Physical realizations of $n \geq 4$-component vector models. III. Phase transitions in Cr, Eu, MnS_2, Ho, Dy, and Tb†

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(Received 4 November 1975)

We use the symmetry arguments of Laudau and Lifshitz, described in the preceding papers, to derive the Landau-Ginzburg-Wilson Hamiltonians corresponding to several antiferromagnetic systems. The phase transitions associated with these models are then studied using the exact renormalization-group technique in $d = 4 - \epsilon$ dimensions. We find that the fcc type-III antiferromagnets $\{\vec{m} \perp [100], \vec{k} = (1/2,0,1)\}$ and the spiral magnets Eu and Cr are described by two $n = 12$ models. For both models, the renormalization-group recursion relations have no stable fixed points. This might explain the existence of first-order transitions in Cr and Eu. We also find that the antiferromagnet MnS_2 is described by the $n = 6$ model derived by Mukamel and Krinsky to represent TbD_2, Nd, and K_2IrCl_6. Since this model has one stable fixed point, it is predicted that the four compounds belong to the same universality class. Similarly, the spiral magnets Tb, Dy, and Ho correspond to the $n = 4$ model which was used to describe NbO_2, $TbAu_2$, and DyC_2, and it is predicted that they all have the same critical behavior. Existing experimental data are discussed and several experiments are suggested.

I. INTRODUCTION

In the preceding papers,[1,2] the Landau theory of phase transitions[3] and the renormalization-group technique[4,5] were used to discuss the phase transitions in several $n \geq 4$-component physcial systems. In this article, we use the symmetry arguments of Landau and Lifshitz to derive the Landau-Ginzburg-Wilson (LGW) Hamiltonians corresponding to the following antiferromagnetic systems: Cr, Eu, MnS_2, Ho, Dy, and Tb. The phase transitions associated with these Hamiltonians are then studied by the exact renormalization-group technique in $d = 4 - \epsilon$ dimensions.

Europium and chromium exhibit first-order antiferromagnetic transitions. The transition in Cr has puzzled theorists for years, and despite numerous attempts to clarify the nature of the transition using various microscopic models together with mean-field-like theories, no satisfactory explanation has yet been reported. We find that the transitions in Cr and Eu are described by an $n = 12$-component LGW Hamiltonian. In studying the model by the renormalization-group technique we find that it possesses no stable fixed points. This lack of stable fixed points might explain the first-order nature of the transition in both metals. Similarly, we find that the type-III antiferromagnets with paramagnetic space group $Fm3m$, propagation vector $\vec{k}$ in the $[\frac{1}{2}01]$ direction and magnetic moments perpendicular to [100] are described by an $n = 12$ model which has no stable fixed point.

MnS_2 is a cubic type-III antiferromagnet. The transition in this compound is described by the $n = 6$ LGW Hamiltonian which has been used by Mukamel and Krinsky[1,2] to discuss the transitions in K_2IrCl_6, Nd, and TbD_2. Since this Hamiltonian has one stable fixed point, it is predicted that MnS_2 belongs to the same universality class as K_2IrCl_6, Nd, and TbD_2. Similarly, we find that the transitions in the spiral magnets Tb, Dy, and Ho are described by the $n = 4$ LGW model which has been used by Mukamel and Krinsky[1,2,6] to discuss the transitions in NbO_2, DyC_2, and $TbAu_2$. It is therefore predicted that all these six compounds exhibit the same critical behavior. We believe it would be of great interest to test these predictions experimentally.

The paper is organized as follows: in Sec. II A we construct the $n = 12$ LGW Hamiltonian which corresponds to the type-III antiferromagnets with $\vec{m} \perp [100]$, and derive the appropriate recursion relations to first order in ϵ. In the Appendix we show that these recursion relations have no stable fixed point. In Sec. II B we derive the appropriate $n = 12$ LGW Hamiltonian for the spiral magnets Eu and Cr, and show that the recursion relations which correspond to this Hamiltonian have no stable fixed point. In Sec. II C we show that the $n = 6$ type-III antiferromagnet MnS_2 belongs to the same universality class as K_2IrCl_6, TbD_2, and Nd. In Sec. II D we show that the $n = 4$ spiral magnets Tb, Dy, and Ho correspond to the same fixed point as NbO_2, $TbAu_2$, and DyC_2. We also discuss the existing experimental data on the critical behavior of Tb and Dy. The main results are summarized and discussed in Sec. III.

II. LANDAU-GINZBURG-WILSON HAMILTONIANS AND CRITICAL BEHAVIOR IN $d = 4 - \epsilon$ DIMENSIONS

A. Fcc type-III antiferromagnets with $\vec{m} \perp [100]$ and $\vec{k} = (\frac{1}{2}, 0, 1)$

In the preceding papers[1,2] Mukamel and Krinsky studied the type-III antiferromagnet with $\vec{m} \parallel [100]$ and $\vec{k} = (\frac{1}{2}, 0, 1)$. They found that the corresponding $n = 6$ LGW Hamiltonian has a unique stable fixed point. In this section we shall show that the phase transition in the fcc type-III antiferromagnet with $\vec{m} \perp [100]$ and space group $Fm3m$ is described by an $n = 12$ Hamiltonian which has *no* stable fixed point. This magnetic structure has not yet been observed in a system with the cubic $Fm3m$ symmetry. The type-III antiferromagnet β-MnS (Ref. 7) has a paramagnetic space group $F\bar{4}3m$, and the analysis described below does not apply to it. It turns out, however, that our proof of the nonexistence of stable fixed points is valid also for the Hamiltonians describing phase transitions in Eu and Cr (Sec. II B).

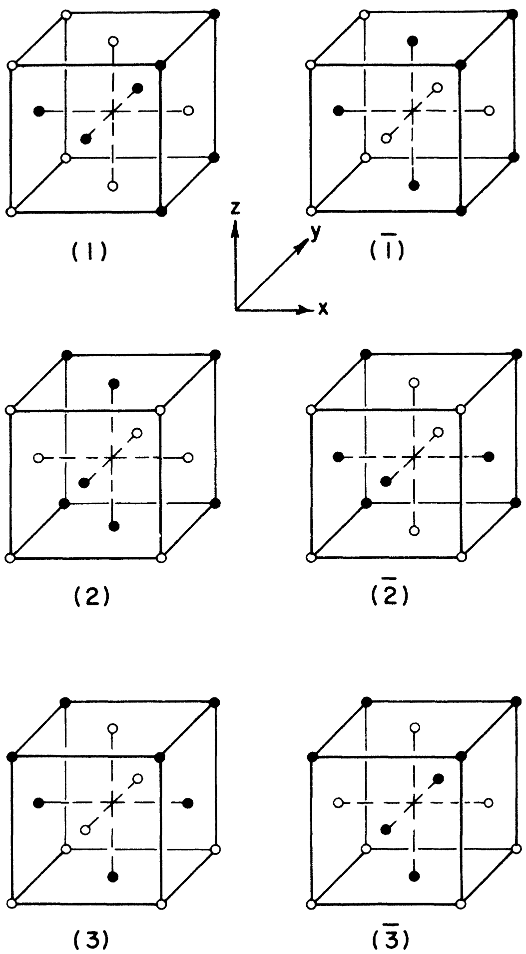

FIG. 1. The six magnetic lattices of type-III antiferromagnets. The lattices (j) and $(\bar{j})$ represent the order parameters ϕ_{j,P_j} and $\bar{\phi}_{j,P_j}$ in the case $\vec{S} \perp \vec{k}_j$ and ϕ_j and $\bar{\phi}_j$ in the case $\vec{S} \parallel \vec{k}_j$. Black dots indicate spins in the $+P_j$ (or $\vec{k}_j$) direction and white dots indicate spins in the $-P_j$ (or $-\vec{k}_j$) direction. The diagrams show half the magnetic unit cell; the full unit cell is obtained by doubling the structure in the j direction.

