad (12)$$

Equation (12) is our final result for the dielectric function at $T=0$. $\epsilon(\vec{q}, \omega, T=0)$ is now in a form which is useful for looking at both the small B and small q limits. From Eq. (12) we can see that in the limit $B\to 0$ ($\omega_B \to 0$) the dielectric function reduces to the required form for $B=0$, as given in Eq. (11).

IV. DISPERSION RELATION

The dielectric function can be used to give the dispersion relation for the frequencies ω of longitudinal density fluctuations in the gas. The dispersion relation at $T=0$ is given by the solution to the equation,

$$\epsilon(\vec{q}, \omega, T=0) = 0. \tag{13}$$

As noted in I, this equation can be solved exactly for $B=0$ to give the Foldy[15] result,

$$\omega^2 = \omega_p^2 + \hbar^2 q^4/4m^2. \tag{14}$$

For $B\neq 0$ the situation is, of course, more complicated but we can obtain asymptotic solutions for ω as was done in I. That is, we look at the long-wavelength $q\to 0$ limit. In the earlier paper, the gas was isotropic and there was no need to consider the relative sizes of the components of $\vec{q}$ when taking the $q\to 0$ limit. However, in this paper, the presence of the magnetic field necessarily demands an anisotropy in the gas, and this means our solution for ω will not only depend on the relative size of the wave number q (in $\hbar q^2/2m$) compared to that of B (in ω_B), but also on the relative size of the component of $\vec{q}$ parallel to the magnetic field compared to the component perpendicular to the magnetic field. All of this is clear when one looks at the form of Eq. (12).

With the various orderings of the three small parameters ω_B, $\hbar q_z^2/2m$, $(\hbar/2m)(q_x^2+q_y^2)$, we are able to obtain the following asymptotic solution for ω,

(i) $\frac{1}{\omega_p}\frac{\hbar q_z^2}{2m}, \frac{\omega_B}{\omega_p} \ll \frac{1}{\omega_p}\frac{\hbar}{2m}(q_x^2+q_y^2) \ll 1$:

$$\omega^2 = \omega_p^2 + \frac{\hbar^2}{4m^2}(q_x^2+q_y^2)^2 + \cdots; \tag{15}$$

(ii) $\frac{1}{\omega_p}\frac{\hbar q_z^2}{2m} \ll \frac{1}{\omega_p}\frac{\hbar}{2m}(q_x^2+q_y^2) \ll \frac{\omega_B}{\omega_p} \ll 1$:

$$\omega^2 = \omega_p^2 + \omega_B^2 + \cdots; \tag{16}$$

(iii) $\frac{1}{\omega_p}\frac{\hbar}{2m}(q_x^2+q_y^2) \ll \frac{1}{\omega_p}\frac{\hbar q_z^2}{2m} \ll \frac{\omega_B}{\omega_p} \ll 1$:

$$\omega^2 = \omega_p^2 + \frac{\hbar^2 q_z^4}{4m^2} + \omega_B^2\left(\frac{q_x^2+q_y^2}{q_z^2}\right) + \cdots; \tag{17}$$

(iv) $\frac{1}{\omega_p}\frac{\hbar}{2m}(q_x^2+q_y^2), \frac{\omega_B}{\omega_p} \ll \frac{1}{\omega_p}\frac{\hbar q_z^2}{2m} \ll 1$:

$$\omega^2 = \omega_p^2 + \frac{\hbar^2 q_z^4}{4m^2} + \cdots. \tag{18}$$

Equations (15) and (18) show, as expected, that in the limit of very weak magnetic fields the two leading order terms in the $T=0$ dispersion relation for the frequency ω are the same as the $B=0$ result for the dispersion relations, namely Eq. (14). Eqs. (16) and (17) show how the dispersion relation is modified as the magnetic field becomes stronger.

V. ELECTROSTATIC POTENTIAL ABOUT A TEST CHARGE

As was done in I, the electrostatic potential $V(\vec{r})$ about a test charge Q immersed in the gas may be investigated by making use of the zero-frequency dielectric function (ZFDF). That is,

$$V(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3q\, e^{i\vec{q}\cdot\vec{r}}\, V(\vec{q}), \tag{19}$$

where

$$V(\vec{q}) = 4\pi Q/q^2\epsilon(\vec{q}, \omega=0). \tag{20}$$

From Eq. (10) the $T=0$ ZFDF is given by,

$$\epsilon(\vec{q}, \omega=0, T=0) = 1 + \frac{2m\omega_p^2}{\hbar q^2}e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{1}{\hbar^2 q_z^4/2m + l\hbar\omega_B}. \tag{21}$$

Unfortunately, when Eqs. (20) and (21) are substituted into Eq. (19), the inherent asymmetry of our gas induced, of course, by the presence of the B field means that the integral obtained which determines $V(\vec{r})$ is extremely complex, and the authors have not been able to solve it in the general anisotropic three-dimensional case.

