Physical realizations of $n \geq 4$-component vector models. II. ϵ-expansion analysis of the critical behavior*

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In the preceding paper, we derived $n \geq 4$-component Landau-Ginzburg-Wilson Hamiltonians describing phase transitions in certain physical systems. Here, we use Wilson's ϵ expansion to study the phase transitions associated with these Hamiltonians. Although the $n = 4$ systems $TbAu_2$, DyC_2, and NbO_2 have different symmetries, they are predicted to have the same critical exponents. Similarly, although the $n = 6$ systems TbD_2, Nd, and K_2IrCl_6 have different symmetries, their critical exponents are predicted to be equal, but different from those of an isotropic $n = 6$ model. We suggest these predictions be tested experimentally. Three of the Hamiltonians we have considered possess no stable fixed points. Two of the materials UO_2 ($n = 6$) and MnO ($n = 8$) described by these Hamiltonians are known to have first-order transitions. We suggest that experiments should be performed to test whether or not the transitions in the $n = 8$ systems ErSb, MnSe, NiO, and in the $n = 4$ systems TbAs, TbP, TbSb are first order.

I. INTRODUCTION

In the preceding paper,[1] we derived $n \geq 4$-component Landau-Ginzburg-Wilson Hamiltonians describing phase transitions in certain physical systems. As discussed by Landau and Lifshitz,[2] the symmetry-breaking order parameter associated with a second-order phase transition transforms according to an irreducible representation of the symmetry group of the high-symmetry phase. The number of independent components of the order parameter is equal to the dimensionality n of the representation according to which it transforms, and the transition is described by an n-component Hamiltonian. When there is no change of the unit cell, the order parameter transforms according to an irreducible representation of the *point group* of the high-symmetry phase, and the dimensionality of these representations always satisfies $n \leq 3$. When the unit cell is doubled in one or more directions, the order parameter transforms according to an irreducible representation of the *space group* of the high-symmetry phase, and the dimensionality of these representations can satisfy[3,4] $n \geq 4$.

In this paper, we use Wilson's ϵ-expansion technique[5] to study the critical phenomena associated with each of our $n \geq 4$-component Hamiltonians. The derivation of exact renormalization-group recursion relations in $4 - \epsilon$ dimensions has been described in detail by Wilson and Kogut,[5] and we will not reproduce their discussion. However, we wish to emphasize that if the initial Hamiltonian $\mathcal{H}$ is invariant under a space group, then the transformed Hamiltonian $\mathcal{H}'$ is also invariant. Suppose the initial Hamiltonian is taken to be of the form

$$\mathcal{H} = -\frac{1}{2}\sum_{i=1}^{n}[r\phi_i^2 + (\nabla\phi_i)^2] - \sum_p u_p \sum_{ijkl}\beta^p_{ijkl}\phi_i\phi_j\phi_k\phi_l,$$

where the sum over p spans the fourth-order invariants of the group. It follows that the transformed Hamiltonian $\mathcal{H}'$ will be of the form

$$\mathcal{H}' = -\frac{1}{2}\sum_{i=1}^{n}[r'\phi_i^2 + (\nabla\phi_i)^2] - \sum_p u'_p \sum_{ijkl}\beta^p_{ijkl}\phi_i\phi_j\phi_k\varphi_l + \text{higher-order invariant terms.}$$

The only second-order invariant that can be constructed from an irreducible representation of a space group is $\sum_{i=1}^{n}\phi_i^2$. As a consequence, even though the fourth-order terms are anisotropic, the renormalization-group transformations will not break the isotropy of the second-order term. Hence, there is only one length scale in the problem. To $O(\epsilon^2)$ it is not necessary to consider sixth- and higher-order invariants, or the momentum dependence of the coupling constants, so the recursion relations for r' and u'_p involve only r and u_p.

Brézin, Le Guillou, and Zinn-Justin[6] have shown that for an $n \leq 3$-component model, the isotropic fixed point is always stable to leading order in ϵ. This is a very interesting result, since it means that whatever anisotropy one chooses consistent with the space group, full rotational symmetry is dynamically generated near the critical point. For $n > 4$, they showed the isotropic fixed point is al-

ways unstable with respect to any anisotropy of the fourth-order terms. The first model we consider is a $2m$-component Hamiltonian with $2m \geq 4$. For each value of m, we find a unique stable fixed point, which is nonisotropic. Hence, full rotational symmetry is not dynamically generated in this case. For the three other Hamiltonians we have considered, we find that to lowest order in ϵ all the fixed points are *unstable*. This nonexistence of a stable fixed point is a new feature that has arisen in the study of $n \geq 4$-component models, and its significance is not yet well established. In Wilson's theory,[5] a critical Hamiltonian is presumed to converge to a stable fixed point upon repeated application of the renormalization-group transformations. The nonexistence of a stable fixed point should therefore mean that there is no critical point for the system (obtainable by varying only the temperature). Hence, one would expect the transition to be discontinuous, i.e., first order. However, the possibility certainly exists that there may be a stable fixed point whose fourth-order coupling is of order unity or does not vanish linearly as $\epsilon \to 0$. Such a stable fixed point would be missed by the ϵ expansion.

