Order in metallic chains. I. The single chain*

H. Gutfreund†
Department of Physics, Stanford University, Stanford, California 94305

R. A. Klemm‡
Department of Applied Physics, Stanford University, Stanford, California 94305
(Received 7 April 1975; revised manuscript received 28 October 1975)

The nature of the order as $T\to 0$ in the one-dimensional interacting electron gas is investigated. We consider both low-momentum-transfer (g_2 and g_4) and large-momentum-transfer (g_1 and g_3) electron-electron interactions, focusing upon the particular value of g_1 for which Luther and Emery have recently found an exact solution. For this value of $g_1 = -6/5\pi v_F$, we calculate the low-frequency behavior of the zero-temperature response functions explicitly. For the singlet superconducting and charge-density-wave responses, our results are consistent with those of Luther and Emery, implying that these response functions are divergent as $\omega\to 0$ for certain values of g_2. For the triplet superconducting and spin-density-wave responses, however, our results differ from those of Luther and Emery, as we show explicitly that these responses have a gap for low frequencies, as was predicted by Lee. Furthermore, they do not diverge at the gap edge for any value of g_2. We also consider the effect of interactions between electrons on the same side of the Fermi "surface," and find that the low-momentum-transfer process of that type (g_4) does not change the response behavior qualitatively. For the large-momentum-transfer process between electrons on the same side of the Fermi "surface" (g_3), we may solve the problem exactly for $g_2 = 0$ and $g_1 = -6/5\pi v_F$, and find that the ground state of the system exhibits only long-range charge-density-wave order. By a mapping onto the classical two-dimensional Coulomb gas problem, we may extend those results for $g_3\neq 0$ to the region $g_1 < 0$ and $g_1 - 2g_2 < |g_3|$.

I. INTRODUCTION

The problem of ordering in one-dimensional metallic chains has been discussed extensively since Little[1] suggested the possibility of superconductivity in one-dimensional organic chains. The interest in this problem has greatly increased recently following the experimental work on TTF-TCNQ (tetrathiafulvalene tetracyanoquinodimethane),[2] KCP [$K_2Pt(CN)_2Br_{0.3}\cdot 3H_2O$],[3] and the discovery of superconductivity in $(SN)_x$ (polysulfur nitride).[4]

The basic difficulty in the theoretical treatment of order in one-dimensional Fermi systems is the fact that such systems possess two inherent instabilities. These show up as divergences in the four-point vertex function $\Gamma(k_1k_2, k_3k_4)$. One divergence occurs in the particle-particle channel (Cooper channel) for $k_1=-k_2$ and it indicates the onset of superconductivity. The other divergence, which is more typical of a one-dimensional system, occurs in the particle-hole channel ("zero-sound" channel) for $k_3-k_1=2k_F$, and it indicates the onset of the Peierls or Overhauser instabilities which result in charge or spin-density waves. There is a competition between these instabilities and it is the role of the theory to specify the conditions under which one, or the other, mode of behavior will prevail.

Bychkov *et al.*[5] were the first to treat the divergences in the two channels simultaneously. They summed the so-called parquet diagrams (which amounts to making the mean-field approximation) and found that in the logarithmic approximation the transition temperatures to the Peierls state (T_P) and to the superconducting state (T_c) are equal. Strictly speaking, however, a single chain cannot undergo a phase transition at any finite temperature, as the fluctuations drive the transition temperature to 0. But real systems are composed of many parallel chains which are in some sense coupled, and may exhibit a phase transition. Any long-range order that might arise in such a system is due solely to its three-dimensional or "coupled" nature. Thus, the discussion of phase transitions involves the solution of the coupled-chain problem. This will be done in the second of this series of papers. In the present paper we restrict our discussion to the single-chain problem. The question of order in a single chain is still meaningful in the sense of a tendency towards long-range order as $T\to 0$.

A one-dimensional Fermi system is characterized by a Fermi "surface" consisting of two points $\pm k_F$. The important electron-electron interaction processes are those which involve electrons with momenta in the neighborhood of these two points; therefore, the momentum transfer is either $q\simeq 0$ or $|q|\simeq 2k_F$. There are four relevant scattering processes (Fig. 1). The $q\simeq 0$, or the forward scattering, processes may involve either electrons on opposite sides of the Fermi "surface"

(g_2) or on the same side (g_4). In the $|q| \simeq 2k_F$, or backward scattering process (g_1), two electrons on opposite sides of the Fermi "surface" interact and exchange momenta. In the case of one electron per atom (or a half-filled band) there is also an umklapp process in which two electrons on the same side of the Fermi "surface" scatter together to the other side (g_3). Various treatments of one-dimensional systems in the literature may be characterized by the assumptions made on these coupling parameters. For example, in the work of Bychkov *et al.*,[5] it is assumed that $g_1 = g_2 = g$ and g_3, g_4 are neglected.

The first discussion of order in a one-dimensional system that went beyond the mean-field approximation was due to Sólyom.[6] He applied the renormalization-group method to calculate the response functions corresponding to three types of order, and found that for $g_1 \geq 0$, the line $g_1 = 2g_2$ separates between the regions of singlet superconducting and charge (spin) density wave behavior (only g_1 and g_2 were taken into account). This approach was extended by Fukuyama *et al.*,[7] who also calculated the triplet superconducting response function. The renormalization-group method used in Refs. 6 and 7 apparently fails for $g_1 < 0$, as the invariant couplings become strong.

Luther and Emery[8] (LE) have recently found a remarkable solution for the one-dimensional electron-gas model with both g_1 and g_2. They extended the Luttinger[9,10] model by adding the backscattering term to the Hamiltonian and were able to diagonalize this Hamiltonian for a particular negative value of g_1. They have also calculated the response functions for this value of g_1. It was then pointed out by Lee[11] that in contrast to their conclusion, there is no possibility of triplet superconductivity or spin-density wave ordering. Chui and Lee[12] mapped the LE Hamiltonian onto the two-dimensional classical Coulomb gas problem and gave arguments extending the regions of charge-density wave and singlet superconductivity behavior to the entire $g_1 < 0$ half-plane.

In Sec. II of the present paper we review the LE solution. Section III presents a detailed calculation of the response functions on the LE "line of solutions" including a novel calculation of the coefficients of the leading divergences. These coefficients will be used in the second paper to calculate the transition temperatures. As the calculation of these response functions has been the source of some confusion in the literature, we present here a rather detailed derivation. In Sec. IV we consider the addition of the g_3 and g_4 interactions. We find that the addition of g_4 is trivial, but should be included on physical grounds, as the case of $g_4 = g_2$ is of particular interest. This addition does not qualitatively change the regions in g_1/g_2 plane of charge-density wave and singlet superconducting behavior, although it distorts them and increases the size of the region in g_1/g_2 space for which the model may be solved. For $g_3 \neq 0$, we may solve the model only for $g_2 = g_4 = 0$ and for the particular value of g_1 for which the LE model has a solution. For this "point" in the g_1/g_2 plane, we calculate the correlation functions, and find that at $T = 0$ there is long-range order of the charge-density-wave type.

II. LUTHER-EMERY SOLUTION

The starting point of the LE model is the Luttinger[9] Hamiltonian with spin

$$H = v_F \sum_{k,s} k[a_{1s}^\dagger(k)a_{1s}(k) - a_{2s}^\dagger(k)a_{2s}(k)] + \frac{2}{L}\sum_k V_k \rho_1(k)\rho_2(-k), \qquad (1)$$

where $a_{is}(k)$ is the fermion annihilation operator for an electron with momentum k, on the branch $i = 1$ (2) with spin $s = \pm 1$. The density operators $\rho_i(k)$ are defined as

$$\rho_i(k) = \frac{1}{\sqrt{2}} \sum_{p,s} a_{is}^\dagger(p+k)a_{is}(p). \qquad (2)$$

The important features of the Luttinger model are the linear dispersion $\epsilon_i^0(k) = \pm v_F k$ for the kinetic energy, where the upper (lower) sign refers to the $i = 1$ (2) branch, and the assumption that in the ground state of the noninteracting system all the states with momentum $k < k_F$ on branch $i = 1$ and $k > -k_F$ on branch $i = 2$ are filled. We also define the spin-density operators

$$\sigma_i(k) = \frac{1}{\sqrt{2}} \sum_{ps} s a_{is}^\dagger(p+k)a_{is}(p). \qquad (3)$$