Although we cannot obtain forms for the electrostatic potential about a test charge in a three-dimensional CBG at $T=0$ in a magnetic field because of the anisotropy of such a system, we may, however, exploit this anisotropy and obtain exact results in two extreme anisotropic cases. Case A looks at the gas when its response is primarily perpendicular to the magnetic field, which is a two-dimensional situation. Case B deals with the gas when its response is primarily parallel to the magnetic field, which is a one-dimensional situation. Case A is characterized by the ordering of the small parameters,

$$\frac{1}{\omega_p}\frac{\hbar q_z^2}{2m}, \frac{\omega_B}{\omega_p} \ll \frac{1}{\omega_p}(q_x^2+q_y^2) \ll 1,$$

associated with Eq. (15). This ordering leads to the dispersion relation

$$\omega^2 \simeq \omega_p^2 + \frac{\hbar^2}{4m^2}(q_x^2+q_y^2)^2,$$

which looks like a two-dimensional version of Eq. (14), which is the three-dimensional $B=0$ result. Similarly, case B is characterized by the ordering,

$$\frac{1}{\omega_p}\frac{\hbar}{2m}(q_x^2+q_y^2),\ \frac{\omega_B}{\omega_p}\ll\frac{\hbar}{2m}q_z^2\ll 1,$$

associated with Eq. (18). This ordering leads to the dispersion relation

$$\omega^2\simeq\omega_p^2+\left(\frac{\hbar^2}{4m^2}\right)q_z^4,$$

which looks like a one-dimensional version of Eq. (14).

Thus for case A it seems reasonable to investigate the two-dimensional version of Eq. (19),

$$V_2(\vec{r})=\frac{1}{(2\pi)^2}\int d^2q\,e^{i\vec{q}\cdot\vec{r}}\frac{4\pi Q}{q^2\epsilon(\vec{q},\omega=0,T=0)},\quad r=(x^2+y^2)^{1/2}\tag{22}$$

where

$$\epsilon(\vec{q},\omega=0,T=0)=1+4m^2\omega_p^2/\hbar^2q^4,\quad q=(q_x^2+q_y^2)^{1/2}\tag{23}$$

which is what we find when we solve Eq. (21) in the limit,

$$\frac{1}{\omega_p}\frac{\hbar q_z^2}{2m},\ \frac{\omega_B}{\omega_p}\ll\frac{\hbar}{2m}(q_x^2+q_y^2)\ll 1,$$

to lowest order in an expansion in terms of these parameters.

Similarly, for case B we write down the one-dimensional integral

$$V_1(\vec{r})=\frac{1}{2\pi}\int dq\,e^{i\vec{q}\cdot\vec{r}}\frac{4\pi Q}{q^2\epsilon(\vec{q},\omega=0,T=0)},\quad r=z\tag{24}$$

where

$$\epsilon(\vec{q},\omega=0,T=0)=1+4m^2\omega_p^2/\hbar^2q^4,\quad q=q_z\tag{25}$$

which is what Eq. (21) yields in the limit

$$\frac{1}{\omega_p}\frac{\hbar}{2m}(q_x^2+q_y^2),\ \frac{\omega_B}{\omega_p}\ll\frac{\hbar q_z^2}{2m}\ll 1,$$

to lowest order in an expansion in terms of these parameters.

For comparison with Eqs. (22) and (24) we write down,

$$V_3(\vec{r})=\frac{1}{(2\pi)^3}\int d^3q\,e^{i\vec{q}\cdot\vec{r}}\frac{4\pi Q}{q^2(1+4m^2\omega_p^2/q^4)},\tag{26}$$

which is the equation in the RPA at $T=0$ for the potential about a test charge Q in a three-dimensional $B=0$ CBG. It is obtained by putting $\omega=0$ in Eq. (11) to obtain

$$\epsilon(\vec{q},\omega=0,T=0)=1+4m^2\omega_p^2/\hbar^2q^4,\tag{27}$$

and then substituting this into Eqs. (19) and (20).