When performing an ϵ expansion, one is studying phase transitions in $4-\epsilon$ dimensions, where ϵ is infinitesimal. This means that predictions of ϵ expansion must be considered as tentative for three-dimensional systems. The ϵ expansion can be used as a phenomenological probe and can serve to raise questions about physical systems which can then be studied by experiment. Predictions of exponents are not precise because $\epsilon = 1$ is not a small parameter. The current "rule of thumb" is to compute the exponents to $O(\epsilon^2)$ and then set $\epsilon = 1$. This procedure has worked quite well for the three-dimensional Ising, XY, and Heisenberg models, for which the answers were known in advance from work using high-temperature series, and from experiment. However, for the $n \geq 4$ physical systems which we are considering, the answers are not known in advance, and experimental study of these systems can serve as a test for the "rule of thumb."

A second difficulty resulting from $\epsilon = 1$ not being a small parameter is that the determination of the stability of a fixed point is not unambiguous. Our procedure for determining the stability of a fixed point is consistent with the work of Brézin *et al.*[6] We consider ϵ to be infinitesimal and determine stability from the term of lowest order in ϵ which does not vanish. Although *ad hoc*, this procedure is systematic. Of course, it is certainly possible that a fixed point which is stable for $d = 4-\epsilon$ (ϵ infinitesimal) will be unstable for $d = 3$, and vice versa. After we have determined the stable fixed point as described above, in order to stimulate experiment, we set $\epsilon = 1$ in the expressions for the critical exponents computed to $O(\epsilon^2)$. We are certainly aware that phase transitions in $d = 3$ may be significantly different than in $d = 4-\epsilon$, but our goal is to create interest in studying the unknown critical behavior of physical systems with $n \geq 4$.

Our paper is organized as follows: In Sec. II, we use the ϵ expansion to study the critical behavior of the $2m$-component Hamiltonian denoted $\mathcal{H}_1$ in the previous paper.[1] For $2m = 4$, this Hamiltonian is relevant to a discussion of the critical behavior of $TbAu_2$, DyC_2, and NbO_2, while for $2m = 6$ it provides a description of the critical phenomena in TbD_2, Nd, and K_2IrCl_6. In Sec. III, we consider the Hamiltonian $\mathcal{H}_4$ which describes[1] the phase transition in the type-I antiferromagnet ($\vec{m} \perp \vec{k}$) UO_2. In Sec. IV, we study the Hamiltonian $\mathcal{H}_3$ corresponding[1] to type-II antiferromagnets with $\vec{m} \parallel \vec{k}$. In Sec. V, we consider the Hamiltonian $\mathcal{H}_2$ appropriate[1] for type-II antiferromagnets with $\vec{m} \perp \vec{k}$. Finally, we summarize our results, and our suggestions for experiments, in Sec. VI.

II. $TbAu_2$, DyC_2, NbO_2, AND TbD_2, Nd, K_2, $IrCl_6$

We consider the $2m$-component Hamiltonian[7]

$$\mathcal{H}_1 = -\frac{1}{2}\sum_{i=1}^{m}[r(\phi_i^2+\bar{\phi}_i^2)+(\nabla\phi_i)^2+(\nabla\bar{\phi}_i)^2] - u\left(\sum_{i=1}^{m}\phi_i^2+\bar{\phi}_i^2\right)^2 - v\sum_{i=1}^{m}(\phi_i^2+\bar{\phi}_i^2)^2 - w\sum_{i=1}^{m}\phi_i^2\bar{\phi}_i^2$$

For $2m = 4$, this Hamiltonian describes[1] critical phenomena in $TbAu_2$, DyC_2 (with $w = 0$), and NbO_2 (with $w \neq 0$), while for $2m = 6$ it describes the critical behavior of TbD_2, Nd (with $w = 0$), and K_2IrCl_6 (with $w \neq 0$). To perform a renormalization-group analysis for this Hamiltonian, it is convenient to rewrite the interaction term as

$$-u_1\sum_{i=1}^{m}(\phi_i^4+\bar{\phi}_i^4) - u_2\sum_{i=1}^{m}\phi_i^2\bar{\phi}_i^2 - u_3\sum_{i<1}(\phi_i^2\phi_j^2+\bar{\phi}_i^2\bar{\phi}_j^2+\phi_i^2\bar{\phi}_j^2+\bar{\phi}_i^2\phi_j^2),$$

where $u = \frac{1}{2}u_3$, $v = u_1 - \frac{1}{2}u_3$, $w = u_2 - 2u_1$. Assuming a small momentum cutoff b^{-1}, we use Wilson's exact renormalization-group technique[5] in dimension $d = 4-\epsilon$, and we keep only terms of order $\ln b$ in the recursion relations for u_1', u_2', u_3', and terms of order $b^2 \ln b$ in the recursion relation for r'. We obtain