The ρ_i and σ_i satisfy the usual boson commutation relations

$$[\rho_i(-k), \rho_j(k')] = \pm\delta_{ij}\delta_{kk'}(Lk/2\pi),$$
$$[\sigma_i(-k), \sigma_j(k')] = \pm\delta_{ij}\delta_{kk'}(Lk/2\pi), \qquad (4a)$$
$$[\sigma_i(-k), \rho_j(k')] = 0,$$

where the upper (lower) sign refers to $i = 1$ (2). These commutation relations hold when evaluated in any excited state of the unperturbed system. A similar model for the interacting one-dimensional electron gas was proposed by Tomonaga.[13] This model assumes that the relevant states of the system are those with electrons and holes in the neighborhood of the Fermi "surface" and that there is no transfer of electrons from one side of the Fermi "surface" to the other. In view of the first assumption Tomonaga linearized the electron kinetic energy near $\pm k_F$ and defined the density operators $\rho^+(k)=\sum_{p>0} a^\dagger(p+k)a(p)$ and $\rho^-(k)=\sum_{p<0} a^\dagger(p+k)a(p)$. These operators satisfy the same commutation relations as the $\rho_1(k)$, $\rho_2(k)$ operators of the Luttinger model, only when evaluated in the subspace of states with electrons and holes in the neighborhood of the Fermi energy. Since it is assumed that these are the only states that matter, the Tomonaga model becomes equivalent to the Luttinger model[14] (if ρ^+ is identified with ρ_1 and ρ^- with ρ_2). Theorists prefer to refer to the Luttinger model, but it is good to keep in mind the equivalence to the Tomonaga model which has a greater physical appeal as it avoids the unrealistic, infinitely deep Fermi-Dirac sea.

The linear dispersion $\epsilon_i^0(k)=\pm v_F k$ leads to the relations

$$[\rho_i(k),H_0]=\mp v_F k\,\rho_i(k)\,,\qquad [\sigma_i(k),H_0]=\mp v_F k\,\sigma_i(k)\,, \tag{4b}$$

where H_0 is the first part of Eq. (1) and, as before, the upper (lower) sign refers to $i=1$ (2). Combining these relations with the fact that all the excited states of H_0 can be obtained by operating with products of the collective operators ρ_i and σ_i on the ground state, it is possible to write H_0 in the form[15,16]

$$H_0=\frac{2\pi v_F}{L}\sum_{k>0}[\rho_1(k)\rho_1(-k)+\rho_2(-k)\rho_2(k)+\sigma_1(k)\sigma_1(-k)+\sigma_2(-k)\sigma_2(k)]\,. \tag{5}$$

The interaction term in Eq. (1) describes the g_2 "forward scattering" interaction shown in Fig. 1.

LE have added to the Hamiltonian the term

$$H_{LE}=\sum_{ss'}\int dx\,\psi_{1s}^\dagger(x)\psi_{2s'}^\dagger(x)\psi_{1s'}(x)\psi_{2s}(x)\times(U_\parallel\delta_{s,s'}+U_\perp\delta_{s,-s'})\,, \tag{6}$$

where

$$\psi_{is}(x)=\frac{1}{\sqrt{L}}\sum_k e^{ikx}a_{is}(k)\,. \tag{7}$$

This term represents the "backward scattering" denoted by g_1 in Fig. 1. The processes in which the spins of the two electrons are parallel or antiparallel play essentially different roles in the

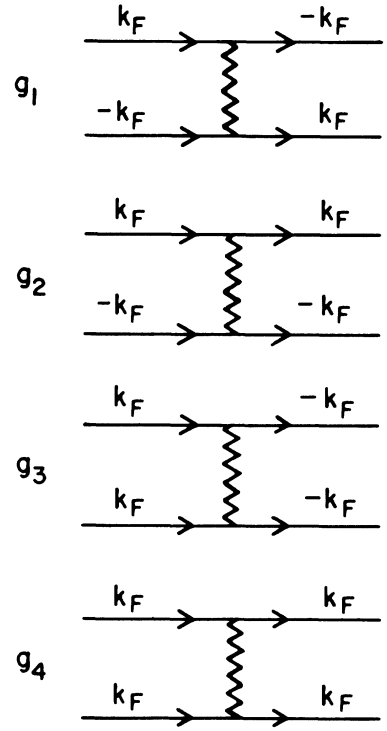

FIG. 1. Shown are diagrammatic representations of the four possible types of electron-electron interactions. Of the two low-momentum-transfer processes, g_4 and g_2 describe the "forward scattering" of electrons on the same and opposite sides of the Fermi surface, respectively. Of the two large-momentum-transfer processes, g_3 and g_1 describe the "backward scattering" of electrons on the same and opposite sides of the Fermi surface, respectively. However, g_3 is only important for an exactly half-filled band.

model, and hence it is advisable to distinguish between them by different coupling strengths $U_\parallel$ and $U_\perp$. We shall keep this notation till the end, when we put $U_\parallel=U_\perp=g_1$. The case of parallel spins can be immediately written as a bilinear form in the density and spin-density operators

$$-\frac{U_\parallel}{L}\sum_k[\rho_1(k)\rho_2(-k)+\sigma_1(k)\sigma_2(-k)]\,. \tag{8}$$

To express the "back-scattering" interaction for electrons with antiparallel spins in these operators, LE make use of the boson representation for the fields

$$\psi_{is}(x)=\frac{1}{(2\pi\alpha)^{1/2}}\exp\left[\pm\left(ik_Fx-\frac{1}{\sqrt{2}}\sum_k A_k(x)[\rho_i(k)+s\sigma_i(k)]\right)\right]\,,$$

where

$$A_k(x)=\frac{2\pi}{Lk}\exp\left(-\frac{\alpha|k|}{2}-ikx\right), \tag{9}$$

the upper (lower) sign in the exponent corresponds to $i=1$ (2), and α^{-1} is a cutoff parameter which is interpreted as a momentum transfer cutoff. When the full Hamiltonian is expressed in terms of the ρ and σ operators, one finds that it separates into two commuting parts,

$$H_\rho=\frac{2\pi v_F}{L}\sum_{k>0}[\rho_1(k)\rho_1(-k)+\rho_2(-k)\rho_2(k)]+\frac{1}{L}\sum_k[2V(k)-U_\parallel]\rho_1(k)\rho_2(-k) \tag{10}$$

and

$$H_\sigma=\frac{2\pi v_F}{L}\sum_{k>0}[\sigma_1(k)\sigma_1(-k)+\sigma_2(-k)\sigma_2(k)]$$
$$-\frac{U_\parallel}{L}\sum_k\sigma_1(k)\sigma_2(-k)+\frac{U_\perp}{(2\pi\alpha)^2}\int dx\left[\exp\left(\sqrt{2}\sum_k A_k(x)[\sigma_1(k)+\sigma_2(k)]\right)+\text{H.c.}\right]. \tag{11}$$

The first part, H_ρ, is simply the Luttinger Hamiltonian and it is readily diagonalized by the canonical transformation $e^{iG_\rho}H_\rho e^{-iG_\rho}$, with

$$G_\rho=\frac{2\pi i}{L}\sum_k\frac{\phi(k)}{k}\rho_1(k)\rho_2(-k), \tag{12}$$

where

$$\tanh 2\phi(k)=-[2V(k)-U_\parallel]/2\pi v_F. \tag{13}$$

One can similarly diagonalize the bilinear terms in H_σ, using a canonical transformation generated by

$$G_\sigma=\frac{2\pi i}{L}\sum_k\frac{\psi}{k}\sigma_1(k)\sigma_2(-k), \tag{14}$$

where

$$\tanh 2\psi=+U_\parallel/2\pi v_F. \tag{15}$$

The transformations in Eqs. (13) and (15) are defined as long as the magnitude of the right-hand side is ≤ 1. This implies that the model breaks down when $V(k)$, $U_\parallel$ are outside the region bounded by the lines $\tilde{U}_\parallel=\pm 2$, $\tilde{U}_\parallel/2-\tilde{V}(k)=\pm 1$, where $\tilde{U}_\parallel=U_\parallel/\pi v_F$ and $\tilde{V}(k)=V(k)/\pi v_F$. The effect of the last transformation on H_σ is to multiply each of the σ operators by e^ψ. LE noted that this term simplifies greatly for the case $2^{1/2}e^\psi=1$, which on account of Eq. (15) implies $\tilde{U}_\parallel=-\frac{6}{5}$. For this particular case, one can express this term as a bilinear form in some fictitious spinless fermion creation and annihilation operators. To do this, one uses Eq. (9) with the factor $(1/\sqrt{2})[\rho_i(k)+s\sigma_i(k)]$ replaced by $\sigma_i(k)$. Writing H_σ in the fictitious fermion representation, we have

