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Critical dynamics of isotropic antiferromagnets using renormalization-group methods: $T \geq T_N$

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We study a model equation of motion describing the dynamics of isotropic antiferromagnets near the critical point. We perform a renormalization-group analysis of this equation of motion correct to $O(\epsilon)$, where $\epsilon = 4 - d$, finding recursion relations and analyzing their fixed points and their stability. For the one stable fixed point we obtain the dynamical index $z = d/2 = 2 - \epsilon/2$. We also calculate the response and correlation functions in the scaling region using perturbation theory correct to $O(\epsilon)$. Our calculation gives analytical forms for the correlation functions for the staggered magnetization and the magnetization. These correlation functions can be written in the dynamical scaling form. The shape function that characterizes the frequency spectrum for the staggered magnetization is Lorentzian in the hydrodynamical regime, but shows fluctuation-induced peaks at and near the Néel temperature. The shape function for the magnetization is essentially Lorentzian for all $x = q\xi$ (where q is the wave number and ξ is the correlation length) and shows a narrow hydrodynamical width as $x \to 0$.

I. INTRODUCTION

A. General comments

We now believe that the generalization of the Landau-Ginzburg-Wilson[1,2] effective-free-energy treatment for static critical phenomena to dynamics corresponds to studying a class of model equations of motion obeyed by the order parameter and the other slowly varying variables of a system near its critical point. These model equations are determined by specifying the Landau-Ginzburg free energy appropriate to describe the statics, choosing a set of bare Onsager or transport coefficients L_i (which requires specifying the conservation laws for the system) and determining certain Poisson-bracket relations among the slowly varying variables.

These equations of motion can be derived from the microscopic equations of motion using ideas similar to those used[3] to justify the use of an effective-free-energy approach to static critical phenomena. A key point is that the L_i are the local and Markoffian limits $(k, \omega \to 0)$ of certain memory functions. The calculation of the bare L_i, which are insensitive to critical behavior, is a formidable task if one begins with a fully microscopic theory.[4] It involves solving the Bethe-Salpeter[5] equation or equivalently a Boltzmann-like equation while being careful to treat the conservation laws properly.[6] We can avoid this microscopic analysis if we use our knowledge from generalized hydrodynamics[7] of which memory functions, on the time and distance scale of interest for critical phenomena (small frequency and wave number) can be replaced by simple constants.[8] These constants are identified with the bare Onsager coefficients and are taken as input in the model. The resultant equations of motion are a considerable simplification on the original microscopic equations of motion since they are directly amenable to renormalization-group (RNG) treatment.

The universality classification[9] for static-critical-phenomena properties of a system is specified by choosing the appropriate free energy F. F specifies the dimensionality d, the vector nature of the order parameter, and the existence of any long-range forces. The universality classification for dynamic critical phenomena depends not only on choosing the effective free energy F, but also on the conservation laws and the Poisson-bracket relations.

The pioneering work in developing these models was carried out by Kawasaki[10-12] building on the earlier work in generalized hydrodynamics by Green,[13] Zwanzig,[14] and Mori[15] and the mode-mode coupling ideas of Fixman.[16] Kawasaki studied nonlinear models arising from the Poisson-bracket or streaming-velocity terms. He[17] and others[18] showed that these "mode" coupling terms can lead to divergent transport coefficients near the critical point. The models studied by Kawasaki ignored nonlinear couplings through the effective free energy. These models were, therefore, not stable below the transition and would not generate the correct static correlation functions beyond the mean-field result. The generalization of these models to include the proper statics with a streaming velocity was first carried out by Halperin, Hohenberg, and Siggia[19] (HHS) for the cases of a binary mixture and a planar ferromagnet. Ma and Mazenko[20] (MM) pointed out that the HHS models were a subset of a large class of models[21] and pointed out the importance of the Poisson-bracket relations in determining the streaming velocity which strongly influences the universality class with respect to dynamical properties. These authors also discussed a number of general properties of these models some of which we summarize in Sec. II.

All of these models result in a nonlinear field theory which requires, for systematic treatment, the use of RNG techniques.

The first application of RNG methods to these model equations of motion was carried out by Halperin, Hohenberg, and Ma.[22] These authors studied the dynamics of systems described by the time-dependent Ginzburg-Landau (TDGL) models.[23] These models ignore streaming velocities. Recently De Dominicis[24] has shown that these models are equivalent to a class of simple Lagrangian models. Subsequently Brézin and De Dominicis[25] have used the Callen-Symanzik and Weinberg formulations of the RNG to study these models to $O(\epsilon^3)$. Thus far there has been only limited work on models including streaming velocities. HHS[17,26] have studied models for binary mixtures, superfluid helium, planar ferromagnets, and isotropic antiferromagnets. For the antiferromagnet these authors find that the dynamical exponent z for the order parameter (the staggered magnetization) is given by $z=2-\frac{1}{2}\epsilon$ ($\epsilon=4-d$) in agreement with our work (see Sec. III). MM[20,27] have reported on a detailed study of the critical dynamics of isotropic ferromagnets in $6-\epsilon$ dimensions. This work is, to our knowledge, the only RNG work on critical dynamics that considers $T<T_c$. More recently Gunton and Kawasaki[28] have studied the critical dynamics of ferromagnets, planar ferromagnets, binary mixtures, helium, and the tricritical behavior of ^{3}He and ^{4}He mixtures using RNG methods.

In this paper we use RNG methods to study model equations of motion for the isotropic antiferromagnet. Our work will be similar in spirit to that of MM and we shall, to a large extent, follow their methods. A number of our results have been reported in a recent letter.[29]

B. Isotropic antiferromagnet

A classical isotropic Heisenberg antiferromagnet can be described by the Hamiltonian[30]

$$H=-\frac{1}{2}\sum_{\alpha\neq\beta}J_{\alpha\beta}\vec{S}_\alpha\cdot\vec{S}_\beta\ , \qquad (1.1)$$

where $J_{\alpha\beta}$ is an antiferromagnetic exchange constant and $\vec{S}_\alpha$ is the spin value on the lattice site α. Let us, for simplicity, consider spins arrayed on a simple-cubic lattice. If we consider only nearest-neighbor interactions then $J_{\alpha\beta}=J$ and, for antiferromagnets, $J<0$. Each spin, therefore, will tend to align itself antiparallel to its six nearest neighbors. One then has the usual picture of two interpenetrating sublattices of spins. On each sublattice the spins are ferromagnetically aligned, but the two sublattice magnetizations are oppositely aligned. The order parameter for the antiferromagnet is the staggered magnetization density defined by

$$\vec{N}(x,t)=\sum_\alpha \eta_\alpha\vec{S}_\alpha(t)\delta(x-x_\alpha)\ , \qquad (1.2)$$

where the sum is over all sites in the magnet and $\eta_\alpha=+1$ for spins on the "up" sublattice and $\eta_\alpha=-1$ for spins on the "down" sublattice. For $T<T_N$ (T_N is the Néel temperature) the spins are antiferromagnetically aligned so that $\langle\vec{N}(x,t)\rangle\neq 0$. We can also define the magnetization density

$$\vec{M}(x,t)=\sum_\alpha \vec{S}_\alpha(t)\delta(x-x_\alpha)\ , \qquad (1.3)$$

where $\langle\vec{M}(x,t)\rangle=0$ for all temperatures. Using the standard commutation relations for spins

$$[S_\alpha^i,S_\beta^j]=i\hbar\sum_k\epsilon_{ijk}\delta_{\alpha,\beta}S_\alpha^k\ , \qquad (1.4)$$

it is easy to show, using (1.1) that

$$[H,\vec{N}]=-\frac{1}{2}\sum_{\alpha\neq\beta}J_{\alpha\beta}(\eta_\alpha-\eta_\beta)(\vec{S}_\alpha\times\vec{S}_\beta)\neq 0\ , \qquad (1.5)$$

where $\vec{N}$ is the total staggered magnetization [i.e., the space integral of (1.2)], so that the order parameter is not conserved in antiferromagnets. However, since $[H,\vec{M}]=0$, the magnetization is conserved. As we shall see in Sec. II the above conservation properties play an important role in determining the form of the equation of motion and therefore strongly affect the dynamical properties of antiferromagnets.