$$r' = b^{2-\eta}\{r + 2[6u_1 + u_2 + 2(m-1)u_3]A(r) - [96u_1^2 + 8u_2^2 + 16(m-1)u_3^2]B(r)\} + O(\epsilon^3),$$

$$u_1' = b^{\epsilon-2\eta}\{u_1 - \tfrac{1}{2}[72u_1^2 + 2u_2^2 + 4(m-1)u_3^2]K_d \ln b + \tfrac{1}{12}[10\,368u_1^3 + 288u_1u_2^2 + 576(m-1)u_1u_3^2 + 96u_2^3 + 192(m-1)u_3^3]K_4^2 \ln b\} + O(\epsilon^4),$$

$$u_2' = b^{\epsilon-2\eta}\{u_2 - \tfrac{1}{2}[48u_1u_2 + 16u_2^2 + 8(m-1)u_3^2]K_d \ln b + \tfrac{1}{12}[3456(u_1^2u_2 + u_1u_2^2) + 576(m-1)u_2u_3^2 + 480u_2^3 + 384(m-1)u_3^3]K_4^2 \ln b\} + O(\epsilon^4),$$

$$u_3' = b^{\epsilon-2\eta}\{u_3 - \tfrac{1}{2}(48u_1u_3 + 8u_2u_3 + 8mu_3^2)K_d \ln b + \tfrac{1}{12}[3456(u_1^2u_3 + u_1u_3^2) + 576u_2u_3^2 + 288u_2^2u_3 + (960m - 1344)u_3^3]K_4^2 \ln b\} + O(\epsilon^4),$$

where

$$\eta = \tfrac{1}{4}K_4^2[96u_1^2 + 8u_2^2 + 16(m-1)u_3^2],$$

$$A(r) = \int_{b^{-1} \le |q| \le 1} \frac{d^dq}{(2\pi)^d} \frac{1}{q^2 + r},$$

$$B(r) = \int\int_{b^{-1} \le |q_i| \le 1} \frac{d^dq_1}{(2\pi)^d} \frac{d^dq_2}{(2\pi)^d} \frac{1}{(q_1^2 + r)(q_2^2 + r)[(q_1 + q_2)^2 + r]},$$

$$K_d = 2^{-(d-1)}\pi^{-d/2}[\Gamma(\tfrac{1}{2}d)]^{-1}.$$

Let us first determine the fixed points to $O(\epsilon)$. Defining $x_i = (K_4/\epsilon)u_i$, the equations for the fixed points become

$$x_1 = 36x_1^2 + x_2^2 + 2(m-1)x_3^2,$$

$$x_2 = 24x_1x_2 + 8x_2^2 + 4(m-1)x_3^2,$$

$$x_3 = 24x_1x_3 + 4x_2x_3 + 4mx_3^2.$$

There are eight real solutions to these equations. We do not consider complex solutions, since if the starting Hamiltonian is real, then it can only approach real fixed points under repeated application of the renormalization-group transformations:

$x_1 = 0,$	$x_2 = 0,$	$x_3 = 0;$	(a)
$x_1 = \frac{1}{40},$	$x_2 = 2x_1,$	$x_3 = 0;$	(b)
$x_1 = \frac{1}{36},$	$x_2 = 0,$	$x_3 = 0;$	(c)
$x_1 = \frac{1}{72},$	$x_2 = \frac{1}{12},$	$x_3 = 0;$	(d)
$x_1 = 1/8(m+4),$	$x_2 = 2x_1,$	$x_3 = 2x_1;$	(e)
$x_1 = (m-1)/8(5m-4),$	$x_2 = 2x_1,$	$x_3 = 2x_1/(m-1);$	(f)
$x_1 = \frac{1}{72}(2 - 1/m),$	$x_2 = 1/12m,$	$x_3 = 1/12m;$	(g)
$x_1 = \frac{1}{72}(1 + 1/m),$	$x_2 = \frac{1}{12}(1 - 1/m),$	$x_3 = 1/12m.$	(h)

Stability is determined by linearizing the recursion relations about each fixed point. For $2m < 4$, the isotropic fixed point (e) is stable, while for $2m > 4$, the fixed point (f) is stable. To determine stability for the special case $2m = 4$, it is necessary to consider the next order in ϵ. This has previously been done by Mukamel.[3]

For $2m > 4$, the stable fixed point to $O(\epsilon^2)$ is given by

$$u_1^* = \frac{(m-1)\epsilon}{8(5m-4)K_d} + \frac{(m-1)(30m^2 - 15m - 27)\epsilon^2}{16(5m-4)^3K_4},$$

$$u_2^* = 2u_1^*,$$

$$u_3^* = \frac{\epsilon}{4(5m-4)K_d} + \frac{(-70m^2 + 205m - 139)\epsilon^2}{8(5m-4)^3K_4}.$$