$$\tilde{H}_\sigma=v_F''\sum_k k[\alpha_1^\dagger(k)\alpha_1(k)-\alpha_2^\dagger(k)\alpha_2(k)]$$
$$+\frac{U_\perp}{2\pi\alpha}\sum_k[\alpha_1^\dagger(k)\alpha_2(k-2k_F)+\text{H.c.}], \tag{16}$$

where $v_F''=\frac{4}{5}v_F$ is the renormalized velocity of the fictitious fermion excitations, resulting from the transformation in Eq. (13). This Hamiltonian may be diagonalized by the rotation

$$\begin{pmatrix}\alpha_1(k+k_F)\\ \alpha_2(k-k_F)\end{pmatrix}=\begin{pmatrix}\cos\theta_k & -\sin\theta_k\\ \sin\theta_k & \cos\theta_k\end{pmatrix}\begin{pmatrix}\overline{\alpha}_1(k)\\ \overline{\alpha}_2(k)\end{pmatrix}, \tag{17}$$

where $\tan 2\theta_k=\Delta/v_F''k$, and $\Delta=|U_\perp|/2\pi\alpha$ is the gap in the energy spectrum

$$\epsilon_{1,2}(k)=v_F''k_F\pm\operatorname{sgn}k[(v_F''k)^2+\Delta^2]^{1/2}. \tag{18}$$

We have defined the transformation in Eq. (17) so that k is measured relative to the Fermi momentum.

The full Hamiltonian may finally be written in the diagonal form

$$H=\frac{2\pi v_F'}{L}\sum_{k>0}[\rho_1(k)\rho_1(-k)+\rho_2(-k)\rho_2(k)]$$
$$+\sum_k[\epsilon_1(k)\overline{\alpha}_1^\dagger(k)\overline{\alpha}_1(k)+\epsilon_2(k)\overline{\alpha}_2^\dagger(k)\overline{\alpha}_2(k)], \tag{19}$$

where $v_F'=v_F\operatorname{sech}2\phi(0)$ is the renormalized velocity due to the transformation in Eq. (12). The spectrum of this Hamiltonian consists of the usual Tomonaga-Luttinger "sound waves" arising from the ρ operators, and of fictitious "single-fermion" types of excitations with gaps, arising from the σ degrees of freedom.

III. RESPONSE FUNCTIONS

Any type of order in the ground state would show up as a singularity in the response of the system to a corresponding generalized external field. Following LE, we shall consider four possible types of one-dimensional ordering as $T\to 0$: charge-density wave (CDW) and antiferromagnetic or spin-density wave (SDW), and singlet and triplet pairing (SS and TS). To each of these types of order there corresponds a generalized susceptibility, or retarded response function;

$$\chi^R_{\rm CDW}(x,t) = -i\Theta(t)\langle[\psi_{1+}(x,t)\psi^\dagger_{2+}(xt),\ \psi_{2+}(0)\psi^\dagger_{1+}(0)]\rangle\,,$$
$$\chi^R_{\rm SDW}(x,t) = -i\Theta(t)\langle[\psi_{1+}(x,t)\psi^\dagger_{2-}(x,t),\ \psi_{2-}(0)\psi^\dagger_{1+}(0)]\rangle\,, \tag{20}$$
$$\chi^R_{\rm SS}(x,t) = -i\Theta(t)\langle[\psi_{1+}(x,t)\psi_{2-}(x,t),\ \psi^\dagger_{2-}(0)\psi^\dagger_{1+}(0)]\rangle\,,$$
$$\chi^R_{\rm TS}(x,t) = -i\Theta(t)\langle[\psi_{1+}(x,t)\psi_{2+}(x,t),\ \psi^\dagger_{2+}(0)\psi^\dagger_{1+}(0)]\rangle\,.$$

One can also write other susceptibilities, involving only operators on one side of the Fermi surface, however, these do not show any singular behavior.

The separation of the Hamiltonian into two independent parts H_ρ and H_σ implies a similar factorization of each of the two terms of the commutators in the response functions. Expressing the field operators in their boson representation and separating the ρ and σ parts, we find that the four response functions can be written in the general form

$$\chi^R(x,t) = [-i\Theta(t)c/(2\pi\alpha)^2][S^\pm_\rho(x,t)S^\pm_\sigma(x,t) - S^\pm_\rho(-x,-t)S^\pm_\sigma(-x,-t)]\,, \tag{21}$$

where $c=1$ for the pairing susceptibilities and $c=e^{2ik_Fx}$ for the charge (spin) density-wave susceptibilities, and

$$S^\pm_\rho(x,t) = \left\langle e^{itH_\rho}\exp\left(-\sum_k \frac{A_k(x)}{\sqrt{2}}[\rho_1(k)\pm\rho_2(k)]\right)e^{-itH_\rho}\exp\left(\sum_k \frac{A_k(0)}{\sqrt{2}}[\rho_1(k)\pm\rho_2(k)]\right)\right\rangle_\rho\,, \tag{22}$$

$$S^\pm_\sigma(x,t) = \left\langle e^{itH_\sigma}\exp\left(-\sum_k \frac{A_k(x)}{\sqrt{2}}[\sigma_1(k)\pm\sigma_2(k)]\right)e^{-itH_\sigma}\exp\left(\sum_k \frac{A_k(0)}{\sqrt{2}}[\sigma_1(k)\pm\sigma_2(k)]\right)\right\rangle_\sigma\,, \tag{23}$$

where the upper (lower) signs in Eq. (22) correspond to CDW and SDW (SS and TS), and in Eq. (23) to CDW and SS (SDW and TS). The S_ρ and S_σ satisfy

$$S_{\rho,\sigma}(-x,t) = S_{\rho,\sigma}(x,t)\,,$$
$$S_{\rho,\sigma}(x,-t) = S^*_{\rho,\sigma}(x,t)\,, \tag{24}$$

as will become apparent from their explicit expressions [Eqs. (26), (34), and (35)]. Using these symmetry relations with Eq. (21), we get

$$\mathrm{Im}\chi^R(\omega) = -\frac{1}{2}\int_{-\infty}^{\infty}\frac{dt}{2\pi\alpha}\int_{-\infty}^{\infty}\frac{dx}{2\pi\alpha}[e^{i\omega t}S^\pm_\rho(x,t)S^\pm_\sigma(x,t) - (\omega\to-\omega)]\,, \tag{25}$$

where $\chi^R(\omega)$ is the Fourier transform of Eq. (21) at momentum $q=0$ for the paring susceptibilities, and at $q=2k_F$ for the charge (spin) density-wave functions.

We shall now calculate the right-hand side of Eq. (25). The functions $S^\pm_\rho(x,t)$ are essentially the same as the corresponding correlation functions in the Luttinger model. The slight differences are the replacement of the forward-scattering coupling strength $2V$ in the Luttinger Hamiltonian by $(2V - U_\parallel)$ in H_ρ, and the appearance of $1/\sqrt{2}$ in Eq. (22). Keeping these in mind, one can obtain $S^\pm_\rho(x,t)$ from the first term in Eq. (24) of Luther and Peschel.[10] At $T=0$, we obtain

$$S^\pm_\rho(x,t) = \alpha^{\delta_\pm}\{[\alpha - i(v'_Ft - x)][\alpha - i(v'_Ft + x)]\}^{-\delta_\pm/2}\,, \tag{26}$$

where

$$\delta_\pm = \left(\frac{1\mp(\tilde g_2+\frac{3}{5})}{1\pm(\tilde g_2+\frac{3}{5})}\right)^{1/2} \tag{27}$$

and $\tilde g_2 = V(0)/\pi v_F$. The exponent δ_+ corresponds to CDW and SDW, and δ_- to SS and TS.

