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Quantum critical phenomena*

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This paper proposes an approach to the study of critical phenomena in quantum-mechanical systems at zero or low temperatures, where classical free-energy functionals of the Landau-Ginzburg-Wilson sort are not valid. The functional integral transformations first proposed by Stratonovich and Hubbard allow one to construct a quantum-mechanical generalization of the Landau-Ginzburg-Wilson functional in which the order-parameter field depends on (imaginary) time as well as space. Since the time variable lies in the finite interval $[0,-i\beta]$, where β is the inverse temperature, the resulting description of a d-dimensional system shares some features with that of a $(d+1)$-dimensional classical system which has finite extent in one dimension. However, the analogy is not complete, in general, since time and space do not necessarily enter the generalized free-energy functional in the same way. The Wilson renormalization group is used here to investigate the critical behavior of several systems for which these generalized functionals can be constructed simply. Of these, the itinerant ferromagnet is studied in greater detail. The principal results of this investigation are (i) at zero temperature, in situations where the ordering is brought about by changing a coupling constant, the dimensionality which separates classical from nonclassical critical-exponent behavior is not 4, as is usually the case in classical statistics, but $4-z$ dimensions, where z depends on the way the frequency enters the generalized free-energy functional. When it does so in the same way that the wave vector does, as happens in the case of interacting magnetic excitons, the effective dimensionality is simply increased by 1; $z=1$. It need not appear in this fashion, however, and in the examples of itinerant antiferromagnetism and clean and dirty itinerant ferromagnetism, one finds $z=2$, 3, and 4, respectively. (ii) At finite temperatures, one finds that a classical statistical-mechanical description holds (and nonclassical exponents, for $d<4$) very close to the critical value of the coupling U_c, when $(U-U_c)/U_c \ll (T/U_c)^{2/z}$. $z/2$ is therefore the quantum-to-classical crossover exponent.

I. INTRODUCTION

The spectacularly successful analysis of critical phenomena in a wide variety of systems using Wilson's renormalization-group ideas[1] has hitherto been limited to *classical* statistical-mechanical models. Such a description is appropriate whenever the critical temperature is finite, provided one is close enough to the instability. Then, when all fluctuation modes have characteristic energies $\ll kT_c$, classical statistics are appropriate. However, one can also think about a phase transition in a zero-temperature system which occurs when, say, a coupling constant reaches a certain threshold. In this case, none of the fluctuation modes have thermal energies, and their statistics will be highly nonclassical. By the same token, in the same system at a finite but low temperature, one should expect quantum effects to be dominant except in a narrow range of coupling strengths near the critical value. (By low temperature, I mean kT much less than characteristic microscopic energies, such as the Fermi energy, bandwidth, Coulomb or exchange energies, etc.)

In addition to quantum effects at low or zero temperature in the equilibrium correlation functions and static-response coefficients, we should expect quite different dynamical properties. In the classical case, one can study dynamical critical phenomena using time-dependent Landau-Ginzburg equations or generalizations thereof.[2] These equations contain as parameters transport coefficients whose existence depends on the presence of collisions to maintain local thermal equilibrium. In a zero-temperature problem, by contrast, there are no collisions, and consequently no transport coefficients and no time-dependent Landau-Ginzburg equations. Similarly, at low T, the dynamics will be effectively collisionless except very close to the critical coupling.

One feature of the classical problem is the separability of the statics and the dynamics—the former may be solved independently of the latter. We shall see here that this, too, breaks down in systems where quantum mechanics is important. Statistics and dynamics are then inextricably connected, and one has to solve for both equilibrium and nonequilibrium properties together in the same formalism, rather than doing the dynamics afterwards. This complication is offset, however, by the fact that the formalism we shall use makes this unified approach the straightforward and natural one.

Our principal formal tool for setting up this class of problems is the functional-integral transformation of Stratonovich and Hubbard.[3] It allows one to construct an exact quantum generalization of the Landau-Ginzburg-Wilson (LGW) free-energy functional used in classical problems. The precise form of this functional will depend on the character of the dynamics of the system in question, but all quantum functionals share the feature that the order-parameter field depends on time as well as space. The time variable is, as one might expect in a quantum-statistical problem, imaginary and in the interval $[0, -i\beta]$ ($\beta = 1/kT$). The Fourier transform of the order parameter (in terms of which it is usually simpler to write the functional) therefore is a function of frequency as well as wave vector, and the frequencies which occur are the (Bose) Matsubara frequencies $i\omega_n = 2\pi i n/\beta$. Section III is devoted to a discussion of the derivation of this functional for the problem of interacting paramagnons in itinerant ferromagnetism.

With this as a starting point, in Sec. III we apply the renormalization group and study the evolution of the parameters in the functional as high wave numbers *and high frequencies* are scaled out of the problem. We show that the quantum LGW functional has a stable Gaussian fixed point under the renormalization group at zero temperature for $d \geq 1$. The $T=0$ critical exponents are thus mean-field-like. We then examine the instability of this fixed point at $T \neq 0$ and calculate the crossover exponent which characterizes the eventual switch to a non-Gaussian fixed point and non-mean-field exponents. In Sec. IV we discuss the utility of approximate solutions of the renormalization-group equations as a substitute for more conventional perturbation-theoretical techniques in problems like this, and examine the effect of the hitherto ignored higher-order terms in the generalized free-energy functional on such solutions. In Sec. V we introduce and apply the renormalization group to several other models in which quantum effects can be important—itinerant antiferromagnetism, interacting magnetic excitons, and the paramagnon problem in the presence of impurities. Finally, Sec. VI is devoted to a somewhat different version of the quantum renormalization group in which all frequency components of the order parameter $\Psi(q, \omega)$ with same q are scaled out of the problem together at each stage of the renormalization-group operation. This procedure is different from that mentioned above, where one scales out high ω and high q together, in that time and space are no longer treated on the same footing. The results are the same, however, and this formulation does have the advantage that it, in principle, allows one to follow the crossover from quantum to classical scaling continuously.

An abbreviated account of part of this work was presented earlier.[4]

II. GENERALIZED LGW FUNCTIONAL FOR INTERACTING PARAMAGNONS

The application of the Stratonovich-Hubbard transformation to itinerant ferromagnetism has been discussed extensively in the literature.[5-7] Here we only outline the steps involved in generating the free-energy functional. We start with a Hubbard interaction Hamiltonian,[8] written in terms of charge- and spin-density variables:

$$H' = U \sum_i n_{i\uparrow} n_{i\downarrow} = \frac{U}{4} \sum_i (n_{i\uparrow} + n_{i\downarrow})^2 - \frac{U}{4} \sum_i (n_{i\uparrow} - n_{i\downarrow})^2 . \quad (2.1)$$

Statistical mechanics requires knowledge of matrix elements of the operator

$$e^{-\beta H} = e^{-\beta H_0} T \exp\left(-\int_0^\beta d\tau H'(\tau) \right) . \quad (2.2)$$

The Stratonovich-Hubbard transformation applies the identity

$$e^{a^2/2} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2 - ax} \quad (2.3)$$

to (2.2) for each imaginary time τ between 0 and β and for every site in the lattice, with the result that

$$Z = \mathrm{Tr}\, e^{-\beta H} = Z_0 \int \delta\Psi \exp\left(-\frac{1}{2} \int_0^\beta d\tau \sum_i \Psi_i^2(\tau) \right) \times \left\langle \mathrm{Tr}\, T \exp\left(-\int_0^\beta d\tau \sum_{i\sigma} \sigma V_i(\tau) n_{i\sigma}(\tau) \right) \right\rangle_0 . \quad (2.4)$$

Here $V_i(\tau) = (\frac{1}{2}U)^{1/2} \Psi_i(\tau)$ is a time-dependent magnetic field acting on site i at "time" τ, and Z_0 is the partition function of the noninteracting system. [In addition to V, there should be another field in the exponential inside the expectation value in (2.4), coupled to the charge density. We ignore it here, since we want to concentrate on the spin fluctuations and expect that charge-density fluctuations will be relatively unimportant.] The expectation value in (2.4) can be expressed in terms of the electron Green's functions of the noninteracting system,

$$G^0_{ij}(\tau,\tau')=\frac{1}{\beta}\sum_{kn}\frac{\exp[i\vec{k}\cdot(\vec{R}_i-\vec{R}_j)-iE_n\tau]}{iE_n-\epsilon_k}, \quad (2.5)$$

so that

$$Z=Z_0\int\delta\Psi\exp\left(-\frac{1}{2}\int_0^\beta d\tau\sum_i\Psi_i^2(\tau)+\sum_\sigma \mathrm{Tr}\ln(1-\sigma VG^0)\right), \quad (2.6)$$

where the matrix V has elements $V_{ij}(\tau,\tau')=V_i(\tau)\delta_{ij}\delta(\tau-\tau')$.

The exponential in Eq. (2.6) is then a formally exact free-energy functional $\Phi[\Psi]$ in which $\Psi_i(\tau)$ [or, in a continuum limit, $\Psi(x,\tau)$] is the order-parameter field. To do much with it, it is generally advantageous to expand it in a power series in Ψ, leading to an expression of the general form

$$\Phi[\Psi]=\frac{1}{2}\sum_{q\omega}v_2(q,\omega)|\Psi(q,\omega)|^2+\frac{1}{4\beta N}\sum_{q_i\omega_i}v_4(q_1\omega_1,q_2\omega_2,q_3\omega_3,q_4\omega_4)\Psi(q_1,\omega_1)\Psi(q_2,\omega_2)\Psi(q_3,\omega_3)\Psi(q_4,\omega_4)\times\delta\left(\sum_{i=1}^4 q_i\right)\delta\left(\sum_{i=1}^4\omega_i\right)+\cdots+\frac{1}{m(\beta N)^{m/2-1}}\sum_{q_i,\omega_i}v_m(q_1\omega_1,\ldots,q_m\omega_m)\prod_{i=1}^m\Psi(q_i,\omega_i)\,\delta\left(\sum_{i=1}^m q_i\right)\delta\left(\sum_{i=1}^m\omega_i\right)+\cdots. \quad (2.7)$$

Clearly the form is analogous to that discussed by Wilson for classical statistics. The effect of quantum mechanics can be traced to the noncommutativity of H_0 and H', which forced us to write $e^{-\beta H}$ in the interaction representation (2.2), requiring the functional averaging identity (2.3) to be applied for each time τ. This makes the order parameter time dependent, with the consequence that (Matsubara) frequencies appear in (2.7) on the same footing as wave vectors. It is as if another dimension were added to the system, but, except at zero temperature, the extent of the system in the extra dimension is finite. We shall examine the effects of the consequent finite spacing between Matsubara frequencies at the end of Sec. III.

The coefficients v_m in (2.7) (irreducible bare-m-point vertices in a diagrammatic perturbation-theoretic development) can in principle be evaluated in terms of the band propagators (2.5); v_m is just proportional to a loop of m electron propagators, with four-momentum transfers $q_1,\omega_1,\ldots,q_m,\omega_m$ between propagator lines.[7] The quadratic coefficient v_2, which we will want to examine here, has an extra term of unity because of the Gaussian weight factor in the functional integral,

$$v_2(q,\omega)=1-U\chi_0(q,\omega), \quad (2.8)$$

where χ_0 is the function evaluated by Lindhard for a free-electron model,[9]

$$\chi_0(q,\omega)=-\frac{1}{\beta}\sum_{kn}G(k,iE_n)G(k+q,iE_n+\omega)=-\sum_k\frac{f(\epsilon_k)-f(\epsilon_{k+q})}{\epsilon_k-\epsilon_{k+q}+\omega}. \quad (2.9)$$

For small q and small ω/qv_F, it has the expansion[10]

$$\chi_0(q,i\omega_m)=N(E_F)[1-\tfrac{1}{3}(q/2k_F)^2-\tfrac{1}{2}\pi(|\omega_m|/qv_F)+\cdots]. \quad (2.10)$$

