New rigorous inequality for critical exponents in the Ising model

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Defining the correlation length by the ϕth moment of the magnetization-magnetization correlation, we give a simple proof of the inequality $\phi\nu(\phi) > 2\Delta_4 - \gamma$ ($\phi > d$) for the ferromagnetic Ising model in d dimensions. We briefly discuss the relevance of this inequality for the constructive ϕ_d^4 approach.

The Josephson inequality[1] $d\nu \geq 2 - \alpha$ is an important inequality in the study of critical phenomena. Although generally believed, its proof is based on assumptions seemingly difficult to motivate.[2] In this note we present a simple proof of a new inequality which will lead to an inequality closely related to Josephson's inequality. More precisely let $\langle ij\rangle \geq 0$ ($i,j \in Z^d$) be the translation-invariant magnetization-magnetization correlation for the ferromagnetic Ising model in d dimensions above the critical temperature such that $\chi = \sum_i \langle 0i\rangle$ is the susceptibility. For any $\phi > 0$ let[2]

$$\xi(\phi) = \left(\frac{\sum_i |i|^\phi \langle 0i\rangle}{\sum_i \langle 0i\rangle}\right)^{1/\phi} \tag{1}$$

be the correlation length and let the critical exponent $\nu(\phi)$ be defined by $\xi(\phi) \sim (\Delta T)^{-\nu(\phi)}$ ($\Delta T = T - T_c$ small). By Hölder's inequality $\xi(\phi) \geq \xi(\phi')$ and $\nu(\phi) \geq \nu(\phi')$ for $\phi \geq \phi'$.

We will prove

$$\phi\nu(\phi) \geq 2\Delta_4 - \gamma \tag{2}$$

for any $\phi > d$. Here γ is the critical exponent for the susceptibility and Δ_4 is a gap exponent.[3] We note that for $\phi = d$ Eq. (2) is an equality if Kadanoff's[4] scaling hypothesis holds. It is also an equality for $d = \phi = 4$ if the critical exponents take their mean-field value. Assuming equality of the gap exponents, the inequality

$$\phi\nu(\phi) \geq (2-\alpha)\frac{(\delta+1)(\delta_s - 1)}{(\delta-1)(\delta_s+1)} \geq 2 - \alpha \tag{3}$$

for any $\phi > d$ is then an immediate consequence of the inequality $-\gamma\delta + \Delta(\delta - 1) \geq 0$ due to Gaunt and Baker[5] and the inequality $\gamma(\delta_s + 1) \geq (2-\alpha)(\delta_s - 1)$ due to Griffiths,[6] who also showed that $\delta \leq \delta_s$ with equality if $\alpha \geq \alpha'$. Hence we may consider (3) to be a weak form of the Josephson inequality.

Now the proof of (2) is quite simple and is based on the Lebowitz inequality[7] for the fourth-order Ursell function

$$\begin{aligned} 0 &\leq -u_4(i,j,k,l) \\ &= -\langle ijkl\rangle + \langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle jk\rangle \end{aligned}$$

for zero external field.

By Griffiths's second inequality[8] we obtain

$$0 \leq -u_4(i,j,k,l) \leq \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle jk\rangle$$

and similar inequalities by cyclic permutation. This gives[7,9]

$$\begin{aligned} -\tilde{u}_4 &= -\sum_{j,k,l} u_4(0,j,k,l) \\ &\leq \sum_{p=1}^{8} \sum_{j,k,l} (A_p)^{1/3}. \end{aligned} \tag{4}$$

Here each A_p is either

$$\langle 0j\rangle\langle 0k\rangle\langle 0l\rangle\langle jk\rangle\langle jl\rangle\langle kl\rangle$$

or else a permutation of

$$\langle 0j\rangle^2\langle kl\rangle^2\langle 0k\rangle\langle jl\rangle .$$

Let $\phi > d$, then

$$\begin{aligned} &\sum_{j,k,l} (\langle 0j\rangle\langle 0k\rangle\langle 0l\rangle\langle jk\rangle\langle jl\rangle\langle kl\rangle)^{1/3} \\ &\quad = \sum_{j,k,l} [(1+|j|)^\phi(1+|l|)^{-\phi}\langle 0j\rangle\langle lk\rangle]^{1/3} \\ &\qquad \times [(1+|l|)^\phi(1+|k|)^{-\phi}\langle 0l\rangle\langle jk\rangle]^{1/3} \\ &\qquad \times [(1+|k|)^\phi(1+|j|)^{-\phi}\langle 0k\rangle\langle jl\rangle]^{1/3} \\ &\quad \leq [1+\xi(\phi)^\phi]\chi^2 2^\phi \sum_l (1+|l|)^{-\phi}, \end{aligned}$$