The magnetic structure of the type-III antiferromagnets is shown in Fig. 1. It belongs to a reciprocal-lattice vector $\vec{k}_1 = (\frac{1}{2}, 1, 0)(2\pi/a)$, where a is the lattice constant of the cubic unit cell. The nonprimitive unit cell is doubled in the x direction and the sublattice magnetization is in the $y-z$ plane. The star of the vector $\vec{k}_1$ consists of six vectors: $\pm\vec{k}_1 = (\pm\frac{1}{2}, 1, 0)(2\pi/a)$, $\pm\vec{k}_2 = (1, \pm\frac{1}{2}, 0)(2\pi/a)$, and $\pm\vec{k}_3 = (0, 1, \pm\frac{1}{2})(2\pi/a)$, and the group of $\vec{k}_1$ is C_{4v}. There are two different order parameters associated with each vector $\vec{k}_j$, and they correspond to the two possible directions of the sublattice magnetization. Thus, the phase transition is described by an $n = 12$-component order parameter. Let $\psi_{\pm j,P_j}$ be the components of the order parameter which belong to the reciprocal-lattice vectors $\pm\vec{k}_j$, $j = 1, 2, 3$, with the sublattice magnetization in the P_j direction, where $P_1 = y, z$, $P_2 = x, z$, and $P_3 = x, y$. We define these components in terms of the twelve real functions ϕ_{j,P_j} and $\bar{\phi}_{j,P_j}$, $j = 1, 2, 3$, which are related to $\psi_{\pm j,P_j}$ through the relations

$$\psi_{\pm j,P_j} \equiv (\phi_{j,P_j} + \bar{\phi}_{j,P_j}) \pm i(\phi_{j,P_j} - \bar{\phi}_{j,P_j}) . \tag{1}$$

The functions ϕ_{j,P_j} and $\bar{\phi}_{j,P_j}$ are

$$\begin{aligned}
\phi_{1,P_1} &= \sum_{\alpha\in\{B\}_1} S_{\alpha,P_1} - \sum_{\alpha\in\{W\}_1} S_{\alpha,P_1} ,\\
\bar{\phi}_{1,P_1} &= \sum_{\alpha\in\{B\}_{\bar{1}}} S_{\alpha,P_1} - \sum_{\alpha\in\{W\}_{\bar{1}}} S_{\alpha,P_1} , \quad P_1 = y, z\\
\phi_{2,P_2} &= \sum_{\alpha\in\{B\}_2} S_{\alpha,P_2} - \sum_{\alpha\in\{W\}_2} S_{\alpha,P_2} ,\\
\bar{\phi}_{2,P_2} &= \sum_{\alpha\in\{B\}_{\bar{2}}} S_{\alpha,P_2} - \sum_{\alpha\in\{W\}_{\bar{2}}} S_{\alpha,P_2} , \quad P_2 = x, z\\
\phi_{3,P_3} &= \sum_{\alpha\in\{B\}_3} S_{\alpha,P_3} - \sum_{\alpha\in\{W\}_3} S_{\alpha,P_3} ,\\
\bar{\phi}_{3,P_3} &= \sum_{\alpha\in\{B\}_{\bar{3}}} S_{\alpha,P_3} - \sum_{\alpha\in\{W\}_{\bar{3}}} S_{\alpha,P_3} , \quad P_3 = x, y
\end{aligned} \tag{2}$$

where the sums $\sum_{\alpha\in\{B\}_j}$ and $\sum_{\alpha\in\{W\}_j}$ are over the sites of the black and white sublattices, respectively, of the lattice j, and S_{α,P_j} is the P_j component of the spin on site α. The lattices j and $\bar{j}$ are defined in Fig. 1. The components of the order parameter form a basis of a twelve-dimensional irreducible representation of the paramagnetic group $Fm3m$. The order parameter transforms under the generators of this group as follows:

$$
\begin{array}{lll}
C_4([001]): & \phi_{1z}\to+\bar\phi_{2z}\,, & \bar\phi_{1z}\to\phi_{2z}\,,\\
& \phi_{1y}\to-\bar\phi_{2x}\,, & \bar\phi_{1y}\to-\phi_{2x}\,,\\
& \phi_{2z}\to-\phi_{1z}\,, & \bar\phi_{2z}\to-\bar\phi_{1z}\,,\\
& \phi_{2x}\to-\phi_{1y}\,, & \bar\phi_{2x}\to-\bar\phi_{1y}\,,\\
& \phi_{3x}\to+\bar\phi_{3y}\,, & \bar\phi_{3x}\to+\phi_{3y}\,,\\
& \phi_{3y}\to-\bar\phi_{3x}\,, & \bar\phi_{3y}\to-\phi_{3x}\,;\\
C_3([111]): & \phi_{1z}\to\phi_{3y}\,, & \bar\phi_{1z}\to\bar\phi_{3y}\,,\\
& \phi_{1y}\to\phi_{3x}\,, & \bar\phi_{1y}\to\bar\phi_{3x}\,,\\
& \phi_{2z}\to\phi_{1y}\,, & \bar\phi_{2z}\to\bar\phi_{1y}\,,\\
& \phi_{2x}\to\phi_{1z}\,, & \bar\phi_{2x}\to\bar\phi_{1z}\,,\\
& \phi_{3x}\to\phi_{2z}\,, & \bar\phi_{3x}\to\bar\phi_{2z}\,,\\
& \phi_{3y}\to\phi_{2x}\,, & \bar\phi_{3y}\to\bar\phi_{2x}\,;\\
C_2([110]): & \phi_{1z}\to-\bar\phi_{2z}\,, & \bar\phi_{1z}\to-\phi_{2z}\,,\\
& \phi_{1y}\to+\bar\phi_{2x}\,, & \bar\phi_{1y}\to+\phi_{2x}\,,\\
& \phi_{2z}\to-\bar\phi_{1z}\,, & \bar\phi_{2z}\to-\phi_{1z}\,,\\
& \phi_{2x}\to+\bar\phi_{1y}\,, & \bar\phi_{2x}\to+\phi_{1y}\,,\\
& \phi_{3x}\to-\phi_{3y}\,, & \bar\phi_{3x}\to-\bar\phi_{3y}\,,\\
& \phi_{3y}\to-\phi_{3x}\,, & \bar\phi_{3y}\to-\bar\phi_{3x}\,;\\
i: & \phi_{i\alpha}\to-\bar\phi_{i\alpha}\,, & \bar\phi_{i\alpha}\to-\phi_{i\alpha}\,,\ \alpha=x,y,z\,;\\
t([\tfrac12 0\tfrac12]): & \phi_{1z}\to\bar\phi_{1z}\,, & \bar\phi_{1z}\to-\phi_{1z}\,,\\
& \phi_{1y}\to\bar\phi_{1y}\,, & \bar\phi_{1y}\to-\phi_{1y}\,,\\
& \phi_{2z}\to-\phi_{2z}\,, & \bar\phi_{2z}\to-\bar\phi_{2z}\,,\\
& \phi_{2x}\to-\phi_{2x}\,, & \bar\phi_{2x}\to-\bar\phi_{2x}\,,\\
& \phi_{3x}\to-\bar\phi_{3x}\,, & \bar\phi_{3x}\to\phi_{3x}\,,\\
& \phi_{3y}\to-\bar\phi_{3y}\,, & \bar\phi_{3y}\to\phi_{3y}\,;
\end{array}
\qquad (3)
$$