As long as we are near the ferromagnetic instability $UN(E_F)=1$, this long-wavelength, low-frequency form of χ_0 gives a good representation of the paramagnon propagator or its spectral weight function

$$\mathrm{Im}\chi(\omega+i\delta)=\mathrm{Im}\left(\frac{\chi_0(q,\omega)}{1-U\chi_0(q,\omega)}\right)\approx\frac{(2/\pi)v_FqN(E_F)\omega}{\omega^2+\{(2/\pi)v_Fq[1-UN(E_F)+\frac{1}{3}(q/2k_F)]\}^2} \quad (2.11)$$

for frequencies where most of the spectral weight lies, since $\mathrm{Im}\chi$ is sizable mostly near a frequency

$$\omega_q=(2/\pi)v_Fq[1-UN(E_F)+\tfrac{1}{3}(q/2k_F)^2]\ll v_Fq. \quad (2.12)$$

The higher-order loops v_m, $m\geq 4$, are complicated functions of the q_i and ω_i, but they share with v_2 the fact that they vary with any q on a scale of $2k_F$ and with ω on a scale of E_F. Put physically, the force between paramagnons has a range $\approx (2k_F)^{-1}$ in space and a retardation $\approx 1/E_F$ in time. When all of the q_i and ω_i vanish, v_m is simply proportional to the $(m-2)$nd derivative of the band density of states at E_F.[7]

In this section we will see an approximation to the full functional in which we use the expansion (2.10) in v_2, ignore all q and ω dependence on v_4, and discard higher-order vertices completely. The finite range and retardation effects will be simulated by cutting off all q sums at $\approx 2k_F$ and all ω sums at $\approx E_F$. Choosing units appropriately, we can write our approximate functional as

$$\Phi[\Psi] = \frac{1}{2}\sum_{q\omega}\left(r_0 + q^2 + \frac{|\omega|}{q}\right)|\Psi(q,\omega)|^2 + \frac{u_0}{4N\beta}\sum_{q_i\omega_i}\Psi(q_1,\omega_1)\Psi(q_2,\omega_2)\Psi(q_3,\omega_3) \times \Psi(-q_1-q_2-q_3, -\omega_1-\omega_2-\omega_3)\,, \tag{2.13}$$

where, in terms of microscopic parameters,

$$r_0 = 1 - UN(E_F)\,, \tag{2.14a}$$

$$u_0 = \tfrac{1}{12}U^2 N''(E_F)\,. \tag{2.14b}$$

This Φ is of almost the same form as the classical LGW functional, except for the presence of the frequency-dependent term in v_2, which contains the essential information about the dynamics. It tells us that the decay mechanism for the paramagnon excitations is Landau damping—the lifetime of a free particle-hole pair of total momentum q is $(v_F q)^{-1}$, and the correlations enhance this lifetime (for small q) by a factor $[1-UN(E_F)]^{-1}$, as reflected in (2.11). This is why ω enters (2.13) in the form $|\omega|/q$. If the dynamics were different, this term would have a different form. We shall examine examples with different dynamics in Sec. V.

Our functional (2.13) therefore describes a set of interacting, weakly-Landau-damped excitations. Terms of higher order in Ψ, as well as higher-order expansions of the coefficients included here in powers of q and ω, contain no essential new physics and in fact are "irrelevant" to the zero-temperature critical behavior in the sense described by Wilson.[1]

It is also possible to write the generalized LGW functional in a form which preserves the rotational invariance of the original Hamiltonian by using a vector paramagnon field $\vec{S}$ in place of Ψ. We shall not dwell at length on the formal derivation of this functional, since this aspect of the problem has been discussed elsewhere.[6,11] Our emphasis is on the form of the coefficients to order S^4 and their physical implications.

The starting point lies in expressing the interaction Hamiltonian as

$$H' = \tfrac{1}{2}U\sum_i (n_{i\uparrow} + n_{i\downarrow}) - \tfrac{2}{3}U\sum_i \vec{S}_i\cdot\vec{S}_i \tag{2.15}$$

instead of (2.1), which only has $S_i^z S_i^z$ terms. Then the application of the identity (2.3) leads to an expression for the partition function [cf. (2.6)]

$$z = z_0\int\delta S\exp\left(-\frac{1}{2}\sum_i\int_0^\beta d\tau\,\vec{S}_i^2(\tau) + \mathrm{Tr}\ln(1-VG^0)\right) \tag{2.16}$$

in which V and G are matrices in spin space as well as in space-time indices, and the Tr indicates a trace over both spin and space-time indices. Explicitly, G^0 is the spin diagonal with elements (2.5), and

$$\langle i,\tau,m|V|j,\tau',m'\rangle = (\tfrac{1}{3}U)^{1/2}\delta_{ij}\delta(\tau-\tau')\vec{S}_i(\tau)\cdot\langle m|\vec{\sigma}|m'\rangle\,. \tag{2.17}$$

When the Tr ln in (2.16) is expanded in powers of V, the quadratic term of the exponent becomes

$$\frac{1}{2}\sum_{q\omega\alpha}[1-\tfrac{2}{3}U\chi_0(q,\omega)]|S_\alpha(q,\omega)|^2 \tag{2.18}$$

and the fourth-order term looks like

$$\frac{1}{4\beta N}\sum_{\substack{q_i,\omega_i\\ \alpha\beta\gamma\delta}} v_4^{\alpha\beta\gamma\delta}(\{q_i,\omega_i\}) \times S_\alpha(q_1\omega_1)S_\beta(q_2\omega_2)S_\gamma(q_3\omega_3)S_\delta(q_4\omega_4) \times\delta\left(\sum_{i=1}^4 q_i\right)\delta\left(\sum_{i=1}^4\omega_i\right)\,, \tag{2.19}$$

where $v_4^{\alpha\beta\gamma\delta}$ is proportional to the v_4 which appeared in the scalar description [Eq. (2.7)],

$$v_4^{\alpha\beta\gamma\delta} = \tfrac{1}{2}(\tfrac{2}{3})^2 v_4\,\mathrm{Tr}(\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta)\,. \tag{2.20}$$

The important point is that the dependence of the quadratic and quartic coefficients on wavevector and frequency arguments is the same as in the scalar case, and the dependence on the polarization labels follows simply from the Pauli spin algebra. So in order that four-paramagnon modes have nonvanishing interactions, the Pauli matrices corresponding to their polarizations must multiply to give the unit matrix. One way to do this is to have $\alpha=\beta$ and $\gamma=\delta$; this part of the interaction is then of the form $(\vec{S}\cdot\vec{S})(\vec{S}\cdot\vec{S})$, as occurs in the usual LGW functional for a vector field. But one can also have $\alpha=\gamma$, $\beta=\delta$ or $\alpha=\delta$,

$\beta=\gamma$, leading to a part of $v_4^{\alpha\beta\gamma\delta}$ proportional to $\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\gamma}\delta_{\beta\delta}=\epsilon_{\alpha\beta\mu}\epsilon_{\gamma\delta\mu}$. This means there is also a part of the S^4 interaction of the form $(\vec{S}\times\vec{S})\cdot(\vec{S}\times\vec{S})$. This term vanishes if all frequency and wave-vector dependence of v_4 is ignored, since we could then write its contribution to Φ as something proportional to

$$v_4 \sum_i \int_0^\beta d\tau [\vec{S}_i(\tau)\times\vec{S}_i(\tau)]^2 = 0\,. \tag{2.21}$$

Nonlocal or retarded terms must therefore be retained in order to see any effects of these parts of the interaction. Such terms, however, are beyond the scope of the present discussion, in which nonlocal effects in v_4 are irrelevant to the phenomena of interest. Accordingly, our vector paramagnon LGW functional is

$$\Phi[\vec{S}]=\frac{1}{2}\sum_{q\omega\alpha}\left(r_0+q^2+\frac{|\omega|}{q}\right)|S_\alpha(q,\omega)|^2 + \frac{u_0}{4\beta N}\sum_{\substack{q_i\omega_i\\ \alpha\beta}} S_\alpha(q_1\omega_1)\,S_\alpha(q_2\omega_2)\,S_\beta(q_3\omega_3) \times S_\beta(-q_1-q_2-q_3,\,-\omega_1-\omega_2-\omega_3)\,. \tag{2.22}$$

The parameters r_0 and u_0 differ from their scalar problem counterparts [as in (2.18) and (2.20)], but this point will not be important here.

There is nothing really new in this section. I have simply collected from various sources the points relevant to establishing the basis of the model functional. The purpose of doing so was purely pedagogical.

III. RENORMALIZATION-GROUP TRANSFORMATION

Beal-Monod was the first to apply the renormalization group to the quantum functional (2.13).[12] She noted that the time acted like an extra dimension, and asserted that the critical behavior would be just the same as that of a $(d+1)$-dimensional system. This is not true, however. The frequency enters (2.13) in the form of a term in Φ proportional to $|\omega|/q$, which is quite different from the way the wave vector occurs. This anisotropy partially destroys the analogy between the present problem and a $(d+1)$-dimensional classical LGW problem, and renders her conclusion invalid. We will see here that it is necessary to generalize the Wilson scaling procedure somewhat to deal with the anisotropic coupling.

The general idea of the renormalization operation is the same as Wilson's. There are three steps: (a) Terms in Φ which have the wave vectors or frequencies of some of the Ψ fields in an "outer shell" are eliminated from the functional integral by carrying out the integration over these $\Psi(q,\omega)$, while holding fixed the $\Psi(q,\omega)$ with smaller q or ω. (b) The variables q and ω, which in the remaining functional integral run up only to a cutoff less than the original one, are rescaled, so that they once again take on the range of values they had in the original problem, before step (a). (c) The fields Ψ are rescaled, so that in terms of the new fields and the rescaled q and ω, the terms with q^2 and $|\omega|/q$ in the quadratic part of Φ look just like those in the original functional. That is, the coefficients of q^2 and $|\omega|/q$ in (2.13) must remain at unity under the group transformation. The only difference will turn out to be that in step (b), q and ω must be rescaled differently, as a consequence of the anisotropy of the functional in the "extra dimension."

To see why this happens, let us try doing the scaling isotropically. We use, as we will throughout this paper, a scaling procedure in which only an infinitesimally thin shell of Ψ's is removed at each stage of the renormalization procedure.[13]

Suppose that in step (a) we have removed Ψ's with $e^{-l}<q<1$ and $e^{-l}<|\omega|<1$, with l infinitesimal. This will affect the quadratic term of (2.12) in two ways: (i) r_0 will be changed to a new value r_0' (the change is of order l), and (ii) the sums on q and ω now have q and $|\omega|$ less than e^{-l}. It is easier to keep track of the manipulations we make if we write the sums as integrals, so the quadratic term (call it Φ_2 originally) now looks like

$$\Phi_2'=\tfrac{1}{2}\beta N\int_0^{e^{-l}}\frac{d^dq\,d\omega}{(2\pi)^{d+1}}\left(r_0'+q^2+\frac{|\omega|}{q}\right)|\Psi(q,\omega)|^2\,. \tag{3.1}$$

Rescaling q and ω (step b) by letting

$$q'=qe^l,\quad \omega'=\omega e^l, \tag{3.2}$$

we can write Φ_2' as

$$\Phi_2'=\tfrac{1}{2}\beta N e^{-(d+1)l}\int_0^1\frac{d^dq'\,d\omega'}{(2\pi)^{d+1}} \times\left(r_0'+q'^2e^{-2l}+\frac{|\omega'|}{q'}\right)|\Psi(q'e^{-l},\omega'e^{-l})|^2\,. \tag{3.3}$$

Step (c) would then have us define a $\Psi'(q',\omega')$, proportional to $\Psi(q'e^{-l},\omega'e^{-l})$ with the coefficient of proportionality chosen so that the coefficients of q'^2 and $|\omega|/q$ are both unity. But this cannot be dome, since the two terms in (3.3) have different coefficients ($e^{-(d+3)l}$ and $e^{-(d+1)l}$, respectively), so any redefinition of Ψ, which multiplies both terms, cannot make *both* coefficients unity. One way to proceed would be to settle for keeping one

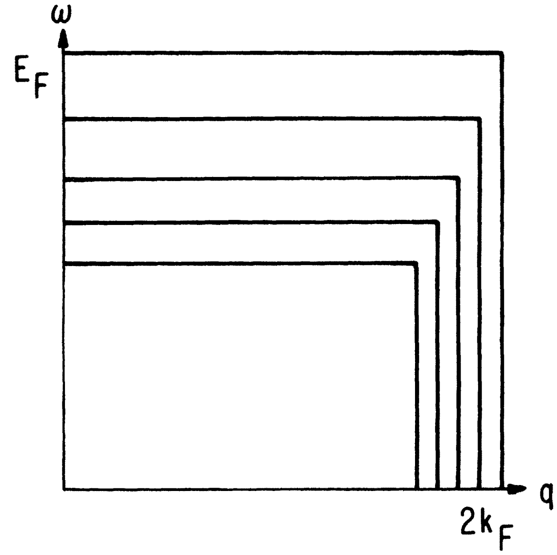

FIG. 1. Scaling procedure (3.4) in q and ω space.

of the coefficients, say that of q'^2, fixed, and letting the other one vary as dictated by that transformation. Here we shall get around the difficulty in a different way, however.