LE argued that the low-frequency behavior of the zero-temperature response functions is determined solely by S_ρ, owing to the finite gap in the excitation spectrum of the σ degrees of freedom. Thus, they found $\mathrm{Im}\chi^R(\omega)_{T=0}\sim\omega^\mu$, where $\mu=-2+\delta_+$ for charge- and spin-density wave behavior and $\mu = -2+\delta_-$ for singlet and triplet superconducting behavior. The fallacy of this argument was pointed out previously by Lee,[11] who has shown that the σ factor in the response functions for CDW and SS is dramatically different from that for SDW and TS. In the first case $S_\sigma(x,t)$ is dominated by a term constant in space and time, so that the Fourier transform of $\chi^R(x,t)$ is indeed determined by $S_\rho(x,t)$ leading to the result of LE. In the second case $S_\sigma(x,t)$ has no such constant term. Careful examination of the space and time behavior of S_σ in this case shows that $\mathrm{Im}\chi^R(\omega)_{T=0}$ never diverges as $\omega\to0$. To see how all this comes about we shall now consider $S^\pm_\sigma(x,t)$ in some detail.

Let us first perform the transformation in Eq. (14) on all the operators in Eq. (23). The result is

$$S_\sigma^\pm(x,t)=\left\langle e^{it\bar{H}_\sigma}\exp\left(-\sum_k \frac{A_k(x)}{2^{1/2}}e^{\pm\psi}[\sigma_1(k)\pm\sigma_2(k)]\right)e^{-it\bar{H}_\sigma}\exp\left(\sum_k\frac{A_k(0)}{2^{1/2}}e^{\pm\psi}[\sigma_1(k)\pm\sigma_2(k)]\right)\right\rangle\ , \qquad (28)$$

where $\bar{H}_\sigma = e^{iG_\sigma}H_\sigma e^{iG_\sigma}$. On the LE line of solutions, namely for $e^{-\psi}=\sqrt{2}$, this expression may be cast in a fermion representation using again Eq. (9) for spinless field operators. One obtains

$$S_\sigma^+(x,t)=(2\pi\alpha)\langle[\Psi_1(x,t)\Psi_2^\dagger(x,t)e^{-2ik_Fx}]^{1/2}\times[\Psi_2(0)\Psi_1^\dagger(0)]^{1/2}\rangle \qquad (29)$$

for CDW and SS, and

$$S_\sigma^-(x,t)=(2\pi\alpha)^2\langle\Psi_1(x,t)\Psi_2(x,t)\Psi_2^\dagger(0)\Psi_1^\dagger(0)\rangle \qquad (30)$$

for SDW and TS. The $\Psi_{1,2}(x,t)\equiv e^{it\bar{H}_\sigma}\Psi_{1,2}(x)e^{-it\bar{H}_\sigma}$ are the fictitious field operators corresponding to the $\bar{\alpha}_{1,2}(k)$ annihilation operators in Eq. (19) and the expectation values are calculated with respect to $\bar{H}_\sigma$. Examination of Eqs. (29) and (30) shows two basic differences between these expectation values. One difference is the powers of the operators, which we shall show to be of minor importance. The major difference is that S_σ^+ is (except for the square roots) a fictitious fermion particle-hole correlation function, whereas S_σ^- is a fictitious fermion particle-particle correlation function. First let us discuss S_σ^+. We are interested in the behavior of S_σ^+ at large x and t and may therefore take $[\psi_1(xt)\psi_2^\dagger(x,t),\psi_2(0)\psi_1^\dagger(0)]=0$, which is true for finite x, t. We can now write

$$S_\sigma^+(xt)\simeq 2\pi\alpha e^{-ik_Fx}\langle[\psi_1(xt)\psi_2^\dagger(xt)\psi_2(0)\psi_1^\dagger(0)]^{1/2}\rangle\ . \qquad (31)$$

At this point we make the classical approximation

$$S_\sigma^+(xt)\simeq 2\pi\alpha e^{-ik_Fx}\langle\psi_1(xt)\psi_2^\dagger(xt)\psi_2(0)\psi_1^\dagger(0)\rangle^{1/2}\ . \qquad (32)$$

The main result of the subsequent calculation of S_σ^+ is that it is a constant for $x, t\to\infty$. This can be shown by general arguments[12] which do not depend on the last approximation. However, the form in Eq. (32) will enable us to compute this constant explicitly and we believe that the approximation involved will give a correct order of magnitude estimate.

We are now ready to calculate $S_\sigma^\pm$. To this end we express the field operators in terms of the creation and annihilation operators $\alpha_{1,2}$, $\alpha_{1,2}^\dagger$, transform to the $\bar{\alpha}_{1,2}$, $\bar{\alpha}_{1,2}^\dagger$ representation [Eq. (17)], and evaluate the expectation values with respect to the $\bar{\alpha}$ part of the Hamiltonian [Eq. (19)]. After a lengthy but straightforward computation (see Appendix A) we get at $T=0$,

$$S_\sigma^+(x,t)\simeq\alpha[C^2+f(x,t)f(-x,t)]^{1/2} \qquad (33)$$

and

$$S_\sigma^-(x,t)=\alpha^2[f(x,t)f(-x,t)-g^2(x,t)]\ , \qquad (34)$$

where (in units of $v_F''=1$)

$$C=\int_0^\infty\frac{dk\,\Delta D(k)}{E(k)}\ , \qquad (35)$$

$$f(x,t)=\int_0^\infty dk\,e^{-iE(k)t}D(k)\left(\cos kx+\frac{ik\sin kx}{E(k)}\right)\ , \qquad (36)$$

$$g(x,t)=\int_0^\infty dk\,e^{-iE(k)t}\frac{D(k)\cos kx\Delta}{E(k)}\ , \qquad (37)$$

where $E(k)=(\Delta^2+k^2)^{1/2}$ and $D(k)$ is the density of states for the σ modes. For simplicity of the final form, we take $D(k)=\exp[-\alpha' E(k)]$, which cuts off the integral at $1/\alpha'=1/\alpha-k_F$, which is the available bandwidth above k_F where the gap appears. Examination of Eqs. (33) and (35) reveals that as $x,t\to\infty$ the dominant term in $S_\sigma^+\simeq\alpha C=(\alpha\Delta)K_0(\alpha'\Delta)$, where K_0 is a Bessel function, and thus the low-frequency behavior of $\mathrm{Im}\chi^R(\omega)$ for CDW and SS is determined by $S_\rho^\pm$. Combining Eqs. (25), (26), and (33), we have

$$\mathrm{Im}\chi^R(\omega)=\frac{C\alpha^{1+\delta_\pm}}{8\pi^2}\int_{-\infty}^\infty dx\int_{-\infty}^\infty dt\frac{e^{i\omega t}}{\{[\alpha+i(v_F't+x)][\alpha+i(v_F't-x)]\}^{\delta_\pm/2}}-(\omega\to-\omega)=f(\delta_\pm)\,\mathrm{sgn}\omega\,|\alpha\omega|^{-2+\delta_\pm}\ , \qquad (38)$$

where

$$f(z)=(\alpha C/\pi^2 2^{-1+z})\Gamma^2(1-\tfrac{1}{2}z)\sin^2\tfrac{1}{2}\pi z\ . \qquad (39)$$

This is the result of LE.

Let us now consider the SDW and TS response functions. The integrals in Eqs. (36) and (37) can be evaluated analytically,[17] and after some algebra we obtain

$$S_\sigma^-(x,t)=(\alpha\Delta)^2[K_1^2(\Delta\sqrt{\xi})-K_0^2(\Delta\sqrt{\xi})]\ , \qquad (40)$$

where

$$\zeta = [\alpha' - i(v_F'' t + x)][\alpha' - i(v_F'' t - x)],$$

and K_0, K_1 are Bessel functions. Combining Eqs. (40), (26), and (25), we get

$$\mathrm{Im}\chi^R(\omega) = + \frac{\Delta^2 \alpha^{\delta_\pm}}{8\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt \frac{e^{i\omega t}}{\{[\alpha + i(v_F' t + x)][\alpha + i(v_F' t - x)]\}^{\delta_\pm}} [K_1^2(\Delta\sqrt{\zeta}\,^*) - K_0^2(\Delta\sqrt{\zeta}\,^*)] - (\omega \to -\omega). \quad (41)$$

It is shown in Appendix B that for low frequencies

$$\mathrm{Im}\chi^R(\omega) \simeq \frac{(2\pi)^{1/2}\alpha^\delta (|\omega| - 2v_F''\Delta)^{\delta+1/2}\Theta(|\omega| - 2v_F''\Delta)\,\mathrm{sgn}\,\omega}{8v_F''(v_F')^\delta \Gamma(\delta + \frac{3}{2})|\omega|^{1/2}}, \quad (42)$$

where $\Gamma(x)$ is the gamma function, $v_F'' = \frac{4}{5} v_F$ and $v_F' = 2\delta v_F/(1+\delta^2)$ and we have set $\delta_\pm = \delta$. Thus we see that the SDW and TS response functions do not diverge as $\omega \to 0$. The gap in the imaginary part of the response function was predicted by Lee[11] on the basis of a convolution argument. The present calculation allows us to investigate the behavior near to $\omega = 2v_F''\Delta$. Equation (42) shows that there is no singularity at the gap edge. We note that this equation is only valid for $\omega - 2v_F''\Delta \ll 1$, or very near the gap edge. As one moves away from the gap edge, corrections of order $(|\omega| - 2v_F''\Delta)^{\delta+3/2}/\omega^{3/2}$ begin to play a role. If in Eq. (42) we take the limit $\omega \gg 2v_F''\Delta$, we obtain the high-temperature frequency dependence given by LE, $\mathrm{Im}\chi^R(\omega) \sim \omega^\delta$, although all of these higher-order corrections will also make a contribution of order ω^δ, so that the overall constant will be different than in Eq. (42).