where $C_l([\cdots])$ is an l-fold rotation axis along the $[\cdots]$ direction, i is the inversion and $t([\frac12\frac12 0])$ is a translation of $(\frac12,\frac12,0)a$.

To set up a Landau-Ginzburg-Wilson Hamiltonian in order to carry out the ϵ expansion, one must know all the invariants of order $l \leqslant 4$ which can be formed by the components of the order parameter. For magnetic transitions the order l must be even, due to time-reversal symmetry, and one has to find all the second- and fourth-order invariants. Each irreducible representation Γ has only one second-order invariant, the sum of the squares of the components of the order parameter. The number of fourth-order invariants is given by the number of times the symmetric part of the representation Γ^4 (denoted by $[\Gamma^4]$) includes the unit representation. The character of the representation $[\Gamma^4]$ is given by:

$$\chi^{[\Gamma^4]}(g)=\tfrac14\chi(g^4)+\tfrac13\chi(g^3)\chi(g)+\tfrac14\chi(g^2)\chi^2(g)+\tfrac18\chi^2(g^2)+\tfrac1{24}\chi^4(g)\,, \qquad (4)$$

where g is a symmetry element of the paramagnetic space group and $\chi(g)$ is the corresponding character of the representation Γ. Using this character table and the orthogonality relations one can find the number of fourth-order invariants which can be formed by the components of the order parameter.

We find that the twelve-dimensional order parameter associated with the phase transition has nine invariants. The LGW Hamiltonian which describes the system is, therefore,

$$\mathcal{H}_5=-\frac12\sum_{j,P_j}\left[r(\phi^2_{j,P_j}+\bar\phi^2_{j,P_j})+(\nabla\phi_{j,P_j})^2+(\nabla\bar\phi_{j,P_j})^2\right]-\sum_{i=1}^{9}u_i O_i\,, \qquad (5)$$

where the invariants O_i are

$$
\begin{aligned}
O_1&=(\phi^2_{1z}+\bar\phi^2_{1z})(\phi^2_{3x}+\bar\phi^2_{3x})+(\phi^2_{2x}+\bar\phi^2_{2x})(\phi^2_{1y}+\bar\phi^2_{1y})+(\phi^2_{3y}+\bar\phi^2_{3y})(\phi^2_{2z}+\bar\phi^2_{2z})\,,\\
O_2&=\phi^2_{1z}\bar\phi^2_{1z}+\phi^2_{1y}\bar\phi^2_{1y}+\phi^2_{2x}\bar\phi^2_{2x}+\phi^2_{2z}\bar\phi^2_{2z}+\phi^2_{3x}\bar\phi^2_{3x}+\phi^2_{3y}\bar\phi^2_{3y}\,,\\
O_3&=\phi^4_{1z}+\bar\phi^4_{1z}+\phi^4_{1y}+\bar\phi^4_{1y}+\phi^4_{2x}+\bar\phi^4_{2x}+\phi^4_{2z}+\bar\phi^4_{2z}+\phi^4_{3x}+\bar\phi^4_{3x}+\phi^4_{3y}+\bar\phi^4_{3y}\,,\\
O_4&=(\phi^2_{1z}+\bar\phi^2_{1z})(\phi^2_{2z}+\bar\phi^2_{2z})+(\phi^2_{2x}+\bar\phi^2_{2x})(\phi^2_{3x}+\bar\phi^2_{3x})+(\phi^2_{3y}+\bar\phi^2_{3y})(\phi^2_{1y}+\bar\phi^2_{1y})\,,\\
O_5&=\phi^2_{1z}\bar\phi^2_{1y}+\phi^2_{1y}\bar\phi^2_{1z}+\phi^2_{2x}\bar\phi^2_{2z}+\phi^2_{2z}\bar\phi^2_{2x}+\phi^2_{3x}\bar\phi^2_{3y}+\phi^2_{3y}\bar\phi^2_{3x}\,,\\
O_6&=\phi^2_{1z}\phi^2_{1y}+\bar\phi^2_{1z}\bar\phi^2_{1y}+\phi^2_{2x}\phi^2_{2z}+\bar\phi^2_{2x}\bar\phi^2_{2z}+\phi^2_{3x}\phi^2_{3y}+\bar\phi^2_{3y}\bar\phi^2_{3x}\,,\\
O_7&=(\phi^2_{1z}+\bar\phi^2_{1z})(\phi^2_{2x}+\bar\phi^2_{2x})+(\phi^2_{1z}+\bar\phi^2_{1z})(\phi^2_{3y}+\bar\phi^2_{3y})+(\phi^2_{2x}+\bar\phi^2_{2x})(\phi^2_{3y}+\bar\phi^2_{3y})\\
&\quad+(\phi^2_{2z}+\bar\phi^2_{2z})(\phi^2_{3x}+\bar\phi^2_{3x})+(\phi^2_{2z}+\bar\phi^2_{2z})(\phi^2_{1y}+\bar\phi^2_{1y})+(\phi^2_{3x}+\bar\phi^2_{3x})(\phi^2_{1y}+\bar\phi^2_{1y})\,,\\
O_8&=(\phi_{1z}\bar\phi_{1y}-\phi_{1y}\bar\phi_{1z})^2+(\phi_{2x}\bar\phi_{2z}-\phi_{2z}\bar\phi_{2x})^2+(\phi_{3y}\bar\phi_{3x}-\phi_{3x}\bar\phi_{3y})^2\,,\\
O_9&=(\phi_{1z}\bar\phi_{1y}-\phi_{1y}\bar\phi_{1z})(\phi_{2x}\bar\phi_{2z}-\phi_{2z}\bar\phi_{2x})+(\phi_{2x}\bar\phi_{2z}-\phi_{2z}\bar\phi_{2x})(\phi_{3y}\bar\phi_{3x}-\phi_{3x}\bar\phi_{3y})\\
&\quad+(\phi_{3y}\bar\phi_{3x}-\phi_{3x}\bar\phi_{3y})(\phi_{1z}\bar\phi_{1y}-\phi_{1y}\bar\phi_{1z})\,.
\end{aligned}
\qquad (6)
$$