The integrals in Eqs. (22), (24), and (26) may be done and we find

$$V_3(\vec{r})=\frac{Q}{r}\cos\left(\frac{Ar}{\sqrt{2}}\right)e^{-Ar/\sqrt{2}},\tag{28}$$

$$V_2(\vec{r})=2Q\,\mathrm{ker}(Ar),\tag{29}$$

$$V_1(\vec{r})=\frac{\pi\sqrt{2}Q}{A}\left[\cos\left(\frac{Ar}{\sqrt{2}}\right)-\sin\left(\frac{Ar}{\sqrt{2}}\right)\right]e^{-Ar/\sqrt{2}},\tag{30}$$

where $A=(4m^2\omega_p^2/\hbar^2)^{1/4}$ and $\mathrm{ker}(x)$ is a Thomson function.[13]

We now take the small- and large-r limits of Eqs. (28)–(30) and we find

$$\lim_{r\to 0}V_3(\vec{r})=Q/r,\tag{31}$$

$$\lim_{r\to 0}V_2(\vec{r})=-2Q\ln(Ar),\tag{32}$$

$$\lim_{r\to 0}V_1(\vec{r})=\mathrm{const}-2\pi Qr,\tag{33}$$

and also,

$$\lim_{r\to\infty}V_3(\vec{r})=Q\cos\left(\frac{Ar}{\sqrt{2}}\right)\frac{e^{-Ar/\sqrt{2}}}{r},\tag{34}$$

$$\lim_{r\to\infty}V_2(\vec{r})=Q\left(\frac{2\pi}{A}\right)^{1/2}\cos\left(\frac{Ar}{\sqrt{2}}\right)\frac{e^{-Ar/\sqrt{2}}}{r^{1/2}},\tag{35}$$

$$\lim_{r\to\infty}V_1(\vec{r})=Q\frac{\pi\sqrt{2}}{A}\left[\cos\left(\frac{Ar}{\sqrt{2}}\right)-\sin\left(\frac{Ar}{\sqrt{2}}\right)\right]e^{-Ar/\sqrt{2}}.\tag{36}$$

We now note the following interesting results. Equations (31)–(33) give exactly the forms of the potential about a free charge in space of the appropriate dimension. That is, if Gauss's law is used to obtain the form of the electric field about a free charge in three, two, and one dimensions, this electric field will yield the potentials of Eqs. (31)–(33). This is just what we would expect if the $V_d(\vec{r})$ were in fact the potentials about a charge Q in a d-dimensional $B=0$ CBG. That is, in the $r\to 0$ limit the screening due to the Bose gas will have no effect on the potential about the test charge and the charge thus behaves as a bare charge.

Also interesting is the fact that in the large r limit, the functions $V_d(\vec{r})$ are damped as $e^{-Ar/\sqrt{2}}/r^{(d-1)/2}$. This is the Ornstein-Zernicke form of the pair correlation function in the large-r limit. In the RPA, the asymptotic form of the potential in d-dimensions in proportional to the pair correlation function, which has just this form. Therefore, if $V_d(\vec{r})$ were the true potentials in d dimensions, then in Eqs. (34)–(36) we have obtained the expected asymptotic form of the potential together with oscillatory factors characteristic of the Bose condensate at $T=0$.

To understand exactly what we have done in obtaining Eqs. (22)–(25), and to know just what these equations mean, we note the following. If one takes $4\pi Q/q^2$ as the Fourier transform of the "Coulomb" potential in both two and one dimensions, and then uses this in a Harris[16] RPA treatment (refer I) of the $B=0$ CBG, one would obtain for the $T=0$ ZFDF exactly

$$\epsilon(\vec{q}, \omega=0, T=0)=1+A^4/q^4 \tag{37}$$

where A is some constant. [Compare this result with Eq. (27) which is the corresponding result for three dimensions.] Using the $4\pi Q/q^2$ propagator implies

$$V_d(\vec{q})=4\pi Q/q^2\epsilon(\vec{q}, \omega=0, T=0), \tag{38}$$

where

$$V_d(\vec{r})=\frac{1}{(2\pi)^d}\int d^dq\, e^{i\vec{q}\cdot\vec{r}}\, V_d(\vec{q}) \tag{39}$$

is the potential about a test charge Q in the gas and $d=2, 1$.