Note that $w^* = u_2^* - 2u_1^* = 0$. When we set $2m = 4$, we

obtain the same stable fixed point as previously obtained by Mukamel.[3]

The critical exponent ν describing the divergence of the correlation length at the critical point is determined by

$$1/\nu = 2 - \eta - 2[6u_1^* + u_2^* + 2(m-1)u_3^*]K_d + \tfrac{3}{2}[96u_1^{*2} + 8u_2^{*2} + 16(m-1)u_3^{*2}]K_4^2.$$

To second order in ϵ, we obtain

$$\nu = \frac{1}{2} + \frac{3(m-1)\epsilon}{4(5m-4)} + \frac{(m-1)(20m^2+253m-334)\epsilon^2}{16(5m-4)^3},$$

$$\eta = \frac{(m-1)(2m-1)\epsilon^2}{4(5m-4)^2}.$$

Expansions for β and γ can be obtained from those for ν and η by using the scaling laws.

The critical behavior of $TbAu_2$, DyC_2, and NbO_2 are all described[1] by $\mathcal{H}_1$ with $2m=4$. Since we have found there exists a unique stable fixed point, our prediction is that these different phase transitions should be described by the same critical exponents. This can be tested experimentally. In addition, if we follow the usual "rule of thumb" and set $\epsilon=1$ in the expansion for the critical exponents up to $O(\epsilon^2)$, we then obtain

$$\beta=0.39,\quad \nu=0.70,\quad \gamma=1.39.$$

These are the same values as for the isotropic $n=4$ model.

The critical behavior of TbD_2, Nd, and K_2IrCl_6 are all described[1] by $\mathcal{H}_1$ with $2m=6$. Since we have found that there exists a unique stable fixed point, our prediction is that the phase transition in these different systems should have the same critical exponents. It is of interest to test this experimentally. Inserting $\epsilon=1$ into the expansions for the exponents calculated to $O(\epsilon^2)$ we obtain

$$\beta=0.38,\quad \nu=0.69,\quad \gamma=1.38.$$

Let us note that these values are significantly different from those obtained by setting $\epsilon=1$ in the expansion to $O(\epsilon^2)$ for the exponents of the isotropic $n=6$ model,

$$\beta=0.41,\quad \nu=0.73,\quad \gamma=1.45.$$

This indicates that anisotropy is affecting the critical exponents in a non-negligible manner. The exponents for $\mathcal{H}_1$ with $2m=4$ and $2m=6$ are very close to those for the isotropic $n=3$ Heisenberg model,

$$\beta=0.38,\quad \nu=0.68,\quad \gamma=1.35.$$

We should emphasize there was no more reason, *a priori*, to have expected the exponents to have $n=3$ Heisenberg values than $n=1$ Ising values.

Some of the physical systems which correspond to the Hamiltonian $\mathcal{H}_1$ may lie outside the domain of attraction of the stable fixed point, in which case the transition is not expected to be second-order. However, if the system exhibits a second-order transition, the critical exponents are predicted to be those given above.

III. TYPE-I ANTIFERROMAGNETS, $\vec{m} \perp \vec{k}$

Let us now consider the six-component Hamiltonian which is relevant[1] for discussion of the phase transition in UO_2:

$$\mathcal{H}_4 = -\frac{1}{2}\sum_{i=1}^{3}[r(\phi_i^2+\bar{\phi}_i^2)+(\nabla\phi_i)^2+(\nabla\bar{\phi}_i)^2] - u_1\sum_{i=1}^{3}(\phi_i^4+\bar{\phi}_i^4) - u_2\sum_{i=1}^{3}\phi_i^2\bar{\phi}_i^2 - u_3\sum_{i<j}(\phi_i^2\phi_j^2+\bar{\phi}_i^2\bar{\phi}_j^2)$$
$$-u_4(\bar{\phi}_1^2\phi_2^2+\bar{\phi}_2^2\phi_3^2+\bar{\phi}_3^2\phi_1^2) - u_5(\phi_1^2\bar{\phi}_2^2+\phi_2^2\bar{\phi}_3^2+\phi_3^2\bar{\phi}_1^2).$$

We will show that to first order in ϵ, the recursion relations derived for this Hamiltonian possess no stable fixed point. The recursion relations for the u_i' are

$$u_1' = b^\epsilon[u_1 - \tfrac{1}{2}(72u_1^2+2u_2^2+4u_3^2+2u_4^2+2u_5^2)K_4\ln b],$$
$$u_2' = b^\epsilon[u_2 - \tfrac{1}{2}(16u_2^2+48u_1u_2+8u_3u_4+8u_3u_5)K_4\ln b],$$
$$u_3' = b^\epsilon[u_3 - \tfrac{1}{2}(20u_3^2+48u_1u_3+4u_2u_4+4u_2u_5+4u_4u_5)K_4\ln b],$$
$$u_4' = b^\epsilon[u_4 - \tfrac{1}{2}(16u_4^2+48u_1u_4+8u_2u_3+8u_3u_5)K_4\ln b],$$
$$u_5' = b^\epsilon[u_5 - \tfrac{1}{2}(16u_5^2+48u_1u_5+8u_2u_3+8u_3u_4)K_4\ln b].$$

