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Magnetic solitons in superfluid ^{3}He†

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Making use of dynamic equations for the spin coordinates in superfluid ^{3}He (valid in the hydrodynamic regime), we study theoretically the nonlinear propagating solution, which describes the moving domain wall in superfluid ^{3}He-A and ^{3}He-B. The dynamic spin susceptibility in the presence of the domain wall is also considered.

I. INTRODUCTION

In spite of a large bulk of theoretical works[1–5] on the spin waves in superfluid ^{3}He, no experimental work has been done on this subject. This may be due to difficulties in creating the spin-wave mode in superfluid ^{3}He: first of all, the spin-wave dispersion has an energy gap excepting a special case in B phase[5] (the wall-pinned configuration), and secondly, in order to excite a propagating mode a large momentum transfer to the spin system is required. This appears to be difficult to achieve by conventional technique (i.e., by inhomogeneous magnetic field). Very recently Wheatley and his co-workers[6] have succeeded in observing a propagating magnetic mode in the B phase of superfluid ^{3}He. Although their result is rather preliminary, it appears rather difficult to identify the observed propagating mode with the spin wave predicted so far, since first, the velocity of this mode is rather small (i.e., about 1/10 of the spin-wave velocity) and secondly, the velocity appears to depend in a complicated way on the exciting magnetic field. Recently Maki and Tsuneto[4] and Maki and Ebisawa (ME)[7] have pointed out that the dynamic equation describing the spin coordinates in superfluid ^{3}He permits a class of nonlinear solutions, which are known as solitons or kink solutions in the other context. In the case of superfluid ^{3}He, the soliton describes generally a moving domain wall at the boundary of two different textures (or the disgyration plane of the spin configurations of the condensate), which can in principle be created experimentally. In the simplest case (the longitudinal perturbation in the A phase) the equation describing the spin coordinates is identical to the sine-Gordon equation.[8] There is large amount of work on this equation in existence in literature.[8] Therefore, in the A phase, relying heavily on the existing literature, we will consider a general soliton problem, two colliding solitons, the scattering of the spin waves from a soliton (or domain wall), and the dynamic spin susceptibility in the presence of a soliton. In the B phase, we will limit ourselves to two distinct classes of solitons in the Leggett configuration. The dynamic spin susceptibility in the presence of a soliton (of the type I and of the type II) is approximately constructed in this case. It is shown that the presence of a soliton can be detected by the resonance technique (or equivalently by the ringing technique), since the intensity of the resonance signal is reduced drastically in the vicinity of the solitons.

II. SOLITONS IN THE A PHASE

In the A phase motion of the spin coordinates in superfluid ^{3}He is expressed in terms of two angles $\beta(\vec{r},t)$ and $\gamma(\vec{r},t)$,[7] where β describes the angle between the $\hat{d}$ vector (i.e., the direction of the pair spin) and the $\hat{l}$ vector (i.e., the symmetry axis of the orbital wave function of the condensate), which is assumed in the z direction and γ is the rotation angle of the $\hat{d}$ vector around the $\hat{l}$ vector.

The Lagrangian describing the motion of the spin coordinates is given by[7]

$$L_A=\tfrac{1}{2}\chi_N\int d^3r\left(\dot\beta^2+\dot\gamma^2-2\omega_0(-\dot\beta\sin\alpha+\dot\gamma\cos\alpha\sin\beta)-\left\{C_\parallel^2\left[\left(\frac{\partial\beta}{\partial z}\right)^2+\left(\frac{\partial\gamma}{\partial z}\right)^2\right]+C_\perp^2\left[\left(\frac{\partial\beta}{\partial x}\right)^2+\left(\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\gamma}{\partial x}\right)^2+\left(\frac{\partial\gamma}{\partial y}\right)^2\right]\right\}-\Omega_A^2\sin^2\beta\right). \tag{1}$$

Here χ_N is the static spin susceptibility in the A phase, α is an additional parameter which is determined by the initial condition, $\omega_0=\gamma_0 H$ is the Larmor frequency due to a static magnetic field H along the x axis, and $C_\parallel$ and $C_\perp$ are the spin-wave velocities in the A phase parallel and perpendicular to the $\hat{l}$ vector.[4] (In the A phase the condensate is a simple product of the spin function and the orbital function and therefore the spin-wave velocity depends only on the propagation direction but not on the spin polarization of the spin wave.) In particular, in the vicinity of the transition temperature we have $C_\perp = \sqrt{2}\,C_\parallel$, while $C_\perp = C_\parallel$ at $T=0$ K.[4] Finally the last term in Eq. (1) provides the dipolar interaction energy and Ω_A is the longitudinal resonance frequency in the A phase. From Eq. (1), the equation of motion is given by

$$\frac{d}{dt}(\dot\gamma - \omega_0 \sin\beta\cos\alpha) - C_\parallel^2\frac{\partial^2\gamma}{\partial z^2} - C_\perp^2\nabla_\perp^2\gamma = 0,$$
$$\ddot\beta + \omega_0\dot\gamma\cos\alpha\cos\beta - C_\parallel^2\frac{\partial^2\beta}{\partial z^2} - C_\perp^2(\nabla_\perp^2\beta) + \Omega_A^2\sin\beta\cos\beta = 0, \tag{2}$$

where

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Furthermore the local magnetization is given by[7]

$$\vec{M} = -\gamma_0\chi_N\vec{\omega}, \tag{3}$$

with

$$\omega_x = -\dot\beta\sin\alpha + \dot\gamma\cos\alpha\sin\beta,$$
$$\omega_y = \dot\beta\cos\alpha + \dot\gamma\sin\alpha\sin\beta, \tag{4}$$
$$\omega_z = \dot\gamma\cos\beta.$$