Applying the ϵ-expansion technique developed by Wilson[4] and retaining only terms of order $\ln b$, where b^{-1} is the momentum cutoff, we get the following recursion relations for the coupling constants r and u_i:

$$\begin{aligned}
r' &= b^2[r + (4u_1 + 2u_2 + 12u_3 + 4u_4 + 2u_5 + 2u_6 + 8u_7 + 2u_8)A(r)],\\
u_1' &= u_1 + [\epsilon u_1 - (8u_1^2 + 4u_1u_2 + 24u_1u_3 + 8u_4u_7 + 4u_5u_7 + 4u_6u_7 + 4u_8u_7 + 0.5u_9^2)K_4]\ln b,\\
u_2' &= u_2 + [\epsilon u_2 - (4u_1^2 + 8u_2^2 + 24u_2u_3 + 4u_4^2 + 4u_5u_6 + 8u_7^2 + 4u_8u_6 + 2u_8^2)K_4]\ln b,\\
u_3' &= u_3 + [\epsilon u_3 - (2u_1^2 + u_2^2 + 36u_3^2 + 2u_4^2 + u_5^2 + u_6^2 + 4u_7^2 + u_8^2 + 2u_8u_5)K_4]\ln b,\\
u_4' &= u_4 + [\epsilon u_4 - (8u_1u_7 + 4u_2u_4 + 8u_4^2 + 4u_5u_7 + 4u_6u_7 + 24u_3u_4 + 4u_8u_7 + 0.5u_9^2)K_4]\ln b,\\
u_5' &= u_5 + [\epsilon u_5 - (8u_1u_7 + 4u_2u_6 + 8u_4u_7 + 8u_5^2 + 24u_3u_5 + 2u_8^2 + 8u_5u_8 - 8u_6u_8 - 8u_8u_2 + 24u_8u_3)K_4]\ln b,\\
u_6' &= u_6 + [\epsilon u_6 - (8u_1u_7 + 4u_2u_5 + 24u_3u_6 + 8u_4u_7 + 8u_6^2 + 2u_8^2 + 4u_2u_8)K_4]\ln b,\\
u_7' &= u_7 + [\epsilon u_7 - (4u_1u_4 + 2u_1u_5 + 2u_1u_6 + 4u_2u_7 + 24u_3u_7 + 2u_4u_5 + 2u_4u_6 + 12u_7^2 + 0.5u_9^2 + 2u_8u_4 + 2u_8u_1)K_4]\ln b,\\
u_8' &= u_8 + [\epsilon u_8 - (8u_2u_8 + 8u_5u_8 + 8u_6u_8 + 8u_8^2 + 2u_9^2)K_4]\ln b,\\
u_9' &= u_9 + [\epsilon u_9 - (4u_1u_9 + 4u_4u_9 + 8u_5u_9 + 8u_7u_9 + 12u_8u_9 + 2u_9^2)K_1]\ln b.
\end{aligned} \tag{7}$$

Here

$$K_4 = \frac{1}{8\pi^2},$$

$$A(r) = \frac{1}{(2\pi)^d}\int_{1/b}^{1} d^dq \frac{1}{q^2 + r}$$

and

$$\epsilon = 4 - d.$$

At a fixed point, $u_i = u_i' - u_i^*$. The fixed point coupling constants are determined by nine coupled second-order equations. To find all the fixed points and test the stability is an intractable task; however, it is possible to show without actually solving the equations, that *any* fixed point of this particular system is unstable with respect to small deviations of at least one linear combination of the coupling constants. A proof is given in the Appendix. The nonexistence of a *stable* fixed point might indicate that the transition is first order. Cubic β-MnS with the zinc-blende structure has the fcc type-III structure,[7] and the magnetic ordering may be described by the twelve-component order parameter defined in Eq. (2). These components form an *irreducible* representation of the space group $Fm3m$, but one can show that they form a *reducible* representation of the space group $F\bar{4}3m$ corresponding to the zinc-blende structure. This representation decomposes into two $n=6$ representations and the phase transition is therefore expected to be first order due to a Landau symmetry argument.[3]

B. $n=12$ vector model corresponding to the spiral magnet Eu and sinusoidal magnet Cr

Europium and chromium are bcc crystals whose paramagnetic space group is $Im3m$. Eu undergoes an antiferromagnetic transition at $T_N = 91\,^\circ\mathrm{K}$. Neutron-diffraction experiments indicate a reciprocal-lattice vector $\vec{k}_1 = (k,0,0)(2\pi/a)$ and magnetic moments in the y-z plane[8] (Fig. 2). Cr undergoes an antiferromagnetic transition at $T_N = 310\,^\circ\mathrm{K}$. It exhibits a transverse sinusoidal magnetic structure[9,10] associated with a reciprocal-lattice vector $\vec{k}_1 = (k,0,0)(2\pi/a)$. Since both transitions are described by the same order parameter, they correspond to the same LGW Hamiltonian. The star of $\vec{k}_1$ consists of the six vectors $\pm\vec{k}_1 = (\pm k,0,0)(2\pi/a)$, $\pm\vec{k}_2 = (0,\pm k,0)(2\pi/a)$, and $\pm\vec{k}_3 = (0,0,\pm k)(2\pi/a)$. The group of $\vec{k}_1$ is C_{4v}. The order parameter belongs to a two-dimensional representation of this group corresponding to the two equivalent perpendicular directions of the magnetic moments. The order parameter has, therefore, twelve linearly independent components:

$$\psi_{\pm j,P_j} = \sum_{\vec{r}} S_{\vec{r},P_j}\, e^{\pm \vec{k}_j\cdot\vec{r}}, \tag{8}$$

where the sum is over the positions of the magnetic ions and $S_{\vec{r},P_j}$ is the P_j component of the spin located at $\vec{r}$ ($p_1 = y,z$, $p_2 = x,z$, $p_3 = x,y$). It turns out to be convenient to introduce the real order parameters ϕ_{j,P_j} and $\bar{\phi}_{j,P_j}$ defined by

$$\begin{aligned}
\phi_{j,P_j} &= \tfrac{1}{2}(\psi_{j,P_j} + \psi_{-j,P_j}),\\
\bar{\phi}_{j,P_j} &= (\psi_{j,P_j} - \psi_{-j,P_j})/2i.
\end{aligned} \tag{9}$$

The fourth-order invariants of the order parameter are constructed by noting that the only fourth-order terms which are translationally invariant are those which can be formed as products of two terms of the form $\psi_{j,P_j}\psi_{-j,P_j'}$. The twelve functions $\psi_{j,P_j}\psi_{-j,P_j'}$ are translationally invariant and transform to one another as a basis of a reducible twelve-dimensional representation of the cubic

