Scattering of spin waves by pores in ferrimagnetic materials

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The differential cross section of the spherical pores in ferrimagnetic materials is calculated by using the Born approximation in scattering theory. Using this scattering cross section, the theory is developed for evaluating the wave-vector dependence of the spin-wave linewidth. The wave-vector dependence of the spin-wave linewidth is $1/k$ in the long-wavelength region and is proportional to k for the short-wavelength region, where k is the wave vector of the spin waves. The present theory indicates that the spin-wave linewidth is a function of the number of pores and their radii. Agreement between the formula and the experimental results is good except in the region very near $k \approx 0$.

I. INTRODUCTION

The influences of grain size and porosity on the effective linewidth and spin-wave linewidth in ferrimagnetic materials are well known. The influence of grain size on the spin-wave linewidth can be qualitatively explained by a model, the grain-size transit-time theory, proposed by Vrehen *et al.*[1] This description, however, is not satisfactory for the porous polycrystalline materials. Scotter[2] has recently observed the effect of inclusions and porosity on spin-wave linewidth. In order to give a simple explanation of the experimental facts, Scotter proposed a more refined theory in which he suggested that a nonmagnetic inclusion or pore rather than grain boundaries limits the mean free path of a spin wave. Hereafter we call this theory a spin-wave transit-time theory.[3]

In this theory he assumed that the expression for the total cross section has the form

$$\sigma = \pi R^2 , \qquad (1)$$

where R is the radius of a spherical nonmagnetic inclusion or pore. However, it is clear that the total cross section is expected to be a more complicated function of radius, frequency, saturation magnetization, dc magnetic field, and exchange constant. The main purpose of Sec. II is to obtain the differential cross section of the spin wave. In Sec. III the results of Sec. II are extended to the evaluation of the relaxation time and the linewidth of the spin wave in polycrystalline materials.

II. THEORY

We begin by considering a ferromagnetic region that is uniformly magnetized to saturation in the z direction by a dc magnetic field, in which the polar angles of the direction of the wave vector $\vec{k}_0$ are θ_0 and ϕ_0. The incident wave is directed toward the scattering region whose properties differ from the surrounding medium. In this case, the incident wave is a spin wave; the scattering region is a spherical cavity. In order to derive the differential cross section of a spherical cavity, it is first necessary to convert the equation of motion into a Schrödinger-type equation of motion for a transverse rf magnetization. The deviation of the magnetization vector from static equilibrium is denoted by $\vec{m}(r,t)$ and it satisfied the linearized equation of motion given by

$$\frac{\partial \vec{m}}{\partial t} = \gamma \vec{m} \times (\vec{H} + D\nabla^2 \vec{m} + \vec{H}_{\text{dipole}}) , \qquad (2)$$

where γ is the gyromagnetic ratio (negative), $\vec{H}$ the total static magnetic field, H_{dipole} the magnetostatic field outside the spherical cavity, and D the exchange constant. From the boundary conditions at the surface of the pore, the magnetostatic potential Φ is given by[4]

$$\Phi = -\frac{4\pi R^3}{3}\left[\frac{1}{2}\left(m^+ e^{j\phi} + m^- e^{-j\phi}\,\frac{\sin\theta}{r^2}\right) + \frac{M_s}{r^2}\cos\theta\right] , \qquad (3)$$

where r is the length of the vector $\vec{r}$ having spherical angles θ and ϕ, $m^{\pm} = m_x \pm j m_y$, R is radius of the spherical cavity, and M_s the saturation magnetization. The magnetic dipole field is given by