Instead of (3.2), we choose a more general scaling,

$$q' = qe^l, \quad \omega' = \omega e^{zl}, \tag{3.4}$$

that is, we scale down at different rates in wave number and frequency. This is pictured schematically in Fig. 1. Then instead of (3.3) we have

$$\Phi_2' = \tfrac{1}{2}\beta N e^{-(d+z)l} \int_0^1 \frac{d^dq'\,d\omega'}{(2\pi)^{d+1}} \times\left(r_0' + q'^2 e^{-2l} + \frac{|\omega'| e^{-zl}}{q'e^{-l}}\right) |\Psi(q'e^{-l}, \omega' e^{-zl})|^2 . \tag{3.5}$$

It is apparent that if we choose $z=3$, the coefficient of $|\omega'|/q'$ inside the large parentheses will be e^{-2l}, the same as the coefficient of q'^2. Then a rescaling of Ψ can make the total coefficient of both of them unity. One demands

$$|\Psi'(q', \omega')|^2 = e^{-(d+z)l}\, e^{-2l} |\Psi(q'e^{-l}, \omega' e^{-zl})|^2$$

or

$$\Psi'(q', \omega') = e^{-(d+z+2)l/2}\Psi(q, \omega). \tag{3.6}$$

Then

$$\Phi_2' = \frac{\beta N}{2} \int_0^1 \frac{d^dq'\,d\omega'}{(2\pi)^{d+1}} \left(r_0' + q'^2 + \frac{|\omega'|}{q'}\right) |\Psi'(q', \omega)|^2 , \tag{3.7}$$

and we see that under the infinitesimal generator of the renormalization group, $r_0 \to r(l) = r_0' e^{2l}$. This is the same behavior found for a classical functional. Under this transformation, however, the quartic term will become

$$\Phi_4' = \tfrac{1}{2}(\beta N)^2 u_0' \int_0^{e^{-l}} \prod_{i=1}^4 \frac{d^dq_i}{(2\pi)^d} \int_0^{e^{-zl}} \prod_{i=1}^4 \frac{d\omega_i}{2\pi} \Psi(q_1, \omega_1)\Psi(q_2, \omega_2)\Psi(q_3, \omega_3)\Psi(-q_1-q_2-q_3, -\omega_1-\omega_2-\omega_3)$$

$$= \exp[-\tfrac{3}{2}(d+z)l + 4(d+z+2)l]\,\tfrac{1}{4}\, u_0'(\beta N)^2 \int_0^1 \prod_{i=1}^4 \frac{d^dq_i'\,d\omega_i'}{(2\pi)^{d+1}} \Psi'(q_1', \omega_1') \times \Psi'(q_2', \omega_2')\Psi'(q_3', \omega_3')\Psi'(-q_1'-q_2'-q_3', -\omega_1'-\omega_2'-\omega_3'), \tag{3.8}$$

where u_0' is different from u_0 because of the elimination of the shell variables. Hence u must transform according to

$$u_0 \to u(l) = u_0' e^{[4-(d+z)]l} = u_0' e^{\epsilon l}, \tag{3.9}$$

where

$$\epsilon = 4 - (d+z). \tag{3.10}$$

As in the classical case, integrating out the Ψ's with q's in the shell will lead to a u_0' different from u_0 by something of order u_0^2. Hence for small u the change in u from rescaling, which is linear in u, will be dominant. We will have

$$\frac{du}{dl} = \epsilon u + O(u^2), \tag{3.11}$$

so that the Gaussian fixed point, with $u=0$, will be stable if ϵ is negative, that is, if $d > 4-z$. One way of putting this is to say that the effective dimensionality is increased by z. In the present example, then, where $z=3$, we should expect a stable Gaussian fixed point and Landau exponents for $d>1$. This result is the central point of this paper.

One should be careful to note, however, that our model functional was derived from the analytic features of a three-dimensional electron gas. Actually, one- and two-dimensional electron gases do not have Lindhard functions which behave like (2.10). In two dimensions the coefficient of q^2 vanishes,[14] while in one dimension χ_0 has its maximum at $q=2k_F$, not $q=0$, indicating that the dominant fluctuations are nearly *antiferromagnetic* paramagnons. It is nevertheless interesting to think about the model (2.13) in arbitrary dimensionality anyway.

In order to complete the derivation of the renormalization-group equations for this problem, it is now necessary only to do the integration out of the outer-shell fields $\Psi(q,\omega)$, with $e^{-l}<q<1$ and $e^{-zl}<|\omega|<1$. (We continue to write the dynamical exponent as z even though we know it to be 3 in this problem because other problems will have other values of z, but will be otherwise very similar.) Consider the quartic term in the functional. The important terms to consider (to lowest order in u) are those with two q_i or ω_i in the outer shell, and where these four-momenta are equal and opposite. (This is because they are positive definite.) Since there are $\frac{1}{2}(4\times3)=6$ ways to pick this pair of Ψ's, we can write

$$\Phi_4=\frac{u_0}{4\beta N}\sum{}''\prod_{i=1}^{4}\Psi(q_i,\omega_i)\,\delta\Big(\sum_{i=1}^{4}q_i\Big)\,\delta\Big(\sum_{i=1}^{4}\omega_i\Big)+\frac{3u_0}{2\beta N}\sum{}''|\Psi(q,\omega)|^2\sum{}'|\Psi(q,\omega)|^2\,, \quad (3.12)$$

where a prime on a sum indicates summing on q's and ω's in the shell and a double prime indicates summing on q's and ω's not in the shell. Errors we make in this approximation will first appear in terms of order Ψ^6 in the new functional[13] and are therefore irrelevant here.

The part of the functional integration over the shell variables is now a product of independent Gaussian integrals

$$\int\frac{d\Psi(q,\omega)\,d\Psi(-q-\omega)}{2\pi}\exp\left[-\frac{1}{2}\left(r_0+q^2+\frac{|\omega|}{q}+\frac{3u_0}{\beta N}\sum_{q'\omega'}{}''|\Psi(q',\omega')|^2\right)|\Psi(q,\omega)|^2\right]=\left(r_0+q^2+\frac{|\omega|}{q}+\frac{3u_0}{\beta N}\sum_{q\omega}{}''|\Psi(q,\omega)|^2\right)^{-1}. \quad (3.13)$$

Hence

$$Z=\int\delta\Psi\exp\left[-\frac{1}{2}\sum_{q,\omega}{}''\left(r_0+q^2+\frac{|\omega|}{q}\right)|\Psi(q,\omega)|^2-\frac{1}{4}\frac{u_0}{\beta N}\sum{}''\prod_{i=1}^{4}\Psi(q_i,\omega_i)\,\delta\Big(\sum_{i=1}^{4}q_i\Big)\,\delta\Big(\sum_{i=1}^{4}\omega_i\Big)-\frac{1}{2}\sum_{q'\omega'}{}'\ln\left(r_0+q'^2+\frac{|\omega'|}{q'}+\frac{3u_0}{\beta N}\sum_{q\omega}{}''|\Psi(q,\omega)|^2\right)\right]. \quad (3.14)$$

The expansion of the ln to fourth order in Ψ then gives a change in r_0,

$$r_0\to r_0'=r_0+\frac{3u_0}{\beta N}\sum_{q\omega}{}'\left(r_0+q^2+\frac{|\omega|}{q}\right)^{-1}, \quad (3.15)$$

and a change in u_0,

$$u_0\to u_0'=u_0-\frac{9u_0^2}{\beta N}\sum_{q\omega}{}'\left(r_0+q^2+\frac{|\omega|}{q}\right)^{-2}. \quad (3.16)$$

In a model with a wave-vector cutoff at $2k_F$ $(=1)$ and a frequency cutoff at E_F $(=1)$ the (primed) sums of q and ω in (3.15) and (3.16) are over an L-shaped region of space, as shown in Fig. 1. We have

$$\frac{1}{\beta N}\sum_{q\omega}{}'=2\int_{e^{-zl}}^{1}\frac{d\omega}{2\pi}\int_0^1 q^{d-1}dq\int\frac{d\Omega_d}{(2\pi)^d}+\int_{-1}^{1}\frac{d\omega}{2\pi}\int_{e^{-l}}^{1}q^{d-1}dq\int\frac{d\Omega_d}{(2\pi)^d}\,. \quad (3.17)$$

This first term is the integration along the horizontal strip ($|\omega|\approx1$, $0<q<1$) and the second is the integration along the vertical strip ($q\approx1$, $0<|\omega|<1$). (Ω_d is the solid angle in d dimensions.) Then ($m=1,2$)

$$\frac{1}{\beta N}\sum_{q\omega}{}'\left(r_0+q^2+\frac{|\omega|}{q}\right)^{-m}=\frac{2z\Omega_d l}{(2\pi)^{d+1}}\int_0^1 q^{d-1}\left(r_0+q^2+\frac{1}{q}\right)^{-m}dq+\frac{2\Omega_d l}{(2\pi)^{d+1}}\int_0^1 d\omega\,(r_0+1+\omega)^{-m}. \quad (3.18)$$

Thus using (3.18) in (3.15) and (3.16), and combining this information with what we discussed earlier about the changes of r and u from rescaling, leads to the renormalization-group equations

$$\frac{dr}{dl}=2r+3C_d u\left[\ln\left(\frac{r+2}{r+1}\right)+z\int_0^1\frac{x^d\,dx}{x^3+rx+1}\right], \tag{3.19}$$

$$\frac{du}{dl}=\epsilon u-9C_d u^2\left[\frac{1}{(r+1)(r+2)}+z\int_0^1\frac{x^{d+1}\,dx}{(x^3+rx+1)^2}\right], \tag{3.20}$$

where $C_d=2\Omega_d/(2\pi)^{d+1}$. In each equation, the first term comes from the rescaling and the second term from the elimination of the outer shell. Within each second term, the first contribution is from the integration along the horizontal strip.

The fact that (3.19) and (3.20) look somewhat messier than their counterparts in the classical problem is just a consequence of the way we chose the cutoffs in our problem—the q cutoff and ω cutoff were independent of each other, so the integration over the shell variables was along two distinct strips. We could alternatively make a model with different cutoffs that simplify the algebra. One simple choice is to choose a q- and ω-dependent cutoff which excludes all $\Psi(q,\omega)$ for which

$$q^2+|\omega|/q>1\,. \tag{3.21}$$

The region of modes in q and ω space included in this model is shown graphically in Fig. 2. The ω cutoff is q dependent,

$$\omega_c(q)=q-q^3\,. \tag{3.22}$$

The outer shell now becomes a strip along the curve $\omega_c(q)$ defined by

$$e^{-2l}<q^2+|\omega|/q\leq 1 \tag{3.23}$$

and

$$\frac{1}{\beta N}\sum_{q\omega}{}'=C_d\int_0^{e^{-l}}q^{d-1}\,dq\int d\omega\,\theta\left(1-q^2-\frac{|\omega|}{q}\right)\times\theta\left(q^2+\frac{|\omega|}{q}-e^{-2l}\right)$$

$$=\frac{2lC_d}{d}\equiv K_d l\,. \tag{3.24}$$

Everywhere along the strip, by the terms of this model, $r_0+q^2+|\omega|/q=r_0+1$, so we get the simpler renormalization-group equations

$$\frac{dr}{dl}=2r+\frac{3K_d u}{1+r}, \tag{3.25}$$

$$\frac{du}{dl}=\epsilon u-\frac{9K_d u^2}{(1+r)^2}, \tag{3.26}$$

which have a form identical to those obtained in

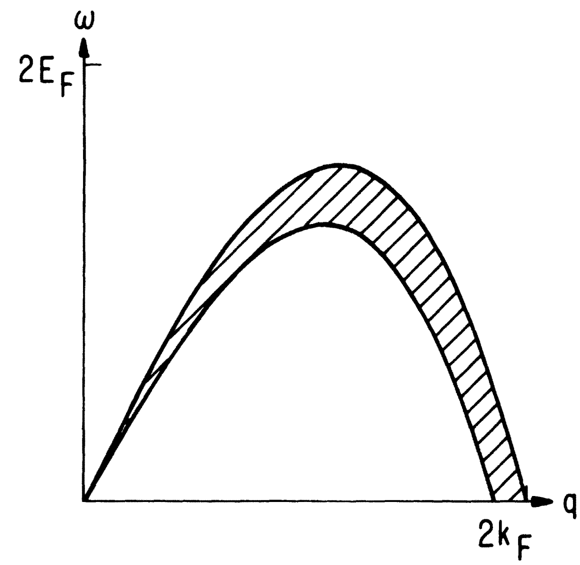

FIG. 2. Region of (q,ω) space contained in the model defined by (3.21) and (3.22). The hatched region is scaled out at each step.

the classical LGW problem, except that ϵ (and K_d) are different. In fact, the seemingly arbitrary cutoff (3.20) is not as contrived as it sounds, since we know that for small q, the microscopic random-phase-approximation (RPA) spectral weight function cuts off abruptly at[9] $\omega>qv_F$, although the Lorentzian spin-fluctuation model (2.11) does not. This cutoff enables the full RPA χ to satisfy the f-sum rule. We can then think of the model with the odd-cutoff rule (3.21) or (3.22) as enforcing an f-sum-rule constraint for small q. This may be irrelevant to critical behavior, but it is certainly no more artificial than the original cutoff procedure.