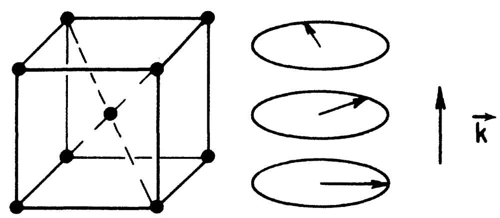

FIG. 2. Helical magnetic structure of Eu. The arrows indicate the ferromagnetic spin alignment in the planes perpendicular to $\vec{k}$.

point group. By finding the second-order invariants of this representation, we obtain all the possible fourth-order invariants of the ϕ_{j,P_j}. The functions $\psi_{j,P_j}\psi_{-j,P'_j}$ transform under the generators of O_h in the following way:

$$
\begin{aligned}
C_4([001]):\ & \psi_{1y}\psi_{-1y} \to \psi_{2x}\psi_{-2x}\,, \quad \psi_{1z}\psi_{-1z} \to \psi_{2z}\psi_{-2z}\,,\\
& \psi_{1y}\psi_{-1z} \to -\psi_{2x}\psi_{-2z}\,, \quad \psi_{1z}\psi_{-1y} \to -\psi_{2z}\psi_{-2x}\,,\\
& \psi_{2z}\psi_{-2z} \to \psi_{1z}\psi_{-1z}\,, \quad \psi_{2x}\psi_{-2x} \to \psi_{1y}\psi_{-1y}\,,\\
& \psi_{2x}\psi_{-2z} \to \psi_{1z}\psi_{-1y}\,, \quad \psi_{2z}\psi_{-2x} \to \psi_{1y}\psi_{-1z}\,,\\
& \psi_{3x}\psi_{-3x} \to \psi_{3y}\psi_{-3y}\,, \quad \psi_{3y}\psi_{-3y} \to \psi_{3x}\psi_{-3x}\,,\\
& \psi_{3x}\psi_{-3y} \to -\psi_{3y}\psi_{-3x}\,, \quad \psi_{3y}\psi_{-3x} \to -\psi_{3x}\psi_{-3y}\,;\\
C_3([111]):\ & \psi_{1y}\psi_{-1y} \to \psi_{2z}\psi_{-2z}\,, \quad \psi_{1z}\psi_{-1z} \to \psi_{2x}\psi_{-2x}\,,\\
& \psi_{1y}\psi_{-1z} \to \psi_{2z}\psi_{-2x}\,, \quad \psi_{1z}\psi_{-1y} \to \psi_{2x}\psi_{-2z}\,,\\
& \psi_{2z}\psi_{-2z} \to \psi_{3x}\psi_{-3x}\,, \quad \psi_{2x}\psi_{-2x} \to \psi_{3y}\psi_{-3y}\,,\\
& \psi_{2z}\psi_{-2x} \to \psi_{3x}\psi_{-3y}\,, \quad \psi_{2x}\psi_{-2z} \to \psi_{3y}\psi_{-3x}\,,\\
& \psi_{3x}\psi_{-3x} \to \psi_{1y}\psi_{-1y}\,, \quad \psi_{3y}\psi_{-3y} \to \psi_{1z}\psi_{-1z}\,,\\
& \psi_{3x}\psi_{-3y} \to \psi_{1y}\psi_{-1z}\,, \quad \psi_{3y}\psi_{-3x} \to \psi_{1z}\psi_{-1y}\,;\\
C_2([110]):\ & \psi_{1y}\psi_{-1y} \to \psi_{2x}\psi_{-2x}\,, \quad \psi_{1z}\psi_{-1z} \to \psi_{2z}\psi_{-2z}\,,\\
& \psi_{1y}\psi_{-1z} \to -\psi_{2x}\psi_{-2z}\,, \quad \psi_{1z}\psi_{-1y} \to -\psi_{2z}\psi_{-2x}\,,\\
& \psi_{2z}\psi_{-2z} \to \psi_{1z}\psi_{-1z}\,, \quad \psi_{2x}\psi_{-2x} \to \psi_{1y}\psi_{-1y}\,,\\
& \psi_{2z}\psi_{-2x} \to -\psi_{1z}\psi_{-1y}\,, \quad \psi_{2x}\psi_{-2z} \to -\psi_{1y}\psi_{-1z}\,,\\
& \psi_{3x}\psi_{-3x} \to \psi_{3y}\psi_{-3y}\,, \quad \psi_{3y}\psi_{-3y} \to \psi_{3x}\psi_{-3x}\,,\\
& \psi_{3x}\psi_{-3y} \to \psi_{3x}\psi_{-3y}\,, \quad \psi_{3y}\psi_{-3x} \to \psi_{3y}\psi_{-3x}\,;\\
i:\ & \psi_{j,P_j}\psi_{-j,P'_j} \to \psi_{j,P'_j}\psi_{-j,P_j}\,. \qquad (10)
\end{aligned}
$$

Using these transformations we found that there are eight second-order invariants of this representation. However, it turns out that when the components ψ_{j,P_j} are inserted, two of these invariants are identical; hence, there are only seven independent fourth-order invariants which can be formed by ψ_{j,P_j} or ϕ_{j,P_j}. We find, that these seven invariants are O_1, $2O_2+O_3$, O_4, O_5+O_6, O_7, O_8, and O_9 defined in Eq. (6). The LGW Hamiltonian is therefore given by

$$
\mathcal{H}_6 = -\frac{1}{2}\sum_{j,P_j}\left[r(\phi^2_{j,P_j}+\bar{\phi}^2_{j,P_j})+(\nabla\phi_{j,P_j})^2+(\nabla\bar{\phi}_{j,P_j})^2\right] - \sum_{i=1}^{9} u_i O_i\,, \qquad (11)
$$

where $u_2 \equiv 2u_3$ and $u_5 \equiv u_6$. The recursion relations for the coupling constants for $\mathcal{H}_6$ are thus closely related to the recursion relations for the Hamiltonian $\mathcal{H}_5$ [Eq. (7)], and the proof in the Appendix for the nonexistence of *stable* fixed points for $\mathcal{H}_5$ is also valid for the Hamiltonian $\mathcal{H}_6$.

More than a decade ago, a first-order transition was discovered in Cr.[11] Despite numerous attempts to clarify the origin of this transition, no satisfactory explanation has been reported yet. This behavior is, however, consistent with the interpretation that a lack of a stable fixed point indicates a first-order transition. Similarly, Mössbauer and specific-heat measurements show that the phase transformation in Eu is first order.[12] Our theory may thus provide a better understanding of the phase transitions on both these elements. Another peculiar feature is that chromium with very small amounts of various impurities exhibits a *second*-order transition.[13] Furthermore, early measurements of the magnetization in Cr and Eu showed a clear tendency towards a second-order behavior.[8,10,14] It would be very interesting to study random $n \geq 4$ models to see if impurities might give rise to stable fixed points.