C. Outline of paper

In Sec. II we discuss the generalized Langevin equations of motion of the type studied by MM. After a brief review of some of the important properties of this equation we specialize our discussion to isotropic antiferromagnets. This leads to the coupled equations of motion describing the dynamics for the staggered magnetization density and the magnetization density. In Sec. III we perform a RNG analysis of these equations of motion correct to $O(\epsilon)$, where $\epsilon=4-d$. We derive recursion relations and their fixed point solutions and their stability are studied. In Sec. IV we develop, to $O(\epsilon)$, perturbation theory for the linear response functions in the scaling region. We then use the classical fluctuation dissipation theorem to obtain $C_N(q,\omega)$, the wave number, and frequency-dependent correlation function for the staggered magnetization, from the linear response function. Our calculation gives an analytical form for $C_N(q,\omega)$ upon which we perform certain exponentiations of logarithmic terms generated by the ϵ expansion. The resulting correlation function has the form predicted by dynamical scaling[31]

$$C_N(q,\omega)=[\chi_N(q)/\omega(q,\xi)]f_x^N(\nu) , \tag{1.6}$$

where $\chi_N(q)$ is the static susceptibility, $\omega(q,\xi)$ is the characteristic frequency, $f_x^N(\nu)$ is the shape function, with $x=q\xi$ (ξ is the correlation length), and $\nu=\omega/\omega(q,\xi)$. We plot the shape function as a function of ν for various values of x. We also carry out a similar analysis for $C_M(q,\omega)$, the correlation function for the magnetization. Finally, in Sec. V. we discuss our results.

II. EQUATIONS OF MOTION

A. Properties of general model

Dynamic critical phenomena are characterized by the slow variation in time of the order parameter and the conserved variables of the system. Thus we are interested in the equation of motion for these variables (which are in general N-component vectors) whose densities we denote by $\Psi_{\alpha i}(x,t)$, where α denotes a particular variable (i.e., energy, magnetization, etc.) and the index i denotes a particular component of this variable. Following MM we assume that the slow variables satisfy a generalized Langevin equation of the form

$$\frac{\partial\Psi_{\alpha i}(x,t)}{\partial t}=V_\alpha^i[x,[\Psi_\alpha]]-\hat{\Gamma}_\alpha\frac{\delta F[\Psi_\alpha]}{\delta\Psi_{\alpha i}(x)}+\eta_\alpha^i(x,t) . \tag{2.1}$$

If $\vec{\Psi}_\alpha$ is not conserved, then $\hat{\Gamma}_\alpha=\Gamma_\alpha$ and Γ_α is assumed to be a simple constant near $T=T_N$, and is called a kinetic coefficient. If $\vec{\Psi}_\alpha$ is a conserved variable, then $\hat{\Gamma}_\alpha=-\Gamma_\alpha\nabla^2$, and Γ_α is a transport coefficient. $\eta_\alpha^i(x,t)$ is a random gaussianly distributed noise source satisfying

$$\langle\eta_\alpha^i(x,t)\rangle=0 \tag{2.2a}$$

and

$$\langle\eta_\alpha^i(x,t)\eta_\beta^j(x't')\rangle=2\delta_{ij}\delta_{\alpha\beta}\hat{\Gamma}_\alpha\delta(x-x')\delta(t-t') . \tag{2.2b}$$

$F[\Psi_\alpha]$ is an appropriately chosen Ginzburg-Landau free-energy functional, which determines the equilibrium properties of the system. $V_\alpha^i[\Psi_\alpha]$ is a streaming velocity in the space of the $\vec{\Psi}_\alpha$. If one neglects this term, then (2.1) reduces to the TDGL equation previously studied by Halperin, Hohenberg, and Ma. The mode-mode coupling effects which give rise to divergent transport coefficients are contained in these streaming velocities. The general form of these streaming velocities, which follow directly from the work of Mori and collaborators,[15] is given by

$$V_\alpha^i(x)=\sum_{x'\beta j}\left(\frac{\delta}{\delta\Psi_{\beta j}(x')}Q_{\alpha\beta}^{ij}(xx')-Q_{\alpha\beta}^{ij}(xx')\frac{\delta F}{\delta\Psi_{\beta j}(x')}\right) . \tag{2.3}$$

The $Q_{\alpha\beta}^{ij}(xx')$ are given by

$$Q_{\alpha\beta}^{ij}(xx')=\lambda_{\alpha\beta}(1/i\hbar)[\hat{\Psi}_{\alpha i}(x),\hat{\Psi}_{\beta j}(x')]_{\hat{\Psi}\to\Psi} , \tag{2.4}$$

where, after evaluating the commutator in (2.4) we are to replace the quantum-mechanical fields by their corresponding classical fields ($\hat{\Psi}\to\Psi$). The $\lambda_{\alpha\beta}$ are coupling constants taken as parameters in the model. It is easy to show that the streaming velocity $V_\alpha^i(x)$ given in (2.3) will satisfy the divergence condition

$$\sum_{x\alpha i}\frac{\delta}{\delta\Psi_{\alpha i}(x)}[V_\alpha^i(x)e^{-F}]=0 \tag{2.5}$$

provided that the $Q_{\alpha\beta}^{ij}(x\bar{x})$ are completely antisymmetric, i.e., $Q_{\alpha\beta}^{ij}(x\bar{x})=-Q_{\beta\alpha}^{ji}(\bar{x}x)$. From (2.4) we see that this only requires that the coupling constants be symmetric: $\lambda_{\alpha\beta}=\lambda_{\beta\alpha}$. MM have shown that the static correlation functions generated by equations of motion of the type (2.1) are the same as those given by the static distribution function e^{-F}. This insures that the equilibrium properties calculated from (2.1) will be correct.

B. Specialization to antiferromagnet

In order to specialize the above results to a study of an isotropic antiferromagnet we must first specify the variables $\Psi_{\alpha i}(x)$ in (2.1). We must, of course, include in our set of dynamical variables the order parameter, the staggered magnetization density $\vec{N}(x,t)$. When we look at the commutator of $N_i(x)$ with $N_j(x')$ we see that the resultant is in terms of the magnetization density $\vec{M}(x,t)$. Thus $\vec{N}$ and $\vec{M}$ are dynamically coupled through the streaming velocity and we must include both in our set $\Psi_{\alpha i}$. A more general model could include a coupling to the energy density.[22,26,32]

The statics of this coupled system can be described by the Ginzburg-Landau free-energy functional

$$F[\vec{N},\vec{M}]=\tfrac{1}{2}\int d^dx\,[r_0\vec{N}^2+(\nabla\vec{N})^2+\tfrac{1}{2}u(\vec{N}^2)^2+r\vec{M}^2] , \tag{2.6}$$

where $r_0 = T - T_N^0$, r, u are positive constants, and, in momentum space, all wave numbers are restricted to be less than a cutoff Λ. Since the magnetization is not critical at T_N, we do not need an M^4 term for stability. We have not considered any coupling between $\vec{M}$ and $\vec{N}$ in $F[\vec{N}, \vec{M}]$ since Allessandrini *et al.*,[33] who studied the static critical properties of antiferromagnets, showed that the coupling between $\vec{N}$ and $\vec{M}$ is irrelevent near the critical point.

Since the staggered magnetization is not conserved, we introduce the bare kinetic coefficient $\hat{\Gamma}_N \equiv \Gamma_N$, and since the magnetization is conserved we introduce the bare transport coefficient Γ_M, via $\hat{\Gamma}_M = -\Gamma_M \nabla^2$.

If we use the standard spin commutation relations (1.4) together with Eqs. (1.2), (1.3), and (2.4) we find

$$Q_{\alpha\beta}^{ij}(xx') = \lambda_{\alpha\beta}\,\delta(x-x') \sum_{k,\gamma} \epsilon_{ijk}\sigma_{\alpha\beta\gamma}\Psi_{\gamma k}(x) , \tag{2.7}$$

where the Greek indices (α, β, γ) can assume the values M or N, i.e., $\Psi_{Ni}(x) \equiv N^i(x)$ and $\Psi_{Mi}(x) \equiv M^i(x)$. In arriving at (2.7) we have found it convenient to introduce $\sigma_{\alpha\beta\gamma}$ defined by[34]

$$\sigma_{\alpha\beta\gamma} = \delta_{\alpha,N}\delta_{\beta,-\gamma} + \delta_{\alpha,M}\delta_{\beta,\gamma} . \tag{2.7a}$$

If we substitute Eq. (2.7) into (2.3) we find, after some simple algebra,

$$\vec{V}_M(x) = \lambda_{MM}\vec{M}\times\vec{H}_M + \lambda_{MN}\vec{N}\times\vec{H}_N \tag{2.8a}$$

and

$$\vec{V}_N = \lambda_{MN}\vec{N}\times\vec{H}_M + \lambda_{NN}\vec{M}\times\vec{H}_N , \tag{2.8b}$$

where we have defined the effective local field

$$\vec{H}_\alpha(x) = \delta F/\delta\vec{\Psi}_\alpha(x) . \tag{2.9}$$

If we put (2.8) into the equation of motion (2.1) we obtain the following pair of coupled equations

$$\frac{\partial\vec{N}}{\partial t} = \lambda_{NN}\vec{M}\times\vec{H}_N + \lambda_{NM}\vec{N}\times\vec{H}_M - \Gamma_N\vec{H}_N + \vec{\eta}_N , \tag{2.10a}$$

$$\frac{\partial\vec{M}}{\partial t} = \lambda_{MM}\vec{M}\times\vec{H}_M + \lambda_{NM}\vec{N}\times\vec{H}_N + \Gamma_M\nabla^2\vec{H}_M + \vec{\eta}_M . \tag{2.10b}$$

Equations (2.10) are the main results of this section and taken together with (2.5) and (2.9) completely define our dynamical model. These equations reduce to those studied by Kawaskai[10] if we set $u=0$, and $\lambda_{NN} = \lambda_{MM} = \lambda_{NM}$. These equations reduce to those of HHS if $\lambda_{NN} = \lambda_{MM} = 0$ and their[35] $S_0 = 0$. We have performed a detailed renormalization group analysis on (2.10). We have found; in agreement with HHS, λ_{MM}, λ_{NN} are irrelevant. Since their inclusion only complicates the model we drop them henceforth.