$$\vec{H}_{\text{dipole}} = -\nabla\Phi . \qquad (4)$$

We treat the problem of scattering of the spin waves by a pore by adopting the fast-passage approximation. When the spin waves go across a pore region with velocity v_g, the transit time is $l/v_g = \tau_g$, where l is the effective length of the pore region. If τ_g is much longer than the relaxation time τ_k of the spin waves, spin-wave energy will be transferred into the pores. For, $\tau_g \ll \tau_k$, there is no energy transfer from the spin waves into the pore, and so the m^+ and m^- terms in the potential Φ can be neglected. For $v_g = 10^4$ cm/sec and $l = 10^{-6}$ cm, we find $\tau_g = 10^{-10}$ sec and $\tau_k = 10^{-8}$ sec, i.e., for $\Delta H_k = 0.5$ Oe. Now we will limit the analysis to the case $\tau_g << \tau_k$. Eq. (3) can be rewritten as

$$\Phi(r,\theta) = -\tfrac{4}{3}\pi R^3 (M_s/r^2)\cos\theta . \quad (5)$$

With this approximation, the equation of motion may be expressed in the form

$$\left[\gamma D\nabla^2 + \omega - \omega_0 - \omega_m\beta\left(\frac{1-3\cos^2\theta}{r^3}\right)\right]\alpha^+ = \omega_m\beta\,\frac{\cos\theta\sin\theta e^{j\phi}}{r^3}, \quad (6)$$

where

$$\alpha^+ = m^+/M_s,\quad \omega_0 = \gamma H,\quad \omega_m = 4\pi\gamma M_s,\quad \beta = \tfrac{4}{3}\pi R^3 .$$

In order to obtain the scattering cross section we shall consider the inhomogeneous equation

$$[L_{op} + \epsilon g(r,\theta,\phi)]\alpha^+ = F(r,\theta,\phi), \quad (7)$$

where L_{op} is an operator expressed as $L_{op} = \gamma D\nabla^2 + \omega - \omega_0$. The total wave function α^+ may be decomposed into a zeroth-order wave and a first-order wave

$$\alpha^+ = \alpha_0^+ + \epsilon\alpha_1^+ ,$$

where ϵ is the covergence factor. Equation (7) then becomes

$$[L_{op} + \epsilon g(r,\theta,\phi)](\alpha_0^+ + \epsilon\alpha_1^+) = F(r,\theta,\phi). \quad (8)$$

Equating the coefficients of equal powers of ϵ we obtain

$$L_{op}\alpha_0^+ = F(\vec{r}). \quad (9)$$

A particular solution of Eq. (9) can be expressed in the form

$$\alpha_0^+(\vec{r}) = \int G^+(\vec{r},\vec{r}\,')F(\vec{r}\,')\,d\vec{r}\,', \quad (10)$$

where the Green's function $G^+(\vec{r},\vec{r}\,')$ satisfies

$$L_{op}G^+(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,') \quad (11)$$

and the solution takes the form, within the infinite medium,

$$G^+(\vec{r},\vec{r}\,') = (1/4\pi\gamma D)(e^{jk|\vec{r}-\vec{r}\,'|}/|\vec{r}-\vec{r}\,'|). \quad (12)$$

With the help of this Green's function, we may write the solution of the wave equation (9) in the form

$$\alpha_0^+(\vec{r}) = e^{j\vec{k}_0\cdot\vec{r}} + \int G^+(\vec{r},\vec{r}\,')F(\vec{r}\,')\,d\vec{r}\,', \quad (13)$$

where the first term in Eq. (13) represents the incident wave normalized to one magnon per unit volume. The first-order solution α_1^+ is given by

$$\alpha_1^+ = -\int G^+(\vec{r},\vec{r}\,')g(\vec{r}\,')\alpha_0^+\,d\vec{r}\,'. \quad (14)$$

By substituting Eq. (13) into Eq. (14), we have

$$\alpha_1^+ = -\int G^+(\vec{r},\vec{r}\,')g(\vec{r}\,')e^{j\vec{k}_0\cdot\vec{r}\,''}\,d\vec{r}\,'' - \int G^+(\vec{r},\vec{r}\,'')g(\vec{r}\,'')G^+(\vec{r}\,'',\vec{r}\,')F(\vec{r}\,')\,d\vec{r}\,'\,d\vec{r}\,''. \quad (15)$$