We will consider in the following a situation in which a magnetic field, which lies in the $x-y$ plane $\Delta\vec{H}(\vec{k}\cdot\vec{x}) = \Delta H(\vec{k}\cdot\vec{x})(\cos\phi, \sin\phi, 0)$, is suddenly turned off at $t=0$. We assume here that the spatial dependence of the magnetic field is one dimensional; ΔH depends only on $(\vec{k}\cdot\vec{x})$. This is because we will study in the following only the one-dimensional solitons. Fortunately, a stable two-dimensional or three-dimensional soliton does not exist, at least in the case of the longitudinal configuration[8] (i.e., $\Delta\vec{H} \,||\, \vec{H}$), and therefore it appears that only the one-dimensional soliton has physical relevance among more complicated solitons. In fact we can extend the nonexistence theorem of two-dimensional solitons for the sine-Gordon equation[8] to the more complicated equation which governs the spin motion in the B phase. Physically this seems to relate to the fact that a domain wall (which is identified with a soliton in the case of superfluid ^{3}He), which closes on itself in the two-dimensional plane, can be continuously shrunk to zero and therefore is always unstable. Coming back to our problem, we assume at $t=0$ the spin coordinates are at equilibrium under a magnetic field $\vec{H} + \Delta\vec{H}(\vec{k}\cdot\vec{x})$, which implies $\hat{d}$ and $\hat{l}$ are parallel along the z axis. Furthermore, we[7] have at $t=0$

$$\dot\gamma\cos\beta = 0,$$
$$-\dot\beta\sin\alpha + \dot\gamma\cos\alpha\sin\beta = \Delta\omega_0\cos\phi, \tag{5}$$
$$\dot\beta\cos\alpha + \dot\gamma\sin\alpha\sin\beta = \Delta\omega_0\sin\phi,$$

where $\Delta\omega_0(\vec{k}\cdot\vec{x}) = \gamma_0\Delta H(\vec{k}\cdot\vec{x})$. The above equation yields

$$\dot\beta = \Delta\omega_0, \quad \dot\gamma = 0, \quad \alpha = \phi - \tfrac{1}{2}\pi \tag{6}$$

at $t=0$.

In order to solve Eq. (2) with the initial condition (6), we assume that a pair of solitons with velocities $\vec{v}$ and $-\vec{v}$ (parallel to $\vec{k}$) are created at $t=0$ around the origin. As we will see later we cannot create a single soliton by turning off any magnetic field. In general the total magnetic energy imparted to the system by turning off the magnetic field is shared by spin waves and pairs of solitons with the opposite helicity. However, after a finite lapse of time, these solitons are well-separated spatially, so that they can be considered as a group of noninteracting solitons. Therefore we will consider first an isolated soliton.

A. A single soliton

In the following we will consider the one-dimensional solution of Eq. (2), where γ and β depend only on a single parameter $s \equiv \vec{k}\cdot\vec{x} - k_0 t$.

Then Eq. (2) reduces to

$$\frac{d^2\gamma}{ds^2} + \frac{\omega_0}{\Omega_A}\frac{U}{(1-U^2)^{1/2}}\sin\phi\frac{d}{ds}(\sin\beta) = 0,$$
$$\frac{d^2\beta}{ds^2} - \sin\beta\cos\beta - \frac{\omega_0}{\Omega_A}\frac{U}{(1-U^2)^{1/2}}\sin\phi\cos\beta\frac{d\gamma}{ds} = 0, \tag{7}$$

where

$$U = k_0/|k|C(n), \quad C(n) = [C_\parallel^2 n_z^2 + C_\perp^2(n_x^2 + n_y^2)]^{1/2},$$
$$\vec{n} = \vec{k}/|k|, \quad |k| = \xi^{-1}(n)(1-U^2)^{-1/2}, \quad \xi(n) = \frac{C(n)}{\Omega_A}. \tag{8}$$

Here $C(n)$ is the spin-wave velocity with the wave vector $\vec{k}$ and $\xi(n)$ is the direction-dependent coherence length in the present problem. The first equation of Eq. (7) is easily integrated to give

$$\frac{d\gamma}{ds} + \frac{\omega_0}{\Omega_A}\frac{U}{(1-U^2)^{1/2}}\sin\phi\sin\beta = 0, \tag{9}$$

since we assumed here $\beta = d\gamma/ds = 0$ for $\vec{x} = \pm\infty$ (i.e., at infinities γ and β take the equilibrium values). Substituting this into the second equation of Eq. (7) we have

$$\frac{d^2\beta}{ds^2} - \left[1 - \left(\frac{\omega_0}{\Omega_A}\right)^2\frac{U}{1-U^2}\sin^2\phi\right]\sin\beta\cos\beta = 0. \tag{10}$$

This is integrated as

$$(d\beta/ds)^2 = \mu^2 \sin^2\beta, \tag{11}$$

and

$$\mu = \left[1 - \left(\frac{\omega_0}{\Omega_A}\right)^2 \frac{U^2}{1-U^2} \sin^2\phi\right]^{1/2}, \tag{12}$$

where we assumed the same boundary conditions as before at infinities. Finally, Eq. (11) gives

$$\tan\tfrac{1}{2}\beta(s) = e^{\pm\mu s}. \tag{13}$$

The soliton with ± in the exponent is called the solution with the helicity ±1. The helicity measures the change in $\beta(s)$ from $s=-\infty$ to $+\infty$. For example, the solution with the + sign describes a domain wall perpendicular to $\hat{k}$, where on the left side of the wall $\beta=0$ (i.e., $\hat{l}$ and $\hat{d}$ are parallel) while on the other side $\beta=\pi$ (i.e., $\hat{l}$ and $\hat{d}$ are antiparallel). This is schematically shown in Fig. 1. If we take the minus sign in Eq. (13), the configurations in the right and left side of the domain wall are reversed. The helicity (N) introduces a new conserved quantity in the present problem. If we impose an initial condition that $\hat{d}$ and $\hat{l}$ are parallel over all space at $t=0$ (which corresponds to a monodomain or the state with the helicity $N=0$), it is clear that by turning off a localized magnetic field we can create a new state with $N=0$. This is most easily achieved by creating a pair of solitons with opposite helicity ($N=\pm1$). The present domain wall is moving with a velocity $UC(n)$ in the direction parallel to $\vec{k}$ (normal to the wall). The moving domain wall accompanies a local magnetization $\vec{M}=\gamma_0\chi_N\vec{\omega}$, where $\vec{\omega}$ has been defined in Eq. (4). Substituting Eqs. (13) and (9) into Eq. (4) we have

$$\omega_x = \pm\Omega_A \frac{U}{(1-U^2)^{1/2}} \times\left(\mu\cos\phi \pm \frac{\omega_0}{\Omega_A}\frac{U}{(1-U^2)^{1/2}}\sin^2\phi\cosh^{-1}(\mu s)\right) \times \cosh^{-1}(\mu s), \tag{14}$$

$$\omega_y = \pm\Omega_A \frac{U\sin\phi}{(1-U^2)^{1/2}} \times\left(\mu \pm \frac{\omega_0}{\Omega_A}\frac{U\cos\phi}{(1-U^2)^{1/2}}\cosh^{-1}(\mu s)\right)\cosh^{-1}(\mu s),$$

$$\omega_z = -\omega_0 \frac{U^2}{1-U^2}\sin\phi\, e^{\pm\mu s}\cosh^{-2}(\mu s).$$