Chui and Lee[12] have recently mapped the LE Hamiltonian for any $U_\parallel$ on the classical two-dimensional Coulomb gas problem. They then argued that the SDW and TS response functions do not diverge as $\omega \to 0$ for any $U_\parallel < 0$, and that the leading term in the σ parts of the CDW and SS response functions is constant. The last conclusion implies that the frequency exponents of these response functions are determined by $S_\rho^\pm$ in the lower half of the $(\tilde{g}_1, \tilde{g}_2)$ plane. The low-frequency behavior is then determined by the exponents

$$\delta_\pm = \left(\frac{1 \mp (\tilde{g}_2 - \tilde{g}_1/2)}{1 \pm (\tilde{g}_2 - \tilde{g}_1/2)}\right)^{1/2}. \quad (43)$$

We have shown that for $\tilde{g}_1 = -\frac{6}{5}$ there is no divergence at $\omega = 2v_F''\Delta$, however, there is the intriguing possibility that such a divergence might appear when one moves away from this value towards less negative $\tilde{g}_1$. Chui and Lee relate the σ part of the TS and SDW response functions to a spin-spin correlation function, claiming that for large x and t

$$S_\sigma^-(x,t) \sim [x^2 - (v_F'' t)^2]^{-\mu/2} \exp\{-2i\Delta[(v_F'' t)^2 - x^2]^{1/2}\}, \quad (44)$$

where

$$\mu = \left(\frac{1 - \tilde{g}_1/2}{1 + \tilde{g}_1/2}\right)^{1/2}.$$

Performing the integrals as in Appendix B, we would find for this form of S_σ^-

$$\mathrm{Im}\chi(\omega) \sim \Theta(|\omega| - 2v_F''\Delta)|\omega|^{-1/2} \times (|\omega| - 2v_F''\Delta)^{-3/2+\delta+\mu}\,\mathrm{sgn}\,\omega, \quad (45)$$

which could be divergent at the gap edge for $-\frac{10}{13} < \tilde{g}_1 < 0$. In order to investigate this interesting possibility, it would be necessary to examine the behavior of the spin-spin correlation function to logarithmic accuracy in the exponent.

The regions of divergence of the various response functions are shown in Fig. 2.

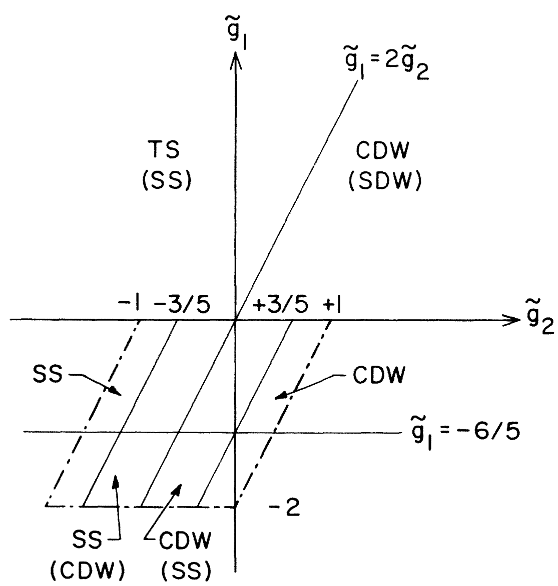

FIG. 2. Shown are the regions in g_1/g_2 space of different types of low-temperature response behavior for $\tilde{g}_4 = 0$. The model can be solved on the lines $\tilde{g}_1 = 0$ and $\tilde{g}_1 = -\frac{6}{5}$, but can be extended into this entire region $-1 < -\tilde{g}_1/2 < 1$ and $-1 < \tilde{g}_2 - \tilde{g}_1/2 < 1$ by a mapping onto the two-dimensional Coulomb gas problem. The line $\tilde{g}_1 = 2\tilde{g}_2$ separates the region in which superconductivity predominates over Peierls-Fröhlich behavior from that region in which the reverse is true.

IV. EFFECT OF OTHER INTERACTION PROCESSES

The interaction term in Eq. (1) contains only the small momentum scattering process corresponding to g_2 (Fig. 1). One can easily include also the other $k\approx 0$ process (g_4). To this end one has to start with the full interaction, as was done by Mattis and Lieb,[9]

$$H_{\text{int}}=\frac{1}{2L}\sum_{k,ss'} V(k)\rho_s(k)\rho_{s'}(-k)\,, \tag{46}$$

where

$$\rho_s(k)=\sum_p a_s^\dagger(p+k)a_s(p)\,. \tag{47}$$

Separating $\rho_s(k)$ into $\rho_s^+(k)$ with summation over $p>0$ and $\rho_s^-(k)$ with summation over $p<0$, and identifying the $\rho^\pm$ operators for small k with the Luttinger $\rho_{1,2}$ operators (see Sec. II), we have

$$H_{\text{int}}=\frac{1}{2L}\sum_{k,s,s'} V(k)[\rho_{1,s}(k)+\rho_{2,s}(k)]\times[\rho_{1,s'}(-k)+\rho_{2,s'}(-k)]\,, \tag{48}$$

where

$$\rho_{i,s}(k)=\sum_k a_{is}^\dagger(p+k)a_{is}(p)\,.$$

The terms which involve ρ operators with different indices yield the g_2 interaction in Eq. (1). The other terms may be written in the form

$$\frac{2}{L}\sum_k V(k)[\rho_1(k)\rho_1(-k)+\rho_2(-k)\rho_2(k)]\,. \tag{49}$$

This expression corresponds to the g_4 interaction in Fig. 1. The $k\to 0$ limit of $V(k)$ in Eq. (48) is denoted by g_4. One can discuss the model for a general $g_4\neq g_2$, however, it is clear that the physically interesting case is $g_2=g_4$. Equation (47) can be added to the kinetic energy in Eq. (5), which results in replacing πv_F by πv_F+g_4, without changing the LE Hamiltonian. Keeping this in mind, we can immediately write down the new exponents

$$\delta'_\pm=\left(1\mp\frac{\tilde g_2+3/5}{1+\tilde g_4}\right)^{1/2}\left(1\pm\frac{\tilde g_2+3/5}{1+\tilde g_4}\right)^{-1/2}\,, \tag{50}$$

where $\tilde g_4=g_4/\pi v_F$. Applying the arguments of Chui and Lee[12] as in the preceding section, we can generalize these exponents to the $\tilde g_1<0$ half plane replacing the $\frac{3}{5}$ by $-\frac{1}{2}\tilde g_1$. The inclusion of the g_4-interaction modifies the picture of Fig. 2. The new regions of divergence of the CDW and SS response functions are shown in Fig. 3 for $\tilde g_4=\tilde g_2$. The broken lines define the region of validity of the model. Beyond these lines the transformations in Eqs. (13) and (15) (with $\pi v_F\to\pi v_F+g_2$) are not defined, because the magnitude of their r.h.s. exceeds unity. Note that the line $g_1=2g_2$ still plays the same role as before. An interesting feature of Fig. 3 is that all the dividing lines meet at a single point.

