PHYSICAL REVIEW B VOLUME 14, NUMBER 3 1 AUGUST 1976

Renormalization-group calculation of the critical exponents for percolation*

Chandan Dasgupta
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174

(Received 2 February 1976)

The relation between the s-state Ashkin-Teller-Potts model and the percolation problem given by Kasteleyn and Fortuin is used to formulate a renormalization-group treatment of the percolation problem. The variational renormalization-group scheme given by Kadanoff is used to treat the s-state model on a square lattice and the $s \rightarrow 1$ limit is taken to get the critical exponents for the percolation problem.

In this paper, we present a renormalization-group (RG) calculation of critical behavior in percolating systems.[1] The systems we study are described as follows.

For the site (bond) percolation problem, we consider a lattice in which sites (bonds) are randomly occupied with probability p and are vacant with probability $q = 1 - p$. For p less than a critical value p_c, occupied sites (bonds) form unconnected clusters. $P(p)$, the fraction of sites (bonds) in an infinitely large cluster is zero for $p < p_c$ and as $p \rightarrow p_c$ from above, $P(p) \sim (p - p_c)^{\beta}$. The mean-square cluster size $S(p)$ diverges as p approaches p_c: $S(p) \sim |p - p_c|^{-\gamma}$ as $p \rightarrow p_c$. Our objective is to calculate the critical exponents for this "phase transition" at $p = p_c$.

In a recent Letter,[2] we presented a RG approach to percolation problems. This was based on a series of papers by Kasteleyn and Fortuin.[3,4] They considered an s-state Ashkin-Teller-Potts (ATP) model defined by the Hamiltonian

$$H^{(s)} = -J \sum_{\langle ij \rangle} (s\delta_{n_i n_j} - 1) - h \sum_i (s\delta_{n_i 1} - 1), \qquad (1)$$

where for each site i, the variable n_i assumes the values $1, 2, \ldots, s$, the sum $\langle ij \rangle$ is over nearest-neighbor pairs and δ_{mn} is the Kronecker delta. This model and the bond percolation problem are related by

$$P(p) = (s-1)^{-1} \left.\frac{\partial f}{\partial h}\right|_{h=s-1=0}, \qquad (2a)$$

$$S(p) = (s-1)^{-1} \left.\frac{\partial^2 f}{\partial h^2}\right|_{h=s-1=0}, \qquad (2b)$$

with $q = e^{-J}$ ($k_B T = 1$), where f is the free energy per site for the s-state ATP model. So, one can use a RG technique to treat the s-state model and then take the $s \rightarrow 1$ limit to get the critical behavior in percolating systems. The nature of the phase transition in the s-state ATP model in three dimensions is not precisely known. However, Baxter[5] has shown that in two dimensions, the s-state model has a continuous phase transition for $s \leq 4$.

Of the various RG techniques, the ϵ expansion is not very suitable for studying the s-state model in two or three dimensions. This is because the critical dimension for the problem is six[2,6] and $\epsilon = 6 - d$ is rather large for both $d = 2$ and $d = 3$. We have treated the s-state ATP model by using the variational RG scheme given by Kadanoff.[7] This scheme gives excellent results[7] for the Ising model and the approximate RG recursion relations in this scheme are relatively easy to obtain. Previously,[2] the finite-cluster method proposed by Niemeyer and van Leeuwen[8] was found to give rather poor results for the two-dimensional percolation problem. Young and Stinchcombe[9] have since obtained better results using a method analogous to the decimation procedure discussed by Kadanoff and Houghton.[10] However, the scheme they used could give only a single critical exponent ν. In contrast, the procedure used here is expected to give reasonable values for ν and δ, from which, using scaling relations, the other static exponents α, β, and γ can be obtained. Briefly, the Kadanoff scheme is as follows.