The domain wall (the disgyration plane) carries the magnetization of the order of $\Omega_A U$. The characteristic thickness of the domain wall is given by $\xi(n)/\mu \sim 0.1$ mm. This enormously large (macroscopic) coherence distance follows from the smallness of the dipolar interaction energy. Since both Ω_A and $C(n)$ vanish like $(1-T/T_c)^{1/2}$ as the temperature approaches the transition temperature T_c, the characteristic length is almost independent of the temperature but rather sensitive to $\vec{n}$. The above result is a simple generalization of ME, where the consideration is limited to the case $\vec{k}\,||\,\vec{Z}$. The energy associated with the soliton is calculated as in ME,

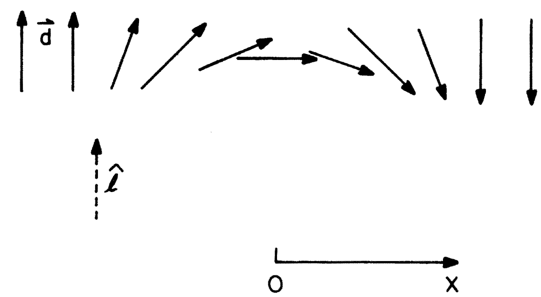

FIG. 1. Spatial variation of the $\hat{d}$ vector in the presence of a soliton (with helicity $N=1$) in the A phase is shown schematically. The $\hat{l}$ vector is taken to be upward over all space.

$$\epsilon_A = \tfrac{1}{2}\chi_N \int_{-\infty}^{\infty} dx' \left\{\dot\gamma^2 + \dot\beta^2 + C_\parallel^2\left[\left(\frac{\partial\beta}{\partial z}\right)^2 + \left(\frac{\partial\gamma}{\partial z}\right)^2\right] + C_\perp^2\left[\left(\frac{\partial\beta}{\partial x}\right)^2 + \left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\gamma}{\partial x}\right)^2 + \left(\frac{\partial\gamma}{\partial y}\right)^2\right] + \Omega_A^2\sin^2\beta\right\}$$
$$= 2\chi_N\mu^{-1}\Omega_A C(n)(1-U^2)^{-1/2}, \tag{15}$$

where $x'=(\vec{k}\cdot\vec{x})/|k|$. The above energy is the energy associated with the moving domain wall per unit area. We note that ϵ_A has a velocity dependence similar to a relativistic particle. This follows simply from the Lorentz invariance of the original Lagrangian (1) when the light velocity is replaced by $C(n)$ the anisotropic spin-wave velocity. As is seen easily from Eq. (15), the surface energy is of the order of $\chi_N\Omega_A C \sim \chi_N\Omega_A^2\xi(n)$. We note also that the excitation energy of a soliton with $\vec{k}\,||\,\hat{l}$ is smaller than that with $\vec{k}\perp\hat{l}$ by a factor $\sqrt{2}$ in the vicinity of T_c, since $C_\perp=\sqrt{2}\,C_\parallel$ in this temperature region.

B. Two-soliton problem

So far we limit ourselves to a single soliton. However, two solitons are more relevant to the turn-off experiment, since we can construct a state with zero helicity by a pair of solitons with opposite helicity. However, in this case the general solution of Eq. (2) appears rather difficult and we consider only the longitudinal case where $\Delta\vec{H}$ is parallel to $\vec{H}$ (i.e., $\phi=0$). In this case the motion of γ and β are completely decoupled and β obeys essentially the sine-Gordon equation;

$$\ddot{\beta} - C_\parallel^2 \frac{\partial^2 \beta}{\partial Z^2} - C_\perp^2 \nabla_\perp^2 \beta + \Omega_A^2 \sin\beta \cos\beta = 0. \quad (16)$$

In this particular equation, the two-soliton problem has been solved by Perring and Skyrme.[9] Assuming that now the solitons are functions of $\vec{k}\cdot\vec{x}$ and t (we still limit to the one-dimensional solitons), we can show Eq. (16) has a solution

$$\tan[\tfrac{1}{2}\beta(\vec{x}, t)] = U^{-1} \frac{\sinh(k_0 t)}{\cosh(\vec{k}\cdot\vec{x})}, \quad (17)$$

where $\vec{k}$, k_0, U, and $C(n)$ have been already defined in Eq. (8). In the limit of a large t, Eq. (17) describes two out going solitons with the opposite helicities and velocities $\vec{v}$ and $-\vec{v}$ $(\vec{v} = k_0\vec{k}/k^2)$;

$$\tan[\tfrac{1}{2}\beta(\vec{x}, t)] \simeq U^{-1}(e^{k_0 t - \vec{k}\cdot\vec{x}} + e^{k_0 t + \vec{k}\cdot\vec{x}})^{-1}. \quad (18)$$

Furthermore, these out-going solitons have a phase shift $\Delta x = -|k|^{-1}\ln U$, due to the attractive interaction between two solitons with the opposite helicity; the positions of the soliton at a time t after the creation are given by $x = \pm|k|^{-1}[k_0 t - \ln(U)]$.

The magnetization associated with the above solution is given by

$$\begin{aligned} M_x(\vec{x}, t) &= -\gamma_0 \chi_N \dot{\beta}(\vec{x}, t) \\ &= -2\gamma_0\chi_N U^{-1} k_0 \cosh(k_0 t)\cosh(\vec{k}\cdot\vec{x}) / \\ &\qquad [\cosh^2(\vec{k}\cdot\vec{x}) + U^{-2}\sinh^2(k_0 t)] \end{aligned} \quad (19)$$

at $t = 0$, Eq. (19) reduces to

$$M_x(\vec{x}, 0) = -2\gamma_0\chi_N U^{-1} k_0 \cosh^{-1}(\vec{k}\cdot\vec{x}). \quad (20)$$