Equations (37)–(39) are exactly Eqs. (22)–(25). This is just what we would expect because, first, Eqs. (23) and (25) are obtained from a ZFDF [Eq. (21)] derived from a RPA treatment in conjunction with a $4\pi Q/q^2$ propagator. (That is, in Sec. II we have a three-dimensional RPA treatment in which has been used $4\pi Q/q^2$ which is the Fourier transform of the three-dimensional Q/r Coulomb potential.) This ZFDF of Eq. (21) has then been solved in the two extreme anisotropic cases where Eqs. (23) and (25) give the appropriate asymptotic expansions for each of these two cases with ω_B/ω_p correction terms and either $q_z^2/(q_x^2+q_y^2)$ or $(q_x^2+q_y^2)/q_z^2$ correction terms dropped off. Therefore, Eqs. (23) and (25) are the same as the $B=0$ reduced-dimension results of Eq. (37). Secondly, in Eqs. (22) and (24) we have again used a $4\pi Q/q^2$ propagator just as we did in Eq. (38).

So, in effect, Eqs. (22)–(25) are the equations for the potential about a test charge Q in a two- and one-dimensional $B=0$ CBG where we assume a $4\pi Q/q^2$ propagator in an RPA treatment.

However, this is unfortunately not the correct way to tackle the exact two- and one-dimensional CBG problems. To do these correctly one must use the correct Fourier transform of the potentials in Eqs. (32) and (33), which are the "Coulomb" potentials in two and one dimensions, respectively. These Fourier transforms are not $4\pi Q/q^2$ because although the transforms are each dimensionally of the form Q/q^2, the coefficient is divergent in each case. This means that $4\pi Q/q^2\epsilon(\vec{q}, \omega=0)$ is not the correct form of $V_{2,1}(\vec{q})$ and that also $\epsilon(\vec{q}, \omega=0, T=0)=1+A^4/q^4$ is not the correct exact expression for the ZFDF of such a gas.

Therefore, if the two- and one-dimensional $B=0$ charged Bose gases were solved exactly, we would not expect Eqs. (29) and (30) to be the expressions obtained for the potential about a test charge in each of the gases. However, as noted before, the $r\to\infty$ limits of these two equations give the expected Ornstein-Zernicke damping form together with a characteristic oscillatory factor and it therefore seems reasonable to think that Eqs. (35) and (36) may be the correct form of the large-r limits of the true potential about a test charge in each of the gases. That is, we might expect that in the $q\to 0$ limit (which corresponds to large r) the q's in the correct expression for $V_{2,1}(\vec{q})$ will conspire to be of the same form as that given by $4\pi Q/q^2(1+A^4/q^4)$ in the small-q limit. The solutions to the purely two- and one-dimensional Bose gases remain unspecified. It is hoped that the above discussion sheds some light on these problems.

Thus, while we have not been able to produce the general anisotropic screening potential for the three-dimensional CBG in an external magnetic field, certain extreme anisotropic limits of this system have given detailed insight into some effective lower-dimensional charged Bose gases. These gases are, as seen above, natural consequences of the extreme anisotropy in the three-dimensional magnetic CBG system.

VI. DISCUSSION

In this paper we have obtained a result [Eq. (4)] in the RPA for the conductivity tensor of a CBG in a magnetic field. This tensor can be used as a means to study both the longitudinal and transverse responses of the gas at all temperatures. In particular, in this paper we have addressed ourselves to the problem of the zero-temperature longitudinal response of the gas and have used the conductivity tensor of Eq. (4) to obtain a closed-form expression, Eq. (10), for the $T=0$ dielectric function of the gas. This is a new result and has not been given before. We note that, as expected, our dielectric function reduces in the $B=0$ limit to the form given in I where the authors studied, also in the RPA, the field-free CBG.

The dielectric function has then been used to obtain, in the case of a weak magnetic field, the dispersion relation for the gas in various limits.

These limits concern the relative size of the component of the wave vector $\vec{q}$ parallel to the magnetic field to that component perpendicular, and the fact that we are forced to obtain solutions in this way is indicative of the inherent anisotropy of the gas.

This anisotropy is further in evidence in Sec. V of the paper where we obtain the form of the electrostatic potential about a test charge in the gas. While we have not been able to obtain a form for this potential for the general three-dimensional gas, we have been able to obtain insight into the forms of the screened potentials in certain extreme anisotropic limits. In these limits, the three-dimensional system behaves as effective lower-dimensional charged Bose gases.