We next summarize the consequences of the renormalization-group equations (3.19) and (3.20) or (3.25) and (3.26). As we mentioned earlier, for $d+z>4$, i.e., $d>1$, ϵ is negative and u is irrelevant, since $u(l)$ dies exponentially with l. The critical properties of the model are therefore Landau-like, and RPA theory is qualitatively correct.[15] For $d<1$ (admittedly a case of only formal relevance) the group equations are correct to first order in ϵ, and corrections to Landau exponents may be calculated from them in the usual way, to order ϵ. At $d=1$, u is marginal, and one obtains logarithmic corrections to power-law critical behavior. That this is true follows trivially in the case of Eqs. (3.25) and (3.26) (the case of the odd cutoff) because their form is just that studied by Fisher *et al.*[16] It is not quite so obvious in the other cutoff model, but I prove it in the appendix.

The results of this section are not dependent on the scalar-field description of the interacting paramagnons. Because the vector-field functional of Eq. (2.22) differs from the scalar version (2.13) only in the number of spin components, it is straightforward to generalize all of the preceding arguments, in direct analogy to the Wilson theory for a vector field. The only difference is that the factor 3 in Eq. (3.19) or (3.25) becomes $n+2=5$, and the factor 9 in (3.20) or (3.26) becomes $n+8=11$. These changes, of course, do not alter the value of ϵ or influence the relevance or irrelevance of the parameters r or u.

Although in this paper we shall not try to do any better than first order in ϵ, it is worth remarking that in generating renormalization-group equations which are correct to order ϵ^2, one finds corrections to the coefficient of q^2 in Φ_2 (which leads to a nonzero η), but no corrections to the $|\omega|/q$ term. Consequently, $z=3-\eta$.

We conclude this section by looking at this problem at finite temperature. The Matsubara frequencies then no longer form a continuum, but are spaced by $2\pi T$. This is of little consequence initially if T is much less than the original frequency cutoff ($\approx E_F$), but as we remove high frequencies from the problem, we eventually reach a point where only a few Matsubara frequencies remain, and we can no longer approximate them very well by a continuum. Finally, only the $\omega=0$ terms will remain in the functional, and we will arrive at the classical LGW problem. Beyond this point, the renormalization-group equations will be of Wilson form, with $\epsilon=4-d$, and close enough to the critical coupling r_c the exponents will be nonclassical. On the other hand, if T is very low, this true critical region will be very narrow. The problem can be phrased in terms of the standard crossover language,[17] where the temperature is the symmetry-breaking parameter.

Actually, one encounters precisely the same sort of situation in a finite classical system, where the wave vectors q also have a finite spacing $2\pi/L$ between them (L is the linear size of the system). The scaling naturally breaks down when the size of the Kadanoff cell exceeds that of the system. Our problem here is analogous to that of a $(d+1)$-dimensional system which is finite (length β) in the extra dimension. Our crossover is analogous to that which occurs between $(d+1)$-dimensional and d-dimensional critical behavior when the correlation length exceeds the length of the system in the finite dimension.

To make this idea more quantitative, note that when the scaling parameter in the renormalization group equations has value l, frequencies between $E_F e^{-zl}$ and E_F have been removed from the problem. (We write the original frequency cutoff E_F explicitly in this section.) Thus the quantum scaling stops, roughly, when just one finite Matsubara frequency remains. That is, $l=\hat{l}$, where

$$2\pi T = E_F e^{-z\hat{l}} \ .$$

The maximum wave vector left in the problem is then

$$q_c \approx k_F e^{-\hat{l}} = k_F (2\pi T/E_F)^{1/z} \ . \quad (3.27)$$

For $q>q_c$ and $\omega>T$, then, the fluctuations of $\Psi(q,\omega)$ are governed by the quantum renormalization-group equations derived above. In three dimensions, our analysis shows that RPA theories will be qualitatively valid. One way to think of q_c is as the inverse of the length over which one has to average microscopic quantities in order to be able to treat them as classical thermodynamic variables.

In the remaining corner of q, ω space, fluctuations will be classical in nature. Figure 3 shows the classical and quantum regions of q, ω space at a particular $T \ll E_F$. It is not obvious from this discussion how to deal with the dynamics when all of the finite Matsubara frequencies have disappeared. This is because I have been too cavalier in treating the ω's as a continuum until all but the very last one was gone. We will see how to do better than this in Sec. VII.

The crossover phenomenon is most simply discussed as follows (we talk only about the case where $\epsilon<0$ before the crossover): For $1 \ll l < \hat{l}$, r grows as

$$r(l) = \bar{r}_0 e^{2l} \quad (3.28)$$

($\bar{r}_0$ differs from r_0 because of the effects of some transient terms in the solutions of the renormalization-group equations). When $r(l)$ gets to unity, we stop the scaling, since we have scaled the problem into one with a small $[O(1)]$ correlation length, which can be treated by perturbation theory. Thus if $r(l)$ gets to unity before l gets to $\hat{l}$, the transition to the classical LGW functional and the Wilson renormalization-group equations never gets a chance to happen. The critical exponents will deffer from their Landau values, then, only if $r(\hat{l})<1$, i.e.,

$$\bar{r}_0 < (2\pi T/E_F)^{2/z} \ . \quad (3.29)$$

In the usual terminology,[16] $\frac{1}{2}z$ is the crossover exponent.

Figure 4 illustrates the consequences of this effect. At each T, we assume we have a different critical Hubbard coupling strength $U_c(T)$; the system is ferromagnetic for $U>U_c(T)$. Stoner theory predicts $U_c(T)=U_c(0)+\alpha T^2$, or $T_c(U) \propto [U-U_c(0)]^{1/2}$. As we approach the transition line at fixed T,

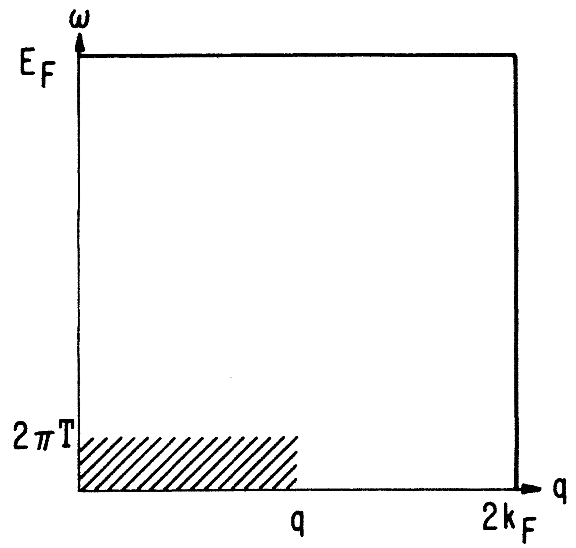

FIG. 3. Regions of the (q, ω) plane where correlation functions are dominated by classical (hatched) and quantum (unhatched) effects.

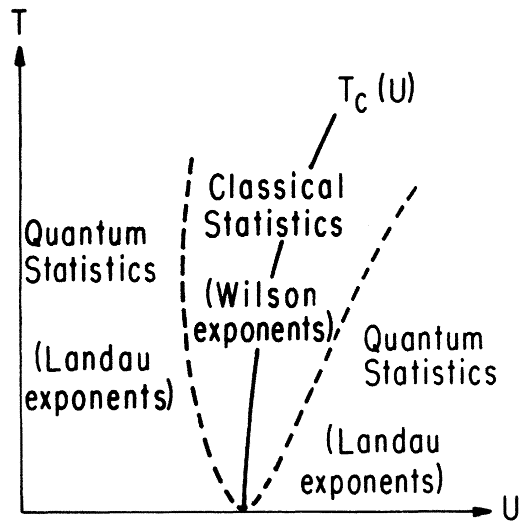

FIG. 4. Crossover diagram showing whether long-wavelength low-frequency critical properties are determined by classical or quantum renormalization groups, as a function of temperature T and coupling constant U, for itinerant ferromagnetic models (2.13) or (2.22).

varying U, $\bar{r}_0 \propto [U - U_c(T)]/U_c(T)$. From (3.29) the crossover occurs when $\bar{r}_0 \approx (T/E_F)^{2/z}$, so exponents are effectively Landau-like outside and Wilson-like inside the region between the dotted lines in Fig. 4.

Since $z > 1$, the temperature region in which, for a given U, the classical LGW functional becomes relevant has a width at least of the order of T_c itself. The quantum-to-classical crossover is therefore more easily observed by sitting at a fixed low T and varying U by alloying or pressure. Note also that just below $U_c(0)$, one may be able to pass from a quantum region to a classical one and back to a quantum one as T is varied.

Another observation worth making is that in the case of fixed $U = U_c(0)$, where $T_c = 0$, as one varies T down to zero, one approaches the instability within a classical region. Therefore non-Landau (Wilson) critical exponents should characterize this transition. This result depends on the fact that $z > 1$, so that the paramagnetic crossover boundary in Fig. 4 moves initially toward the left as T increases from zero.

One may also note that the quantum renormalization-group procedure for $l < \hat{l}$ provides a way of explicitly deriving the LGW functional microscopically. One will usually want to transform back to the original scale, letting the fact that the momentum cutoff in this functional is generally much smaller than the microscopic characteristic inverse length, $q_c \ll k_F$, appear explicitly.

Finally, the description of the quantum-classical crossover is independent of the number of components n of the order parameter (for physically relevant dimensionalities). This is because the quantum problem has $\epsilon < 0$ with ϵ independent of n, and the determination of the crossover value of l requires only asking when the renormalization-group equations in the quantum region break down. (Inside the classical region, critical indices will depend on n, but that is not our concern here.)

IV. APPROXIMATE SOLUTIONS OF THE RENORMALIZATION-GROUP EQUATIONS

In Sec. III, we looked at the properties of the renormalization group we had derived insofar as they bore on the critical properties of the model. Here we show how we can make approximate solutions of these equations to give more qualitative information about the nature of the correlation functions. The form of these solutions is similar, but not identical to that obtained by simple perturbation theory in U. We start with the model (2.13) and its associated renormalization-group equations (3.25) and (3.26). As explained in Ref. 7, this model affords a credible description of weak itinerant ferromagnetism and strong paramagnetism when the band density of states is not too rapidly varying near E_F. Equivalently, one must have u_0 (2.14b) fairly small. (If the density of states is rapidly varying, one must keep higher-order terms than $u_0\Psi^4$.)

We deal with the physically relevant case of $\epsilon(= -2) < 0$, so the Gaussian fixed point is stable. Our approximation procedure is to linearize the general renormalization-group equations around this fixed point and solve the subject to the initial conditions $r(0)=r_0$, $u(0)=u_0$. We are therefore ignoring the effects of Ψ^6 and higher-order terms which are generated n the exact renormalized LGW functional as transient terms but die out (faster than r and u) for large l. The only kinds of terms we include are those which are present in the original functional. For u_0 small, we can ignore the second term in the equation for du/dl, giving

$$u(l)=u_0 e^{-|\epsilon| l}. \tag{4.1}$$

In (3.25) for dr/dl, we ignore the r in the denominator of the second term since corrections to this would be $O(ru)$. We therefore have to solve

$$\frac{dr}{dl}-2r=3K_d u_0 e^{-|\epsilon| l}. \tag{4.2}$$

The solution is standard:

$$r(l)=[r_0+3K_d u_0/(2+|\epsilon|)]e^{2l} - [3K_d u_0/(2+|\epsilon|)]e^{-|\epsilon| l}. \tag{4.3}$$

To apply this to the zero-temperature problem, we scale (that is, carry out the renormalization-group transformations) (4.1) and (4.3) until $r(l)=1$. Beyond this point, the problem is one whose Kadanoff cell size exceeds the correlation length, so it can be dealt with in perturbation theory. We have $r(l)=1$ when

$$l=l_1=\tfrac{1}{2}\ln\left[\left(\frac{1}{r_0+3K_d u_0/2+|\epsilon|}\right)^{-1}\right], \tag{4.4}$$

provided this $l_1 \gg 1$ [that is, $r_0+3K_d u_0/(2+|\epsilon|) \ll 1$]. At this l the correlation length is the Kadanoff cell size, that is, the factor by which distances have been scaled, which is e^l. Thus

$$\xi=(r_0+3K_d u_0/2+|\epsilon|)^{-1/2}\equiv\bar{r}_0^{-1/2}. \tag{4.5}$$

This just tells us that the effect of the anharmonicity is to shift the instability point from $r_0=0$ to $\bar{r}_0=0$. Such a result could be obtained straightforwardly by perturbation theory in u_0, but it is enlightening to see how it emerges from the linearized renormalization-group analysis.