Furthermore, Eq. (17) yields $\beta(\vec{x}, 0) = 0$. Therefore Eq. (17) is consistent with the initial condition at $t = 0$ [see Eq. (6)], if $\Delta H(\vec{k}\cdot\vec{x})$ is given by

$$\Delta\vec{H} = -\gamma_0^{-1}\Omega_A(1 - U^2)^{-1/2}\cosh^{-1}(\vec{k}\cdot\vec{x}), \quad (21)$$

which yields the maximum value of $\Delta\vec{H}$ at $\vec{k} = 0$ $[\Delta H_{max} = |\gamma_0|^{-1}\Omega_A(1 - U^2)^{-1/2}]$. The temporal variation of the magnetization given in Eq. (20) has been calculated and plotted in Fig. 2. As expected the magnetization has a sharp peak at $\vec{x} = 0$ at $t = 0$. The peak first broadens and then splits into two outgoing peaks as the time increases. Until now we are concerned with general solutions of Eq. (2). In a turn-off experiment, where a local magnetic field is turned off suddenly, it is very likely that the total magnetic energy imparted to the system is converted into spin waves (which are the collective mode with the helicity $N = 0$) and solitons. However, it appears rather difficult to establish the production rate of these two modes. As to the necessary condition[7] for the soliton creation, we may state that the total magnetic energy imparted to the system has to be greater than twice the energy given in Eq. (15) $[\epsilon_{tot} \geq 4\chi_N\Omega_A C(n)]$. However, this is not likely to be adequate, since Eq.

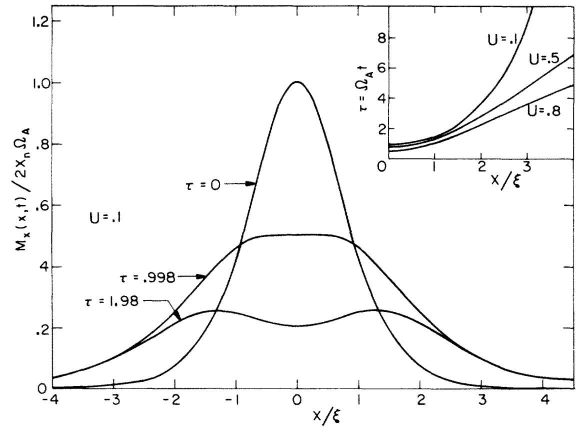

FIG. 2. Local magnetizations associated with a pair of solitons are shown for time $\tau(\equiv\Omega_A t) = 0$, 0.998 and 1.98 for $U = 0.1$ after the creation of the solitons. Here $U = v/c(n)$ the velocity of the outgoing solitons normalized by the spin-wave velocity. In the insert the trajectories of one of the solitons (the peak position of the magnetizations) are shown for various U's.

(21) seems to require a rather sharp field distribution to create a pair of solitons. Another possible condition will be that the rotation angle of the $\hat{d}$ vector at the center of the turn-off field exceeds π. This is equivalent to say that $\Delta\omega_0|_{max} \geq \Omega_A$ in the present case. Although it is difficult to formulate a precise condition for the soliton creation, we believe that it is very likely that a pair of solitons can be created, if both the above conditions are met in the experiment.

III. SOLITONS IN THE B PHASE

In the B phase, where the condensate is described in terms of the Balian-Werthamer state, the general discussion of the soliton is rather difficult. Therefore we will limit ourselves to the longitudinal solitons in the Leggett configuration, where we concentrated on the angle of rotation α around the symmetry breaking axis $\hat{\omega}$. The basic equation describing the spin dynamic in this special case is derived from the Lagrangian[5,7]

$$L_B = \tfrac{1}{2}\chi_B \int d^3r \{\dot{\alpha}^2 - [C_\perp^2(\nabla_\perp\alpha)^2 + C_\parallel^2(\partial\alpha/\partial z)^2] + \tfrac{16}{15}\Omega_B^2(\cos\alpha + \tfrac{1}{4})^2\}, \quad (22)$$

where χ_B is the static susceptibility in the B phase, $\alpha = \theta$ the angle of the rotation around the $\hat{\omega}$ vector, which is parallel to the external magnetic field in the Leggett configuration, and $C_\perp$ and $C_\parallel$ are the perpendicular and the parallel component of the spin-wave velocity to the $\hat{\omega}$ axis in the B phase given by[5]

$$C_\perp = \sqrt{2}\, C_\parallel = \left[\frac{2N}{5m^*}(\rho_s/\rho)\Big/\chi_B\right]^{1/2}, \quad (23)$$

where

$$\rho_s/\rho = 2\pi T \sum_{n=0}^{\infty} \frac{\Delta^2}{(\omega_n^2+\Delta^2)^{3/2}},$$

the superfluid density in the BCS superconductor, m^* is the effective mass of the quasiparticle.

Finally Ω_B is the longitudinal resonance frequency in the B phase. The equation of the motion for α is given by

$$\ddot{\alpha} - C_\perp^2\left(\nabla_\perp^2\alpha + \frac{1}{2}\frac{\partial^2}{\partial z^2}\alpha\right) = \tfrac{16}{15}\Omega_B^2(\cos\alpha + \tfrac{1}{4})\sin\alpha\,. \quad (24)$$

As before we will consider the one-dimensional solution of Eq. (24). We assume that the solution is given by $\alpha = \alpha(s)$, and $s = \vec{k}\cdot\vec{x} - k_0 t$. Then Eq. (24) reduces to

$$\frac{d^2\alpha}{ds^2} = \tfrac{16}{15}(\cos\alpha + \tfrac{1}{4})\sin\alpha\,, \quad (25)$$

where we took

$$|k| = \Omega_B/C(n)(1-U^2)^{1/2}\,,$$
$$C(n)^2 = C_\perp^2(n_\perp^2 + \tfrac{1}{2}n_z^2) = C_\perp^2(1-\tfrac{1}{2}n_z^2)\,, \quad (26)$$
$$k_0 = U|k|\,, \quad \hat{n} = \vec{k}/|k|\,.$$

Equation (25) is integrated to yield

$$\left(\frac{d\alpha}{ds}\right)^2 = \tfrac{16}{15}(\cos\alpha + \tfrac{1}{4})^2\,. \quad (27)$$

Here we assumed that for $s = \pm\infty$, $\cos\alpha = -\frac{1}{4}$ and $d\alpha/ds = 0$ (i.e., the system is at equilibrium at infinities). Then Eq. (27) yields

$$\frac{d\alpha}{ds} = \pm 4/\sqrt{15}\,(\cos\alpha + \tfrac{1}{4})\,, \quad (28)$$

which can be integrated and we have

$$\tan\frac{\alpha}{2} = \pm\sqrt{\tfrac{5}{3}}\,\coth\tfrac{1}{2}s \quad (29)$$

or

$$\tan\frac{\alpha}{2} = \pm\sqrt{\tfrac{5}{3}}\,\tanh\tfrac{1}{2}s\,. \quad (30)$$