C. Symmetrized equation of motion

Before turning to a RNG analysis of these equations it will be useful, for developing graphical techniques, to write (2.10) as a single symmetrized vector equation. First we define the space-time Fourier transform (in d dimensions)

$$\vec{\Psi}_\alpha(q, \omega) = \int d^d x\, dt\, \vec{\Psi}_\alpha(x, t) e^{-i(\vec{q}\cdot\vec{x} - \omega t)} \tag{2.11}$$

Then, after Fourier transforming (2.10) and using (2.6), we can write the equation of motion in the form

$$\Psi_{\alpha i}(q,\omega) = \Psi_{\alpha i}^{(0)}(q,\omega) + G_\alpha^{(0)}(q,\omega)\tilde{V}_{ijk\alpha\beta\gamma}(q,q_1,q_2,\omega,\omega_1,\omega_2)\Psi_{j\beta}(q_1,\omega_1)\Psi_{k\gamma}(q_2,\omega_2)$$
$$+ G_\alpha^{(0)}(q,\omega)\tilde{B}_{ijkl\alpha\beta\gamma\delta}(q,q_1,q_2,q_3,\omega,\omega_1,\omega_2,\omega_3)\Psi_{j\beta}(q_1,\omega_1)\Psi_{k\gamma}(q_2,\omega_2)\Psi_{l\gamma}(q_3,\omega_3) , \tag{2.12}$$

where summations or integrations over all repeated indices are to be understood. In (2.12) we have introduced several symmetrized vertices. The three-point vertex, [of $O(\lambda_{\alpha\beta})$] is given by

$$\tilde{V}_{ijk,\alpha\beta\gamma}(q,q_1,q_2;\omega,\omega_1,\omega_2) = (2\pi)^{d+1}\delta^{(d)}(q-q_1-q_2)\delta(\omega-\omega_1-\omega_2)\tilde{V}_{ijk,\alpha\beta\gamma}(q,q_1) , \tag{2.13}$$

where

$$\tilde{V}_{ijk,\alpha\beta\gamma}(q,q_1) = [\epsilon_{ijk}/2\Gamma_\alpha(q)]W_{\alpha\beta\gamma}(q,q_1) , \tag{2.14}$$

$$\Gamma_\alpha(q) = \delta_{\alpha N}\Gamma_N + \delta_{\alpha M}\Gamma_M q^2 , \tag{2.15a}$$

$$W_{\alpha\beta\gamma}(q,q_1) = \sigma_{\alpha\beta\gamma}[\lambda_{\alpha\gamma}\chi_\gamma^{(0)^{-1}}(q-q_1) - \lambda_{\alpha\beta}\chi_\beta^{(0)^{-1}}(q_1)] , \tag{2.15b}$$

with the zeroth-order static susceptibilities $\chi_\alpha^{(0)}(q)$ are given by

$$\chi_\alpha^{(0)^{-1}}(q) = \delta_{\alpha N}(r_0 + q^2) + \delta_{\alpha M} r , \tag{2.15c}$$

and $\sigma_{\alpha\beta\gamma}$ is defined by (2.7a).

The symmetrized four-point vertex [of $O(u)$] is defined by

$$\tilde{B}_{ijkl\alpha\beta\gamma\delta}(q, q_1, q_2, q_3, \omega, \omega_1, \omega_2, \omega_3) = (2\pi)^{d+1}\delta^{(d)}(q - q_1 - q_2 - q_3)\delta(\omega - \omega_1 - \omega_2 - \omega_3)\tilde{B}_{ijkl\alpha\beta\gamma\delta}(q, q_1, q_2) , \tag{2.16}$$

with

$$\tilde{B}_{ijkl\alpha\beta\gamma\delta}(q, q_1, q_2) = -\tfrac{1}{3}u\delta_{\alpha N}\delta_{\beta N}\delta_{\gamma N}\delta_{\delta N}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl}) . \tag{2.17}$$

In (2.12) we have also defined, for $\alpha = M$, N and $i = x, y, z$,

$$\Psi_{i\alpha}^{(0)}(q, \omega) = G_\alpha^{(0)}(q, \omega)\eta_\alpha^i(q, \omega)/\Gamma_\alpha(q) , \tag{2.18}$$

where $G_\alpha^{(0)}(q, \omega)$ is the zeroth-order (i.e., $\lambda_{\alpha\beta} = 0$, $u = 0$) linear response function,

$$G_\alpha^{(0)-1}(q, \omega) = -i\omega/\Gamma_\alpha(q) + \chi_\alpha^{(0)-1}(q) . \tag{2.19}$$

Using the above definitions it is straightforward to demonstrate that (2.12) correctly reproduces the coupled equations (2.10).

D. Graphs

It will be convenient for the following analysis if we represent our equation of motion (2.14) graphically. We shall use the following conventions. Let us denote $\Psi_{i\alpha}(q, \omega)$ by a wiggly line, $\Psi_{i\alpha}^{(0)}(q,\omega)$ by a broken line, $G_\alpha^{(0)}(q, \omega)$ by a heavy solid line with an arrow. An n-point vertex (here $n = 3, 4$) will be represented by a dot with one heavy arrowed line entering and $n - 1$ lines leaving. Each of the n lines entering and leaving a vertex has associated with it four indices $(i\alpha q\omega)$. We have symmetrized the vertices defined in (2.14) and (2.16) so that the vertices are symmetric under the interchange of all the indices associated with any two of the $n - 1$ lines leaving an n-point vertex. This will enable us to recognize graphs which are equivalent and will reduce the number of graphs we need to consider. The δ functions in (2.13) and (2.16) indicate that energy and momentum are conserved at each vertex. Following the above conventions our equation of motion (2.12) can be represented graphically as shown in Fig. 1.

III. RENORMALIZATION GROUP ANALYSIS

A. RNG transformations

The equation of motion (2.12) is specified by the set of parameters $\mu = (r_0, r, u, \Gamma_M, \Gamma_N, \lambda_{MN})$. Under the RNG transformation[36] $R_b\,\mu = \mu'$ these parameters are transformed to a new set denoted by μ'. The operations implied by R_b consist of two steps. If we write $R_b = R_b^s R_b^i$, then under R_b^i we integrate out of our equation of motion those components of $\Psi_{i\alpha}(q, \omega)$ with wave numbers in the range $\Lambda/b < q < \Lambda$, where Λ is the ultraviolet cutoff in wave-number space and $b > 1$ is a scale change. The second step of the RNG R_b^s is a rescaling of our new equation of motion so that we end up with the original cutoff Λ.

Ideas concerning fixed point structure of the RNG has been extensively discussed in the literature.[36] The indices y_i [see Eq. (3.5) in Ref. 2] describe the RNG transformation of the ith variable near its fixed point value. If $y_i > 0$, the variable is relevant, if $y_i < 0$, the variable is irrelevant. Only t_1, which we associate with the temperature, is relevant ($y_i > 0$) in our case. The difficulty is to find the stable fixed point where all of the other variables are irrelevant ($y_i < 0$).

In practice the RNG transformation R_b^i can be carried out only if the nonlinear couplings in our equation of motion are small. For our equations of motion (2.12) this (as we shall see) requires that d be near 4, or $\epsilon = 4 - d$ be small. In carrying out an expansion in ϵ, a main concern is to treat the "slow transients,"[37] t_i's with y_i of $O(\epsilon)$, carefully. In our calculation there are several of these slow transients.