We can also ask about the finite-temperature problem and the form of the LGW functional after the quantum-to-classical crossover. The crossover occurs when $e^{-zl}\approx T/E_F$, i.e., at l equal to

$$\hat{l}=z^{-1}\ln(E_F/T)\gg 1. \tag{4.6}$$

Thus the Landau-Ginzburg parameters at this point are

$$r(\hat{l})=\bar{r}_0(E_F/T)^{2/z}, \tag{4.7}$$

$$u(\hat{l})=u_0(T/E_F)|\epsilon|/z. \tag{4.8}$$

When this problem is expressed back in the scale of the original one, we simply have a problem with $r_{\text{eff}}=\bar{r}_0$, $u_{\text{eff}}=u_0$, and a cutoff at $q_c\approx k_F(T/E_F)^{1/z}$. Another way to express this result is to say that perturbation theory is sufficient to calculate the change in effective Landau-Ginzburg parameters which comes from scaling the finite Matsubara frequencies out of the problem (provided the quantum region ϵ is negative). The situation would be more complicated (and more interesting) if the Gaussian fixed point were not stable for $l<\hat{l}$. A hint of what can happen then can be seen in the $\epsilon=0$ case discussed in the Appendix.

It is not difficult to generalize this discussion to a generalized LGW functional with local couplings of all (even) orders in Ψ, such as appear in (2.7), rather than just a Ψ^4 term. We then need an infinite set of renormalization-group equations, rather than just two of them. To derive them, we follow the same argument we used in Sec. III. The nth-order anharmonic term in Φ has the form

$$\Phi_n=\frac{u_0^{(n)}}{n!}(\beta N)^{1-n/2} \times\sum_{q_i\omega_i}\prod_{i=1}^{n}\Psi(q_i,\omega_i)\,\delta\left(\sum_{i=1}^{n}q_i\right)\delta\left(\sum_{i=1}^{n}\omega_i\right). \tag{4.9}$$

Under the rescaling of wave vectors, frequencies, and fields (3.4) and (3.6), this becomes

$$\Phi_n=\frac{u_0^{(n)}}{n!}(\beta N)^{1-n/2}(e^{(d+z+2)l/2})^n(e^{-(d+z)l})^{(n-1)} \times\sum_{q_i\omega_i}\sum_{i=1}^{n}\Psi'(q_i',\omega_i')\,\delta\left(\sum_{i=1}^{n}q_i'\right)\delta\left(\sum_{i=1}^{n}\omega_i'\right). \tag{4.10}$$

(The first exponential factor comes from the n fields rescaled, the second from the rescaling of the variables of integration.) Thus the rescaling gives a contribution to

$$\left(\frac{du^{(n)}}{dl}\right)_{\text{rescaling}}=\epsilon_n u^{(n)}, \tag{4.11}$$

where

$$\epsilon_n=n-(d+z)(\tfrac{1}{2}n-1). \tag{4.12}$$

The fact that all ϵ_n are negative for $n\geq 4$ means that the Gaussian fixed point $u^{(n)}=0$ is at least metastable. We shall not consider the possibility of a different true stable fixed point.

We turn then to the elimination of the shell fields $\Psi(q,\omega)$ with $e^{-l}<q\leq 1$ and $e^{-zl}<\omega\leq 1$. In each term (4.9) we separate out the terms with factors $\Psi(q,\omega)\Psi(-q,-\omega)$, with q and ω in the outer shell. As in previous discussion, the "paired" terms like this are important because they are positive definite, so they dominate those in which only one Ψ has its (q, ω) in the shell as well as those with two Ψ's with arguments in the shell but not equal and opposite. The fact that the shell is thin, on the other hand, permits us to ignore any terms with more than two Ψ's with arguments in the shell; if l is the thickness of the shell, they contribute only to order l^2. Furthermore, there are $\frac{1}{2}n(n-1)$ ways to pick the pair. Thus the part of Φ_n involving shell variables is

$$\Phi_n' = \frac{u_0^{(n)}}{2(n-2)!}(\beta N)^{1-n/2} \sum_{q_i\omega_i}'' \prod_{i=1}^{n-2} \Psi(q_i, \omega_i) \times \delta\left(\sum_{i=1}^{n-2} q_i\right) \delta\left(\sum_{i=1}^{n-2} \omega_i\right) \sum_{q\omega}' |\Psi(q, \omega)|^2 . \quad (4.13)$$

Now the coefficient of the shell Ψ's is no longer just a quadratic function of the nonshell Ψ's, but includes the sum of all of the expressions like (4.14) over all n. In place of (3.14) we find

$$Z = \int \delta\Psi \exp\left\{-\frac{1}{2}\sum_{q\omega}''\left(r_0 + q^2 + \frac{|\omega|}{q}\right)|\Psi(q,\omega)|^2 - \sum_{n=4}^{\infty}\frac{u_0^{(n)}}{n!}(\beta N)^{1-n/2}\sum_{q_i\omega_i}''\prod_{i=1}^{n}\Psi(q_i,\omega_i)\,\delta\left(\sum_{i=1}^{n} q_i\right)\delta\left(\sum_{i=1}^{n}\omega_i\right) - \frac{1}{2}\sum_{q'\omega'}\ln\left[r_0 + q'^2 + \frac{|\omega'|}{q'} + \sum_{n=2}^{\infty}\frac{u_0^{(n+2)}}{n!}(\beta N)^{-n/2}\sum_{q_i,\omega_i}''\prod_{i=1}^{n}\Psi(q_i,\omega_i)\,\delta\left(\sum_{i=1}^{n} q_i\right)\delta\left(\sum_{i=1}^{n}\omega_i\right)\right]\right\}. \quad (4.14)$$

(In this section, all sums on n are taken over *even* n only.) In words, the new functional has two parts, the first of which is just the old functional restricted to the nonshell Ψ's and the second of which is an average over the shell of the logarithm of the inverse RPA propagator plus the second functional derivative of the original functional with respect to the shell Ψ's.

As in the preceding discussion of the Ψ^4-only problem, we linearize the group equations around the Gaussian fixed point $u^{(n)} = 0$, and solve them subject to the initial conditions $u^{(n)}(0) = u_0^{(n)}$. Things simplify enormously on linearization. On expanding the logarithm, we get a change in $u^{(n)}$ proportional to $u^{(n+2)}$,

$$\delta u^{(n)} = \tfrac{1}{2}u^{(n+2)}(\beta N)^{-1}\sum_{q'\omega'}'\left(q^2 + \frac{|\omega|}{q}\right)^{-1} = \tfrac{1}{2}lC_d\left(\ln 2 + z\int_0^1 \frac{x^d\,dx}{x^3+1}\right)u^{(n+2)} \equiv \tfrac{1}{2}lC_d\lambda u^{(n+2)} . \quad (4.15)$$

The other cutoff model (3.21) gives

$$\delta u^{(n)} = \tfrac{1}{2}lK_d u^{(n+2)} . \quad (4.16)$$

We shall use (4.16) rather than (4.15); the difference is only quantitative. Combining the changes in $u^{(n)}$ from rescaling (4.11) with (4.16) gives the renormalization-group equations

$$\frac{du^{(n)}}{dl} = \epsilon_n u^{(n)} + \tfrac{1}{2}K_d u^{(n+2)} . \quad (4.17)$$

We solve them by Laplace transform. Define

$$\bar{u}^{(n)}(S) = \int_0^\infty dl\, e^{-lS}u^{(n)}(l) \quad (4.18)$$

and take the Laplace transform of (4.17),

$$S\bar{u}^{(n)}(S) - u_0^{(n)} = \epsilon_n\bar{u}^{(n)}(S) + \tfrac{1}{2}K_d\bar{u}^{(n+2)}(S) . \quad (4.19)$$

Then

$$\bar{u}^{(2)}(S) = \frac{u_0^{(2)}}{S-\epsilon_2} + \frac{\frac{1}{2}K_d\bar{u}^{(4)}(S)}{S-\epsilon_2} = \frac{u_0^{(2)}}{S-\epsilon_2} + \frac{\frac{1}{2}K_d u_0^{(4)}}{(S-\epsilon_2)(S-\epsilon_4)} + \frac{(\frac{1}{2}K_d)^2 u_0^{(6)}}{(S-\epsilon_2)(S-\epsilon_4)(S-\epsilon_6)} + \cdots . \quad (4.20)$$

In taking the inverse Laplace transform

$$r(l) = u^{(2)}(l) = \frac{1}{2\pi i}\int_C dS\, e^{lS}\left(\frac{u_0^{(2)}}{S-\epsilon_2} + \frac{\frac{1}{2}K_d u_0^{(4)}}{(S-\epsilon_2)(S-\epsilon_4)} + \cdots\right), \quad (4.21)$$

we are interested only in the large-l behavior, so we need worry only about the poles at the largest ϵ_n, that is, $\epsilon_2 = 2$. Then

$$r(l) \sim e^{2l}\left[u_0^{(2)} + \frac{\frac{1}{2}K_d u_0^{(4)}}{\epsilon_2-\epsilon_4} + \frac{(\frac{1}{2}K_d)^2 u_0^{(6)}}{(\epsilon_2-\epsilon_4)(\epsilon_2-\epsilon_6)} + \cdots\right]. \quad (4.22)$$

But since $\epsilon_n - \epsilon_{n+2} = d + z - 2 \equiv \Delta$, this is

$$r(l) \sim e^{2l}\left(u_0^{(2)} + \frac{\frac{1}{2}K_d u_0^{(4)}}{\Delta} + \frac{(\frac{1}{2}K_d)^2 u_0^{(6)}}{2\Delta^2} + \frac{(\frac{1}{2}K_d)^3 u_0^{(8)}}{3\times 2\Delta^3} + \cdots\right) = e^{2l}\sum_{n=0}^{\infty}\left(\frac{K_d}{2\Delta}\right)^{n/2}\frac{u_0^{(n)}}{(\frac{1}{2}n)!} . \quad (4.23)$$

There is a simple way of looking at this result.[7] The coefficient of e^{2l} is just the average of the second derivative of the function

$$u(x)=\sum_{n=0}^{\infty}\frac{1}{n!}u_0^{(n)}x^n \tag{4.24}$$

over a Gaussian distribution of x with variance K_d/Δ. To see this, just evaluate

$$\begin{aligned}\langle u''(x)\rangle &= \sum_{n=2}^{\infty}\frac{1}{(n-2)!}u_0^{(n)}\langle x^{n-2}\rangle\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}u_0^{(n+2)}\langle x^n\rangle\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}u_0^{(n+2)}(n-1)!!\left(\frac{K_d}{\Delta}\right)^{n/2}\\ &= \sum_{n=0}^{\infty}\frac{u_0^{(n+2)}}{2^{n/2}(\frac{1}{2}n)!}\left(\frac{K_d}{\Delta}\right)^{n/2},\end{aligned} \tag{4.25}$$

which is just what appears in (4.23). Hence the arguments which led to (4.5) give a correlation length

$$\xi=(\langle u''\rangle)^{-1/2}=[r_0+\langle(u''-r_0)\rangle]^{-1/2}. \tag{4.26}$$

A dimensionless parameter measuring the deviation from the pure RPA result is

$$\sigma=\left(\frac{K_d}{\Delta}\right)^{1/2}=\left(\frac{4\Omega_d}{(2\pi)^{d+1}d(d+z-2)}\right)^{1/2}. \tag{4.27}$$

In three dimensions, $\sigma=0.0518$.