There are two types of solitons (or the domain walls) in the B phase. We may introduce the two types of the helicity numbers N_1 and N_2. In the first type of the domain wall [i.e., Eq. (29)] α changes from α_0 to $2\pi - \alpha_0$ from the left to the right or vice versa, where $\alpha_0 = \cos^{-1}(-\frac{1}{4})$. We may assign to those solutions the helicity $N_1 = \pm 1$, respectively. In the second type of the domain wall [i.e., Eq. (30)] α changes from $-\alpha_0$ to $+\alpha_0$ from the left to the right, for example, and we assign the helicity $N_2 = \pm 1$ for these two solutions. These two helicities are separately conserved in the B phase. The spatial variation of the angle α, for these two types of solitons (domain walls) are shown in Fig. 3. Since the second type of solitons include the region of α with the larger dipolar interaction energy, the second type of soliton has the larger excitation energy. The surface energy associated with two types of solitons are given by

$$\epsilon_B^{\mathrm{I}} = \tfrac{1}{2}\chi_B \int dx'\left\{\dot{\alpha}^2 + C_\perp^2[(\nabla_\perp\alpha)^2 + \tfrac{1}{2}(d\alpha/dz)^2] + \tfrac{16}{15}\Omega_B^2(\cos\alpha + \tfrac{1}{4})^2\right\}$$
$$= 2[1 + (1/\sqrt{15})(\alpha_0 - \pi)]\chi_B\Omega_B C(n)(1-U^2)^{-1/2} \quad (31)$$

and

$$\epsilon_B^{\mathrm{II}} = 2(1 + 1/\sqrt{15}\,\alpha_0)\chi_B\Omega_B C(n)(1-U^2)^{-1/2}, \quad (32)$$

respectively.

The excitation energy for these solitons depend on Ω_B and $C(n)$ similarly to the one in the A phase except changes in the numerical factor. Therefore we can formulate in a similar way the problem of the soliton excitation by turning off a local magnetic field. The local magnetization due to the moving solitons are given by

$$M_z(\vec{x}) = \gamma_0\chi_B\tfrac{1}{4}\sqrt{15}\,\Omega_B U(1-U^2)^{-1/2}(\cosh s + \tfrac{1}{4})^{-1} \quad (33)$$

and

$$M_z(\vec{x}) = \gamma_0\chi_B\tfrac{1}{4}\sqrt{15}\,\Omega_B U(1-U^2)^{-1/2}(\cosh s - \tfrac{1}{4})^{-1}, \quad (34)$$

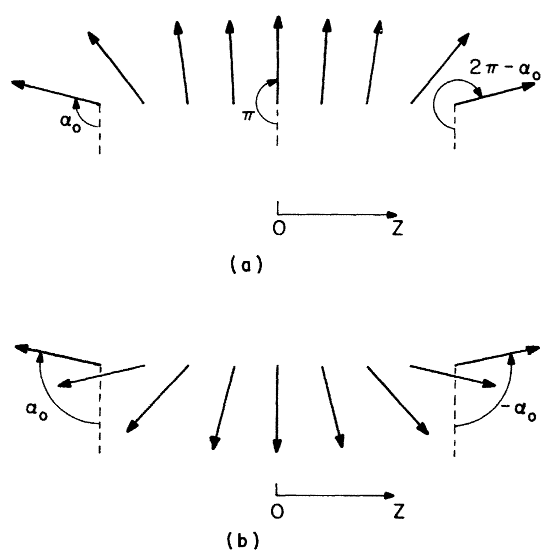

FIG. 3. Spatial variation of the rotation angle α of the $\hat{\omega}$ vector associated with two types of solitons in the B phase are shown. In (a) and (b) the α variations associated with the type-I soliton and the type-II soliton are shown, respectively. Here $\alpha_0 = \cos^{-1}(-\frac{1}{4})$.

respectively, for the type-I and the type-II solitons.

We have also looked for a two-soliton solution of Eq. (24) but we are so far unable to find an analytical solution. It appears that the equation in the A phase is the only known simple case, where a number of exact solutions are available.

IV. SCATTERING OF SPIN WAVES FROM THE DOMAIN WALL (A PHASE)

So far we have considered one-dimensional moving solutions (i.e., the domain wall or the disgyration plane) in both superfluid ^{3}He-A and ^{3}He-B. As is well known, the nonlinear equation of the spin coordinates contains another class of solutions, the spin waves describing a small fluctuation of the spin coordinate from the equilibrium configuration. We will describe in this section in some details the scattering of spin waves from a static soliton in the A phase. This poses no restriction in our formulation, since the scattering of spin waves from a moving soliton can be easily obtained from a pseudo-Lorentz transformation, where the velocity of the light has to be replaced by the corresponding spin-wave velocity $C(n)$. Furthermore, if we include in the dynamic equation the effect of the normal dissipation of the soliton due to the spin-diffusion term, the velocity of the soliton slows down exponentially in time[7] and after a while (say $t \gtrsim 10^{-2}$ sec) the soliton stops completely somewhere in the liquid ^{3}He. Then it stays there forever, unless it is annihilated with another soliton with the opposite helicity, at the wall of the container or by simply raising the temperature above the transition temperature. Therefore the problem considered here has a practical significance, although it may be very difficult to observe the predicted scattering phenomena experimentally at the time of writing. Furthermore, the bound-state solution and the scattering solutions thus determined enable us to construct the dynamic spin susceptibility in the presence of a soliton. In fact we will show that the presence of soliton decreases the intensity of the longitudinal resonance, which is measurable by the resonance technique.