The RG transformation takes a Hamiltonian $H_{\vec{K}}(\sigma)$ in which σ represents the initial set of spin variables and $\vec{K}$ represents a set of coupling constants, to a new Hamiltonian $H_{\vec{K}'}(\mu)$ which involves a new set of coupling constants $\vec{K}'$ and a smaller number of variables, μ. This transformation is taken to be of the form

$$H_{\vec{K}'}(\mu) = -\ln \mathrm{Tr}_{\sigma} \exp[S(\mu, \sigma) - H_{\vec{K}}(\sigma)], \qquad (3)$$

where S satisfies $\mathrm{Tr}_{\mu} \exp[S(\mu, \sigma)] = 1$, so that the total free energy F of the system remains invariant under the transformation. The transformation (3) can also be considered to be a transformation upon the K's of the form

$$\vec{K}' = R(\vec{K}). \qquad (4)$$

In this scheme, one constructs a "lower bounding" approximation to R, denoted by R^L, which has the

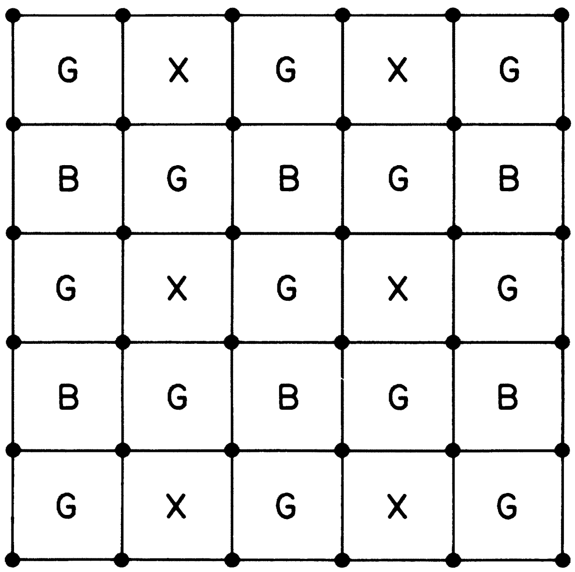

FIG. 1. Kadanoff's renormalization scheme for the square lattice. ●: σ variables. Each square marked **×** contains a μ variable.

property

$$F(R^L(\vec{K}))\leq F(\vec{K}). \tag{5}$$

This is done by writing

$$H_{R^L(\vec{K})}(\mu)=-\ln\mathrm{Tr}_\sigma\exp[S(\mu,\sigma)-H_{\vec{K}}(\sigma)+V(\mu,\sigma)], \tag{6}$$

where $V(\mu,\sigma)$ is a sum of terms which are each odd under a lattice symmetry operation for H. [For the proof that R^L defined in (6) satisfies (5), see Ref. (7).] A suitable choice of V factorizes the sum in (6) into a product of independent terms each of which can be evaluated exactly. Any parameter in the transformation may then be varied to get an optimal bound for the free energy and the corresponding "optimal lower bounding" recursion relation $\vec{K}'=R^L(\vec{K})$ is used to calculate the critical exponents.

We have applied this technique to the s-state ATP model on a square lattice. The Hamiltonian is written in the form

$$H_{\vec{K}}(\sigma)=-\sum_{\text{squares}}\sum_i K_i s_i, \tag{7}$$

where, within each square, the set s_i includes all the possible interactions which satisfy the point group symmetries of the square. When all the s states $1,2,\ldots,s$ are equivalent, we have to consider the following seven interactions:

$$s_0=1,$$

$$s_1=\sum_p(s\delta_{ab}-1),$$

$$s_2=\frac{1}{2}\sum_p(s\delta_{ac}-1),$$

$$s_3=\sum_p[s^2\delta_{abc}-s(\delta_{ab}+\delta_{bc}+\delta_{ca})+2],$$

$$s_4=\sum_p[\tfrac{1}{4}s^3\delta_{abcd}-s^2\delta_{abc}+s(\delta_{ab}+\tfrac{1}{2}\delta_{ac})-\tfrac{3}{4}],$$