B. Explicit evaluation of RNG

1. R_b^i

In order to implement R_b^i it is convenient to separate $\Psi_{i\alpha}(q, \omega)$ into its low and high wave-number components by writing

$$\Psi_{i\alpha}(q, \omega) = \Psi_{i\alpha}^{<}(q, \omega) + \Psi_{i\alpha}^{>}(q, \omega) , \tag{3.1a}$$

where we have defined

$$\Psi_{i\alpha}^{<}(q, \omega) = \Psi_{i\alpha}(q, \omega)\Theta(\Lambda/b - q) \tag{3.1b}$$

and

$$\Psi_{i\alpha}^{>}(q, \omega) = \Psi_{i\alpha}(q, \omega)\Theta(q - \Lambda/b) , \tag{3.1c}$$

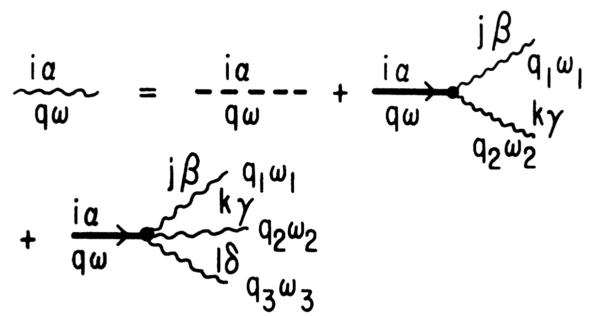

FIG. 1. Graphical representation of the equation of motion (2.12).

where $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$. If we denote $\Psi^{<}_{i\alpha}(q,\omega)$ by a wiggly line with an L and $\Psi^{>}_{i\alpha}(q,\omega)$ by a wiggly line with an H, then after substituting (3.1) into (2.12), we obtain two coupled equations for $\Psi^{<}_{i\alpha}(q,\omega)$ and $\Psi^{>}_{i\alpha}(q,\omega)$ which are shown graphically in Fig. 2.

As we mentioned above we can only carry out R^{i}_{b} if the nonlinear couplings are small. For consistency we must assume $u\sim O(\epsilon)$ and $\lambda\sim O(\epsilon^{1/2})$. We then need to iterate the equation for $\Psi^{>}_{i\alpha}(q,\omega)$ [Fig. 2(a)] to $O(\epsilon^{3/2})$. This means that we keep terms of $O(\lambda^3,\lambda u)$. If we then substitute the iterated equation for $\Psi^{>}_{i\alpha}$ into the equation for $\Psi^{<}_{i\alpha}$ and keep terms of $O(\epsilon^2)$, we obtain the equation shown graphically in Fig. 3. We need to keep terms of $O(\epsilon^2)$ here because we are doing a self-consistent calculation to $O(\epsilon)$ and therefore in correcting quantities which are of $O(\epsilon)$ we must keep corrections of $O(\epsilon^2)$. Figure 3 does not include diagrams which will vanish on averaging over $\Lambda/b<q<\Lambda$ (i.e., those containing an odd number of $\Psi^{<(0)}_{i\alpha}$ will vanish because of the statistical properties of the noise). The next step is to average over the components of the noise limited to the region $\Lambda/b<q<\Lambda$. In principle, since $\Psi^{<}_{i\alpha}$ is a functional of $\Psi^{>}_{i\alpha}$, we expand $\Psi^{<}_{i\alpha}$ in terms of $\Psi^{(0)>}_{i\alpha}$ and $\Psi^{(0)<}_{i\alpha}$ average over the $\Psi^{(0)>}_{i\alpha}$ and then resum the resulting series to obtain equations in terms of the full

$$\langle\Psi^{<}_{i\alpha}\rangle_H\,,$$

where we have carried out the high momentum average. We can carry out this resummation to

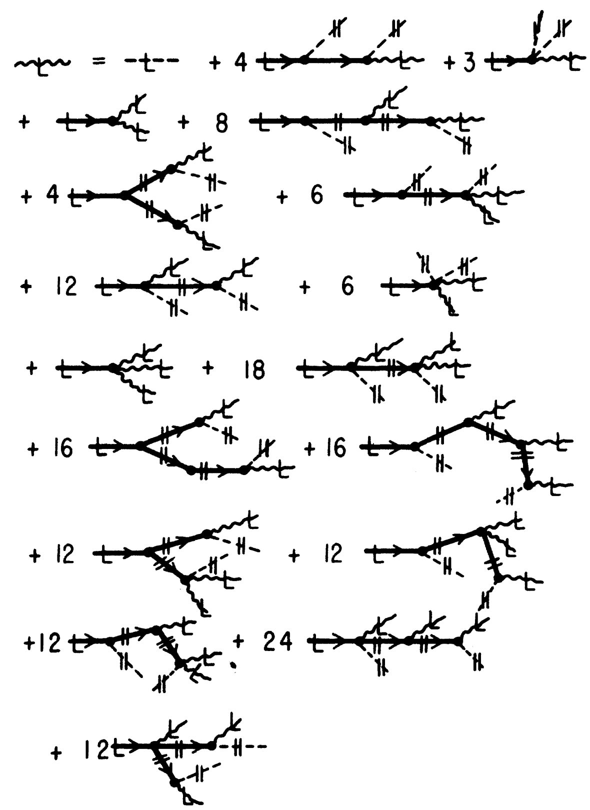

FIG. 3. Equation for $\Psi^{<}_{i\alpha}(q,\omega)$ iterated to $O(\epsilon^2)$.

the lowest order in the interaction by simply writing, for example,

$$\langle\Psi^{<}_{i\alpha}\Psi^{<}_{j\beta}\rangle_H=\langle\Psi^{<}_{i\alpha}\rangle_H\langle\Psi_{j\beta}\rangle_H+O(u,\lambda)\,,$$

but if one works to higher order, one must be more careful. To save writing we will still denote $\langle\Psi^{<}_{i\alpha}\rangle_H$ by $\Psi^{<}_{i\alpha}$ after the averaging. When we average $\Psi^{<}_{i\alpha}(q,\omega)$ (Fig. 3) over $\Lambda/b<q<\Lambda$ we find averages of the form $\langle\Psi^{(0)}_{i\alpha}(q,\omega)\Psi^{(0)}_{j\beta}(q',\omega')\rangle$ which are easily evaluated using (2.18) and (2.2b):

$$\langle\Psi^{(0)}_{i\alpha}(q,\omega)\Psi^{(0)}_{j\beta}(q',\omega')\rangle=\delta_{ij}\delta_{\alpha\beta}(2\pi)^{d+1}\delta(q+q')\times\delta(\omega+\omega')S^{(0)}_{\alpha}(q,\omega)\,, \tag{3.2}$$

where we have defined the zeroth-order correlation function

$$S^{(0)}_{\alpha}(q,\omega)=(2/\omega)\,\mathrm{Im}\,G^{(0)}_{\alpha}(q,\omega)\,. \tag{3.3}$$

The average in (3.2) corresponds graphically to joining together pairs of dashed lines with H's and are denoted by lines with a circle. Using this convention we obtain, on averaging $\Psi^{<}_{i\alpha}(q,\omega)$ (Fig. 3), the equation shown graphically in Fig. 4. Note that the application of R^{i}_{b} to our original equation of motion (Fig. 1) has generated a new equation of motion (Fig. 4) which has 16 additional terms.

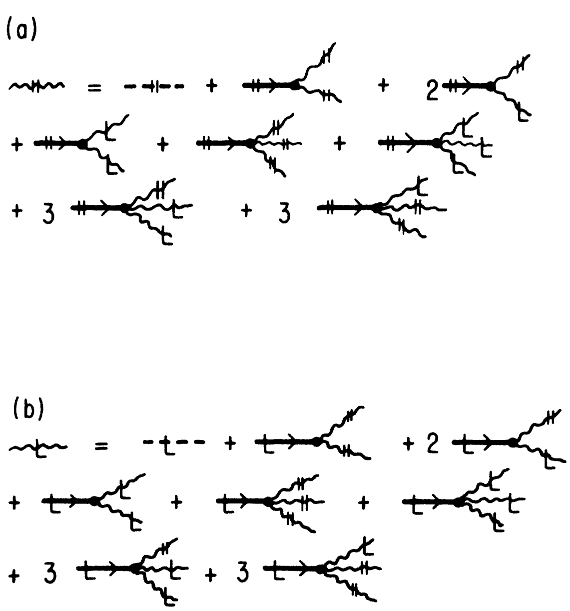

FIG. 2. Coupled equations for (a) $\Psi^{>}_{i\alpha}(q,\omega)$ and (b) $\Psi^{<}_{i\alpha}(q,\omega)$.