If in the Hamiltonian $\mathcal{H}_4$, we make the interchange $\phi_1 \to \bar{\phi}_2$, $\phi_2 \to \bar{\phi}_3$, $\phi_3 \to \bar{\phi}_1$, then $u_2 \to u_4$. Similarly, under the interchange $\phi_1 \to \bar{\phi}_3$, $\phi_2 \to \bar{\phi}_1$, $\phi_3 \to \bar{\phi}_2$, we see that $u_2 \to u_5$. Finally, if we let $\phi_i \to \bar{\phi}_i$, then $u_4 \to u_5$. This is reflected in the invariance of the recursion relations under any permutation of u_2, u_4, and u_5. As a consequence, if there exists a fixed point with $u_2 \neq u_4$, then there

also exists another fixed point with u_2 and u_4 interchanged. We list below all fixed points not related by a permutation of u_2, u_4, u_5.

We again define $x_i = (K_4/\epsilon)u_i$. There exist eight fixed points with $x_3 = x_4 = x_5$ which correspond to those of $\mathcal{H}_1$ with $2m = 6$. In addition, there exist four new fixed points with $x_2 = x_4 = x_5$. These are

$$x_1 = \tfrac{1}{44}, \quad x_2 = 0, \quad x_3 = \tfrac{1}{22};$$
$$x_1 = \tfrac{1}{54}, \quad x_2 = 0, \quad x_3 = \tfrac{1}{18};$$
$$x_1 = \tfrac{3}{136}, \quad x_2 = \tfrac{1}{68}, \quad x_3 = \tfrac{3}{68};$$
$$x_1 = \tfrac{7}{360}, \quad x_2 = \tfrac{1}{60}, \quad x_3 = \tfrac{1}{20}.$$

All the fixed points can be seen to be unstable by noting

$$u_5' - u_4' \cong b^{\lambda\epsilon}(u_5 - u_4),$$

where

$$\lambda = 1 + 4x_3^* - 24x_1^* - 8x_4^* - 8x_5^*,$$

and that $\lambda > 0$ for all of the fixed points.

The phase transition in UO_2 is known to be first order.[8] This is consistent with the interpretation that the nonexistence of a stable fixed point means there is no critical point. We raise the question as to whether the lack of a stable fixed point can be used as a "rule of thumb" to predict that a phase transition should be discontinuous.

IV. TYPE-II ANTIFERROMAGNETS, $\vec{m} \perp \vec{k}$

The phase transitions in TbAs, TbP, and TbSb are described[1] by the four-component Hamiltonian[9]

$$\mathcal{H}_3 = -\frac{1}{2}\sum_{i=1}^{4}[r\phi_i^2 + (\nabla\phi_i)^2] - u\left(\sum_{i=1}^{4}\phi_i^2\right)^2 - v\sum_{i=1}^{4}\phi_i^4 - w\phi_1\phi_2\phi_3\phi_4.$$

The recursion relations are

$$u' = b^{6-2\eta}[u - \tfrac{1}{2}(96u^2 + 48uv + \tfrac{1}{2}w^2)K_d \ln b + \tfrac{1}{12}(16\,128u^3 + 13\,824u^2v + 3456uv^2 + 180uw^2 + 36vw^2)K_4^2 \ln b],$$
$$v' = b^{6-2\eta}[v - \tfrac{1}{2}(72v^2 + 96uv - \tfrac{1}{2}w^2)K_d \ln b + \tfrac{1}{12}(10\,368v^3 + 27\,648uv^2 + 20\,736u^2v - 144uw^2 - 36vw^2)K_4^2 \ln b],$$
$$w' = b^{6-2\eta}[w - \tfrac{1}{2}(96uw)K_d \ln b + \tfrac{1}{12}(20\,736u^2w + 6912uvw + 36w^3)K_4^2 \ln b],$$

where

$$\eta = \tfrac{1}{4}K_4^2(192u^2 + 96v^2 + 192uv + w^2).$$

To first order in ϵ, we find the four unstable fixed points

$$u = 0, \quad v = 0, \quad w = 0;$$
$$u = 0, \quad v = \epsilon/36K_4, \quad w = 0;$$
$$u = \epsilon/48K_4, \quad v = -\epsilon/72K_4, \quad w = \pm\epsilon/6K_4.$$

in addition to the isotropic fixed point,

$$u = \epsilon/48K_4, \quad v = 0, \quad w = 0.$$

Linearizing about the isotropic fixed point, we obtain two zero eigenvalues, so it is necessary to go to second order in ϵ to determine stability. The isotropic fixed point is degenerate and will be shown to split into four different fixed points, all unstable, when the calculation is carried to second order.