C. Type-III antiferromagnets: MnS_2

MnS_2 crystalizes in the pyrite structure which is a NaCl-like arrangement of Mn and S_2 groups, with the axes of the S_2 groups along the body diagonals. The paramagnetic space group is $Pa3$. The magnetic structure below the transition temperature is type-III antiferromagnet.[15] The magnetic lattice is shown in Fig. 1. The unit cell is doubled in the x direction, and the sublattice magnetization is along the x axis. The phase transition in K_2IrCl_6, which has the same magnetic structure, has been discussed by Mukamel and Krinsky.[1,2] They found that this transition is described by an $n=6$ LGW Hamiltonian with three fourth-order invariants. The paramagnetic space group of MnS_2 ($Pa3$) is a subgroup of the paramagnetic space group of K_2IrCl_6 ($Fm3m$). This might give rise to additional fourth-order invariants besides the three found by Mukamel and Krinsky. However, it turns out that if one applies the formula (4) for the number of invariants using only the classes of the space group $Pa3$, the number of invariants is unchanged. This is in sharp contrast to the fcc type-III antiferromagnet β-MnS, where the lowering of the symmetry gives rise to four new invariants. The corresponding Hamiltonian is

$$\mathcal{H}_7 = -\frac{1}{2}\sum_{i=1}^{3}\left[r(\phi_i^2+\bar{\phi}_i^2)+(\nabla\phi_i)^2+(\nabla\bar{\phi}_i)^2\right] - u\left(\sum_{i=1}^{3}\phi_i^2+\bar{\phi}_i^2\right)^2 - v\sum_{i=1}^{3}(\phi_i^2+\bar{\phi}_i^2)^2 - w\sum_{i=1}^{3}\phi_i^2\bar{\phi}_i^2 \qquad (12)$$

with a stable fixed point corresponding to $w^*=0$, $u^*\neq 0, v^*\neq 0$. The exponents associated with this fixed point are

$$\nu = \tfrac{1}{2} + \tfrac{3}{22}\epsilon + (605/8\times 11^3)\epsilon^2, \quad \eta = \tfrac{5}{242}\epsilon^2 . \qquad (13)$$

Using scaling relations the following exponents are obtained for $d=3$ ($\epsilon=1$): $\nu=0.69, \beta=0.38, \gamma=1.38, \alpha=-0.14$.

MnS_2 thus belongs to a large class of magnetic systems with $n=6$, which to second order in ϵ are predicted to exhibit the same critical behavior. Besides MnS_2 and K_2IrCl_6, this class includes systems with fundamentally different symmetries such as TbD_2 and Nd. It would be of great interest to test this universality by comparing experimental data on the critical properties of these compounds.

D. $n=4$ model of the spiral magnetic systems Tb, Dy, and Ho

The rare-earth elements Tb, Dy, and Ho crystallize in the hcp structure with space group $P6_3/mmc$. Below T_N they exhibit a spiral magnetic structure with magnetic moments in the basal plane and the propagation vector $\vec{k}$ along the c direction.[16-18] This magnetic structure is indicated in Fig. 3. The star of the $\vec{k}$ vector consists of the two vectors $\pm\vec{k} = \pm(0,0,k)(2\pi/a)$. For each of these vectors, there are two equivalent perpendicular directions of the spins in the basal plane, so that the number of independent components of the order parameter is four:

$$\psi_{\pm k,P} = \phi_P \pm i\bar{\phi}_P = \sum_{\vec{r}} S_{\vec{r},P}\, e^{\pm i\vec{k}\cdot\vec{r}}, \quad P=x,y. \qquad (14)$$

Here the sum is over the positions of the magnetic ions, and $S_{\vec{r},P}$ is the P component of the spin located at site $\vec{r}$. Because of translational invariance, the only possible fourth-order invariants in these components has to be of second order in the functions $\psi_{k,P}\psi_{-k,P'}$, which clearly are translationally invariant and transform to one another as a basis of a representation of the hexagonal point group D_{6h}. We find that there are three second-order invariants which can be formed from this basis. These invariants are

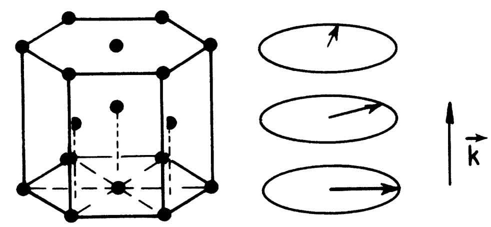

FIG. 3. Spiral magnetic structure of Tb, Dy, and Ho. The arrows indicate the spin alignment within the hexagonal planes.

$$\begin{aligned} O_1 &= [(\psi_{kx}\psi_{-kx})+(\psi_{ky}\psi_{-ky})]^2, \\ O_2 &= \tfrac{1}{2}(\psi_{kx}\psi_{-ky})(\psi_{-kx}\psi_{ky}) - \tfrac{1}{4}[(\psi_{ky}\psi_{-kx})^2+(\psi_{kx}\psi_{-ky})^2], \\ O_3 &= \tfrac{1}{2}(\psi_{kx}\psi_{-kx})(\psi_{ky}\psi_{-ky}) - \tfrac{1}{4}[(\psi_{ky}\psi_{-kx})^2+(\psi_{kx}\psi_{-ky})^2]. \end{aligned} \qquad (15)$$

However, the invariants O_2 and O_3 are clearly identical, so that the number of fourth-order invariants in the components of the order parameters is two. Using the real order parameters φ_P and $\bar{\phi}_P$ these invariants may be written

$$\begin{aligned} O_1 &= (\phi_x^2+\bar{\phi}_x^2+\phi_y^2+\bar{\phi}_y^2)^2, \\ O_2 &= (\phi_x\bar{\phi}_y-\phi_y\bar{\phi}_x)^2, \end{aligned} \qquad (16)$$

and the corresponding LGW Hamiltonian is

$$\begin{aligned} \mathcal{H}_8' = &-\tfrac{1}{2}[r(\phi_x^2+\bar{\phi}_x^2+\varphi_y^2+\bar{\phi}_y^2) + (\nabla\phi_x)^2+(\nabla\bar{\phi}_x)^2+(\nabla\phi_y)^2+(\nabla\bar{\phi}_y)^2] \\ &- u_1(\phi_x^2+\bar{\phi}_x^2+\phi_y^2+\bar{\phi}_y^2)^2 - u_2(\phi_x\bar{\phi}_y-\phi_y\bar{\phi}_x)^2. \end{aligned} \qquad (17)$$

Let us rewrite this Hamiltonian in terms of the following variables:

$$\begin{aligned} \eta_1 &= (\phi_x+\bar{\phi}_y)/\sqrt{2}, \quad \eta_2 = (\phi_x-\bar{\phi}_y)/\sqrt{2}, \\ \bar{\eta}_2 &= (\bar{\phi}_x+\phi_y)/\sqrt{2}, \quad \bar{\eta}_1 = (\bar{\phi}_x-\phi_y)/\sqrt{2}. \end{aligned} \qquad (18)$$

These variables describe the "spiral" components of the magnetization, i.e.,

$$\eta_{1,2} \propto \sum_{\vec{r}} S_{\vec{r}x}\cos\vec{k}\cdot\vec{r} \pm S_{\vec{r}y}\sin\vec{k}\cdot\vec{r},$$

$$\bar{\eta}_{1,2} \propto \sum_{\vec{r}} \pm S_{\vec{r}y}\cos\vec{k}\cdot\vec{r} + S_{\vec{r}x}\sin\vec{k}\cdot\vec{r}.$$