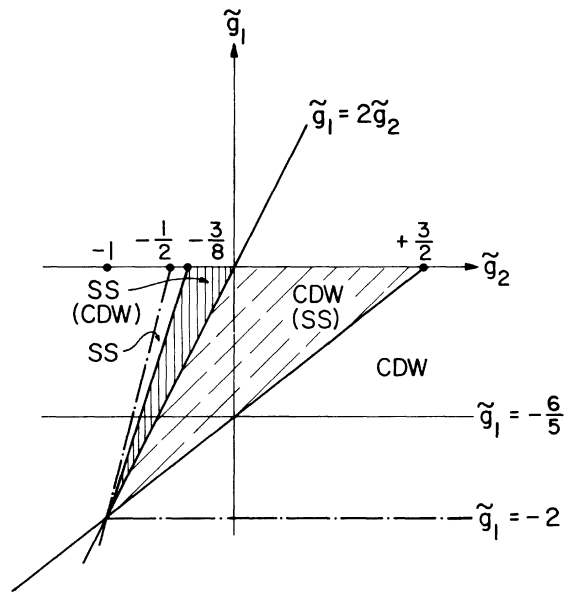

FIG. 3. Shown are the regions in g_1/g_2 for $\tilde g_1<0$ and $\tilde g_4=\tilde g_2$ of the different types of low-temperature behavior of the response functions. The dotted and dashed lines represent the limits for which the model can be solved. The line $\tilde g_1=2\tilde g_2$ still separates the region in which superconductivity predominates over Peierls-Fröhlich behavior from the region in which the reverse is true.

The inclusion of the umklapp process (g_3) in the case of a half-filled band into the LE model is not so straightforward. Since the process involves a short-range interaction, we can represent it by a contact interaction similar to the backscattering Hamiltonian

$$H_u=g_3\sum_s\int dx\,\psi_{1s}^\dagger(x)\psi_{1-s}^\dagger(x)\psi_{2-s}(x)\psi_{2s}(x)\,. \tag{51}$$

Using Eq. (9) to express H_u in the boson representation we find[18]

$$H_u=\frac{g_3}{(2\pi\alpha)^2}\times\int dx\left[\exp\left(-4ik_Fx+\sqrt{2}\sum_k A_k(x)[\rho_1(k)+\rho_2(k)]\right)+\text{H.c.}\right]\,. \tag{52}$$

This is exactly of the same form as the boson representation of H_{LE}, except that it depends on the ρ operators instead of the σ operators. The transformation in Eq. (12) now diagonalizes the bilinear part of H_ρ and introduces the factor $e^{\phi(k)}$ in the exponent of the umklapp term [Eq. (50)]. This term can be reduced[12] to a bilinear form in fictitious spinless fermion fields for the case $e^{\phi(k)}/\sqrt{2}=1$ just as in the LE case and the spectrum of H_ρ shows a gap $\Delta'=|g_3|/2\pi\alpha$ at $\pm k_F$. In view of Eq. (13) and the fact that the diagonalization of H_σ requires $-U_\parallel/2\pi v_F=\frac{3}{5}$, we find that H_ρ with the term H_u can be diagonalized when $V(k)=0$. Thus, including g_3 we can solve the problem at the point $\tilde{g}_1=-\frac{6}{5}$, $\tilde{g}_2=0$. To calculate the response function at this point, we note that the functions $S_\rho^\pm$ are now identical to $S_\sigma^\pm$, except for a different value of the gap. The CDW response functions involves the product $S_\rho^+S_\sigma^+$, which is a constant at large x, t, which indicates the presence of long-range order of the CDW type.[19] For the other response functions we get

$$\begin{aligned}
\text{Im}\chi^R_{\text{SDW}}(\omega) &\sim \text{sgn}\omega\Theta(|\omega|-2v_F'\Delta)\frac{[\alpha(|\omega|-2v_F'\Delta)]^{1/2}}{|\alpha\omega|^{1/2}},\\
\text{Im}\chi^R_{\text{SS}}(\omega) &\sim \text{sgn}\omega\Theta(|\omega|-2v_F''\Delta')\frac{[\alpha(|\omega|-2v_F''\Delta')]^{1/2}}{|\alpha\omega|^{1/2}},\\
\text{Im}\chi^R_{\text{TS}}(\omega) &\sim \text{sgn}\omega\Theta[|\omega|-2(v_F'\Delta+v_F''\Delta')]\\
&\quad\times\frac{\{\alpha[|\omega|-2(v_F'\Delta+v_F''\Delta')]\}^{1/2}}{|\alpha\omega|^{1/2}}.
\end{aligned} \tag{53}$$

Thus, we find that none of these three response functions diverges as $\omega\to 0$. We can apply the argument of Chui and Lee[12] to H_u and find that S_ρ^+ is a constant and S_ρ^- has the same asymptotic form as S_σ^- for $g_1-2g_2<|g_3|$. However, this is mostly in the region in g_1/g_2 space for which the CDW response functions is dominant even in the absence of g_3, so it is still possible that for $g_3\neq 0$, the SS response function might diverge for sufficiently positive $\tilde{g}_1-2\tilde{g}_2$. However, we note that Dzyaloshinskii and Larkin[20] found that in the parquet approximation, the system exhibits a metallic ground state for $g_1>0$ and $g_1-2g_2>|g_3|$. If we may speculate on extrapolating their results to $g_1<0$, their results suggest that g_3 may destroy the tendency towards long-range SS order as $T\to 0$. However, as this extrapolation can only be made for small g_1 and g_2, it cannot be trusted in the region for $g_1<0$ where in the absence of g_3, the SS response function is the only divergent one.

V. DISCUSSION

We have investigated the low-temperature properties of the one-dimensional electron gas with a variety of electron-electron interactions. For the case of only low-momentum-transfer interactions ($g_2, g_4\neq 0, g_1=g_3=0$), the model reduces to the Luttinger model, and the response behavior of the system has been found by Luther and Peschel.[10] For that case, we have $\text{Im}\chi(\omega)_{T=0}\sim\omega^{-\mu}$, where $\mu=1-\delta_\pm'$, and $\delta_\pm'$ is given by Eq. (50) with the $\frac{3}{5}$ replaced by 0. The SS and TS response functions are identical, as are the SDW and CDW response functions, and the regions of g_2 for which they diverge do not overlap. We note that since Luther and Peschel[10] did not include spin in their calculation, they found $\text{Im}\chi(\omega)_{T=0}\sim\omega^{-2\mu}$, which has the same regions of different response behavior, but is slightly different in form.

The inclusion of the large-momentum-transfer process between electrons on opposite sides of the Fermi "surface" (g_1) changes the response behavior of the system dramatically. Luther and Emery[8] found a remarkable solution to this problem for $\tilde{g}_1=-\frac{6}{5}$, and we have explicitly calculated the four response functions. We find that in striking contrast to the case $g_1=0$, the response functions for SS and TS (and for CDW and SDW) behavior are completely different. The SS and CDW response functions are each characterized by an exponent, $\text{Im}\chi(\omega)\sim\omega^{-\mu}$, where $\mu=2-\delta_\pm'$, and $\delta_\pm'$ is given by Eq. (50). For this value of g_1, there is now a region where both the SS and CDW response functions diverge, but the point $\tilde{g}_2=-\frac{3}{5}$ separates the region in which the CDW response function diverges more strongly than the SS response function does, from the region in which the reverse is true. There are also regions in which just one of these two response functions diverges.

The TS and SDW response functions, however, are never divergent on the LE line of solutions. Their low-frequency behavior is characterized by a gap, as was first pointed out by Lee.[11] However, we have also shown that they do not even diverge at the gap edge for $\tilde{g}_1=-\frac{6}{5}$. However, there is the intriguing possibility that for $0>\tilde{g}_1>-\frac{10}{13}$, these response functions might diverge at the gap edge for $T=0$, unless there are some additional logarithmic corrections to the exponential "gap" factor of the σ mode correlation function. This would not imply static long-range order at $T=0$ for $q=0$ (or $q=2k_F$), since for finite temperatures, the gap would contribute a factor $\sim\exp(-2v_F''\Delta/T)$ for $q=0$ (or $2k_F$), insuring that these response functions would vanish as $T\to 0$. However, the divergence at the gap edge for the zero-temperature response functions would show up as a peak in the finite-temperature response functions in the neighborhood of $T\sim 2v_F''\Delta$, so that the excited states of the system would contain a large amount of TS and/or SDW character, depending upon the value of g_2.