So far it appears that the complete eigenfunctions associated the scattering of the spin wave from a soliton (or the domain wall) are known only for the sine-Gordon equation and therefore we limit ourselves to the longitudinal situation in the A phase. We assume as in Sec. I, that a static field H_0 is applied in the x direction. Furthermore, we assume that all spin waves carry the magnetic moment along the x direction. In this situation we can concentrate on the motion of β. The equilibrium values of α, β, and γ are given by $\alpha = \frac{1}{2}\pi$, $\gamma = 0$, and $\beta = \beta_0(x)$, where

$$\beta_0(x) = 2\tan^{-1}(e^{x/\xi}) . \tag{35}$$

Although we can consider the case in which the domain wall has an arbitrary angle to the static field without any further complication, we limit ourselves to the case that $\beta_0(x)$ depends only on x for simplicity. The above $\beta_0(x)$ implies that the domain wall lies in the y-z plane with the origin at $x=0$. The scattering of the spin waves from the domain wall is treated by Eq. (2), where we put $\alpha = \frac{1}{2}\pi$.

$$\ddot{\beta} - C_\parallel^2 \frac{\partial^2}{\partial z^2}\beta - C_\perp^2(\nabla_\perp^2 \beta) + \Omega_A^2 \sin\beta\cos\beta = 0 . \tag{36}$$

We will consider a small deviation of β from $\beta_0(x)$. Putting $\beta_1 = \beta - \beta_0(x)$ and expanding in β_1, we have

$$\ddot{\beta}_1 - C_\perp^2 \frac{\partial}{\partial x^2}\beta_1 + \Omega_A^2 \cos^2\beta_0(x)\beta_1 = 0 , \tag{37}$$

where we neglected the y and z dependence of β_1, which is redundant. Equation (37) has a set of eigenfunctions, which satisfy

$$-\lambda_n f_n(x) - C_\perp^2 \frac{\partial^2}{\partial x^2} f_n(x) + [V(x) + \Omega_A^2] f_n(x) = 0 , \tag{38}$$

where

$$V(x) = \Omega_A^2[\cos 2\beta_0(x) - 1] = -2\Omega_A^2 \cosh^{-2}(x/\xi) . \tag{39}$$

It is known that Eq. (38) has one bound state[10]

$$f_b(x) = (1/\sqrt{2\xi})\cosh^{-1}(x/\xi) \tag{40}$$

with $\lambda_b = 0$ and a set of scattering states[11,12]

$$f_k(x) = (2\pi)^{-1/2}\xi^{-1/2}(\Omega_A/\omega)e^{ikx}[\xi k + i\tanh(x/\xi)] \tag{41}$$

with

$$\lambda_k = \Omega_A^2 + C^2k^2 (\equiv \omega^2) .$$

In Fig. 4 the potential $V(x)$ and the bound-state wave function $f_b(x)$ are shown.

Equation (41) tells that the domain wall is transparent to the spin waves; all spin waves, which hit the domain wall, go through the domain wall.[12] The bound state, on the other hand, describes a small oscillation of the domain wall itself. So far we limit ourselves to the problem in the A phase. In fact it appears that the situation is quite different in the B phase (in the Leggett configuration). In the presence of the domain wall the equilibrium value of α is given by either $\alpha = \alpha_0^{\text{I}}(z)$ or $\alpha_0^{\text{II}}(z)$, where α_0^{I} and α_0^{II} are given in Eqs. (29) and (30). The scattering of the spin waves due to the domain wall can be recast in terms of the eigenvalue

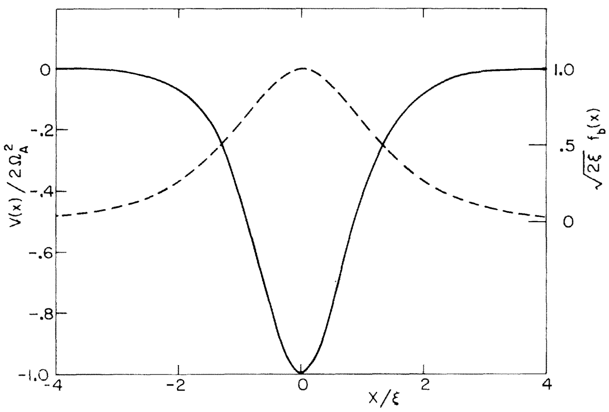

FIG. 4. Potential due to a static soliton (solid curve) in the A phase, together with the bound-state wave function (broken curve) are shown.

problem;

$$-\lambda_n f_n(z) - C_\parallel^2 \frac{\partial^2}{\partial z^2} f_n(z) + [V_{\mathrm{I,II}}(z) + \Omega_B^2] f_n(z) = 0\,, \quad (42)$$

where

$$\begin{aligned} V_{\mathrm{I}}(z) &= -\tfrac{16}{15}\Omega_B^2[\cos(2\alpha_0^{\mathrm{I}}) + \tfrac{1}{4}\cos\alpha_0^{\mathrm{I}}] - \Omega_B^2 \\ &= -\Omega_B^2\{\tfrac{15}{8}[\cosh(z/\xi) + \tfrac{1}{4}]^{-2} + \tfrac{3}{4}[\cosh(z/\xi) + \tfrac{1}{4}]^{-1}\} \end{aligned}$$

and

$$\begin{aligned} V_{\mathrm{II}}(z) &= -\tfrac{16}{15}\Omega_B^2[\cos(2\alpha_0^{\mathrm{II}}) + \tfrac{1}{4}\cos\alpha_0^{\mathrm{II}}] - \Omega_B^2 \\ &= -\Omega_B^2\{\tfrac{15}{8}[\cosh(z/\xi) - \tfrac{1}{4}]^{-2} \\ &\quad - \tfrac{3}{4}[\cosh(z/\xi) - \tfrac{1}{4}]^{-1}\}. \end{aligned} \quad (43)$$

It can be easily shown Eq. (42) has a zero-energy bound state

$$(\lambda_b = 0),\ f_b^{\mathrm{I,II}}(z) \propto \cos[\alpha_0^{\mathrm{I,II}}(z)] + \tfrac{1}{4}.$$

More explicitly we have

$$f_b^{\mathrm{I}}(z) = \frac{\sqrt{15}}{4\sqrt{2\xi}}\left(1 + \frac{\alpha_0 - \pi}{\sqrt{15}}\right)^{-1/2}[\cosh(z/\xi) + \tfrac{1}{4}]^{-1}$$

and

$$f_b^{\mathrm{II}}(z) = \frac{\sqrt{15}}{4\sqrt{2\xi}}\left(1 + \frac{\alpha_0}{\sqrt{15}}\right)^{-1/2}[\cosh(z/\xi) - \tfrac{1}{4}]^{-1},$$

respectively. The potential $V_{\mathrm{I}}(z)$ and $V_{\mathrm{II}}(z)$ together with the corresponding bound-state wave functions $f_b^{\mathrm{I}}(z)$ and $f_b^{\mathrm{II}}(z)$ are shown in Fig. 5. We note that in the case of the second type soliton in the B phase the potential $V_{\mathrm{II}}(z)$ has small shoulders at both edges.

However, we were so far unable to determine the scattering states. Furthermore making use of the criterion formulated by Kay and Moses,[11] the domain wall in the B phase is no longer transparent; the spin wave which collides with the domain wall in the B phase is partially reflected.