The second term of Eq. (15) is of second order and is negligibly small, while comparison of Eq. (6) with Eq. (7) shows that

$$g(\vec{r}) = -\omega_m\beta(1-3\cos^2\theta)/r^3, \quad (16)$$

$$F(\vec{r}) = \omega_m\beta(\cos\theta\sin\theta/r^3)e^{j\phi}. \quad (17)$$

If $\vec{n}$ is a unit vector pointing in the direction of scattering given by $\vec{k}$, the Green's function takes the form

$$G^+(\vec{r},\vec{r}\,') = (1/4\pi\gamma D)(e^{jkr}/r)e^{-jk\vec{n}\cdot\vec{r}\,'}, \quad (18)$$

where we have used the approximate relation

$$|\vec{r}\,' - \vec{r}| = r - \vec{n}\cdot\vec{r}\,' + O(r'/r). \quad (19)$$

Equation (13) is

$$\alpha_0^+ = e^{j\vec{k}_0\cdot\vec{r}} + \frac{\omega_m\beta}{4\pi\gamma D}\frac{e^{jkr}}{r} \times \int e^{-jk\vec{n}\cdot\vec{r}\,'}\frac{\cos\theta'\sin\theta'}{r'^3}e^{j\phi}\,dr'. \quad (20)$$

Equation (15) is

$$\alpha_1^+ = \frac{\omega_m\beta}{4\pi\gamma D}\frac{e^{jkr}}{r}\int e^{-jk\vec{n}\cdot\vec{r}\,''}\frac{1-3\cos^2\theta''}{r''^3} \times e^{jk\vec{n}_0\cdot\vec{r}\,''}\,d\vec{r}\,'' + O\left(\left(\frac{\omega_m\beta}{4\pi\gamma D}\right)^2\right). \quad (21)$$

The relationship of the associated vectors $\vec{r}\,'$ and $\vec{n}$ to $\vec{n}_0$ is indicated in Fig. 1. The integral in Eqs. (20) and (21) can be calculated by using the coordinate system shown in Fig. 1. In order to perform the integration in Eq. (20), we expand $e^{-jk\vec{n}\cdot\vec{r}\,'}$ in Legendre polynomials,

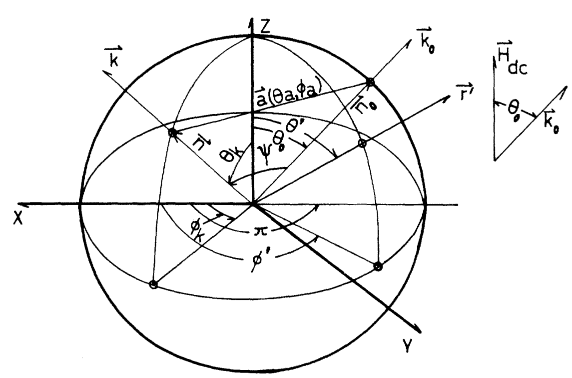

FIG. 1. Coordinate system as used in calculations pertaining to Eqs. (20) and (21).

$$e^{-j k \vec{n} \cdot \vec{r}'} = \sum_{n=0}^{\infty} (2n+1)(-j)^n \times \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos m(\phi' - \phi) P_n^m(\cos\theta') \times P_n^m(\cos\theta_k) j_n(kr'), \quad (22)$$

where ϵ_n is the Neuman factor $\epsilon_0 = 1$, $\epsilon_n = 2(n = 1, 2, \ldots)$. The integration in Eq. (21) is also easily done using this expansion. We then have

$$\alpha_0^+ = e^{j\vec{k}_0 \cdot \vec{r}} + (e^{jkr}/r) f_0(\theta_k), \quad (23)$$

where

$$f_0(\theta_k) = -(\omega_m \beta / 4\pi\gamma D) \tfrac{4}{9} \pi P_2^1(\cos\theta_k) e^{j\phi_k}. \quad (24)$$