$$s_5=\sum_p(\tfrac{1}{2}s^2\delta_{ab}\delta_{cd}-s\delta_{ab}+\tfrac{1}{2}),$$

$$s_6=[s^2\delta_{ac}\delta_{bd}-s(\delta_{ac}+\delta_{bd})+1].$$

Here a, b, c, and d represent in cyclic order the four σ's at the corners of the square; $\sum_p$ means a sum over the four cyclic permutations of the four quantities a, b, c, and d; and $\delta_{a_1a_2\cdots a_n}\equiv\delta_{a_1a_2}\delta_{a_2a_3}\cdots\delta_{a_{n-1}a_n}$. All these interactions have been made traceless with respect to each of the variables involved. If we introduce an "external field" in one of the s "directions," ten more interactions (s_7 through s_{16}) have to be taken into account. Every fourth square (which has been labeled by X in Fig. 1) contains a new variable μ. We use an S which depends on a parameter x:

$$S(\mu,\sigma)=\sum_{\text{X squares}}\left(x[s(\delta_{\mu A}+\delta_{\mu B}+\delta_{\mu C}+\delta_{\mu D})-4]-\sum_i U_i s_i\right), \tag{8}$$

where A, B, C, and D are the four σ's at the corners of the X square and the coefficients U_i are determined from the condition $\mathrm{Tr}_\mu\exp[S(\mu,\sigma)]=1$. This is the most straightforward generalization of the transformation used by Kadanoff to treat the Ising model. [The Ising model is identical with the $s=2$ ATP model and for $s=2$, the transformation (8) reduces to the one used by Kadanoff.]

Following Kadanoff, we now write $V(\mu,\sigma)=\sum_{\text{squares}}\sum_i v_i s_i$, where v_i is chosen to be $(-K_i+U_i)$ in squares marked X, $(-K_i)$ in squares marked G, and $(3K_i-U_i)$ in squares marked B. Then we have, from (6),

$$\exp[-H_{R_x^L(\vec{K})}(\mu)]=\mathrm{Tr}_\sigma\exp\left(\sum_{\text{X squares}}x[s(\delta_{\mu A}+\delta_{\mu B}+\delta_{\mu C}+\delta_{\mu D})-4]+\sum_{\text{B squares}}\sum_i(4K_i-U_i)s_i\right). \tag{9}$$

The right-hand side of Eq. (9) is a product of independent sums over B squares. These sums are easily evaluated and the lower bound recursion relation $\vec{K}'=R_x^L(\vec{K})$ is thus obtained for any x. This recursion relation has s as a parameter in it and we can take the $s\to1$ limit analytically. The resulting recursion relation has a second-order fixed point. The location of this fixed point in the

TABLE I. Numerical results for the critical exponents.

Exponent	This work	Value obtained from Series expansion	Monte Carlo simulation
α	−0.686	−0.7 ±0.2 [a]	−0.65 ±0.1 [e]
β	0.140	0.148 ±0.004 [a] 0.14 ±0.03 [b]	0.14 ±0.02 [f]
γ	2.406	2.43 ±0.03 [c] 2.38 ±0.02 [d]	2.23 ±0.2 [g] 2.3 ±0.1 [e]
ν	1.343	1.34 ±0.02 [d]	1.5 ±0.2 [a]
δ	18.249	17^{+4}_{-3} [d]	

[a] See Ref. 2.
[b] See Ref. 13.
[c] See Ref. 15.
[d] See Ref. 14.
[e] See Ref. 18.
[f] This was obtained in Ref. 2 by analyzing the data of Ref. 16.
[g] This was obtained in Ref. 17 by analyzing the data of Ref. 16.

coupling constant space depends on the value of the variational parameter x. x^*, the value of x appropriate for this fixed point is determined by extremizing the lower bound to the free energy for the fixed-point Hamiltonian. This is done in the following way.