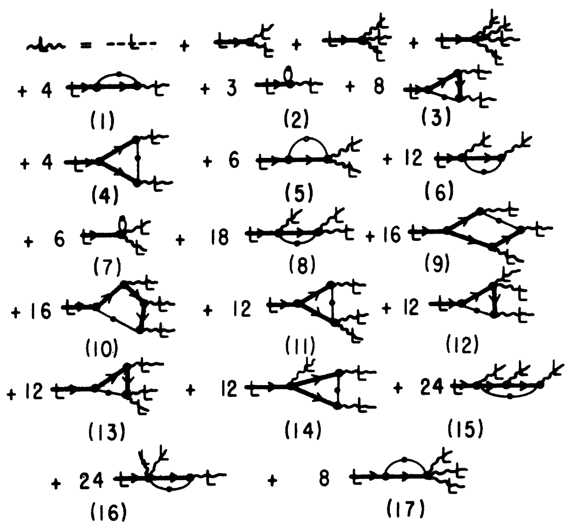

FIG. 4. Equation of motion for $\Psi^<_{i\alpha}(q,\omega)$ after averaging over $\Lambda/b < q < \Lambda$. For ease of discussion we number the terms generated in perturbation theory.

The evaluation of these diagrams is straightforward, although too lengthy to present here. After a few simple algebraic manipulations, we can see that the new terms can be grouped together to give an equation of motion of the same form as (2.12). One difference is that the corrections to the various vertices are now frequency and wave-number dependent. We can however expand these vertices in a power series in q and ω. The lowest-order terms in q expansion have the same form as the original vertices. One can then show that the coefficients of the q- and ω-dependent terms are irrelevant in the scaling region, and we need keep only the lowest-order terms. We find then that under R^i_b our original equation of motion (2.12) has been transformed to an equation of identical form, the only change being new parameters.

2. R^s_b

Under R^s_b, the second step of the RNG transformation, we rescale our equations of motion according to the following prescription:

$$t \to t' = b^{-z} t , \tag{3.4a}$$

$$x \to x' = x/b , \tag{3.4b}$$

and

$$\Psi_{i\alpha}(x,t) \to \Psi'_{i\alpha}(x',t') = b^{-d/2+1-\eta_\alpha/2}\Psi_{i\alpha}(x/b, b^{-z}t) , \tag{3.4c}$$

where the exponents z and η_α are, in principle, determined by the fixed-point equation $R_b\mu^* = \mu^*$. However, since we work to $O(\epsilon)$ and the magnetization is not critical the "anomalous dimensions," η_α can be set equal to zero. It is a simple matter of dimensional analysis to show, using (2.10) and (3.4), that the parameters $\lambda_{\alpha\beta}$ and Γ_β transform in the following way under R^s_b:

$$\Gamma'_\alpha = \Gamma_M b^{z-4}\delta_{\alpha M} + \Gamma_N b^{z-2}\delta_{\alpha N} , \tag{3.5a}$$

where the extra factor of b^{-2} in (3.5a) for $\alpha = M$ arises from the conservation of the magnetization. We also find that

$$\lambda'_{\alpha\beta} = \lambda_{\alpha\beta} b^{z-d/2-1} \tag{3.5b}$$

and

$$u' = b^{4-d} u , \tag{3.5c}$$

$$r'_\alpha = b^2 r_\alpha . \tag{3.5d}$$

C. Recursion formulas, fixed points, and stability

Using (3.5) and the results of the explicit evaluation of the diagrams in Fig. 4, we find, for the full RNG $\mu' = R^s_b R^i_b \mu$, the recursion formulas,

$$\Gamma'_M = b^{z-4}\Gamma_M(1 + \tfrac{1}{2} f_{MN} K_4 \ln b) , \tag{3.6a}$$

$$\Gamma'_N = b^{z-2}\Gamma_N[1 + 2a f_{NM} K_4 \ln b/(1+a)] , \tag{3.6b}$$

$$\lambda'_{MN} = b^{z-d/2-1}\lambda_{MN} , \tag{3.6c}$$

$$u' = b^{4-d} u(1 - 11 K_4 u \ln b) , \tag{3.6d}$$

$$r' = b^2 r , \tag{3.6e}$$

and

$$r'_0 = b^2[r_0 + \tfrac{5}{2} u K_4 \Lambda^2 (1 - b^{-2})] , \tag{3.6f}$$

where we have defined

$$K_4 = (8\pi^2)^{-1}$$

$$f_{NM} = \lambda^2_{NM}\Lambda^{-\epsilon}/\Gamma_M\Gamma_N , \tag{3.7}$$

$$a = \Gamma_M r/\Gamma_N .$$

In obtaining (3.6a) and (3.6b) we have dropped corrections of $O(1/r)$ since from (3.6e) it is clear that near the fixed point ($b \to \infty$) these terms become vanishingly small. The recursion formulas for Γ'_M, Γ'_N, and λ'_{MN} can be written in terms of the dimensionless quantity $a = \Gamma_M r/\Gamma_N$ and f_{MN}. The fixed-point equations for these quantities are

$$a^* = a^*[1 + f^*_{MN}\tfrac{1}{2}K_4[1 - 4a^*/(1+a^*)]\ln b] \tag{3.8a}$$

and

$$f^*_{MN} = b^{\epsilon - x^*} f^*_{MN} , \tag{3.8b}$$

with

$$x^* = \tfrac{1}{2} f^*_{MN} K_4 [1 + 4a^*/(1+a^*)] . \tag{3.8c}$$

We therefore see from (3.8b) that there are two fixed-point solutions for f^*_{MN} given by

$$f^*_{MN} = 0 \tag{3.9a}$$

or

$$f^*_{MN} = 2\epsilon/K_4[1+4a^*/(1+a^*)] \; . \tag{3.9b}$$

We now consider (3.8a). For the case $f^*_{MN}=0$ we have that a^* is arbitrary. For the case Eq. (3.9b) we find the solutions $a^*=0$ and $a^*=\frac{1}{3}$. Therefore the three fixed-point solutions of (3.8) are given by

$$f^*_{MN}=0 \; , \quad a^* \text{ arbitrary} \tag{3.10a}$$

$$f^*_{MN}=2\epsilon/K_4 \, , \quad a^*=0 \tag{3.10b}$$

$$f^*_{MN}=\epsilon/K_4 \, , \quad a^*=\tfrac{1}{3} \; . \tag{3.10c}$$

We now consider the stability of the fixed points given in (3.10). The linearized recursion formula for a and f_{MN} at the fixed point $f^*_{MN}=0$ are easily found to be

$$\delta a' = \delta a + \tfrac{1}{2}a^*K_4[1-4a^*/(1+a^*)]\ln b\, \delta f_{MN} \tag{3.11a}$$

and

$$\delta f'_{MN} = b^{4-d}\delta f_{MN} \; , \tag{3.11b}$$

which is clearly unstable. In the second case $a^*=0, f^*_{MN}=2\epsilon/K_4$, we have

$$\delta a' = b^{\epsilon}\delta a \tag{3.12a}$$

and

$$\delta f'_{MN} = \delta f_{MN} \; , \tag{3.12b}$$

so that this fixed point is also unstable. Finally we consider the fixed point $a^*=\frac{1}{3}$, $f^*_{MN}=\epsilon/K_4$, for which we find

$$\delta a' = b^{-3/8\epsilon}\delta a \tag{3.13}$$

and

$$\delta f'_{MN} = b^{-\epsilon}\delta f_{MN} \; , \tag{3.14}$$

so that this fixed point is stable. Finally we consider the static quantities u, r, and r_0. From (3.6e) we see that $r^* \to \infty$ in agreement with the calculation of Allesandrini *et al.*[33] The fixed-point values of u and r_0 have been determined by Wilson and Fisher.[38] The results are summarized in Tables I and II.

We can determine the dynamical index z by requiring that the equation of motion be invariant at the fixed point. We see therefore that this is accomplished if Γ_N and $\Gamma_M r$ are invariant under R_b. From Eqs. (3.6a), (3.6b), and (3.6e) we find[39]

$$(\Gamma_M r)' = (\Gamma_M r)b^{z-2+f_{MN}K_4/2} \; , \tag{3.15}$$

$$\Gamma'_N = \Gamma_N b^{z-2+2af_{MN}K_4/(1+a)} \; . \tag{3.16}$$

The results for z for the various dynamical fixed points are given in Table I. We note that the fixed point $a^*=0$ requires $\Gamma_M r=0$ and z is determined by the equation for Γ_N. The value of z associated with the stable dynamical fixed point

$$z = 2 - \tfrac{1}{2}\epsilon = \tfrac{1}{2}d \tag{3.17}$$

is in agreement with the results of Kawasaki[40] and dynamical scaling.[31]

From the above analysis we have seen that the parameters a and f_{MN} transform among themselves under R_b and determine a stable dynamical fixed point [see (3.10c)]. This concludes our discussion of the RNG and the main results of this section are summarized in Table I.