The solution of the Eq. (10) is

$$\alpha_1^+ = (e^{jkr}/r) f_1(\theta_a) + O((\omega_m \beta / 4\pi\gamma D)^2), \quad (25)$$

where

$$f_1(\theta_a) = \tfrac{4}{3} \pi (\omega_m \beta / 4\pi\gamma D) P_2^0(\cos\theta_a), \quad (26)$$

where $\cos\theta_a$ is the directional cosine of the vector $\vec{a}$ and $\vec{a} = \vec{n} - \vec{n}_0$. It follows from the geometry of the situation that θ_a is related to θ_k and θ_0 by

$$\cos\theta_a = (\cos\theta_k - \cos\theta_0)/a, \quad (27)$$

where a is given by

$$|\vec{a}| = a = [2(1 - \cos\psi)]^{1/2} \quad (28)$$

and

$$\cos\psi = -\sin\theta_K \sin\theta_0 \cos\phi_K + \cos\theta_K \cos\theta_0. \quad (29)$$

In Eqs. (24) and (26), the functions P_2^0 and P_2^1 are Legendre functions depending only on angles

$$P_2^0(\cos\theta_a) = \tfrac{1}{2}(3\cos^2\theta_a - 1), \quad (30)$$

$$P_2^1(\cos\theta_K) = 3\cos\theta_K \sin\theta_K. \quad (31)$$

In the calculation of the integrals Eqs. (20) and (21) we have used the formula

$$\int_0^\infty \frac{j_2(x)}{x} dx = \tfrac{1}{3}. \quad (32)$$

III. RELAXATION TIME

In this section we shall consider the problem from the viewpoint that spin waves behave like particles (magnons). Scotter[3] proposed a phenomenological theory of the spin-wave transit time in which nonmagnetic inclusions in polycrystalline yttrium iron garnet (YIG) limit the mean-free-path length of spin-waves.

Our theory of the relaxation times follows along the general lines suggested by Scotter. The nonmagnetic inclusions are assumed to have a total scattering cross section $\sigma^T(\theta_0)$, where θ_0 is the polar angle of an incident spin-wave, and to be arranged with separation s on a cubic lattice. The probability P that a spin wave will hit N pores within a distance s is given by

$$P = N\sigma(\theta_0)/s^2. \quad (33)$$

Hence the mean free path l is defined by

$$l = s/P = s^3/N\sigma^T(\theta_0) = \beta/p\sigma^T(\theta_0), \quad (34)$$

where

$$p = N\beta/s^3 \quad (35)$$

is the porosity. The present theory is based on the most plausible assumption that the relaxation times of the spin waves are expressed by averaging the transit time over all the incident angles of the spin waves. Using Eq. (34), the spin-wave linewidth ΔH_K is given by[5] (see Appendix)

$$\tau_k = \frac{1}{\gamma \Delta H_k} = \frac{1}{4\pi} \int \frac{l}{v_g(\theta_0)} d\Omega = \frac{\beta}{4\pi p} \int \frac{1}{v_g(\theta_0)\sigma^T(\theta_0)} d\Omega. \quad (36)$$

From the definition of cross section, it follows that

$$\sigma(\theta_k, \phi_k, \theta_0) = |f(\theta_k, \phi_k, \theta_0)|^2, \quad (37)$$

where

$$f(\theta_k, \phi_k, \theta_0) = f_0(\theta_k, \phi_k, \theta_0) + f_1(\theta_k, \phi_k, \theta_0) \quad (38a)$$

and

$$v_g^2(\theta_0) = (2\gamma Dk)^2 + (\omega_m \sin\theta_0 \cos\theta_0 / k)^2. \quad (38b)$$