For the case of a half-filled band, the presence of g_3 tends to enhance the tendency towards long-range CDW order as $T\to 0$, and to suppress the ordering of the SS and TS types. By a method analogous to that of Luther and Emery,[8] we were able to solve the problem for $g_3\neq 0$, $g_2=0$, and $\tilde{g}_1=-\frac{6}{5}$, and by arguments similar to those of Chui and Lee,[12] we were able to state that for $-2<\tilde{g}_1<0$ and for $(\tilde{g}_1-2\tilde{g}_2)/(1+\tilde{g}_4)<|\tilde{g}_3|$, there is a long-range order of the CDW type at $T=0$, and all of the other response functions have gaps for small frequencies. This result is not inconsistent with renormalization group results of Dzyaloshinskii and Larkin,[19] who find the possibility of a metallic ground state for $g_1>0$ and $\tilde{g}_1-2\tilde{g}_2<|\tilde{g}_3|$, where g_4 was set =0. For $g_1>0$ and g_3 small, this is entirely in the region in g_1/g_2 space where in the absence of g_3 the system would have exhibited superconducting ordering as $T\to 0$. Thus, we may speculate that for $g_1<0$, the presence of g_3 for a half-filled band might remove the possibility of superconducting ordering as $T\to 0$.

We remark that we have not given any consideration to the case $g_1>0$. However, that regime is attainable by the renormalization group calculations. Since there are only two regions of different physical behavior (instead of four for $g_1<0$), the expansion for small interaction strengths appears to give a full physical description of the system. For $g_1<0$, the renormalization group cannot fully describe the system, as a mapping onto the two-dimensional classical Coulomb gas reveals that the boundaries of the regions in which only one response function diverges occur for $\tilde{g}_2$ of order unity when $\tilde{g}_1$ is small in magnitude. We note, however, that Sólyom[6] predicted from renormalization group calculations that the ground state of the system for $g_1<0$ was a "period-doubled singlet superconductor," or that the response functions for SS and CDW behavior both diverge for $g_1<0$, which is consistent with the Coulomb-gas mapping for small coupling constants.

We have considered the important types of electron-electron interactions in the one-dimensional electron gas, assuming the interactions to be static, or arising only from the electrons themselves. We have not considered more physical models, such as might more properly treat the electron-phonon interaction. We have also neglected the effect of impurities, as well as many other possible physical situations which could give important corrections to these results. Perhaps the most important of the interactions that will be present in a real material are interchain couplings, which can be responsible for long-range order at a finite temperature in these systems. These effects will be discussed in the following paper.

ACKNOWLEDGMENTS

The authors would like to thank S. Doniach, P. A. Lee, W. A. Little, J. Zittartz, V. J. Emery, and I. Peschel for many helpful dicusssions. We would also like to thank the National Science Foundation, International Business Machines, and the U. S. Army Research Council for support.

APPENDIX A

We wish to calculate explicitly the quantities $S_\sigma^\pm(x,t)$, given by Eqs. (30) and (32) of the text. At $T=0$, we have

$$[S_\sigma^+(x,t)]^2=(2\pi\alpha)^2\langle\psi_1(xt)\psi_2^\dagger(xt)e^{-2ik_Fx}\psi_2(00)\psi_1^\dagger(00)\rangle\,.$$

By Fourier transformation of the fields, we obtain

$$[S_\sigma^+(x,t)]^2=\left(\frac{2\pi}{L}\right)^2\alpha^2\sum_{kk'k''k'''}e^{i(k-k')x}\langle e^{itH_\sigma}\alpha_{1,k-k_F}\alpha^\dagger_{2,k'+k_F}e^{-itH_\sigma}\alpha_{2,k''+k_F}\alpha^\dagger_{1,k''-k_F}\rangle_{H_\sigma}\,. \tag{A2}$$

We must now perform the rotation of the α_i operators to the diagonal $\overline{\alpha}_i$ representation [see Eq. (17)],

$$\begin{aligned}[S_\sigma^+(x,t)]^2=\left(\frac{2\pi}{L}\right)^2\alpha^2\sum_{kk'k''k'''}&e^{i(k-k')x}\langle(\overline{\alpha}_{1k}e^{it\epsilon_1(k)}\cos\theta_k-\overline{\alpha}_{2k}e^{it\epsilon_2(k)}\sin\theta_k)\\ &\times(\overline{\alpha}^\dagger_{2k'}e^{-it\epsilon_2(k')}\cos\theta_{k'}+\overline{\alpha}^\dagger_{1k'}e^{-it\epsilon_1(k')}\sin\theta_{k'})(\overline{\alpha}_{2k''}\cos\theta_{k''}+\overline{\alpha}_{1k''}\sin\theta_{k''})\\ &\times(\overline{\alpha}^\dagger_{1k'''}\cos\theta_{k'''}-\overline{\alpha}^\dagger_{2k'''}\sin\theta_{k'''})\rangle_{\tilde{H}_\sigma}\,.\end{aligned} \tag{A3}$$

We use the zero-temperature results for the free fermion operators $\overline{\alpha}_i$,

$$\begin{aligned}&\langle\overline{\alpha}_{ik}\overline{\alpha}^\dagger_{jk'}\rangle=\delta_{ij}\delta_{kk'}\Theta(\mp k)\,,\\ &\langle\overline{\alpha}^\dagger_{ik}\overline{\alpha}_{jk'}\rangle=\delta_{ij}\delta_{kk'}\Theta(\pm k)\,,\\ &\langle\overline{\alpha}_{ik}\overline{\alpha}_{jk'}\rangle=0\,,\end{aligned} \tag{A4}$$

where the upper (lower) sign refers to $i=1$ (2). In evaluating the expectation values, the chemical potential μ is chosen to be $v_F'' k_F$. After some straightforward algebra, we have

$$S_\sigma^+(xt) = \alpha[C^2 + f(x,t)f(-x,t)]^{1/2}, \tag{A5}$$

where

$$C = \frac{2\pi}{L}\sum_{k>0} D(k)\sin 2\theta_k, \tag{A6}$$

$$f(x,t) = \frac{2\pi}{L}\sum_{k>0} D(k)e^{-itE(k)}\,(\cos^2\theta_k e^{-ikx} + \sin^2\theta_k e^{+ikx}), \tag{A7}$$

and $E(k) = (k^2+\Delta^2)^{1/2}$. In Eqs. (A6) and (A7), we have introduced the density of states, $D(k)$, as S_σ^+ is otherwise logarithmically divergent. Letting

$$\frac{2\pi}{L}\sum_k \to \int_0^\infty dk,$$

and rearranging the terms in f, we obtain Eqs. (35) and (36) of the text.

Similarly, for $S_\sigma^-(x,t)$ we have

$$S_\sigma^-(x,t) = \alpha^2\left(\frac{2\pi}{L}\right)^2 \sum_{kk'k''k'''} e^{i(k+k')x}\langle e^{itH_\sigma}\alpha_{1,k-k_F}\alpha_{2,k'+k_F}e^{-itH_\sigma}\alpha^\dagger_{2,k''+k_F}\alpha^\dagger_{1,k'''-k_F}\rangle_{H_\sigma}. \tag{A8}$$

Going to the diagonal representation,

$$S_\sigma^-(x,t) = \alpha^2\left(\frac{2\pi}{L}\right)^2 \sum_{kk'k''k'''} e^{i(k+k')x}\langle(\overline{\alpha}_{1k}e^{it\epsilon_1(k)}\cos\theta_k - \overline{\alpha}_{2k}e^{it\epsilon_2(k)}\sin\theta_k)(\overline{\alpha}_{2k'}e^{it\epsilon_2(k')}\cos\theta_{k'} + \overline{\alpha}_{1k'}e^{it\epsilon_1(k')}\sin\theta_{k'})$$
$$\times(\overline{\alpha}^\dagger_{2k''}\cos\theta_{k''} + \overline{\alpha}^\dagger_{1k''}\sin\theta_{k''})(\overline{\alpha}^\dagger_{1k'''}\cos\theta_{k'''} - \overline{\alpha}^\dagger_{2k'''}\sin\theta_{k'''})\rangle_{\overline{H}_\sigma}. \tag{A9}$$

We finally obtain

$$S_\sigma^-(x,t) = \alpha^2[f(x,t)f(-x,t) - g^2(x,t)], \tag{A10}$$

where

$$g(x,t) = \frac{2\pi}{L}\sum_{k>0} D(k)e^{-iE(k)t}\cos kx\,\sin 2\theta_k. \tag{A11}$$

Again rearranging the terms in f, we obtain Eq. (37) of the text.

Clearly, the functions $S_\sigma^\pm(x,t)$ satisfy

$$S_\sigma^\pm(-x,t) = S_\sigma^\pm(x,t)$$

and (A12)

$$S_\sigma^\pm(x,-t) = S_\sigma^\pm(x,t)^*.$$

We note that our expression [Eq. (A10)] for $S_\sigma^-(x,t)$ differs slightly from the expression given by Eq. (12) in the comment by Lee.