To proceed, we write

$$u = \epsilon/48K_4 + u_2\epsilon^2,$$
$$v = v_2\epsilon^2,$$
$$w = w_2\epsilon^2,$$

and we determine u_2, v_2, and w_2 from the fixed point equations. From terms of $O(\epsilon^3)$ in the equation $u' = u$, we obtain

$$0 = -v_2 - 2u_2 + 13/(24)(24)K_4 - (K_d - K_4)/24K_4^2.$$

Naively, one would expect to obtain two additional constraints by considering terms of $O(\epsilon^3)$ in the equations $v' = v$ and $w' = w$. However, no new constraints are obtained in this manner. Let us now note that any diagram contributing to the recursion relation for v' or w' possesses at least one v or w vertex. Since $v = O(\epsilon^2)$ and $w = O(\epsilon^2)$, a diagram with four vertices contributing to v' or w' must be at least of $O(\epsilon^5)$. It follows that although we have neglected diagrams with more than three vertices in deriving the recursion relations, we are justified in using the $O(\epsilon^4)$ terms in the equations $v' = v$ and $w' = w$ to derive constraints. One can check that the $O(\epsilon^3)$ contributions to u, v, w cancel out. From $v' = v$ and $w' = w$, we obtain

$$0 = \frac{17}{24K_4}v_2 - \frac{K_d - K_4}{K_4^2}v_2 - 36v_2^2 - 48u_2v_2 + \tfrac{1}{4}w_2^2,$$
$$0 = \frac{17}{24K_4}w_2 - 48u_2w_2 - \frac{K_d - K_4}{K_4^2}w_2.$$

There exist four solutions to these equations:

$$u^* = \frac{\epsilon}{48K_d} + \frac{13\epsilon^2}{(24)(48)K_4}, \quad v^* = 0, \quad w^* = 0,$$

with eigenvalues $\lambda_u = -\epsilon + \frac{13}{24}\epsilon^2$, $\lambda_v = \frac{1}{6}\epsilon^2$, $\lambda_w = \frac{1}{6}\epsilon^2$;

$$u^* = \frac{\epsilon}{48K_d} + \frac{5\epsilon^2}{(24)(48)K_4},\ v^* = \frac{\epsilon^2}{72K_4},\ w^* = 0,$$

with eigenvalues $\lambda_1 = -\epsilon + \frac{13}{24}\epsilon^2, \lambda_2 = -\frac{1}{6}\epsilon^2, \lambda_3 = \frac{1}{2}\epsilon^2$;

$$u^* = \frac{\epsilon}{48K_d} + \frac{17\epsilon^2}{(24)(48)K_4},\ v^* = \frac{-\epsilon^2}{144K_4},\ w^* = \frac{\pm\epsilon^2}{12K_4},$$

with eigenvalues $\lambda_1 = -\epsilon + \frac{13}{24}\epsilon^2, \lambda_2 = -\frac{1}{6}\epsilon^2, \lambda_3 = \frac{1}{2}\epsilon^2$. Therefore, we see that all of the fixed points are unstable.

Careful measurements close to the transition temperature of TbAs, TbP, and TbSb have not yet been made. Previous work[10] was done well below the transition temperature in order to determine the nature of the ordered state. We suggest that these substances be studied near their transition temperature in order to test the "rule of thumb" that absence of a stable fixed point means the transition is discontinuous.

V. TYPE-II ANTIFERROMAGNETS, $\vec{m} \perp \vec{k}$

Finally, we consider the eight-component Hamiltonian appropriate[1] to the study of the transitions in MnO, MnSe, NiO, and ErSb.