Finally, we remark that the CBG in a magnetic field has been studied in the ideal Bose gas case by Schafroth,[3] and in the interacting Bose gas case by Fetter.[4] Both these authors used linear-response theory to show the existence of a Meissner-Ochsenfeld effect at zero temperature and slightly above zero. Because the Meissner-Ochsenfeld effect has been exhibited for both interacting and ideal charged Bose gases, in this paper the authors have not concerned themselves with this issue but deal with the situation where the applied magnetic field $\vec{H}$ is strong enough to penetrate the gas and set up a uniform magnetic field $\vec{B}$ inside the gas. This situation has not been investigated before for the CBG, although such studies have been carried out for the electron gas, as mentioned in Sec. I. We have worked exclusively at $T=0$ where the analysis simplifies somewhat because of the complete Bose condensate. However, as was done in I for the $B=0$ CBG, the work may be extended to $T>0$ and as mentioned above, in this paper we have set out the machinery to enable this to be done. The discussion of this problem is left for future study.

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APPENDIX A

We need to evaluate matrix elements of the form $\langle b| e^{i\vec{q}\cdot\vec{x}} | b'\rangle$. We do this as follows,

$$\langle b| e^{i\vec{q}\cdot\vec{x}} | b'\rangle = \langle n, k_x, k_z | e^{i\vec{q}\cdot\vec{x}} | n', k_x', k_z'\rangle = \int d^3x\, \chi^*_{n,k_x,k_z}(\vec{x}) e^{i\vec{q}\cdot\vec{x}} \chi_{n',k_x',k_z'}(\vec{x}).$$

Substituting Eq. (2) yields,

$$\langle b| e^{i\vec{q}\cdot\vec{x}} | b'\rangle = I_x I_y I_z, \tag{A1}$$

where

$$I_x = \int_{-\Omega^{1/3}/2}^{\Omega^{1/3}/2} dx\, e^{i(k_x'-k_x+q_x)x}, \quad I_z = \int_{-\Omega^{1/3}/2}^{\Omega^{1/3}/2} dz\, e^{i(k_z'-k_z+q_z)z},$$

$$\begin{aligned} I_y = {} & \frac{1}{\Omega^{2/3}} \left(\frac{1}{2^n n!\, 2^{n'}(n')!}\right)^{1/2} \left(\frac{m\omega_B}{\pi\hbar}\right)^{1/2} \\ & \times \int_{-\Omega^{1/3}/2}^{\Omega^{1/3}/2} dy\, e^{iq_y y} \exp\left\{-\frac{m\omega_B}{2\hbar}\left[\left(y+\frac{\hbar k_x}{m\alpha_B}\right)^2 + \left(y+\frac{\hbar k_x'}{m\alpha_B}\right)^2\right]\right\} \\ & \times H_n\left[\left(\frac{m\omega_B}{\hbar}\right)^{1/2}\left(y+\frac{\hbar k_x}{m\alpha_B}\right)\right] H_{n'}\left[\left(\frac{m\omega_B}{\hbar}\right)^{1/2}\left(y+\frac{\hbar k_x'}{m\alpha_B}\right)\right]. \end{aligned} \tag{A2}$$

Because we have imposed periodic boundary conditions, I_x and I_z can be simply evaluated to yield,

$$I_x = \Omega^{1/3}\delta_{k_x, k_x'+q_x}, \tag{A3}$$

$$I_z = \Omega^{1/3}\delta_{k_z, k_z'+q_z}, \tag{A4}$$

where $\delta_{\alpha,\beta}$ is the Kronecker delta function. We may use these results to replace k_x' by $k_x - q_x$ in Eq. (A2). Thus,

$$\begin{aligned} I_y = {} & \frac{1}{\Omega^{2/3}} \left(\frac{1}{2^n n!\, 2^{n'}(n')!}\right)^{1/2} \left(\frac{m\omega_B}{\pi\hbar}\right)^{1/2} \\ & \times \int_{-\infty}^{\infty} dy\, e^{iq_y y} \exp\left\{-\frac{m\omega_B}{2\hbar}\left[\left(y+\frac{\hbar k_x}{m\alpha_B}-\frac{\hbar q_x}{m\alpha_B}\right)^2 + \left(y+\frac{\hbar k_x}{m\alpha_B}\right)^2\right]\right\} \\ & \times H_n\left[\left(\frac{m\omega_B}{\hbar}\right)^{1/2}\left(y+\frac{\hbar k_x}{m\alpha_B}\right)\right] H_{n'}\left[\left(\frac{m\omega_B}{\hbar}\right)^{1/2}\left(y+\frac{\hbar k_x}{m\alpha_B}-\frac{\hbar q_x}{m\alpha_B}\right)\right], \end{aligned}$$

where we have allowed $\Omega\to\infty$ in the limits of integration.