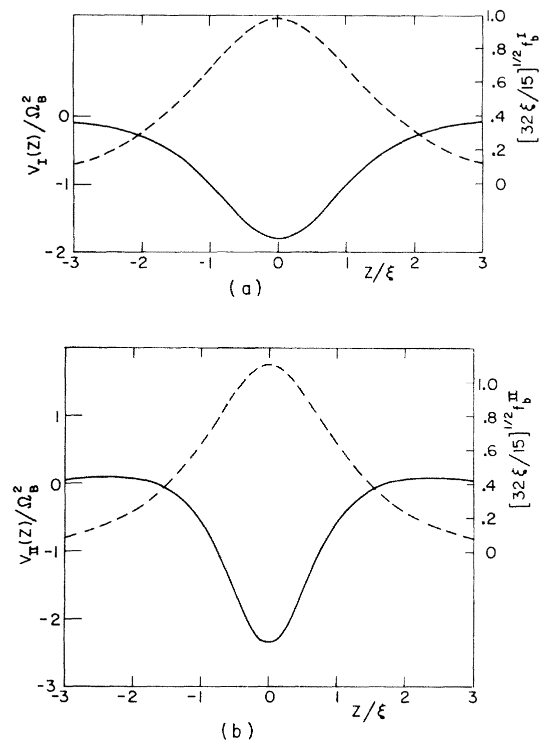

FIG. 5. Potential due to two types of solitons (solid curves) in the B phase, together with the corresponding bound-state wave functions (broken curve) are shown. In (a) and (b) are shown the cases of the type I and the type II, respectively.

V. LONGITUDINAL SPIN SUSCEPTIBILITY (IN THE A PHASE)

Making use of the complete set of the eigenfunctions discussed in Sec. IV, we can determine the longitudinal spin susceptibility in the A phase. We imagine that an oscillatory magnetic field $H_\omega(x)$ is applied parallel to the static field H_0 (along the x axis, Sec. II); the motion of β is then given by

$$\ddot{\beta} - C_\perp^2 \frac{\partial^2}{\partial x^2}\beta - \Omega_A^2 \sin\beta\cos\beta = -i\omega\gamma_0 H_\omega(x). \quad (44)$$

On the other hand the longitudinal magnetization is given by

$$M_x(x) = -\gamma_0\chi_N[\dot{\beta} - \gamma_0 H_\omega(x)]. \quad (45)$$

In the presence of a soliton (i.e., the domain wall), we linearize Eq. (44) in $\beta_1 = \beta - \beta_0(x)$ [where $\beta_0(x)$ has been already given in Eq. (35)] and we have

$$\ddot{\beta}_1 - C_\perp^2 \frac{\partial^2}{\partial x^2}\beta_1 + \Omega_A^2 \cos 2\beta_0(x)\beta_1 = -i\omega\gamma_0 H_\omega(x). \quad (46)$$

The above equation is solved in terms of the eigenfunctions Eqs. (40) and (41) as

$$\beta_1(x) = -i\omega\gamma_0 \int g(x,x')H_\omega(x')\,dx', \tag{47}$$

where

$$g(x,x') = -\frac{1}{\omega^2} f_b(x) f_b(x') + \frac{1}{2k\omega^2}\left[\sin k|x-x'|(k^2+\xi^{-2}\tanh x/\xi \tanh x'/\xi) + \xi^{-1}k\cos k(x-x')|\tanh x/\xi - \tanh x'/\xi|\right], \tag{48}$$

and $k=(1/C)(\omega^2-\Omega_A^2)^{1/2}$.

In terms of the above Green's function $g(x,x')$, we can express the magnetization as

$$M_x(x) = \gamma_0^2 \chi_N \int_\infty^\infty G_A(x,x')H_\omega(x')\,dx', \tag{49}$$

where

$$G_A(x,x') = \delta(x-x') + \omega^2 g(x,x') \tag{50}$$

$$= \frac{i\Gamma_A}{i\Gamma_A - \omega} f_b(x) f_b(x') + \frac{1}{2k}\left(\xi^{-2}\tanh\frac{x}{\xi}\tanh\frac{x'}{\xi}\sin k|x-x'| + \xi^{-1}k\left|\tanh\frac{x}{\xi} - \tanh\frac{x'}{\xi}\right|\cos k|x-x'|\right). \tag{51}$$

Here we made use of a relation[11]

$$\delta(x-x') = f_b(x) f_b(x') + \int_{-\infty}^{\infty} dk\, f_k^*(x) f_k(x').$$

In the first term of Eq. (47), the lifetime of the zero-energy mode has been included (see Appendix). The lifetime arises from the spin-diffusion term[7]; Γ_A can be expressed in terms of the spin-diffusion constant D in the A phase as

$$\Gamma_A = D_\perp \int_{-\infty}^{\infty} dx \left(\frac{\partial f_b}{\partial x}(x)\right)^2 = \tfrac{1}{3} D_\perp \xi^{-2}. \tag{52}$$

The first term in Eq. (51) gives rise to a central peak in the resonance experiment at $\omega=0$. While the second term gives rise to the possible excitation of the spin wave with the wave vector k due to the oscillatory field in the presence of the soliton. We note that the existence of the domain wall and the central peak associated with the oscillation of the domain wall was discussed recently in the case of the Ginzburg-Landau field theory of Krumhansl and Schrieffer.[13] When $H_\omega(x)$ is almost homogeneous, the contribution from the second term in Eq. (51) vanishes identically; this implies that if the soliton sits in the middle of the cavity, the longitudinal resonance at $\omega=\Omega_A$ disappears completely. In general the existence of the soliton in the cavity appears to reduce the signal intensity enormously.