TABLE I. Summary of the main results of the RNG analysis performed in Sec. III for the dynamical fixed points. The exponents y_i are related to the eigenvalues of the linear transformation $R_b{}^L$ appearing in the linearized recursion formulas.

a^*	f^*_{MN}	z	y_a	$y_{f_{MN}}$	Stable
arbitrary	0	2	0	ϵ	No
0	$2\epsilon/K_4$	2	ϵ	0	No
$\frac{1}{3}$	ϵ/K_4	$\frac{1}{2}d$	$\frac{-3}{8}\epsilon$	$-\epsilon$	Yes

IV. PERTURBATION THEORY $T \geq T_N$

A. Equation of motion in external field

In developing perturbation theory it is convenient to calculate the linear response to an external field. Following the general development in Sec. II D of MM we can introduce this external field into the model through the replacement

$$F \to F' = F - \sum_{i\alpha}\int d^dx\, h_{i\alpha}(x,t)\Psi_{i\alpha}(x) \tag{4.1}$$

in the equation of motion. The linear response function $G_{ij\alpha\beta}(x-x',t-t')$ is then defined by

$$\langle\Psi_{i\alpha}(xt)\rangle = \sum_{\beta j}\int d^d\bar{x}\, d\bar{t}\, G_{ij\alpha\beta}(x-\bar{x},t-\bar{t}) \times h_{j\beta}(\bar{x},\bar{t}) + O(h^2) \; . \tag{4.2}$$

We introduce G because it has a more convenient perturbation-theory expansion than the correlation function. We can calculate the correlation function

$$C_{ij\alpha\beta}(x-x',t-t') = \langle\Psi_{i\alpha}(x,t)\Psi_{j\beta}(x't')\rangle \tag{4.3}$$

TABLE II. Summary of the results of Sec. III for the static parameters r, r_0, and u (we find it convenient here to consider r^{-1}).

u^*	r_0^*	${r^{-1}}^*$
$\epsilon/11K_4$	$-\frac{5}{22}\epsilon\Lambda^2$	0
$y_u=-\epsilon$	$y_{r_0}=2$	$y_{r^{-1}}=2$

using the classical fluctuation dissipation theorem

$$C_{ij\alpha\beta}(k,\omega)=(2/\omega)\,\mathrm{Im}G_{ij\alpha\beta}(k,\omega)\ , \tag{4.4}$$

where $C_{ij\alpha\beta}(k,\omega)$ is the Fourier transform

$$C_{ij\alpha\beta}(k,\omega)=\int d^dx\,dt\,e^{-i\vec{k}\cdot\vec{x}+i\omega t}C_{ij\alpha\beta}(x,t)\ . \tag{4.5}$$

In terms of Fourier transforms

$$\langle\Psi_{i\alpha}(k,\omega)\rangle=\sum_{\beta j}G_{ij\alpha\beta}(k,\omega)h_{j\beta}(k,\omega)+O(h^2)\ . \tag{4.6}$$

After we include the external field, our equation of motion takes the form

$$\frac{\partial\vec{\Psi}_\alpha(x,t)}{\partial t}=\vec{V}_\alpha(x,t)-\hat{\Gamma}_\alpha\vec{H}_\alpha+\hat{\Gamma}_\alpha\vec{h}_\alpha(x,t)-\sum_{\beta\gamma}\lambda_{\alpha\beta}\sigma_{\alpha\beta\gamma}[\vec{\Psi}_\gamma(x,t)\times\vec{h}_\beta(x,t)]+\vec{\eta}(xt)\ , \tag{4.7}$$

where the last two terms contain the dependence on the external field. The contribution of these new terms to the equation of motion are represented graphically in Fig. 5 and are to be added to those in Fig. 1. The first term in Fig. 5 corresponds to the term $\hat{\Gamma}_\alpha h_{\alpha i}(x,t)$ in the equation of motion. The x at the end of a propagator represents a factor of $h_{\alpha i}(q,\omega)$. The second term comes from replacing F with $\Psi\ h$ in the streaming velocity (2.3) and gives rise to a Larmor precession term in the resulting equation of motion. The three point vertex with an external field line (represented by an x attached to the vertex) is given

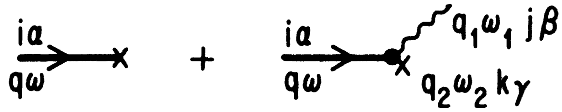

FIG. 5. Two terms which must be added to our equation of motion (see Fig. 1) in the presence of an external magnetic field (represented by a cross).

by

$$Z_{ijk\alpha\beta\gamma}(q,q_1,q_2,\omega,\omega_1,\omega_2)=(2\pi)^{d+1}\delta^{(d)}(q-q_1-q_2)\delta(\omega-\omega_1-\omega_2)\times\tilde{Z}_{ijk\alpha\beta\gamma}(q,q_1)\ , \tag{4.8}$$

where

$$\tilde{Z}_{ijk\alpha\beta\gamma}(q,q_1)=-\lambda_{\alpha\beta}\sigma_{\alpha\beta\gamma}\epsilon_{ijk}/\Gamma_\alpha(q)\ . \tag{4.8a}$$

B. Iteration to first order in ϵ

We can calculate the response function in the scaling region if we fix $a=\Gamma_M r/\Gamma_N$, $f=\lambda_{NM}^2\Lambda^{-\epsilon}/\Gamma_M\Gamma_N$, and u at their fixed-point values.[41] Thus we can take $\lambda\sim\epsilon^{1/2}$ and $u\sim\epsilon$ and expand G in powers of λ and u. After iterating the equation given in Figs. 1 and 5 to first order in ϵ (second order in λ and first order in u) we find, after averaging over the noise, that $\langle\Psi_{i\alpha}(q,\omega)\rangle$ is given, to first order in the external field, by the graphs in Fig. 6. We note that graphs with a closed loop on a three-point vertex vanish by symmetry. If we now define the self-energy via the equation

$$G_{ij\alpha\beta}(q,\omega)=G_\alpha^0(q,\omega)\delta_{\alpha\beta}\delta_{ij}+\sum_{k\gamma}G_\alpha^0(q,\omega)\Sigma_{\alpha\gamma}^{ik}(q,\omega)G_{kj\beta\gamma}(q,\omega)\ , \tag{4.9}$$

we find to $O(\epsilon)$,

$$\Sigma_{\alpha\beta}^{ij}(q,\omega)=\delta_{ij}\delta_{\alpha\beta}[\Sigma_\alpha^{(1)}(q,\omega)+\Sigma_\alpha^{(2)}(q,\omega)+\Sigma_\alpha^H]\ , \tag{4.10a}$$

where

$$\Sigma_\alpha^{(1)}(q,\omega)=\sum_{\beta\gamma jk}\left(\int\frac{d\omega}{2\pi}\frac{d^4\overline{q}}{(2\pi)^4}\,4\tilde{V}_{\alpha\beta\gamma ijk}(q,\overline{q})G_\beta^0(\overline{q},\overline{\omega})S_\gamma^0(q-\overline{q},\omega-\overline{\omega})\tilde{V}_{jik\beta\alpha\gamma}(\overline{q},q)\right)\ , \tag{4.10b}$$

$$\Sigma_\alpha^{(2)}(q,\omega)=\sum_{\beta\gamma jk}\left(\int\frac{d\omega}{2\pi}\frac{d^4\overline{q}}{(2\pi)^4}\,2\tilde{V}_{\alpha\beta\gamma ijk}(q,\overline{q})G_\gamma^0(q-\overline{q},\omega-\overline{\omega})S_\beta^0(\overline{q},\overline{\omega})\tilde{Z}_{\gamma\beta\alpha kji}(q-\overline{q},\overline{q})G_\alpha^{0-1}(q,\omega)\right)\ , \tag{4.10c}$$

$$\Sigma_\alpha^H=3\sum_{\beta k}\tilde{B}_{\alpha\beta\beta\alpha ikki}\left(\int\frac{d\omega\,d^4q}{(2\pi)^5}S_\beta^0(q,\omega)\right)\ . \tag{4.10d}$$

The last term is just the usual Hartree term that shifts the value of T_N from its zeroth-order value. It is easy to show after integrating over an internal frequency that the sum of $\Sigma^{(1)}$ and $\Sigma^{(2)}$ take the simple form

$$\Sigma_\alpha^T(q,\omega)=\int\frac{d^4\overline{q}}{(2\pi)^4}\sum_{\mu\rho}\lambda_{\mu\alpha}\frac{W_{\alpha\mu\rho}(q,\overline{q})\chi_\rho^{(0)}(q-\overline{q})\sigma_{\mu\rho\alpha}}{-i\,\omega+L_\rho(q-\overline{q})+L_\mu(\overline{q})}\frac{2i\omega}{\Gamma_\alpha^2(q)}\ , \tag{4.11}$$

where $W_{\alpha\mu\rho}(q_1,q_2)$ is defined by (2.15b), the $\chi_\alpha^{(0)-1}(q)$ are the zeroth-order inverse static susceptibilities (2.15c) and

$$L_\alpha(q) \equiv \Gamma_\alpha(q)\chi_\alpha^{(0)-1}(q) \ . \tag{4.12}$$

Note that Σ^T is proportional to ω and vanishes for zero frequency. This is in agreement with the theorem, discussed by MM, that the streaming terms (which are proportional to λ) can not contribute to the static correlation functions.