If we now make the change of variable,

$$y=t\left(\frac{\hbar}{m\omega_B}\right)^{1/2}-\frac{\hbar}{m\omega_B}[k_x-\tfrac{1}{2}q_x-i\tfrac{1}{2}q_y\,\mathrm{sgn}(e)],$$

we obtain,

$$I_y=\frac{1}{\Omega^{2/3}}\frac{1}{\sqrt{\pi}}\left(\frac{1}{2^n n!\,2^{n'}(n')!}\right)^{1/2}\exp\left(-\frac{\hbar}{m\omega_B}[\tfrac{1}{4}(q_x^2+q_y^2)+i(q_yk_x-\tfrac{1}{2}q_xq_y)\,\mathrm{sgn}(e)]\right)$$
$$\times\int_{-\infty}^{\infty}dt\,e^{-t^2}H_n\left[t+\tfrac{1}{2}\left(\frac{\hbar}{m\omega_B}\right)^{1/2}[q_x\,\mathrm{sgn}(e)+iq_y]\right]H_{n'}\left[t-\frac{1}{2}\left(\frac{\hbar}{m\omega_B}\right)^{1/2}[q_x\,\mathrm{sgn}(e)-iq_y]\right]. \tag{A5}$$

The integral in Eq. (A2) may be done[13] to give the following result for I_y:

$$I_y=\frac{1}{\Omega^{2/3}}\,{}^nF_{n'}\exp\left(-\frac{\hbar}{m\omega_B}[\tfrac{1}{4}(q_x^2+q_y^2)+i(q_yk_x-\tfrac{1}{2}q_xq_y)\,\mathrm{sgn}(e)]\right), \tag{A6}$$

where

$${}^nF_{n'}=\left(\frac{2^{n'}(n')!}{2^n n!}\right)^{1/2}\left(\frac{\hbar}{m\omega_B}\right)^{(n-n')/2}[q_x\,\mathrm{sgn}(e)+iq_y]^{n-n'}L_{n'}^{n-n'}\left(\frac{\hbar}{2m\omega_B}(q_x^2+q_y^2)\right)\quad\text{if } n'\le n,$$
$$=\left(\frac{2^n n!}{2^{n'}(n')!}\right)^{1/2}\left(\frac{\hbar}{m\omega_B}\right)^{(n'-n)/2}[-q_x\,\mathrm{sgn}(e)+iq_y]^{n'-n}L_{n'}^{n-n'}\left(\frac{\hbar}{2m\omega_B}(q_x^2+q_y^2)\right)\quad\text{if } n'\ge n,$$

and $L_\alpha^\beta(x)$ is a Laguerre polynomial.[13]

We note at this stage that a very useful recurrence relation,

$$\left(\frac{n'+1}{2}\right)^{1/2}[q_x\,\mathrm{sgn}(e)+iq_y]\,{}^nF_{n'+1}+\left(\frac{n'}{2}\right)^{1/2}[q_x\,\mathrm{sgn}(e)-iq_y]\,{}^nF_{n'-1}=\left(\frac{m}{\hbar\omega_B}\right)^{1/2}(n-n')\omega_B-\frac{\hbar}{2m}(q_x^2+q_y^2)\,{}^nF_{n'} \tag{A7}$$

may be derived from the two recurrent relations for the Laguerre polynomials,[13] namely; $xL_n^{\alpha+1}(x)=(n+\alpha+1)L_n^\alpha(x)-(n+1)L_{n+1}^\alpha(x)$ and $L_n^{\alpha-1}(x)=L_n^\alpha(x)-L_{n-1}^\alpha(x)$.

Substituting Eqs. (A3), (A4), and (A6) into Eq. (A1) gives the required matrix element:

$$\langle b|e^{i\vec{q}\cdot\vec{x}}|b'\rangle=\langle n,k_x,k_z|e^{i\vec{q}\cdot\vec{x}}|n',k_x',k_z'\rangle$$
$$={}^nF_{n'}\exp\left(-\frac{\hbar}{m\omega_B}[\tfrac{1}{4}(q_x^2+q_y^2)+i(q_yk_x-\tfrac{1}{2}q_xq_y)\,\mathrm{sgn}(e)]\right)\delta_{k_x,k_x'+q_x}\delta_{k_z,k_z'+q_z}. \tag{A8}$$