V. OTHER MODELS

It is apparent from the discussion in Sec. III that the value of the dynamical exponent z is crucial in determining the qualitative structure of the renormalization-group equations, and that its value (i.e., 3) in the paramagnon problem is a consequence of the fact that frequency occurs in the generalized LGW functional in the form of a term in the quadratic part of Φ proportional to $|\omega|/q$. In this section we examine some other systems, finding their z's and discussing the consequences for the $T=0$ critical behavior and the low-T crossover to normal critical exponents. We shall not derive the quantum LGW functionals for these systems; rather, we shall appeal to the physical interpretation of the LGW coefficients to argue what qualitative form they should have.

Example 1: dirty itinerant ferromagnet. Fulde and Luther have shown how impurities lead to spin diffusion in the RPA,[18] that is, the spin-fluctuation spectral weight function has a form

$$\chi''(q,\omega)=\chi(q)Dq^2\omega/[\omega^2+(Dq^2)^2] \tag{5.1}$$

when $ql\ll 1$, where l is the electronic mean free path. This should be contrasted with the Landau-damping form (2.11). We can incorporate these effects into our formalism by using a quadratic part of Φ of the form

$$\Phi_2=\frac{1}{2}\sum_{q\omega}\left(r_0+q^2+\frac{|\omega|}{D_0q^2}\right)|\Psi(q,\omega)|^2 \tag{5.2}$$

instead of the expression in (2.13). The imaginary part of the reciprocal of the coefficient in (5.2) is then of the form (5.1). The only significant difference between this problem and the previous one is in the rescaling. When $q\to q'=qe^l$ and we let $\omega\to\omega'=\omega e^{zl}$, $q^2=q'^2e^{-2l}$ and $|\omega|/D_0q^2=(|\omega'|/D_0q'^2)e^{(z-2)l}$; thus we must choose $z=4$ to make the two coefficients identical, so that they can be made equal to unity by a scale change in Ψ. Therefore one expects Landau critical exponents in any positive dimensionality for this problem.

There will also be minor quantitative differences in the form of the group equations for this case, since in eliminating the shell variables, one now encounters integrations of the form ($D_0=1$)

$$I_m=\frac{1}{\beta N}\sum_{q\omega}{}'\left(r_0+q^2+\frac{|\omega|}{q^2}\right)^{-m} \tag{5.3}$$

instead of (3.18). The second terms in the brackets in (3.19) and (3.20) are therefore replaced by

$$z\int_0^1\frac{x^{d+1}dx}{x^4+rx^2+1}\quad\text{and}\quad z\int_0^1\frac{x^{d+3}dx}{(x^4+rx^2+1)^2}, \tag{5.4}$$

respectively. [If we construct a model with a cutoff in analogy to (3.22), we get equations like (3.25) and (3.26), except that K_d is multiplied by a factor $d/(d+1)$.]

Actually, the introduction of randomness should actually be regarded as having a more fundamental effect on the LGW functional than just making the dynamics diffusive. The parameters r_0 and u_0 should become random functions of position with specified probability distributions, as in the work of Lubensky and Harris[19] and of Grinstein and Luther.[19] However, their work indicates that this has little effect on critical properties when the order parameter is a three-vector, as it really should be in this case.

Example 2: itinerant antiferromagnet. In this case the instability is at a finite wave vector Q, so fluctuations near this wave vector have no special q dependence like those imposed by rotational invariance on the ferromagnetic fluctuations of long wavelength in the previous examples. We characterize their decay by a single relaxation time τ, and express this in writing the quadratic part of Φ as

$$\Phi_2=\frac{1}{2}\sum_{q\omega}(r_0+q^2+|\omega|\tau)|\Psi(q,\omega)|^2. \tag{5.6}$$

In order that the second and third terms here scale in the same way, we must take $z=2$. The Gaussian fixed point will therefore be stable in greater than two dimensions in this case. There will also be minor changes in the form of the group equations, as in the previous example, but we will not write these here. This model also describes incipient charge-density wave or superconducting fluctuations.

The order parameter should, strictly speaking, be a vector in this problem as well. In fact, if there are m inequivalent values of Q at which the instability can happen, Ψ should be a $(3m)$-component vector. Similarly, for a charge-density wave instability, where the charge density is a scalar, the order parameter has m components if there are m inequivalent values of the instability wave vector, and for a superconductor, Ψ has two components. But, again, the points we discussed in Sec. III do not depend on the number of components of Ψ.

One must use extreme caution in applying the model (5.6) to one-dimensional metallic models, or to higher-dimensional ones with one-dimensional features such as flat pieces of Fermi surface. In these cases, the coefficients r_0, u_0, and all higher-order $u^{(n)}$ have a singular temperature dependence as $T\to 0$, and each $u^{(n)}$ is more singular than $u^{(n-1)}$. One therefore may not truncate Φ at any finite order at low temperatures.

Example 3. singlet–ground-state magnet (singlet-singlet model). In these systems, a nonvanishing matrix element of the z component of the total atomic angular momentum between two crystal-field-split levels leads to a magnetic state for sufficient exchange strength.[20] (The problem is isomorphic to that of an Ising model in a transverse field equal to the crystal-field splitting.) Even in the absence of magnetic order the exchange allows the crystal-field excitons to propagate in the lattice; the excitation structure is reflected in the RPA susceptibility

$$\chi(q,\omega)\propto\frac{1}{\omega^2-\omega^2(q)}\quad(\omega\text{ real}),\tag{5.7}$$

where $\omega^2(q)$ is an even function of q whose q-independent part goes to zero as the instability is approached. This is often described as a soft magnetic exciton. In properly chosen units, and for sufficiently small q,

$$\omega^2(q)=r_0+q^2.\tag{5.8}$$

We can put this information into a generalized LGW functional by writing a quadratic part of Φ,

$$\Phi_2=\frac{1}{2}\sum_{q\omega}(r_0+q^2+|\omega|^2)\,|\Psi(q,\omega)|^2.\tag{5.9}$$

[The change in relative sign of q^2 and ω^2 between (5.7) and (5.9) is because the ω's in (5.9) are (imaginary) Matsubara frequencies.] This heuristic procedure has been justified microscopically by Klenin and the author.[21] The quadratic term looks like (5.9), and at zero temperature the quartic term is of the form we have been using here.[21] At finite T, there are anomalous singular quartic terms proportional to $e^{-\text{const}/T}$, which we can ignore here for very low temperatures. In this problem, then, time acts just like another dimension [(5.9) is Lorentz invariant], the dynamical exponent $z=1$, and three dimensions is the dividing line between Landau and Wilson critical exponents.[22,23]

The singlet-triplet model, which has very different critical dynamics in the classical statistical region at finite temperature because of its rotational symmetry,[24] is not expected to behave very differently from the singlet-singlet model in the quantum scaling region, since as we emphasized above, the vector or scalar character of the order parameter is irrelevant to the critical dimensionality and crossover index.

One way to look at this problem or the preceding example is to think of the frequency label of Ψ as labeling different components of an infinite-dimensional vector field,

$$\Phi_2=\frac{1}{2}\sum_{qm}(r_0^m+q^2)\,|\Psi_m(q)|^2,\tag{5.10}$$

with $r_0^m=r_0+|\omega_m|^2$ in this example, or $r_0^m=r_0+|\omega_m|$ in the antiferromagnet. (These frequencies are measured in units of the high-frequency cutoff.) This point has been made independently by Young.[23] In carrying out the renormalization group on this anisotropic classical problem, initially the behavior of the group transformations is as if the spin dimensionality is infinite, provided that the anisotropy is small. As the scaling proceeds, however, all m values except $m=0$ become irrelevant, and there is an eventual crossover to a scalar-field problem. The crossover exponent can be extracted simply in this picture by the same sort of arguments we used in obtaining (3.29). The scaling stops when $r^0(l)=r_0^0e^{2l}\approx 1$, and if the other $r^m(l)=r_0^me^{2l}$ have not reached unity well before this, the critical behavior will not be characteristic of a scalar field. Whether this has happened depends then on whether $r_0^m\gg r_0^0$, that is, $|\omega_m|^2\gtrless r_0$ (example 3) or $|\omega_m|\gtrless r_0$ (example 2). This gives a crossover exponent of 1 for the antiferromagnet and $\frac{1}{2}$ for the singlet–ground-state problem, in agreement with the prediction $\frac{1}{2}z$ obtained from the $(d+1)$-dimensional scaling procedure in Sec. III.

The crossover diagram looks slightly different from Fig. 4 in this case, and is shown in Fig. 5. From mean-field theory, we find that the curve $T_c(J)$ rises from zero at the critical exchange J_c with all of its derivatives infinite. Since the crossover exponent is $\frac{1}{2}z = \frac{1}{2}$, the crossover boundaries approach the $T=0$ instability point with infinite slope, in contrast with the itinerant ferromagnetic case of Fig. 4, where they came in with zero slope. In both cases, however, the approach to zero temperature with coupling fixed so that $T_c = 0$ is in a classical statistical region.

This is not necessarily universally true, however, as the following example shows: Suppose that we designate the variable coupling parameter (U or J in previous examples) by x, i.e., $\bar{r} = x - x_c$. Suppose further that the phase boundary $T_c(x)$ rises from zero at $x_c(0)$ with a power law $[x - x_c(0)]^{\tilde{\beta}}$. Then if the crossover exponent ϕ is greater than $\tilde{\beta}$ (or equivalently, if $z < 2/\tilde{\beta}$) the paramagnetic crossover boundary will move leftward from x_c as T rises, as in the previous examples, but if $\phi < \tilde{\beta}$, it will begin by moving rightward instead. The simplest way to see this is to look at the crossover boundaries as functions of T,

$$x_{\pm}(T) = x_c(T) \pm \alpha T^{2/z} = x_c(0) + \alpha' T^{1/\tilde{\beta}} \pm \alpha T^{2/z} \tag{5.11}$$

(α and α' are constants). If $2/z < 1/\tilde{\beta}$, the $T^{1/\tilde{\beta}}$ term is initially negligible relative to the $T^{2/z}$ term, so the net sign is necessarily negative for the paramagnetic crossover boundary. But if $2/z > 1/\tilde{\beta}$, the $T^{2/z}$ term is a small correction on the $T^{1/\tilde{\beta}}$ piece so the total term (5.11) is necessarily positive for small enough T. The latter

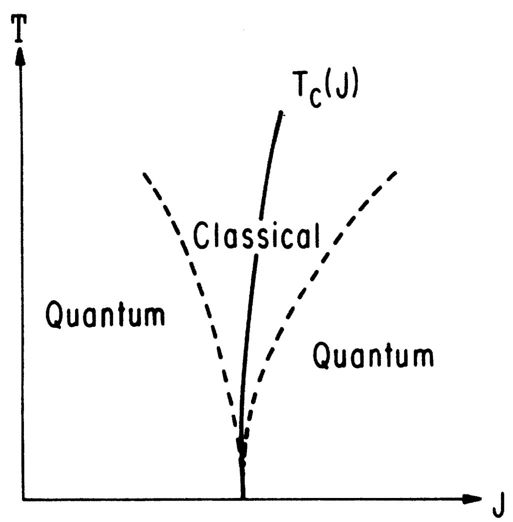

FIG. 5. Crossover diagram for the singlet-ground-state ferromagnet (Ising model in a transverse field). (J is the exchange coupling.)

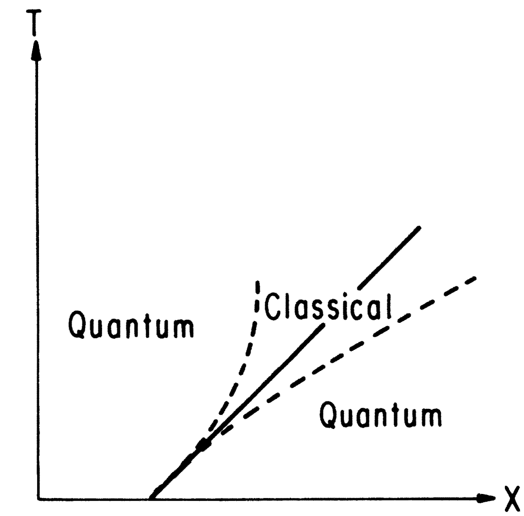

FIG. 6. Crossover diagram for a hypothetical system with $\beta = 1$ and $\phi = \frac{1}{2}$. X is a coupling-strength parameter.

situation is shown in Fig. 6, where we take $\tilde{\beta}$ and $z=1$ for illustrative purposes.