We can then write the linear response function in the form

$$G_{\alpha\beta ij}(k,\omega) = \delta_{\alpha\beta}\delta_{ij}G_\alpha(k,\omega) \ , \tag{4.13}$$

$$G_\alpha^{-1}(k,\omega) = [-i\omega/\bar{\Gamma}_\alpha(k,\omega) + \chi_\alpha^{-1}(k)] \ , \tag{4.14}$$

where

$$\chi_\alpha^{-1}(k) = \chi_\alpha^{(0)-1}(k) - \Sigma_\alpha^H \ , \tag{4.15}$$

$$\bar{\Gamma}_\alpha(k,\omega) = \Gamma_\alpha(q)[1 + Q_\alpha(q,\omega)] \ , \tag{4.16}$$

and

$$Q_\alpha(q,\omega) = \frac{1}{\Gamma_\alpha(q)} \int \frac{d^4\bar{q}}{(2\pi)^4} \sum_{\rho\mu} \frac{\chi_\rho(q-\bar{q})\chi_\mu(\bar{q})W^2_{\alpha\mu\rho}(q,\bar{q})}{-i\omega + L_\rho(q-\bar{q}) + L_\mu(\bar{q})} \ . \tag{4.17}$$

The last equation follows from some simple rearrangements of the integrand of (4.11). Since χ_α^{-1} differs from $\chi_\alpha^{(0)-1}$ only by a shift in T_N of $O(\epsilon)$, we can consistently use χ_α in place of $\chi_\alpha^{(0)}$ in the Q_α's. Note that χ_α^{-1} is still of the Ornstein-Zernike form. We obtain more explicitly,

$$Q_N(q,\omega) = \frac{2\lambda_{NM}^2}{\Gamma_N} \int \frac{d^4\bar{q}}{(2\pi)^4} \frac{\chi_N(q-\bar{q})\chi_M^{-1}(\bar{q})}{-i\omega + L_N(q-\bar{q}) + L_M(\bar{q})} \tag{4.18}$$

and

$$Q_M(q,\omega) = \frac{\lambda_{MN}^2}{\Gamma_M q^2} \int \frac{d^4\bar{q}}{(2\pi)^4} \frac{\chi_N(q-\bar{q})\chi_N(\bar{q})[\chi_N^{-1}(q-\bar{q}) - \chi_N^{-1}(\bar{q})]^2}{-i\omega + L_N(q-\bar{q}) + L_N(\bar{q})} \ . \tag{4.19}$$

The calculation of the correlation functions centers on the explicit calculation of Q_N and Q_M.

C. Calculation of $C_N(q,\omega)$

1. Evaluation of $Q_N(q,\omega)$

We can write the integral for $Q_N(q,\omega)$ in the form (remembering $r\to\infty$ in the scaling region)

$$Q_N(q,\omega) = \frac{4\epsilon}{3\pi}\int_0^\Lambda \frac{\bar{q}^3\,d\bar{q}}{\bar{q}^2+\xi^{-2}} \times \int_0^\pi \frac{\sin^2\theta\,d\theta}{-i\omega/\Gamma_N + \bar{q}^2 + \frac{1}{3}(q^2+\bar{q}^2-2q\bar{q}\cos\theta)} \ . \tag{4.20}$$

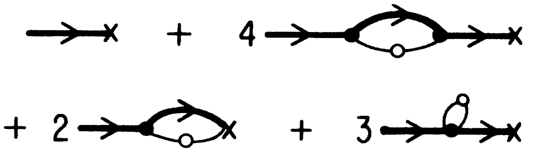

FIG. 6. Graphs which contribute to $\langle\Psi_{i\alpha}(q,\omega)\rangle$ to $O(\epsilon)$.

These integrals can be evaluated explicitly. If we neglect, as we must for consistency, contributions of order q/Λ, and $\omega/\Gamma_N\Lambda^2$ relative to 1, we obtain

$$Q_N(q,\omega) = (2\epsilon/x^2)\{\tfrac{1}{8}x^2[1+\ln(\xi\Lambda)^2] - t\Delta\ln\sigma + (t-\tfrac{1}{2}b)\ln(1+t-\tfrac{1}{4}b)\} \ , \tag{4.21}$$

where

$$x = q\xi \ , \tag{4.22}$$

$$\nu = \omega/\Gamma_N q^2(1+x^{-2}) \ , \tag{4.23}$$

$$t = -\tfrac{1}{4}[(1-x^2) + 3i\nu(1+x^2)] \ , \tag{4.24}$$

$$b = \tfrac{1}{4}x^2 \ , \tag{4.25}$$

$$\Delta = (1+b/t^2)^{1/2} \ , \tag{4.26}$$

$$\sigma = [2t(1+t)(1+\Delta)+b]/[2t(1+\Delta)-b] \ . \tag{4.27}$$

One simple check on our result is to calculate the integral in (4.20) for $q=\omega=0$ directly and compare with the corresponding limit for (4.21). One finds in both cases

$$Q_N(0,0)=\tfrac{1}{4}\epsilon[2\ln(\xi\Lambda)-3\ln\tfrac{4}{3}]. \qquad (4.28)$$

2. Renormalization, dynamic scaling, and exponentiation

Combining (4.16) and (4.28) we easily calculate the physical kinetic coefficient as

$$\tilde{\Gamma}_N(0,0)=\Gamma_N[1+Q_N(0,0)]$$
$$=\Gamma_N\{1+\tfrac{1}{4}\epsilon[2\ln(\xi\Lambda)-3\ln\tfrac{4}{3}]\}. \qquad (4.29)$$

According to (4.29) as $T\to T_N$ the physical kinetic coefficient $\tilde{\Gamma}_N$ diverges logarithmically. This result must be modified if it is to agree with the form predicted by general RNG arguments. We know that in the scaling region[42]

$$G^{-1}(q,\omega,\xi^{-1})=b^{-2+\eta}G^{-1}(qb,\omega b^{z},\ b/\xi). \qquad (4.30)$$

If we use (4.14) and note the static result

$$\chi^{-1}(q,\xi^{-1})=b^{-2+\eta}\chi^{-1}(qb,b/\xi), \qquad (4.31)$$

we find

$$\Gamma(q,\omega,\xi^{-1})=b^{-z+2-\eta}\Gamma(qb,\omega b^{z},b/\xi). \qquad (4.32)$$

If we successively choose to set two of the three variables q, ω, ξ^{-1} to zero and make appropriate choices for b, we obtain the scaling relations

$$\Gamma(q,0,\xi=\infty)=(q/\Lambda)^{z+\eta-2}\Gamma(\Lambda,0,\xi=\infty), \qquad (4.33a)$$

$$\Gamma(0,\omega,\xi=\infty)=(\omega/\Gamma_N\Lambda^2)^{(z-2+\eta)/z}\Gamma(0,\Gamma\Lambda^2,\xi=\infty), \qquad (4.33b)$$

$$\Gamma(0,0,\xi)=(\xi\Lambda)^{-z+2-\eta}\Gamma(0,0,1/\Lambda). \qquad (4.33c)$$

If (4.29) is to agree with (4.33c), we must perform the first of several exponentiations. To $O(\epsilon)$ we can rewrite (4.29) as

$$\tilde{\Gamma}_N(\xi)\equiv\tilde{\Gamma}_N(0,0)\equiv\tilde{\Gamma}_N(\xi\Lambda)^{\epsilon/2}(1-\tfrac{3}{4}\epsilon\ln\tfrac{4}{3}), \qquad (4.34)$$

which agrees with (4.33c) to $O(\epsilon)$. $\tilde{\Gamma}_N(\xi)$ is the physical kinetic coefficient for the staggered magnetization and the inverse relaxation time $\tau_N^{-1}(\xi)\equiv\tilde{\Gamma}_N(\xi)\chi_N^{-1}(0)$ governs the long time decay for uniform ($q=0$) modes in the system. The conventional van Hove[43] theory of critical slowing down predicts that $\tau_N(\xi)\sim\xi^2$. We see that a diverging kinetic coefficient $\tilde{\Gamma}_N(\xi)$ corresponds to a shorter relaxation time than that predicted by conventional theory. It is worth noting that $\tilde{\Gamma}_N$ is related to the correlation function by

$$\tilde{\Gamma}_N(\xi)=\lim_{\omega\to0}\lim_{q\to0}[\tfrac{1}{2}C_N(q,\omega)\chi_N^{-2}(q,\xi)]^{-1}, \qquad (4.35)$$

and the response function by

$$\tilde{\Gamma}_N(\xi)=\lim_{\omega\to0}\lim_{q\to0}\left(\frac{\partial}{\partial(-i\omega)}G^{-1}(q,\omega)\right)^{-1}. \qquad (4.36)$$