We also need to evaluate matrix elements of the form $\langle b|e^{i\vec{q}\cdot\vec{x}}\vec{v}|b'\rangle$. We do this by breaking up $\vec{v}$ into its components. Now,

$$\langle b|e^{i\vec{q}\cdot\vec{x}}v_x|b'\rangle=\langle b|e^{i\vec{q}\cdot\vec{x}}\frac{1}{m}\left(p_x-\frac{e}{c}A_x\right)|b'\rangle$$
$$=\int d^3x\,\chi_b^*(\vec{x})e^{i\vec{q}\cdot\vec{x}}\left(\frac{\hbar}{im}\frac{\partial}{\partial x}+y\alpha_B\right)\chi_{b'}(\vec{x})$$
$$=\int d^3x\,\chi_b^*(\vec{x})e^{i\vec{q}\cdot\vec{x}}\left(y\alpha_B+\frac{\hbar k_x'}{m}\right)\chi_{b'}(\vec{x}).$$

By using the same change of variable as for $\langle b|e^{i\vec{q}\cdot\vec{x}}|b'\rangle$ together with the recurrence relation for Hermite polynomials,[13] $xH_n(x)=\tfrac{1}{2}H_{n+1}(x)+nH_{n-1}(x)$, the above integral can be evaluated to give,

$$\langle b|e^{i\vec{q}\cdot\vec{x}}v_x|b'\rangle=\langle n,k_x,k_z|e^{i\vec{q}\cdot\vec{x}}v_x|n',k_x',k_z'\rangle$$
$$=\mathrm{sgn}(e)\left(\frac{\hbar\omega_B}{m}\right)^{1/2}\left[\left(\frac{n'+1}{2}\right)^{1/2}{}^nF_{n'+1}+\left(\frac{n'}{2}\right)^{1/2}{}^nF_{n'-1}\right]$$
$$\times\exp\left(-\frac{\hbar}{m\omega_B}[\tfrac{1}{4}(q_x^2+q_y^2)+i(q_yk_x-\tfrac{1}{2}q_xq_y)\,\mathrm{sgn}(e)]\right)\delta_{k_x,k_x'+q_x}\delta_{k_z,k_z'+q_z},$$

$$\langle b|e^{i\vec{q}\cdot\vec{x}}v_y|b'\rangle=\langle b|e^{i\vec{q}\cdot\vec{x}}\frac{1}{m}\left(p_y-\frac{e}{c}A_y\right)|b'\rangle$$
$$=\int d^3x\,\chi_b^*(\vec{x})e^{i\vec{q}\cdot\vec{x}}\frac{\hbar}{im}\frac{\partial}{\partial y}\chi_{b'}(\vec{x}).$$

This integral can be done as for $\langle b| e^{i\vec{q}\cdot\vec{x}}| b'\rangle$ with the aid of another recurrence relation for Hermite polynomials,[13] namely, $dH_n/dx = 2nH_{n-1}(x)$. We obtain

$$\langle b| e^{i\vec{q}\cdot\vec{x}} v_y | b'\rangle = \langle n, k_x, k_z | e^{i\vec{q}\cdot\vec{x}} v_y | n', k_x', k_z'\rangle$$

$$= i\left(\frac{\hbar\omega_B}{m}\right)^{1/2}\left[\left(\frac{n'+1}{2}\right)^{1/2} {}^nF_{n'+1} - \left(\frac{n'}{2}\right)^{1/2} {}^nF_{n'-1}\right]$$

$$\times \exp\left(-\frac{\hbar}{m\omega_B}\left[\tfrac{1}{4}(q_x^2+q_y^2)+i(q_yk_x-\tfrac{1}{2}q_xq_y)\,\mathrm{sgn}(e)\right]\right)\delta_{k_x,k_x'+q_x}\,\delta_{k_z,k_z'+q_z},$$

$$\langle b| e^{i\vec{q}\cdot\vec{x}} v_z | b'\rangle = \langle b| e^{i\vec{q}\cdot\vec{x}}\frac{1}{m}\left(p_z - \frac{e}{c}A_z\right)| b'\rangle$$

$$= \int d^3x\,\chi_b^*(\vec{x})e^{i\vec{q}\cdot\vec{x}}\frac{\hbar}{im}\frac{\partial}{\partial z}\chi_{b'}(\vec{x})$$

$$= \frac{\hbar k_z'}{m}\int d^3x\,\chi_b^*(\vec{x})e^{i\vec{q}\cdot\vec{x}}\chi_{b'}(\vec{x})$$

$$= \frac{\hbar}{m}(k_z-q_z)\,{}^nF_{n'}\exp\left(-\frac{\hbar}{m\omega_B}\left[\tfrac{1}{4}(q_x^2+q_y^2)+i(q_yk_x-\tfrac{1}{2}q_xq_y)\,\mathrm{sgn}(e)\right]\right)\delta_{k_x,k_x'+q_x}\,\delta_{k_z,k_z'+q_z}.$$