The set of interactions s_i contains a constant term $s_0 = 1$, K_0 being the corresponding coupling constant. This constant term grows at each renormalization step, and after a large number of iterations dominates the free energy. Therefore, x^* is that value of x for which $K_0^{(n)}$, the constant term per square after n applications of the lower bound transformation to the fixed-point Hamiltonian, is an extremum for large n. For $s = 1$, however, $K_0^{(n)} = 4^n K_0^{(0)}$ for all x. So, in the $s \to 1$ limit, we take x^* to be that x which extremizes (minimizes)

$$\lim_{s\to 1}(s-1)^{-1}(K_0^{(n)} - 4^n K_0^{(0)}) = \left.\frac{dK_0^{(n)}}{ds}\right|_{s=1} \quad \text{for large } n.$$

Numerically, we find that $x^* = 1.250$. For $x = x^*$, the fixed point is located at

$$K_1 = K_2 = 0.247\,15, \quad K_3 = 0.033\,72, \quad K_4 = -0.019\,19,$$

$$K_5 = K_6 = -0.002\,74, \quad K_7 = \cdots = K_{16} = 0.$$

The standard procedure[11,12] is used to calculate the critical exponents ν and δ from this optimal lower bounding recursion relation $\vec{\mathrm{K}}' = R^L_{x^*}(\vec{\mathrm{K}})$. We then use the scaling relations to obtain the values of the exponents α, β, and γ. The results are shown in Table I, where we have also listed the values obtained from series expansion[2,13-15] and from Monte Carlo simulation.[2,16-18] It is readily seen that the numerical results of our RG calculation are in excellent agreement with the results obtained from the other approaches.

In order to calculate p_c, we must find the intersection of the K_1 axis with the critical surface. This cannot be done in this scheme because, for the transformation (8), the critical surface does not intersect the K_1 axis. However, starting from a Hamiltonian containing only nearest-neighbor coupling $J = J^*$, the fixed-point can be approached by first applying an exact decimation transformation[10] in which every other spin variable on the lattice is summed over, and then repeatedly applying the lower bound transformation. Numerically, we find that $J^* = 0.73$. This gives

$$p_c = 1 - e^{-J^*} = 0.518,$$

which is in good agreement with the exact value,[19] $p_c = 0.5$.

ACKNOWLEDGMENTS

The author would like to thank Professor A. B. Harris for many illuminating discussions and for a critical reading of the manuscript, and Professor T. C. Lubensky for suggesting the problem and for many helpful discussions.

---

*Supported in part by the Office of Naval Research under grant number N000 14-67-A-0216-0032 and by the National Science Foundation.

[1] For a review, see, V. K. S. Shante and S. Kirkpatrick,

Adv. Phys. 20, 325 (1971).

[2] A. B. Harris, T. C. Lubensky, W. K. Holcomb, and C. Dasgupta, Phys. Rev. Lett. 35, 327 (1975); 35, 1397 (E) (1975).

[3] P. W. Kasteleyn and C. M. Fortuin, J. Phys. Soc. Jpn. Suppl. 26, 11 (1969).

[4] C. M. Fortuin and P. W. Kasteleyn, Physica (Utr.) 57, 536 (1972).

[5] R. J. Baxter, J. Phys. C 6, L445 (1973).

[6] G. Toulouse, Nuovo Cimento B 23, 234 (1974).

[7] L. P. Kadanoff, Phys. Rev. Lett. 34, 1005 (1975).

[8] T. Niemeyer and J. M. J. van Leeuwen, Physica (Utr.) 71, 17 (1974).

[9] A. P. Young and R. B. Stinchcombe, J. Phys. C 8, L535 (1975).

[10] L. P. Kadanoff and A. Houghton, Phys. Rev. B 11, 377 (1975).

[11] K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974).

[12] S.-K. Ma, Rev. Mod. Phys. 45, 589 (1973).

[13] M. F. Sykes, M. Glen and D. S. Gaunt, J. Phys. A 7, L105 (1974).

[14] A. G. Dunn, J. W. Essam, and D. S. Ritchie, J. Phys. C8, 4219 (1975).

[15] M. F. Sykes, D. S. Gaunt, and M. Glen, J. Phys. A 9, 97 (1976).

[16] P. Dean and N. F. Bird, National Physical Laboratory Report No. Ma61 (unpublished).

[17] D. Stauffer, Phys. Rev. Lett. 35, 394 (1975).

[18] S. Kirkpatrick, Phys. Rev. Lett. 36, 69 (1976).

[19] M. F. Sykes and J. W. Essam, J. Math. Phys. 5, 117 (1964).