$$\begin{aligned}
\mathcal{H}_2 = &-\frac{1}{2}\sum_{i=1}^{4}[r(\phi_i^2+\overline{\phi}_i^2)+(\nabla\phi_i)^2+(\nabla\overline{\phi}_i)^2] - u_1\sum_{i=1}^{4}(\phi_i^2+\overline{\phi}_i^2)^2 - u_2\sum_{i<j}(\phi_i^2\phi_j^2+\overline{\phi}_i^2\overline{\phi}_j^2+\phi_i^2\overline{\phi}_j^2+\overline{\phi}_i^2\phi_j^2)\\
&- u_3(\phi_1\phi_2\overline{\phi}_3\overline{\phi}_4+\overline{\phi}_1\overline{\phi}_2\phi_3\phi_4+\phi_1\overline{\phi}_2\overline{\phi}_3\phi_4+\overline{\phi}_1\phi_2\phi_3\overline{\phi}_4-\phi_1\overline{\phi}_2\phi_3\overline{\phi}_4-\overline{\phi}_1\phi_2\overline{\phi}_3\phi_4-3\phi_1\phi_2\phi_3\phi_4-3\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3\overline{\phi}_4)\\
&- u_4\Big(\sum_{i<j}(\phi_i^2\phi_j^2+\overline{\phi}_i^2\overline{\phi}_j^2)-2(\phi_1\overline{\phi}_1\phi_2\overline{\phi}_2+\phi_2\overline{\phi}_2\phi_3\overline{\phi}_3+\phi_1\overline{\phi}_1\phi_4\overline{\phi}_4+\phi_3\overline{\phi}_3\phi_4\overline{\phi}_4-\phi_1\overline{\phi}_1\phi_3\overline{\phi}_3-\phi_2\overline{\phi}_2\phi_4\overline{\phi}_4)\Big)\\
&- u_5[2(\phi_1^2\phi_3^2-\overline{\phi}_1^2\overline{\phi}_3^2)+2(\phi_2^2\phi_4^2-\overline{\phi}_2^2\overline{\phi}_4^2)-(\phi_1^2\phi_2^2-\overline{\phi}_1^2\overline{\phi}_2^2)-(\phi_1^2\phi_4^2-\overline{\phi}_1^2\overline{\phi}_4^2)-(\phi_2^2\phi_3^2-\overline{\phi}_2^2\overline{\phi}_3^2)-(\phi_3^2\phi_4^2-\overline{\phi}_3^2\overline{\phi}_4^2)\\
&\quad+\sqrt{3}(\phi_1\overline{\phi}_1-\phi_3\overline{\phi}_3)(\phi_4^2+\overline{\phi}_4^2-\phi_2^2-\overline{\phi}_2^2)+\sqrt{3}(\phi_2\overline{\phi}_2-\phi_4\overline{\phi}_4)(\phi_1^2+\overline{\phi}_1^2-\phi_3^2-\overline{\phi}_3^2)]\\
&- u_6[4(\phi_1^2\phi_2^2+\phi_2^2\phi_3^2+\phi_3^2\phi_4^2+\phi_4^2\phi_1^2+\overline{\phi}_1^2\overline{\phi}_2^2+\overline{\phi}_2^2\overline{\phi}_3^2+\overline{\phi}_3^2\overline{\phi}_4^2+\overline{\phi}_4^2\overline{\phi}_1^2)\\
&\quad+(\phi_1^2\phi_3^2+\phi_2^2\phi_4^2+\overline{\phi}_1^2\overline{\phi}_3^2+\overline{\phi}_2^2\overline{\phi}_4^2)+12(\phi_1\overline{\phi}_1\phi_3\overline{\phi}_3+\phi_2\overline{\phi}_2\phi_4\overline{\phi}_4)\\
&\quad+3(\phi_1^2\overline{\phi}_3^2+\overline{\phi}_1^2\phi_3^2+\phi_2^2\overline{\phi}_4^2+\overline{\phi}_2^2\phi_4^2)+2\sqrt{3}(\phi_1\overline{\phi}_1-\phi_3\overline{\phi}_3)(\phi_4^2-\overline{\phi}_4^2-\phi_2^2+\overline{\phi}_2^2)\\
&\quad+2\sqrt{3}(\phi_2\overline{\phi}_2-\phi_4\overline{\phi}_4)(\phi_1^2-\overline{\phi}_1^2-\phi_3^2+\overline{\phi}_3^2)].
\end{aligned}$$

The recursion relations for the u_i' are

$$\begin{aligned}
u_1' &= b^\epsilon[u_1 - \tfrac{1}{2}(80u_1^2+12u_2^2+6u_4^2+18u_5^2+108u_6^2+12u_2u_4+48u_2u_6+36u_4u_6)K_4\ln b],\\
u_2' &= b^\epsilon[u_2 - \tfrac{1}{2}(32u_2^2+2u_3^2+4u_4^2+24u_5^2+192u_6^2+64u_1u_2+16u_1u_4+64u_1u_6+16u_2u_4+64u_2u_6+24u_4u_6)K_4\ln b],\\
u_3' &= b^\epsilon[u_3 - \tfrac{1}{2}(48u_2u_3+56u_3u_4+160u_3u_6)K_4\ln b],\\
u_4' &= b^\epsilon[u_4 - \tfrac{1}{2}(8u_3^2+24u_4^2+48u_5^2+32u_1u_4+32u_2u_4+112u_4u_6)K_4\ln b],\\
u_5' &= b^\epsilon[u_5 - \tfrac{1}{2}(48u_1u_5+24u_2u_5+24u_4u_5-24u_5u_6)K_4\ln b],\\
u_6' &= b^\epsilon[u_6 - \tfrac{1}{2}(-12u_5^2+48u_6^2+32u_1u_6+32u_2u_6+8u_4u_6)K_4\ln b].
\end{aligned}$$