η_1 and $\bar{\eta}_1$ correspond to right-handed spirals and η_2 and $\bar{\eta}_2$ correspond to left-handed spirals. $\bar{\eta}_1$ and $\bar{\eta}_2$ are related to η_1 and η_2 through rotations of 90° along the propagation vector. In this representation, the Hamiltonian is

$$\mathcal{H}_8 = -\frac{1}{2}\sum_{i=1}^{2}\left[r(\eta_i^2+\bar{\eta}_i^2)+(\nabla\eta_i)^2+(\nabla\bar{\eta}_i)^2\right] - u\left(\sum_{i=1}^{2}\eta_i^2+\bar{\eta}_i^2\right)^2 - v\sum_{i=1}^{2}(\eta_i^2+\bar{\eta}_i^2)^2, \quad (19)$$

where $u = u_1 - \frac{1}{4}u_2$ and $v = \frac{1}{2}u_2$. This Hamiltonian was studied by Mukamel and Krinsky[1,2,6] to describe the phase transition in NbO_2, $TbAu_2$, and DyC_2. It was found to have one stable fixed point, and we therefore predict, that the six compounds Ho, Dy, Tb, NbO_2, $TbAu_2$, and DyC_2 *all* have the same critical behavior, despite the fundamental differences in the symmetries and underlying physical nature of the phase transitions. The critical exponents are[6]

$$\nu = 0.70, \eta = 0.02, \beta = 0.39, \gamma = 1.39, \alpha = -0.17 \quad (20)$$

As pointed out by Mukamel,[6] these exponents are identical to those of the $n=4$ isotropic model.

We now proceed to compare these exponents with experimental results. For Tb, the exponent β has been measured by Dietrich and Als-Nielsen.[16] They found $\beta \sim 0.25$ within the interval $0.001 < 1 - T/T_N < 0.025$. However, this value may not be reliable due to imperfections of the crystal and extinction.[19] For Dy, Chien *et al.* obtained $\beta \sim 0.335$ within the temperature range $0.01 < 1 - T/T_N < 0.3$. using the Mössbauer effect.[20] It would be interesting to measure this critical exponent by neutron diffraction. The specific heat exponents α and α' above and below T_N, respectively, were measured by Lederman and Salamon.[21] Using a "*conventional*" fitting method they found $\alpha = \alpha' = 0.13$, and a two region fit gave $\alpha = \alpha = 0.18$ close to T_N. This result is in disagreement with the predictions of the ϵ expansion, and it would be interesting to study the critical behavior of the other compounds in this universality group to see whether or not they agree with the experiments on Dy.

III. SUMMARY AND CONCLUSIONS

We derived the Landau-Ginzburg-Wilson Hamiltonian corresponding to several $n \geq 4$ physical systems, and studied the phase transitions of these models by the exact renormalization group in $d = 4-\epsilon$ dimensions. We found that the type-III antiferromagnet MnS_2 is described by the $n=6$ LGW Hamiltonian which has previously been studied by Mukamel and Krinsky to discuss the critical behavior of K_2IrCL_6, Nd, and TbD_2. Since this Hamiltonian has one stable fixed point, it is predicted that these four compounds belong to the same universality class, despite the fact that their symmetries are different. We therefore suggest a systematic experimental study of the critical behavior of these systems. Even if the ϵ expansion may not give reliable numerical estimates of the critical exponents at $d=3$, such experiments may provide a crucial test of the universality, predicted by the renormalization-group calculations. The critical exponents, to second order in ϵ, for $d=3$ were found to be

$$\beta = 0.38, \gamma = 1.38, \nu = 0.69, \alpha = -0.14. \quad (21)$$

The effect of the anisotropy on the critical behavior of this model is easily seen by comparing these exponents to those corresponding to the isotropic $n=6$ model

$$\beta = 0.41, \gamma = 1.45, \nu = 0.73, \alpha = -0.27. \quad (22)$$

We also found that the spiral magnetic systems Tb, Dy, and Ho are described by the $n=4$ LGW Hamiltonian, which was studied by Mukamel and Krinsky to discuss the critical behavior of NbO_2, $TbAu_2$, and DyC_2. It is thus predicted that these six systems have the same critical behavior. The critical exponents to second order in ϵ and $d=3$ are

$$\beta = 0.39, \gamma = 1.39, \nu = 0.70, \alpha = -0.17. \quad (23)$$

We suggest experiments be performed to test these exponents. The type-III antiferromagnets with $\vec{m} \perp [100]$, $\vec{k} = (\frac{1}{2}, 0, 1)$ and the spiral magnets Eu and Cr are described by two $n=12$ LGW Hamiltonians. These models were found to have no stable fixed points. This is consistent with the experimental result that the transitions in Eu and Cr are first order. We suggest further experimental studies of these systems in order to throw light on the nature of the first-order transitions.

ACKNOWLEDGMENTS

We wish to thank J. D. Axe, G. A. Baker, Jr., M. Blume, K. Carneiro, L. Corliss, D. E. Cox, V. J. Every, J. Hastings, Y. Imry, S. Krinsky, R. Pynn, S. M. Shapiro, G. Shirane, and S. Shtrikman for many illuminating discussions.

APPENDIX

In this appendix, we shall prove that there is no *stable* fixed point of the recursion relations, Eq. (7).

We introduce

$$x_i = K_4 u_i^* / \epsilon \quad (A1)$$

and get the following equations, which must be satisfied at the fixed point:

$$x_1 = 8x_1^2 + 4x_1x_2 + 24x_1x_3 + 8x_4x_7 + 4x_5x_7 + 4x_6x_7 + 4x_8x_7 + 0.5x_9^2, \tag{A2}$$

$$x_2 = 4x_1^2 + 8x_2^2 + 24x_2x_3 + 4x_4^2 + 8x_7^2 + 4x_5x_6 + 4x_6x_8 + 2x_8^2, \tag{A3}$$

$$x_3 = 2x_1^2 + x_2^2 + 36x_3^2 + 2x_4^2 + 4x_7^2 + x_6^2 + (x_5 + x_8)^2, \tag{A4}$$

$$x_4 = 8x_1x_7 + 4x_2x_4 + 8x_4^2 + 4x_5x_7 + 4x_6x_7 + 24x_3x_4 + 4x_8x_7 + 0.5x_9^2, \tag{A5}$$

$$x_5 = 8x_1x_7 + 4x_2x_6 + 8x_4x_7 + 8x_5^2 + 24x_3x_5 + 2x_8^2 + 8x_5x_8 - 8x_6x_8 - 8x_8x_2 + 24x_8x_3, \tag{A6}$$

$$x_6 = 8x_1x_7 + 4x_2x_5 + 24x_3x_6 + 8x_4x_7 + 8x_6^2 + 2x_8^2 + 4x_2x_8, \tag{A7}$$

$$x_7 = 4x_1x_4 + 2x_1x_5 + 2x_1x_6 + 4x_2x_7 + 24x_3x_7 + 2x_4x_5 + 2x_4x_6 + 12x_7^2 + 0.5x_9^2 + 2x_8x_4 + 2x_8x_1, \tag{A8}$$

$$x_8 = 8(x_2 + x_8')x_8 + 2x_9^2, \tag{A9}$$

$$x_9 = (4x_1 + 4x_4 + 8x_5 + 8x_7 + 12x_8 + 2x_9)x_9, \tag{A10}$$

where $x_8' = x_5 + x_6 + x_8$.