We can carry over formally the present result to the longitudinal resonance in the B phase. However, since we can not solve the scattering problem in the B phase, we have no simple expression for the contribution from the continuum. If we limit to the bound state, we have the contribution to $G_B^{\mathrm{I}}(z,z)$ as

$$G_B^{\mathrm{I},0}(z,z') = \frac{i\Gamma_B^{\mathrm{I}}}{i\Gamma_B^{\mathrm{I}} - \omega} f_b^{\mathrm{I}}(z) f_b^{\mathrm{I}}(z'),$$

$$G_B^{\mathrm{II},0}(z,z') = \frac{i\Gamma_B^{\mathrm{II}}}{i\Gamma_B^{\mathrm{II}} - \omega} f_b^{\mathrm{II}}(z) f_b^{\mathrm{II}}(z'), \tag{53}$$

where now $\chi_z(z,z)$ (see Sec. III) is given by

$$\chi_z(z,z) = \chi_B G_B^{\mathrm{I,II}}(z,z)$$

for the type-I soliton and the type-II soliton, respectively. The lifetimes of the central peaks are given by

$$\Gamma_B^{\mathrm{I}} = D_B \int \left(\frac{\partial f_b^{\mathrm{I}}}{\partial z}(z)\right)^2 dz = \frac{1}{30}\,\frac{11 + 16/\sqrt{15}\,(\alpha_0 - \pi)}{1 + (1/\sqrt{15})(\alpha_0 - \pi)}\, D_B \xi_B^{-2}$$

and

$$\Gamma_B^{\mathrm{II}} = D_B \int \left(\frac{\partial f_b^{\mathrm{II}}}{\partial z}(z)\right)^2 dz = \frac{1}{30}\,\frac{11 + 16/\sqrt{15}\,\alpha_0}{1 + (1/\sqrt{15})\alpha_0}\, D_B \xi_B^{-2}, \tag{54}$$

respectively, where D_B is the spin-diffusion constant in the B phase. We note that Γ_B^{I} appears elsewhere[7] as the characteristic damping constant for the type-I soliton. Although we cannot construct the contribution from the continuum to $G_B^{\mathrm{I,II}}$, we believe that the contribution disappears when $H_\omega(z)$ is homogeneous and the soliton (or the domain wall) lies in the middle of the cavity. Therefore we expect a large reduction of the resonance signal at $\omega=\Omega_B$ in the presence of solitons.

VI. CONCLUDING REMARKS

We have studied the one-dimensional stationary solutions of the nonlinear equation governing the spin dynamics in superfluid ^{3}He. We find that the stationary solution represents the domain wall between two energetically degenerate textures in superfluid ^{3}He. The moving domain walls or solitons can be created by turning off a local magnetic field, if the field is strong enough $[\Delta H_{\max} \simeq (1/\gamma_0)\Omega_A$ and $(1/\gamma_0)\Omega_B$ for the A phase and the B phase, respectively$]$. We studied also the scattering of the spin waves by the domain wall. It is shown that the domain wall in the A phase is transparent, while the domain wall in the B phase partially reflects the spin waves. Furthermore, the existence of the domain wall reduces significantly the normal longitudinal resonance at $\omega = \Omega_A$ (or Ω_B) in the A phase (or in the B phase) and gives rise to a central peak in the resonance at $\omega = 0$, associated with the oscillation of the domain wall itself in the presence of the oscillatory field.

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APPENDIX: EFFECT OF THE SPIN-DIFFUSION TERM ON THE SPIN SUSCEPTIBILITY

We will study here the effect of the spin-diffusion term on the susceptibility based on a phenomenological model.[7]

A. A phase

In the presence of the spin-diffusion term Eq. (44) in the text is modified by

$$\ddot{\beta} - C_\perp^2 \frac{\partial^2}{\partial x^2}\beta - D_\perp \frac{\partial^2 \dot{\beta}}{\partial x^2} + \Omega_A^2 \sin\beta\cos\beta = -i\omega\gamma_0 H_\omega(x)\,, \tag{A1}$$

where $D_\perp$ is the spin-diffusion constant. We have neglected here the damping due to the intrinsic spin relaxation[14] in the present situation since this term gives smaller correction to the zero-frequency mode by a factor ω/Ω_A, where ω is the frequency of the oscillatory field. After linearizing in $\beta_1 = \beta - \beta_0$, we have an eigenequation in analogy to Eq. (38),

$$-\lambda_n f_n(x) - C^2{}_\perp \frac{\partial^2}{\partial x^2} f_n(x) - i\omega D_\perp \frac{\partial^2}{\partial x^2} f_n(x) + [V(x) + \Omega_A^2] f_n(x) = 0, \tag{A2}$$

where ω is the oscillation frequency. We can treat the effect of the spin-diffusion term by perturbation. For the bound state, substituting in (A2) $f_n(x) = f_b(x)$, we have

$$-\lambda_b f_b(x) - i\omega D_\perp \frac{\partial^2}{\partial x^2} f_b(x) = 0\,; \tag{A3}$$

multiplying (A3) by $f_b(x)$ and taking the integral over x, we have,

$$\begin{aligned}\lambda_b &= -i\omega D_\perp \int_{-\infty}^{\infty} f_b(x) \frac{\partial f_b(x)}{\partial x^2}\,dx \\ &= i\omega D_\perp \int_{-\infty}^{\infty} \left(\frac{\partial f_b}{\partial x}\right)^2 dx = i\omega\Gamma_A\,,\end{aligned} \tag{A4}$$

where Γ_A has been given in the text. Therefore in the presence of the relaxation, the double pole in ω in Eq. (48) is replaced by

$$-\frac{1}{\omega^2} \to \frac{1}{\lambda_b - \omega^2} = \frac{1}{\omega(i\Gamma_A - \omega)} \tag{A5}$$

which yields the first term in Eq. (51).

Within the same approximation we can calculate the effect of the spin-diffusion term to λ_n of the scattering states in Eq. (41)

$$\lambda_k = \Omega_A^2 + C_\perp^2 k^2 + i\omega D_1 \int_{-\infty}^{\infty} \left(\frac{\partial f_k}{\partial x}\right)^2 dx \Big/ \int_{-\infty}^{\infty} |f_k|^2\,dx\,. \tag{A6}$$

Although in this case the damping due to the intrinsic spin relaxation is as important (since now $\omega \sim \Omega_A$).

B. B phase

In the B phase Eq. (42) in the presence of the spin-diffusion term is similarly modified;

$$-\lambda_n f_n(z) - C_\parallel^2 \frac{\partial^2}{\partial z^2} f_n(z) - i\omega D_B \frac{\partial^2}{\partial z^2} f_n(z) + [V_{\mathrm{I,II}}(z) + \Omega_B^2] f_n(z) = 0\,. \tag{B1}$$

For the bound state, within the similar approximation as before we have

$$\lambda_b^{\mathrm{I,II}} = i\omega D_B \int_{-\infty}^{\infty} \left(\frac{\partial f_b^{\mathrm{I,II}}}{\partial z}(z)\right)^2 (dz), \tag{B2}$$

where the bound-state wave function has to be normalized $\{$i.e., $\int_{-\infty}^{\infty} [f_b^{\mathrm{I,II}}(z)]^2\,dz = 1\}$.

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