We write the result for $\langle b| e^{i\vec{q}\cdot\vec{x}}\vec{v}| b'\rangle$ in matrix form,

$$\langle n, k_x, k_z | e^{i\vec{q}\cdot\vec{x}}\vec{v}| n', k_x', k_z'\rangle = \begin{bmatrix} \mathrm{sgn}(e)\left(\frac{\hbar\omega_B}{m}\right)^{1/2}\left[\left(\frac{n'+1}{2}\right)^{1/2} {}^nF_{n'+1} + \left(\frac{n'}{2}\right)^{1/2} {}^nF_{n'-1}\right] \\ i\left(\frac{\hbar\omega_B}{m}\right)^{1/2}\left[\left(\frac{n'+1}{2}\right)^{1/2} {}^nF_{n'+1} - \left(\frac{n'}{2}\right)^{1/2} {}^nF_{n'-1}\right] \\ \frac{\hbar}{m}(k_z-q_z)\,{}^nF_{n'} \end{bmatrix}$$

$$\times \exp\left(-\frac{\hbar}{m\omega_B}\left[\tfrac{1}{4}(q_x^2+q_y^2)+i(q_yk_x-\tfrac{1}{2}q_yq_x)\,\mathrm{sgn}(e)\right]\right)\delta_{k_x,k_x'+q_x}\,\delta_{k_z,k_z'+q_z}. \tag{A9}$$

The final type of matrix element we require is of the form $\langle b| \vec{v}e^{i\vec{q}\cdot\vec{x}}| b'\rangle$. Using the product rule for derivatives we readily obtain,

$$\langle b| \vec{v}e^{i\vec{q}\cdot\vec{x}}| b'\rangle = \frac{\hbar\vec{q}}{m}\langle b| e^{i\vec{q}\cdot\vec{x}}| b'\rangle + \langle b| e^{i\vec{q}\cdot\vec{x}}\vec{v}| b'\rangle. \tag{A10}$$

APPENDIX B

In Eq. (10) we have forms like

$$e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{1}{a+bl}.$$

Now,

$$e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{1}{a+bl} = e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\int_0^{\infty}e^{-(a+bl)t}\,dt,\qquad a>0,\ b>0$$

$$= e^{-x}\int_0^{\infty}e^{-at}\sum_{l=0}^{\infty}\frac{x^l}{l!}(e^{-bt})^l\,dt$$

$$= e^{-x}\int_0^{\infty}e^{-at}\exp(xe^{-bt})\,dt.$$

Making the change of variable $s = 1-e^{-bt}$ yields

$$e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{1}{a+bl} = \frac{1}{b}\int_0^1 e^{-xs}(1-s)^{a/b-1}\,ds$$

$$= \frac{1}{b}\sum_{p=0}^{\infty}\frac{(-x)^p}{p!}\int_0^1 s^p(1-s)^{a/b-1}\,ds$$

$$= \frac{1}{b}\sum_{p=0}^{\infty}\frac{(-x)^p}{p!}B(p+1, a/b),$$

where $B(x,y)$ is the beta function.[13] Using $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ where $\Gamma(x)$ is the gamma function,[13] and also $\Gamma(x+1) = x\Gamma(x)$ we find

$$e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{1}{a+bl} = \sum_{p=0}^{\infty}\frac{(-bx)^p}{\prod_{n=0}^{p}(a+nb)}. \tag{B1}$$

The right-hand side of Eq. (B1) is in quite a useful form but we will find it useful to add and subtract a term to give,

$$e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{1}{a+bl}$$

$$=\frac{1}{a+bx}+\sum_{p=0}^{\infty}(-bx)^p\left(\frac{1}{\prod_{n=0}^{p}(a+nb)}-\frac{1}{a^{p+1}}\right). \tag{B2}$$

This result has been derived for $a>0$, $b>0$ but we may analytically continue into the region of convergence of the right hand side of Eq. (B2), which is all a and all b. So Eq. (B2) is valid for all a and b.

In Eq. (10) we also have forms like

$$e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{l}{a+bl}.$$

It is a simple matter to show,

$$e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{l}{a+bl}=\frac{1}{b}-\frac{a}{b}e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{1}{a+bl}. \tag{B3}$$

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