One sees that here at $T=0$ critical point [with $x = x_c(0)$] is approached entirely in the *quantum* region as T is lowered to zero. Furthermore, in this case the width of the temperature region in which classical statistics hold (for a fixed x) also becomes negligibly small relative to T_c itself, unlike the previously encountered situations, where no crossover could be observed as a function of T for temperatures near T_c. The nature of the phase transition when $T_c = 0$ and the existence or nonexistence of a crossover near T_c as T is varied both depend on the relative magnitudes of φ and $\tilde{\beta}$.

One can generalize the discussion of this section to any quantum LGW functional with a Ψ^4 interaction which is local in space and time and whose quadratic part is of the form

$$\Phi_2 = \sum_{q\omega} \left(r_0 + q^{\sigma} + \frac{|\omega|^m}{q^{m'}} \right) |\Psi(q,\omega)|^2 . \tag{5.12}$$

Under the rescaling transformation, this becomes

$$\Phi_2 = e^{-(d+z)l} \sum_{q'\omega'} \left(r_0 + q'^{\sigma} e^{-\sigma l} + \frac{|\omega'|^m}{(q')^{m'}} e^{(m'\sigma - mz)l} \right) \times |\Psi(q'e^{-l}, \omega' e^{-zl})|^2 , \tag{5.13}$$

so the fields must rescale like

$$\Psi'(q', \omega') = e^{-(d+z+\sigma)l/2} \Psi(q'e^{-l}, \omega' e^{-zl}) \tag{5.14}$$

and z must be chosen to satisfy $\sigma = mz - m'\sigma$, i.e.,

$$z = (\sigma + m')/m . \tag{5.15}$$

When the rescaled fields are substituted into the quartic term in Φ, one finds $u_0 \to u_0 e^{\epsilon l}$, where

$$\epsilon = 2\sigma - d - z = (2 - 1/m)\sigma - m'/m - d, \qquad (5.16)$$

in contrast to the classical result $\epsilon = 2\sigma - d$.[16] The dimensionality is effectively increased by z, as before, and z will always be positive in the physically relevant cases m, $m' \geq 0$. One example of a model in which z is negative would be the case with $m = 1$, $m' < 0$, in which the decay rate of fluctuations of wave vector q is proportional to $q^{-|m'|}$—that is, long-wavelength fluctuations decay faster than short-wavelength ones. (I am not aware of any physical situations with this property.)

A feature shared by all of the examples discussed here is that they are characterized by a competition between two parts of the Hamiltonian, one of which wants the system to order and the other of which minimized when there is no long-range order. If the former is just barely strong enough to counteract the latter, T_c will be much lower than the characteristic energies of either part, and many finite-ω Ψ's must be integrated out of the problem before it can be cast into classical LGW form. Our discussions here are relevant to any such situation. An example of a system where these effects are negligible is an ordinary Heisenberg magnet, since there T_c is of the order of the only characteristic microscopic energy J, hence near T_c no modes have characteristic energies $\gg T$.

VI. ALTERNATIVE FORMULATION OF THE RENORMALIZATION GROUP

In this section, we set up the renormalization group in a diagrammatic language[1,25] and perform the scaling in a different way. We will scale out shell variables explicitly only in q; the scaling in frequency will be taken care of implicitly by Bose functions which appear in the integrals. We start again with the scalar paramagnon problem. The elimination of the shell variables is expressed as a Hartree (single-loop) self-energy correction to the free propagator $(r_0 + q^2 + |\omega|/q)^{-1}$ and internal-loop corrections to the four-point interaction vertex u_0 (Fig. 7). In both of these corrections, the internal loops have their momenta between e^{-l} and 1. If we also restricted their Matsubara frequencies to lie between e^{-zl} and 1, these self-energy and vertex corrections would lead to the terms in brackets in (3.19) and (3.20). Here, however, we choose to sum over all frequencies. The scaling is thus in vertical strips in (q, ω) space (Fig. 8). We therefore have a self-energy correction

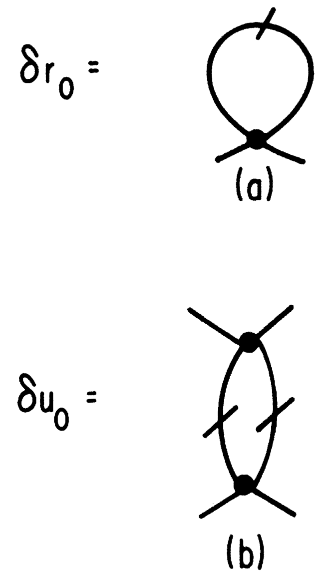

FIG. 7. Lowest-order diagrams for the change in parameters $r(a)$ and $u(b)$ under the renormalization group. A bar through a propagator indicates that its q is in the outer shell $e^{-l} < q \leq 1$.

$$\delta r = \frac{3u_0}{\beta N} \sum_{q\omega} {}' \chi(q, \omega) \qquad (6.1)$$

and a vertex correction

$$\delta u = -\frac{9u_0^2}{\beta N} \sum_{q\omega} {}' \chi^2(q, \omega), \qquad (6.2)$$

where $\chi(q, \omega) = (r_0 + q^2 + |\omega|/q)^{-1}$ is the free-paramagnon propagator. (Here the prime on the

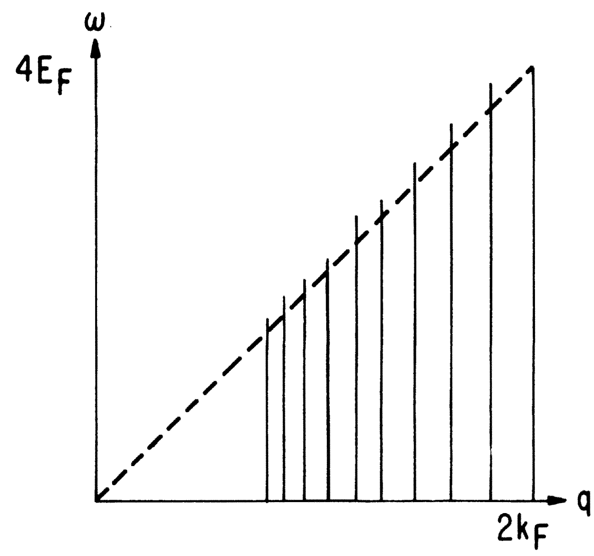

FIG. 8. Scaling procedure used in Sec. VII. The dashed line marks the f-sum-rule cutoff imposed to make (7.4) finite.

sum indicates that q's, but not ω's, lie in the shell $e^{-l} < q \le 1$.) The Matsubara sums can be converted to integrals over real frequency, using the spectral representation

$$\chi(q, \omega_m) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\chi''(q, \omega)}{\omega - \omega_m}, \tag{6.3}$$

so that

$$\delta r = \frac{3u_0}{N} \sum_q \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \chi''(q, \omega)[n(\omega) + \tfrac{1}{2}] \tag{6.4}$$

and

$$\delta u = -\frac{9u_0^2}{N} \sum_q{}' \int_{-\infty}^{\infty} \frac{d\omega\, d\omega'}{\pi^2} \frac{\chi''(q, \omega)\chi''(q, \omega')}{\omega' - \omega} \times [n(\omega) - n(\omega')]. \tag{6.5}$$

The first of these is formally logarithmically divergent as it stands, since the Bose function approaches zero and χ'' goes like $1/\omega$ for large ω. We remedy this deficiency by noting that the true spectral weight function from which our model (2.11) was derived has an abrupt cutoff when $\omega = qv_F$. We therefore cut the integrals off at $\omega = q = 1$.

Two limiting cases are apparent: In the high-T limit, the Bose function may be approximated by T/ω for $\omega < 1$. Then the frequency integral (6.4) just gives $T\chi(q, 0)$, the equal-time correlation function in this classical limit. Similarly, the frequency integral in the vertex correction (6.5) is just proportional to $T\chi^2(q, 0)$. Only static susceptibilities enter the renormalization-group equations. In this way, the dynamics become irrelevant to the static critical behavior in the high-temperature problem.

In the low-T limit, the Bose function is $-\theta(-\omega)$, or $n(\omega) + \frac{1}{2} = \frac{1}{2}\,\mathrm{sgn}\omega$. Here we recover the quantum limit we have discussed above, except for numerical factors which are a consequence of the cutoff model we use here. (It is different from both of the models described in Sec. III.) We obtain

$$\delta r = \frac{3u_0}{N} \sum_q{}' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \chi''(q, \omega)\,\mathrm{sgn}\omega \tag{6.6}$$

and

$$\delta u = -\frac{9u_0^2}{N} \sum_q{}' \int_{-\infty}^{\infty} \frac{d\omega\, d\omega'}{2\pi^2} \frac{\chi''(q, \omega)\chi''(q, \omega')}{\omega' - \omega} \times (\mathrm{sgn}\omega - \mathrm{sgn}\omega'). \tag{6.7}$$

These answers are just what one gets by treating the Matsubara frequencies in (6.1) and (6.2) as a continuum and integrating from zero to unity.

The point to note here is that we can in principle deal with the more general expressions (6.4) and (6.5). But as we scale down in frequency, the argument of the Bose functions must also be scaled by a factor e^{-zl}. Explicitly, the renormalization-group equations are

$$\frac{dr}{dl} = 2r + 3uC_d \int_{-1}^{1} d\omega\, \chi''(1, \omega)[n(\omega e^{-zl}) + \tfrac{1}{2})], \tag{6.8}$$

$$\frac{du}{dl} = \epsilon(l)u - 9u^2 C_d \int_{-1}^{1} \frac{d\omega_1\, d\omega_2}{\pi} \chi''(1, \omega_1)\chi''(1, \omega_2) \times \left(\frac{n(\omega_1 e^{-zl}) - n(\omega_2 e^{-zl})}{\omega_2 - \omega_1}\right). \tag{6.9}$$

Unlike previous renormalization-group equations, they depend unavoidably on l on their right-hand sides, through $\epsilon(l)$ and the Bose factors. The l dependence of ϵ has the following origin: In the diagrammatic perturbation theory, each successive order in u_0 involves two propagators and one integration over k and summation over ω. The rules are given in the original scale with the unrenormalized propagators. Now in terms of the rescaled wave vectors and frequencies (q', ω') the propagator is

$$\chi(q, \omega) = [r_0 + (q'^2 + |\omega|/q')e^{-2l}]^{-1}. \tag{6.10}$$

But we want to write things in terms of the renormalized propagator

$$\tilde{\chi}(q', \omega') = (r_0' e^{2l} + q'^2 + |\omega'|/q')^{-1}, \tag{6.11}$$

which differs from (6.10) by a factor e^{2l}. This, together with the phase-space renormalization, forces us to renormalize the interaction to $\tilde{u}_0$, so that the physically relevant quantity

$$u_0 \sum_{q\omega} \chi^2 = \tilde{u}_0 \sum_{q'\omega'} \tilde{\chi}^2 \tag{6.12}$$

remains invariant. For example, in the diagram of Fig. 7(b), we want to renormalize the interaction so that

$$u_0 \sum_{q\omega}{}' \chi^2(q, \omega) = u_0 \sum_q{}' \int \frac{d\omega_1\, d\omega_2}{\pi^2} \chi''(q, \omega_1)\chi''(q, \omega_2) \times \left(\frac{n(\omega_1) - n(\omega_2)}{\omega_2 - \omega_1}\right) \tag{6.13}$$

remains constant. One can see that this renormalization depends on the temperature. For the high-T case, this just becomes

$$Tu_0 \sum_q \chi^2(q, 0) = Tu_0 e^{-dl} \sum_{q'}{}' [e^{2l}\tilde{\chi}(q', 0)]^2 = T^2 \tilde{u}_0 \sum_{q'}{}' \tilde{\chi}^2(q', 0), \tag{6.14}$$

so we must choose $\tilde{u}_0 = u_0 e^{(4-d)l}$. Then $\epsilon = 4 - d$, as in Wilson theory. But at zero T, (6.13) is

$$u_0 e^{-dl} \sum_q{}' \int \frac{e^{-2zl}\, d\omega_1' d\omega_2'}{2\pi^2} [e^{2l}\tilde{\chi}''(q,\omega_1')][e^{2l}\tilde{\chi}''(q,\omega_2')] \left(\frac{\mathrm{sgn}\omega_1' - \mathrm{sgn}\omega_2'}{e^{-zl}(\omega_1' - \omega_1')}\right)$$

$$= \tilde{u}_0 \sum_q{}' \int \frac{d\omega_1' d\omega_2'}{2\pi^2} \tilde{\chi}''(q,\omega_1')\tilde{\chi}''(q,\omega_2') \left(\frac{\mathrm{sgn}\omega_1' - \mathrm{sgn}\omega_2'}{\omega_2' - \omega_1'}\right), \tag{6.15}$$

so that $\tilde{u}_0 = u_0 e^{(4-d-z)l}$, or $\epsilon = 4-d-z$. That is, ϵ changes from $4-d-z$ to $4-d$ as l passes through $\hat{l}$ (3.28). In the intermediate region, one can formally determine $\epsilon(l)$ by defining

$$\tilde{u}_0 = u_0 \exp\left(\int_0^l \epsilon(l')\, dl'\right).$$

That is,

$$\epsilon(l) = 4 - d - z + \frac{\int d\omega_1 d\omega_2\, \chi''(\omega_1)\chi''(\omega_2)\{[n(\omega_1 e^{-zl}) - n(\omega_2 e^{-zl})]/(\omega_2 - \omega_1)\}}{\int d\omega_1 d\omega_2\, \chi''(\omega_1)\chi''(\omega_2)\{[n(\omega_1) - n(\omega_2)]/(\omega_2 - \omega_1)\}}, \tag{6.16}$$

where $\chi''(\omega)$ means $\chi''(1,\omega)$. In either limit, $l \ll \hat{l}$, we could of course have found $\epsilon(l)$ without recourse to analysis of any particular diagram, since for $l \ll \hat{l}$ the Matsubara frequencies effectively form a continuum and the sum of ω may be approximated as an integral, while for $l \gg \hat{l}$ only the $\omega = 0$ terms matter. But the full formalism [(6.8) and (6.9), with (6.15)] gives one a well-defined, if messy, calculational procedure for treating the crossover region.