The total cross section is given by

$$\sigma^T(\theta_0) = \int |f(\theta_k, \phi_k, \theta_0)|^2 d\Omega_k. \quad (39)$$

From Eqs. (24) and (26), we have

$$\sigma^T = (\omega_m \beta / 4\pi\gamma D)^2 \pi^3 \times 2.726. \quad (40)$$

It is interesting to note that the total cross section is independent of the angle of incidence θ_0. The θ_0 independence of the total cross section is not surprising. The first-order solution α_1^+ in Eq. (21) contains terms of first and second order in the factor of $\omega_m \beta / 4\pi\gamma D$. The θ_0 dependence of the total cross section arises in second order in the factor of $\omega_m \beta / 4\pi\gamma D$. However, in the pore scattering the first order perturbation term $O(\omega_m \beta / 4\pi\gamma D)$ reduces to a constant when integrated over all solid angles, and the θ_0 dependence thereby disappears. As a numerical example we consider YIG at room temperature. Using $D = 5 \times 10^{-9}$ Oe cm^2 and $\omega_m = 3 \times 10^{10}$ sec^{-1}, finally we have

$$\Delta H_k = p\beta 1.075 \times 10^{26} \frac{(1 + 1.37 \times 10^{-22} k^4)^{1/4}}{k F(\sin^{-1}\eta, K)}, \quad (41)$$

where the $F(\sin^{-1}\eta, K)$ is the elliptic integral and is defined by

$$F(\sin^{-1}\eta, K) = \int_0^{\sin^{-1}\eta} \frac{d\phi}{(1 - K^2\sin^2\phi)^{1/2}}, \tag{42}$$

where

$$K^2 = \frac{1 + [1 + (4\gamma D/\omega_m)^2 k^4]^{1/2}}{2[1 + (4\gamma D/\omega_m)^2 k^4]^{1/2}} = \frac{1 + (1 + 1.37\times 10^{-22} k^4)^{1/2}}{2(1 + 1.37\times 10^{-22} k^4)^{1/2}} \tag{43}$$

and

$$\eta = \frac{2[1 + (4\gamma D/\omega_m)^2 k^4]^{1/4}}{1 + [1 + (4\gamma D/\omega_m)^2 k^4]^{1/2}} = \frac{2(1 + 1.37\times 10^{-22} k^4)^{1/4}}{1 + (1 + 1.37\times 10^{-22} k^4)^{1/2}}. \tag{44}$$

For pores of radius $R = 5.4\times 10^{-7}$ cm, we get for sufficiently large k

$$\Delta H_k = 4\times 10^{-6} k \text{ (Oe)} \quad \text{for } k \gg 10^5 \text{ cm}^{-1}, \tag{45}$$

where we have used $N = 1.86\times 10^{16}$ per unit volume (1 cm^3) for the number of pores. The porosity p is defined as

$$p = N\beta/V, \tag{46}$$

where $\beta = \frac{4}{3}\pi R^3$. In this estimation we have $p = 1\%$. The curve for the spin-wave linewidth versus wave vector is shown in Fig. 2 together with the experimental results.[6] After comparing their experimental results with this theoretically derived linewidth ΔH_K we conclude that the relation between the group velocity and the relaxation time of the spin wave is probably the most important relation although it does not completely explain the data particularly in the low-k region. The wave-vector dependence of the spin-wave linewidth is $1/k$ in the long-wavelength region and is proportional to k to the short-wavelength region. These wave-vector dependences arise from the k dependence of the group velocity. The relaxation time is obtained by dividing the mean free path by the group velocity of the spin waves. The group velocity is proportional to k in the short-wavelength region and varies as $1/k$ in the long-wavelength region, while the mean free path is independent of k.