where the last inequality follows from Hölder's inequality and the trivial estimate $(1+|j|)^\phi \leq 2^\phi(1+|j|^\phi)$. Similarly

$$\begin{aligned} &\sum_{j,k,l} (\langle 0j\rangle^2\langle 0k\rangle\langle kl\rangle^2\langle jl\rangle)^{1/3} \\ &\quad = \sum_{j,k,l} [(1+|l-j|)^\phi(1+|k-l|)^{-\phi}\langle 0j\rangle\langle jl\rangle]^{1/3} \\ &\qquad \times [(1+|j|)^\phi(1+|l-j|)^{-\phi}\langle 0j\rangle\langle kl\rangle]^{1/3} \\ &\qquad \times [(1+|k-l|)^\phi(1+|j|)^{-\phi}\langle 0k\rangle\langle kl\rangle]^{1/3} \\ &\quad \leq [1+\xi(\phi)^\phi]\chi^2 2^\phi \sum_l (1+|l|)^{-\phi}, \end{aligned}$$

such that finally

$$-\bar{u}_4 \leq 8 \times 2^{\phi} \sum_l (1+|l|)^{-\phi}[1+\xi(\phi)^{\phi}]\chi^2 \tag{5}$$

from which (2) follows since $-\bar{u}_4 \sim (\Delta T)^{-\gamma-2\Delta_4}$ by definition.

We note that without further information, it is not possible to neglect the ϕ dependence of ν. Indeed for the (unphysical) case $\langle ij\rangle \sim |i-j|^{-(d+2+\epsilon)}$ ($\epsilon < \frac{1}{3}$) the right-hand side of Eq. (4) is ∞ for $d>4$, whereas $\chi<\infty$, $\xi(2)<\infty$. However, by the results of Lebowitz and Penrose[10] $\langle ij\rangle$ decreases exponentially for $|i-j|$ large above the critical temperature, provided the interaction is of finite range. If ξ denotes the corresponding decay length, then under reasonable assumptions it is easy to see that $\xi(\phi)=C(\phi)\xi$, where, near the critical point, $C(\phi)$ does not depend on the temperature in a singular way. Then in particular

$$\nu(\phi)=\nu=\text{const}, \quad \phi>0. \tag{6}$$

Based on assumptions similar to relation (6), Glimm and Jaffe have also given a discussion of Eq. (2).[11]

We note that the preceding arguments may be easily applied to systems other than the Ising model provided the corresponding Lebowitz and Griffiths inequalities are satisfied. In particular we may apply it to the Euclidean ϕ_d^4 field theory and redo the arguments in Ref. 7. For the corresponding quantities estimate (5) is the analog of relation (2.7) in Ref. 9, if we interpret the mass as an inverse correlation length. Thus we obtain an *a priori* bound on the renormalized coupling constant and get away without the assumptions of the Euclidean axioms.[12] Inequality (2) for the Ising model combined with assumption (6) has also a direct application to ϕ_d^4 due to a discussion in Ref. 13, which is closely related to the one in Ref. 9. There the ϕ^4 theories are parametrized by a triple $y_i>0$ ($i=1,2,3$) of normalization constants. Specifically y_2/y_1 and y_3 play the role of the square of the correlation length and the truncated four-point function at momentum zero, respectively. In the lattice approximation with lattice constant a, this set of possible normalization constants is bounded by a two-dimensional "Ising surface" which is parametrized by $\gamma_3>0$ and T and is of the form

$$y_1 = y_1(\gamma_3, T) = a^2\gamma_3\chi(T),$$
$$y_2 = y_2(\gamma_3, T) = a^4\gamma_3\xi^2(T)\chi(T),$$
$$y_3 = y_3(\gamma_3, T) = -a^{4+d}\gamma_3^2\bar{u}_4(T),$$

where $\chi(T)$, $\xi^2(T)=\xi^2(T, \phi=2)$ and $\bar{u}_4(T)$ are our Ising model quantities. We discuss the case $T>T_c$. For $a\to 0$, we have to let $T\to T_c$ such that $y_2y_1^{-1} = a^2\xi^2(T)$ stays finite $\neq 0$. Eliminating a gives

$$y_3 = \frac{-\bar{u}_4(T)}{\xi^d(T)\chi^2(T)}\, y_1^2\left(\frac{y_2}{y_1}\right)^{d/2}.$$

For $T\to T_c$ by (6) y_3 stays greater than 0 only if (2) is actually an equality and again Eq. (2.7) of Ref. 9 is recovered. Then also a nontrivial ϕ_d^4 is expected to exist.[13]

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