It is also worth pointing out in passing, although this limit is not the subject of this paper, that in the high-temperature problem the procedure outlined here gives one a handle on the dynamical problem without recourse to the Langevin-equation approach generally used to discuss critical dynamics. The results are the same, of course. The perturbation series which comes out of the generalized LGW functional here is equivalent, after analytic continuation of frequencies to the real axis, to the perturbative solutions of the Langevin equations of time-dependent Landau-Ginzberg theory,[2] provided that the Bose functions are always replaced by their classical limits T/ω.

ACKNOWLEDGMENTS

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APPENDIX: THE ONE-DIMENSIONAL PARAMAGNON MODEL

The solution of the problem expressed by (2.13) on its borderline dimensionality (1) makes a nice example of how the quantum renormalization works, even though, as I mentioned in Sec. III, it has nothing to do with one-dimensional interacting electrons. Part of the model, of course, is the specification of the cutoff, and here we use the first of the two schemes discussed in Sec. III, in which the wave-vector and frequency cutoffs are each taken (independently) to be unity. The renormalization-group equations come out slightly different from (3.18) and (3.19) because it does not make much sense to talk about a solid angle in one dimension. Instead of (3.16) we have

$$\frac{1}{\beta N} \sum_{q\omega}{}' = 2\int_{e^{-zl}}^1 \frac{d\omega}{2\pi} \int_{-1}^1 \frac{dq}{2\pi} + 2\int_{-1}^1 \frac{d\omega}{2\pi} \int_{e^{-l}}^1 \frac{dq}{2\pi}, \tag{A1}$$

so

$$\frac{1}{\beta N} \sum_{q\omega}{}' \left(r_0 + q^2 + \frac{|\omega|}{q}\right)^{-m} = \frac{zl}{\pi^2} \int_0^1 dq \left(r_0 + q^2 + \frac{1}{q}\right)^{-m} + \frac{l}{\pi^2} \int_0^1 \frac{d\omega}{(r_0 + 2 + \omega)^m}. \tag{A2}$$

Thus (3.18) and (3.19) become ($\epsilon = 0$, $z = 3$)

$$\frac{dr}{dl} = 2r + \frac{3u}{\pi^2}\left[\ln\left(\frac{r+2}{r+1}\right) + 3\int_0^1 \frac{x^2\, dx}{(x^3 + rx + 1)^2}\right], \tag{A3}$$

$$\frac{du}{dl} = \frac{9u^2}{\pi^2}\left(\frac{1}{(r+1)(r+2)} + 3\int_0^1 \frac{x^2\, dx}{(x^3 + rx + 1)^2}\right). \tag{A4}$$

For small r and u we can ignore the r in the denominator of the equation for du/dl, so that

$$\frac{du}{dl} = -\frac{9u^2}{\pi^2}, \tag{A5}$$

whose solution is

$$u(l) = (u_0^{-1} + 9l/\pi^2)^{-1}. \tag{A6}$$

We then substitute this expression into (A3), and keep terms to first order in r on the right-hand side.[16] The expansion of the integral gives

$$\int_0^1 \frac{x\,dx}{1+x^3+rx} = \int_0^1 \frac{x\,dx}{1+x^3} - \tfrac{1}{6} r + O(r^2) \ , \qquad \text{(A7)}$$

so

$$\frac{dr}{dl} = 2r + \frac{3}{\pi^2(u_0^{-1} + 9l/\pi^2)}(\lambda - r) \ , \qquad \text{(A8)}$$

where

$$\lambda = \ln 2 + \int_0^1 \frac{x\,dx}{x^3+1} \ . \qquad \text{(A9)}$$

The solution of (A8), ignoring transients which decay as $1/l$, is

$$r(l) = \bar{r}_0 e^{2l}[l_0/(l_0+l)]^{1/3} \ , \qquad \text{(A10)}$$

where $l_0 = \pi^2/9u_0$. As in Sec. IV, $\bar{r}_0$ differs from r_0 because of the transients. The linearized equation is valid up to the $l = l_1$ where $r = 1$, but we need go no further, since at this point the effective Kadanoff cell size is equal to the correlation length, and perturbation theory will suffice to finish the problem. To find the correlation length, then, we set $r(l_1) = 1$ in (A10) and use $e^{l_1} = \xi$. Then

$$\xi = \bar{r}_0^{-1/2}(\ln 1/r_0)^{1/6} \ , \qquad \text{(A11)}$$

which is exactly the $\epsilon = 0$ result found from the more usual sort of renormalization-group equations (like 3.25 and 3.26).[16] This is not surprising, since the linearized equations in the two cases are the same.

This analysis tells us that there is an ordered state at $T = 0$ if the coupling strength is large enough. It is interesting to look at this case ($\bar{r}_0 < 0$) at finite T, since we know there can then be no order. When the last finite Matsubara frequency is scaled out, $l = \hat{l} = \frac{1}{3}\ln(E_F/T)$, so ($\hat{l} \gg l_0$)

$$r(\hat{l}) = -|\bar{r}_0|\left(\frac{E_F}{T}\right)^{2/3}\left(\frac{3l_0}{\ln(E_F/T)}\right)^{1/3} \ , \qquad \text{(A12)}$$

$$u(\hat{l}) = \frac{\pi^2}{3\ln(E_F/T)} \ . \qquad \text{(A13)}$$

I have not found any reasonable way to use the renormalization group (or at least its momentum-space version) on the classical one-dimensional LGW problem that remains at this point. However, this problem has been studied using other methods.[26] One finds, for $|r| \gg u$, a correlation length

$$\begin{aligned} \xi &= |r(l)|^{-1/2}\exp[|r(l)|^{3/2}/u(l)] \\ &= |\bar{r}_0|^{-1/2}\left(\frac{T}{E_F}\right)^{1/3}\left(\frac{\ln(E_F/T)}{3l_0}\right)^{1/6} \\ &\quad \times \exp\left[\frac{(3|\bar{r}_0|)^{3/2}}{\pi^2}\left(\frac{E_F}{T}\right)\left(l_0 \ln\frac{E_F}{T}\right)^{1/2}\right] . \end{aligned} \qquad \text{(A14)}$$

However, this is the correlation length in a problem whose scale differs from that of the original problem by a factor $e^l = (E_F/T)^{1/3}$. Thus the true ξ is larger by a factor e^{2l},

$$\begin{aligned} \xi &= |\bar{r}_0|^{-1/2}\left(\frac{3u_0\ln(E_F/T)}{\pi^2}\right)^{1/6} \\ &\quad \times \exp\left[\frac{\sqrt{3}|\bar{r}_0|^{3/2}}{\pi u_0^{1/2}}\left(\frac{E_F}{T}\right)\left(\ln\frac{E_F}{T}\right)^{1/2}\right] . \end{aligned} \qquad \text{(A15)}$$

The quantum effects manifest themselves in the fractional powers of $\ln(E_F/T)$ which produce corrections to classical one-dimensional LGW behavior.

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[1] K. Wilson and J. Kogut, Phys. Rep. C 12, 75 (1974); S. Ma, Rev. Mod. Phys. 45, 589 (1973); M. E. Fisher, *ibid.* 46, 597 (1974).
[2] B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. Lett. 29, 1548 (1972); Phys. Rev. B 10, 137 (1974); B. I. Halperin, P. C. Hohenberg, and E. Siggia, Phys. Rev. Lett. 28, 548 (1974); S. Ma and G. Mazenko, *ibid.* 33, 1384 (1974); Phys. Rev. B 11, 4077 (1975); R. Freedman and G. Mazenko, Phys. Rev. Lett. 34, 1575 (1975).
[3] R. L. Stratonovich, Dokl. Akad. Nauk SSSR 2, 1097 (1957) [Sov. Phys.-Doklady 2, 416 (1957)]; J. Hubbard, Phys. Rev. Lett. 3, 77 (1959).
[4] J. Hertz, AIP Conf. Proc. 24, 298 (1975).
[5] S. Q. Wang, W. E. Evenson and J. R. Schrieffer, Phys. Rev. Lett. 23, 92 (1969); J. R. Schrieffer, *ibid.* 23, 92 (1969); J. R. Schrieffer, W. E. Evenson and S. Q. Wang, J. Phys. (Paris) 32, C1, (1971).
[6] J. R. Schrieffer (unpublished); W. E. Evenson, J. R. Schrieffer, and S. Q. Wang, J. Appl. Phys. 41, 1199 (1970).
[7] J. A. Hertz and M.A. Klenin, Phys. Rev. B 10, 1084 (1974).
[8] J. Hubbard, Proc. R. Soc. A 276, 238 (1963).
[9] J. Lindhard, Dan. Vidensk. Selsk. Mat.-Fys. Medd. 28, 8 (1954).
[10] S. Doniach and S. Engelsberg, Phys. Rev. Lett. 17, 750 (1966).
[11] M. T. Beal-Monod, K. Maki, and J. P. Hurault, J. Low Temp. Phys. 17, 439 (1974).
[12] M. T. Beal-Monod, Solid State Commun. 14, 677 (1974).
[13] F. J. Wegner and A. Houghton, Phys. Rev. A 8, 401 (1973); J. Hertz, J. Low Temp. Phys. 5, 123 (1971).
[14] C. Kittel, in *Solid State Physics,* edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic, New York, 1969), Vol. XXII, p. 14.
[15] This result was also obtained in a different way by M. T. Beal-Monod, J. Low Temp. Phys. 17, 467 (1974), and M. T. Beal-Monod and K. Maki, Phys. Rev. Lett. 34, 1461 (1975).
[16] M. E. Fisher, S. Ma, and B. G. Nickel, Phys. Rev.

Lett. 29, 917 (1972); see also F. J. Wegner and E. K. Riedel, Phys. Rev. B 7, 248 (1973).
[17]E. K. Riedel and F. J. Wegner, Z. Phys. 225, 195 (1969); Phys. Rev. Lett. 24, 730 (1970).
[18]P. Fulde and A. Luther, Phys. Rev. 170, 570 (1968).
[19]T. Lubensky, Phys. Rev. B 11, 3573 (1975); T. Lubensky and A. B. Harris, AIP Conf. Proc. 24, 311 (1975); G. Grinstein, *ibid.* 24, 313 (1975); G. Grinstein and A. Luther (unpublished).
[20]See, e.g., P. Fulde and I. Peschel, Adv. Phys. 21, 1 (1972).
[21]M. A. Klenin and J. A. Hertz, AIP Conf. Proc. 24, 242 (1975), and unpublished.
[22]This was also noticed for this system by J. Lajzerowicz and P. Pfeuty, Phys. Rev. B 11, 4560 (1975).
[23]A. P. Young, Oxford report (unpublished).
[24]P. C. Hohenberg and J. Swift (unpublished).
[25]J. Rudnick, Phys. Rev. B 11, 363 (1975) has used a similar diagrammatic procedure in the classical LGW problem.
[26]D. J. Scalapino, M. Sears, and R. Ferrell, Phys. Rev. B 6, 3409 (1972); J. Krumhansl and J. R. Schrieffer, *ibid.* 11, 3535 (1975).