From (A2) and (A5) we derive the equation

$$x_1 - x_4 = (x_1 - x_4)[\,8x_1 + 8x_4 + 4x_2 - 8x_7 + 24x_3\,]. \tag{A11}$$

Now, either $x_1 - x_4$ is different from zero, and the quantity in the square brackets is one, or $x_1 - x_4$ is zero. In the latter case, any solution of the fixed point equations must be stable with respect to small perturbations of $x_1 - x_4$, i.e., the quantity in the brackets must be greater than one. Hence, we get the condition

$$8x_1 + 8x_4 + 4x_2 - 8x_7 + 24x_3 = 1 + C, \quad C \geq 0. \tag{A12}$$

From Eq. (A10) we get a similar condition

$$4x_1 + 4x_4 + 8x_7 + 8x_5 + 12x_8 + 2x_9 \geq 1. \tag{A13}$$

It turns out to be convenient to introduce the linear combinations

$$(\text{A3}) + 2(\text{A4}):\quad (72x_3^2 + 10x_2^2 + 24x_2x_3 - 2x_3 - x_2) + 8x_1^2 + 8x_4^2 + 16x_7^2 + 2x_8'^2 + 2x_8^2 = 0, \tag{A14}$$

$$(\text{A3}) + 2(\text{A4}) + (\text{A6}) + (\text{A7}) + (\text{A9}):\quad (72x_3^2 + 10x_2^2 + 24x_2x_3 - 2x_3 - x_2) + x_8'(24x_3 + 4x_2 - 1)$$
$$+ 16x_7^2 + 8x_1^2 + 8x_4^2 + 16(x_1 + x_4)x_7 + 2x_8'^2 + 6x_8^2 + 8x_6^2 + 8(x_5 + x_8)^2 + 2x_9^2 = 0, \tag{A15}$$

$$(\text{A6}) + (\text{A7}) + (\text{A9}):\quad x_8'(24x_3 + 4x_2 - 1) + 8x_7(2x_1 + 2x_4) + 8x_6^2 + 8(x_5 + x_8)^2 + 4x_8^2 + 2x_9^2 = 0, \tag{A16a}$$

(A12) is inserted in (A16a):

$$2x_8'(x_1 + x_4) = 2x_7x_8' + x_7(4x_1 + 4x_4) + 2x_6^2 + 2(x_5 + x_8)^2 + x_8^2 + \tfrac{1}{2}x_9^2 + \tfrac{1}{4}Cx_8', \tag{A16b}$$

$$(\text{A2}) + (\text{A5}):\quad (x_1 + x_4)(24x_3 + 4x_2 - 1 + 8x_7) + 8x_7x_8' + 8x_1^2 + 8x_4^2 + x_9^2 = 0. \tag{A17a}$$

This equation togehter with (A12) gives

$$4x_1x_4 = 2x_7x_8' + x_7(4x_1 + 4x_4) + \tfrac{1}{4}x_9^2 + \tfrac{1}{4}C(x_1 + x_4) \tag{A17b}$$

(A12) and (A17b) are inserted in (A8),

$$12x_7^2 - x_7(4x_1 + 4x_4 - 8x_7 - 2x_8') + 2x_8'(x_1 + x_4) + \tfrac{1}{4}(x_1 + x_4 + 4x_7)C + \tfrac{3}{4}x_9^2 = 0 \tag{A18}$$

and (A16b) is inserted in (A18),

$$20x_7^2 - x_7(4x_8' + C) + 2x_6^2 + 2(x_5 + x_8)^2 + x_8^2 + \tfrac{5}{4}x_9^2 + \tfrac{1}{4}C(x_1 + x_4 + x_8') = 0. \tag{A19}$$

It is possible to show that (A14) implies the following inequalities

$$24x_3 + 4x_2 < 1, \tag{A20}$$

$$24x_3 + 4x_2 - 8x_7 < 1, \tag{A21a}$$

or [with (A12)]

$$x_1 + x_4 > 0. \tag{A21b}$$

This is most easily shown using the Lagrange multiplier formalism. For instance, to prove (A20) we form the expression

$$24x_3 + 4x_2 - \lambda f_1(x_1 \cdots x_8'), \tag{A22}$$

where f_1 is the left-hand side of Eq. (A14), minimize it with respect to all the parameters $x_1 \cdots x_8'$, and determine λ from the condition $f_1 = 0$.

We find that the extremum value of $24x_3+4x_2$ is $\frac{1}{3}+\frac{1}{3}(2)^{1/2}$ which is less than 1.

We also prove, that for $C=0$,

$$f_2=4x_1+4x_4-8x_7-2x_8'>0. \tag{A23}$$

In this case we form the expression

$$f_2(x_1,x_4,x_7,x_8')-\lambda_1(8x_1+8x_4-8x_7+24x_3+4x_2) -\lambda_2 f_1(x_1,\ldots,x_8'), \tag{A24}$$

and again we minimize with respect to $x_1\cdots x_8'$ and determine λ_1 and λ_2 from the conditions (A12) and (A14). We find that $f_2>0$;

As the next step we shall prove that $x_8'<0$. It is convenient to write Eq. (A8) in the form

$$12x_7^2-x_7(1-24x_3-4x_2)+2(x_1+x_4)x_8' +4x_1x_4+0.5x_9^2=0. \tag{A8$'$}$$

Let us assume that $x_8'>0$. If $C>0$, then $x_1=x_4$. Using the conditions (A12b) and (A20), we see immediately that $x_7>0$. If $C=0$, then Eq. (A18) implies $x_7>0$ if we consider the inequality (A23). But now all the terms on the left-hand side of (A19) are positive; hence we have a contradiction, and we have proved that x_8' can not be positive.

Now consider Eq. (A15). It is easy to show that

$$72x_3^2+10x_2^2+24x_2x_3-2x_3-x_2\geqslant-\tfrac{1}{36}, \tag{A25}$$

and we have shown that $x_8'(24x_3+4x_2-1)\geqslant 0$, so

$$16x_7^2+8x_1^2+8x_4^2+16(x_1+x_4)x_7+2x_8'^2+8x_6^2 +2x_9^2+6x_8^2+8(x_5+x_8)^2\leqslant\tfrac{1}{36}. \tag{A26}$$

But the condition (A13) implies that the quantity on the left-hand side of (A26) is greater than $\frac{3}{50}>\frac{1}{36}$. The proof is easily carried out using the Lagrange formalism. Hence our system of equations have no solutions subject to the constraints (A12) and (A13), and the corresponding Hamiltonian has no stable fixed point.

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†Work supported by Energy Research and Development Administration.

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