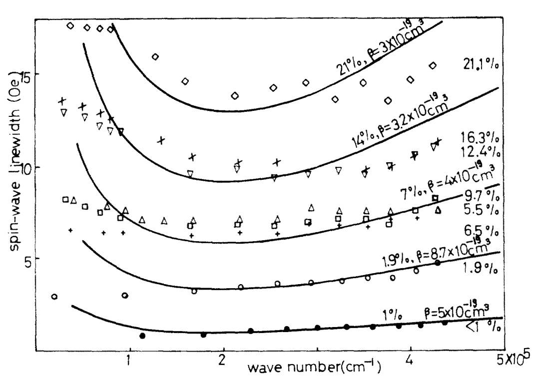

FIG. 2. Experimental values (Ref. 3) for the ΔH_K are shown for comparison. The plotted theoretical curve is incorrect very near $k \approx 0$.

It has been generally believed that the linewidth is linearly proportional to the porosity; in fact, the physical mechanism is far from that simple and as shown in the present theory the spin-wave linewidth is a function of both the number of pores and their radii. Scotter[2] has suggested that the effect of porosity on the spin-wave linewidth is influenced by the size of the pores present in polycrystalline YIG and is not simply related to porosity alone, and that a more complicated procedure is necessary for the experimental investigation, which permits changes of pore size at constant porosity. The present analysis serves to support that contention. That is to say, it is impossible to form a physical picture of the spin-wave linewidth without a theoretical or experimental investigation of the number of pores and the reasonable value of their radii.

APPENDIX: SPIN-WAVE LINEWIDTH AND RELAXATION TIME (REF. 5)

The relaxation time of waves can be defined as as integral over the time-dependent decay function $f(t)e^{j\omega_0 t}$

$$f(0)\tau = \int f(t)\, dt. \tag{A1}$$

The linewidth of this decay is given by

$$F(\omega_0)\Delta\omega = \int F(\omega)\, d\omega, \tag{A2}$$

where $F(\omega)$ and $f(t)$ are related to each other by Fourier transforms, and therefore

$$f(t)e^{j\omega_0 t} = \frac{1}{2\pi}\int F(\omega)e^{j\omega t}\, d\omega \tag{A3}$$

and

$$F(\omega) = \int f(t)e^{+j\omega_0 t}e^{-j\omega t}\, dt. \tag{A4}$$

Let us assume the existence of N identical time-varying processes. The measured quantities, such as the linewidth or the relaxation time, are equal to the sum of identically and independently existing phenomena started at time $t = 0$. Thus the over-all measured observables $f_T(t)$ should be given by

$$f_T(t)=\sum_{i=1}^{N} f_i(t). \tag{A5}$$

Defining the relaxation time τ_i ($i=1,2,3,\ldots,N$) by

$$\tau_i=\left(\int f_i(t)\,dt\right)\Big/f_i(0), \tag{A6}$$

from the requirement of N-identical time-varying processes it follows that

$$f_1(0)=f_2(0)=f_3(0)=\cdots=f_N(0). \tag{A7}$$

From Eqs. (A5)–(A7), we have the over-all relaxation time

$$\tau_T=\left(\int f_T(t)\,dt\right)\Big/f_T(0)$$

$$=\left(\sum_{i=1}^{N}\int f_i(t)\,dt\right)\Big/Nf_1(0)=\langle\tau\rangle. \tag{A8}$$

The τ_T is nothing but the average relaxation time of N identical phenomena started at time $t=0$. Using this relaxation time, the linewidth $\Delta\omega_T$ should be given by

$$\Delta\omega_T=\left(\sum_{i=1}^{N}\int F_i(\omega)\,d\omega\right)\Big/\sum_{i=1}^{N} F_i(\omega_0)$$

$$=\left(\sum_{i=1}^{N} 2\pi f_i(0)\right)\Big/\sum_{i=1}^{N}\int f_i(t)\,dt=\frac{2\pi}{\langle\tau\rangle}. \tag{A9}$$

We thus come to the following conclusion. The over-all linewidth $\Delta\omega_T$ is proportional to the inverse of the average value of the relaxation time τ.

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