APPENDIX B

In the present appendix we derive the low-frequency behavior of the TS and SDW response functions at zero temperature, Eq. (41) of the text,

$$\mathrm{Im}\chi(\omega) = \frac{\alpha^\delta\Delta^2}{8\pi^2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dt\,e^{i\omega t}\{[\alpha + i(v_F't - x)][\alpha + i(v_F't + x)]\}^{-\delta/2}$$
$$\times(K_1^2[\Delta\{[\alpha' + i(v_F''t - x)][\alpha' + i(v_F''t + x)]\}^{1/2}]$$
$$- K_0^2[\Delta\{[\alpha' + i(v_F''t - x)][\alpha' + i(v_F''t + x)]\}^{1/2}]) - (\omega \to -\omega), \tag{B1}$$

where α^{-1} and $(\alpha')^{-1}$ are the available bandwidths in momentum units for the plasmon and fictitious fermion modes, respectively, and v_F' and v_F'' are the renormalized Fermi velocities for the plasmon and fictitious fermion modes, respectively, as are defined in the text. In Eq. (B1), $\delta=\delta_-$ for TS, and $\delta=\delta_+$ for SDW responses. Letting $s=t-x/v_F''$ and $s'=t+x/v_F''$, and using the asymptotic form for the Bessel functions, we have

$$\operatorname{Im}\chi(\omega)=\frac{\alpha^\delta}{16\pi v_F''}\int_{-\infty}^{\infty}\int\frac{ds\,ds'\exp\{i\omega(s+s')/2v_F''-2\Delta[(\alpha'+is)(\alpha'+is')]^{1/2}\}}{(\alpha'+is)(\alpha'+is')\{[\alpha+i(us+vs')][\alpha+i(vs+us')]\}^{\delta/2}}-(\omega\to-\omega)\,. \tag{B2}$$

where $u=(v_F'+v_F'')/2v_F''$ and $v=(v_F'-v_F'')/2v_F''$ are dimensionless constants. We observe that the integrals in Eqs. (B1) and (B2) have no poles or branch cuts in the lower half s and s' planes. In addition, the asymptotic form of the Bessel functions is never exponentially growing in the lower-half s and s' planes or on the real s and s' axes. Therefore if we choose $\omega>0$, the $(\omega\to-\omega)$ term in Eq. (B2) vanishes identically. Similarly, if $\omega<0$, the first term vanishes, so we know that $\operatorname{Im}\chi(\omega)$ is odd in ω. Let us now assume $\omega>0$. We may evaluate the integrals by closing the contours along hemicircles of radii R and R' in the lower-half s,s' planes, and then let $R,R'\to\infty$. Since the integral is symmetric in s and s', the result does not depend upon the order in which R and R' are allowed to go to infinity, so we may choose $R'=R$. Thus along the hemicircle parts of the contours, we let $s=Re^{-i\varphi}$, and $s'=Re^{-i\varphi'}$, where $0\le\varphi,\ \varphi'\le\pi$. We have

$$\operatorname{Im}\chi(\omega>0)=\lim_{R\to\infty}\frac{\alpha^\delta}{16\pi v_F''(iR)^\delta}\int_0^\pi d\varphi\int_0^\pi d\varphi'\,e^{i\omega R(e^{-i\varphi}+e^{-i\varphi'})/2v_F''}\frac{e^{-2i\Delta Re^{-i(\varphi+\varphi')/2}}}{[(ue^{-i\varphi}+ve^{-i\varphi'})(ve^{-i\varphi}+ue^{-i\varphi'})]^{\delta/2}}\,. \tag{B3}$$

Changing variables to $\theta=\frac{1}{2}(\varphi+\varphi')$ and $\theta'=\frac{1}{2}(\varphi-\varphi')$, we obtain

$$\operatorname{Im}\chi(\omega>0)=\lim_{R\to\infty}\frac{\alpha^\delta}{16\pi v_F''(iR)^\delta}\int_0^\pi d\theta\,e^{i\delta\theta-2i\Delta Re^{-i\theta}}\int_{-\pi/2}^{\pi/2}\frac{d\theta'e^{i\omega Re^{-i\theta}\cos\theta'/v_F''}}{(u^2+v^2+2uv\cos2\theta')^{\delta/2}}\,. \tag{B4}$$

Recalling from Eq. (B2) that if ω had been negative, the integral would have vanished, we observe that we may only obtain a finite contribution to the integral in Eq. (B4) if $\omega\cos\theta'>2v_F''\Delta$, and we thus observe that $\operatorname{Im}\chi(\omega>0)=0$ for $\omega<2v_F''\Delta$. However, the nature of the behavior of $\operatorname{Im}\chi(\omega)$ near $\omega=2v_F''\Delta$ is still of interest, as it is important to determine whether or not it diverges there. We shall thus perform an expansion in powers of $(\omega-2v_F''\Delta)$. Let us examine the θ' integral

$$I_{\theta'}=2\int_0^{\pi/2}\frac{dx\,e^{i\beta\cos x}}{(u^2+v^2+2uv\cos2x)^{\delta/2}}, \tag{B5}$$

where $\beta=R\omega e^{-i\theta}/v_F''$. Since $|2uv\cos2x|<u^2+v^2$ for finite renormalized Fermi velocities, we may safely expand the denominator in a Taylor series

$$I_{\theta'}=2(u^2+v^2)^{-\delta/2}\sum_{k=0}^{\infty}\frac{(-y)^k\Gamma(\frac{1}{2}\delta+k)I_k(\beta)}{k!\,\Gamma(\delta/2)}, \tag{B6}$$

where $y=2uv/(u^2+v^2)$, and

$$I_k(\beta)=\int_0^{\pi/2}dx\,e^{i\beta\cos x}(\cos2x)^k\,. \tag{B7}$$

Since we are interested in ω near to $2v_F''$ and R large, we require only the asymptotic form of $I_k(\beta)$ for large β. By writing $(\cos2x)^k$ in terms of $\cos2mx$, where $0\le m\le k$, the integrals may be performed, and we find

$$I_k(\beta)\cong e^{i(\beta-\pi/4)}[(\pi/2\beta)^{1/2}+O(\beta^{-3/2})] \tag{B8}$$

independent of k in the leading order. There are also some terms of order β^{-1}, but those terms do not contain an exponential factor, and consequently do not contribute to the θ integral. The terms of order $e^{i\beta}\beta^{-3/2}$ can be shown to give a contribution to $\operatorname{Im}\chi(\omega)$ which is smaller by a factor of order $(\omega-2\Delta v_F'')/\Delta v_F''$ relative to the leading term, and can therefore be neglected. We therefore have

$$I_{\theta'}\cong(2\pi/i\beta)^{1/2}e^{i\beta}(v_F''/v_F')^\delta\,. \tag{B9}$$

Combining Eqs. (B4) and (B9), we have

$$\operatorname{Im}\chi(\omega>0)\cong\frac{\alpha^\delta(v_F'')^{\delta-1/2}}{8(v_F')^\delta(2\pi\omega)^{1/2}}\times\lim_{R\to\infty}(iR)^{-(\delta+1/2)}\int_0^\pi d\theta\exp[i(\delta+\tfrac{1}{2})\theta+iR\,e^{-i\theta}(\omega/v_F''-2\Delta)] \tag{B10}$$

and

$$\operatorname{Im}\chi(\omega)\cong\frac{(2\pi)^{1/2}\alpha^\delta(|\omega|-2v_F''\Delta)^{\delta+1/2}\Theta(|\omega|-2v_F''\Delta)\operatorname{sgn}\omega}{8v_F''(v_F')^\delta\Gamma(\delta+3/2)|\omega|^{1/2}}, \tag{B11}$$

where $\Gamma(x)$ is the gamma function, $v_F' = 2\delta v_F/(1+\delta^2)$ and $v_F'' = \frac{4}{5} v_F$. Equation (B11) can be obtained from Eq. (B10) by relating the θ integral to

$$\int_{-\infty}^{\infty} ds\, e^{i(\omega/v_F'' - 2\Delta)s} (\alpha + is)^{-(\delta+3/2)}$$

which is given in the tables.

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*Supported in part by the National Science Foundation through Grant No. GH-41213 and by the U. S. Army Research Council through Grant No. DAHCO4-74-G-02-22.

† Permanent address, The Racah Institute of Physics, The Hebrew University of Jerusalem, Israel.

‡IBM Postdoctoral Fellow.

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