Defining $x_i = (K_4/\epsilon)u_i$, the fixed points are

$$\begin{aligned}
&x_1 = \cdots = x_6 = 0,\\
&x_1 = \tfrac{1}{40},\ x_2 = \cdots = x_6 = 0,\\
&x_1 = \tfrac{1}{64},\ x_2 = \tfrac{1}{32},\ x_3 = \cdots = x_6 = 0,\\
&x_1 = \tfrac{3}{128},\ x_2 = \tfrac{1}{64},\ x_3 = \cdots = x_6 = 0,\\
&x_1 = \tfrac{1}{160},\ x_2 = \tfrac{1}{80},\ x_3 = \pm\tfrac{1}{20},\ x_4 = \tfrac{1}{40},\ x_5 = x_6 = 0,\\
&x_1 = \tfrac{3}{256},\ x_2 = \tfrac{3}{128},\ x_3 = \pm\tfrac{1}{32},\ x_4 = \tfrac{1}{64},\ x_5 = x_6 = 0.
\end{aligned}$$

All of the fixed points are unstable, since

$$u_5' \cong b^{\lambda\epsilon}u_5,$$

with

$$\lambda = 1 - 24x_1^* - 12x_2^* - 12x_4^*,$$

and for all the fixed points $\lambda > 0$.

The transition in MnO has recently been shown[11] to be first order. The nature of the transitions in MnSe and NiO is less well established.[12] Measurement[13] of the order parameter of ErSb indicates the transition is second order (continuous), but specific-heat measurements[14] indicate there is a latent heat. We believe it is of interest to pursue the study of the nature of the transition in ErSb.

VI. SUMMARY AND CONCLUSIONS

The relation between a system's symmetry properties and its phase transitions has been long studied using Landau's[2] phenomenological theory. Recently, Wilson[5] showed how to extend the Landau theory to include effects of critical fluctuations by using an expansion in $\epsilon = 4 - d$. Within the framework of the ϵ expansion, Brézin *et al.*[6] showed that for n-component systems with $n \leq 3$, the critical exponents are independent of anisotropy, while for $n \geq 4$ the exponents do depend on the anisotropy. In this respect, universality is predicted to be weaker for $n \geq 4$ systems. With this in mind, we have proposed[1,3] that it is of great interest to study experimentally the critical behavior of systems with $n \geq 4$. It was noted by Mukamel[3] that certain phase transitions involving an increase in the unit cell are described by n-component Landau-Ginzburg-Wilson Hamiltonians with $n \geq 4$. In the previous paper[1] we used the group-theoretical methods of Landau and Lifshitz[2] to construct effective Hamiltonians appropriate for the discussion of phase transitions in specific physical systems. In this paper, we used the ϵ expansion to study the critical phenomena associated with these Hamiltonians.

We have noted that the phase transitions in $TbAu_2$, DyC_2, and NbO_2 are all described by four-component order parameters. Each of these order parameters corresponds to the breaking of different symmetries; in fact, the transition in NbO_2 is structural, while those in $TbAu_2$ and DyC_2 are antiferromagnetic. We have shown that the transitions in each of these materials are described by the same effective Hamiltonian. Using the ϵ expansion, we have found that there exists a unique stable fixed point, so we predict that the critical exponents for these different transitions should be the same. Similarly, we have found that the transitions in TbD_2, Nd, and K_2IrCl_6 are described by six-component order parameters. Again, each order parameter corresponds to the breaking of a different symmetry, but all the systems are described by the same effective Hamiltonian. Within the ϵ expansion we have found a unique stable fixed point corresponding to this Hamiltonian, so we predict these different systems should have the same critical exponents. We hope that experimental work will be done to determine the critical exponents of these four- and six-component systems.

The type-I antiferromagnet ($\vec{m} \perp \vec{k}$) UO_2 has been shown to be described by a six-component order parameter. The appropriate effective Hamiltonian has no stable fixed points within the ϵ expansion. This is consistent with the known result that UO_2 has a first-order (discontinuous) transition.[8] The type-II antiferromagnets ($\vec{m} \parallel \vec{k}$) TbAs, TbP, and TbSb are described by four-component order parameters. The corresponding effective Hamiltonian has no stable fixed points within the ϵ expansion. Previous measurements[10] on these systems have been performed far below the transition temperature with the objective of determining the nature of the ordered state. We suggest that it is worthwhile to go back and study these substances very near the transition temperature in order to determine whether or not the transitions are first order. The type-II antiferromagnets ($\vec{m} \perp \vec{k}$) MnO, MnSe, NiO, and ErSb are described by eight-component order parameters. The corresponding effective Hamiltonian has been found to have no stable fixed point within the ϵ expansion. It has recently been shown that MnO has a first-order transition.[11] The nature of the transitions[12] in MnSe and NiO are not yet well established. Measurements[13] of the order parameter for ErSb indicate the transition is second-order (continuous), but the specific-heat measurements[14] indicate there may exist a latent heat. We believe it is of interest to determine whether or not the transitions in MnSe, NiO, and ErSb are first order.

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