$\tilde{\Gamma}_N(\xi)$ is therefore real and observable since both $C_N(q,\omega)$ and the static susceptibility $\chi_N(q)$ are measurable. The quantity Γ_N and the cutoff Λ are *not* observable. We can eliminate them in terms of $\tilde{\Gamma}_N(\xi)$ and assume $\tilde{\Gamma}_N(\xi)$ can be obtained from experiment. Once we make this replacement Γ_N and Λ are removed from the theory. Note, however, that our theory predicts that the physical kinetic coefficient $\tilde{\Gamma}_N(\xi)$ has the nontrivial temperature dependence $\sim\xi^{\epsilon/2}$. In principle $\Gamma_N\Lambda^{\epsilon/2}\times(1-\frac{3}{4}\epsilon\ln\frac{4}{3})$ can be determined experimentally as the coefficient of the $\xi^{\epsilon/2}$ dependence. We note that our result for $\Gamma_N(\xi)$ disagrees with that found by HHS.[44] We do not know the reason for this discrepancy. If we now go back to our general expression for $\tilde{\Gamma}_N(q,\omega)$, we can write, to $O(\epsilon)$,

$$\tilde{\Gamma}_N(q,\omega)$$
$$=\tilde{\Gamma}_N(0,0)\{1+(2\epsilon/x^2)[\tfrac{1}{8}x^2+6x^2\ln\tfrac{4}{3}-t\Delta\ln\sigma$$
$$+(t-\tfrac{1}{2}b)\ln(1+t-\tfrac{1}{4}b)]\}. \qquad (4.37)$$

Our result for $\Gamma_N(q,\omega)$ is still not completely satisfactory. The reason is that the limit $T\to T_N,\nu\to0$ will lead to logarithmic divergences. This will violate equations (4.33a) and (4.33b). Further exponentiations are therefore necessary. These exponentiations are not unique, consequently we discuss the method we use in the Appendix. We find, following the exponentiation procedure described in the Appendix, that[45]

$$\tilde{\Gamma}_N(q,\omega)=\tilde{\Gamma}_N(0,0)(1+x^2)^{-\epsilon/4}(1-4i\nu)^{3\epsilon/4}$$
$$\times(1-3i\nu)^{-\epsilon}W_N(x,\nu), \qquad (4.38)$$

where

$$W_N(x,\nu)=1+(2\epsilon/x^2)[\tfrac{3}{8}x^2(\tfrac{1}{3}+\ln\tfrac{4}{3})-t\Delta\ln\sigma$$
$$+(t-\tfrac{1}{2}b)\ln(1+t-\tfrac{1}{4}b)$$
$$+\tfrac{1}{4}x^2\ln\bar{t}^{\,2}/\tau-\tfrac{1}{8}x^2\ln\tau], \qquad (4.39)$$

$$\bar{t}=(1+x^2)(1-3i\nu), \qquad (4.40a)$$

$$\tau=(1+x^2)(1-4i\nu). \qquad (4.40b)$$

Although the function $W_N(x,\nu)$ looks very complicated it is a rather slowly varying function of x and ν that has the simple limiting behavior

$$W_N(0,0)=1, \qquad (4.41a)$$

$$W_N(x=\infty,\nu=0)=1+\tfrac{1}{4}\epsilon(1-2\ln4+3\ln3)$$
$$(=1.38\ \text{for}\ \epsilon=1), \qquad (4.41b)$$

$$W_N(x=\infty,\nu=\infty)=1+\tfrac{1}{4}\epsilon(2\ln4-3\ln3)$$
$$(=1.131\ \text{for}\ \epsilon=1), \qquad (4.41c)$$

$$W_N(x=0,\ \nu=\infty)=1+\epsilon(1+\ln 3-\tfrac{4}{3}\ln 4)$$
$$(=2.883 \text{ for } \epsilon=1)\,. \quad (4.41\text{d})$$

We see that W_N remains close to 1 except for small x and large ν. This corresponds to a region where the shape function is small, so our results should be reasonable for all x and ν.

3. *Characteristic frequencies*

In order to present our results in the usual dynamical scaling form we need to define a characteristic frequency $\omega(q,\xi)$. There are several ways of introducing $\omega(q,\xi)$. In MM the characteristic frequency was identified with the poles in the response function

$$G^{-1}(q,\omega)=0\,, \quad (4.42)$$

which can be solved to give a frequency $\omega_c=\omega(q,\xi)$. This is a convenient definition if the correlation function is essentially Lorentzian but other alternatives are preferable if one has a non-Lorentzian form. We can also define the characteristic frequency as

$$\bar{\omega}(q,\xi)\equiv\left(\frac{\partial}{\partial(-i\omega)}\,G^{-1}(q,\omega,\xi)\right)^{-1}_{\omega=0}\chi^{-1}(q,\xi)\,. \quad (4.43)$$

If the spectrum is Lorentzian this definition is essentially the same as obtained from (4.42) and $\omega(q,\xi)$ is a measure of the width of the Lorentzian. According to dynamical scaling we can write

$$\bar{\omega}(q,\xi)=q^{z}\Omega(x)\,, \quad (4.44)$$

where $\Omega(x)$ is the scaling function. We can easily compute from (4.14), (4.36), and (4.38) that

$$\bar{\omega}_N(q,\xi)=q^{2-\epsilon/2}\Omega^N(x)\,, \quad (4.45)$$

where

$$\Omega^N(x)=\Gamma_N\Lambda^{\epsilon/2}(1-\tfrac{3}{4}\epsilon\ln\tfrac{4}{3})(1+x^{-2})^{1-\epsilon/4}W_N(x,0) \quad (4.46)$$

and

$$W_N(x,0)=1+\tfrac{1}{4}\epsilon\{1-(4/x^2)\ln(1+\tfrac{1}{4}x^2)+\ln[(1+x^2)/(1+\tfrac{1}{4}x^2)]\}\,. \quad (4.47)$$

It is convenient to plot

$$\Delta^N(x)=\Omega^N(x)/\Omega^N(\infty)\,, \quad (4.48)$$

where

$$\Omega^N(\infty)=\Gamma_N\Lambda^{\epsilon/2}(1-\tfrac{3}{4}\epsilon\ln\tfrac{4}{3})\,[1+\tfrac{1}{4}\epsilon(1+\ln 4)]\,. \quad (4.49)$$

We have plotted $\Delta^N(x)$ in Fig. 7. Note that $\Delta^N(x)$ has the same qualitative behavior as that calculated by Resibois and Pyette[46] and Huber and Krueger[47] and compared with the neutron scattering data of Nathans *et al.* and Lau *et al.*[48]

We note that if the spectrum is not Lorentzian in character, these definitions of $\omega(q,\xi)$ are not necessarily the most convenient since they lose a simple physical interpretation. In the present case, where the spectrum turns out to be non-Lorentzian, it is more convenient and more natural to choose $\omega_c(q,\xi)\chi(q,\xi)$ in terms of the factor in $\Gamma_N(q,\omega)$ that is independent of ν. This choice leads to a characteristic frequency

$$\omega(q,\xi)=\Gamma_N\Lambda^{\epsilon/2}q^{2-\epsilon/2}(1+x^{-2})^{1-\epsilon/4}\,(1-\tfrac{3}{4}\epsilon\ln\tfrac{4}{3})\,. \quad (4.50)$$

This choice for a characteristic frequency is much simpler than $\bar{\omega}_N$. While $\omega(q,\xi)$ and $\bar{\omega}_N$ are equal for small x, they differ for large x by

$$\lim_{x\to\infty}\frac{\bar{\omega}_N(q,\xi)}{\omega(q,\xi)}=W_N(\infty,0)$$
$$=[1+\tfrac{1}{4}\epsilon(1+\ln 4)]\quad(=1.597 \text{ for } \epsilon=1)\,. \quad (4.51)$$

4. *Shape function*

Using the characteristic frequency (4.50) we can write (4.38) in the form

$$\tilde{\Gamma}_N(q,\omega)=\chi_N(q,\xi)\omega_N(q,\xi)\gamma_N(x,\nu)\,, \quad (4.52)$$

where

$$\gamma_N(x,\nu)=(1-4i\nu)^{3\epsilon/4}(1-3i\nu)^{-\epsilon}W_N(x,\nu)\,. \quad (4.53)$$

We can then write the response function in the form

$$G_N(q,\omega)=\chi_N(q)\gamma_N(x,\nu)/[-i\nu+\gamma_N(x,\nu)]\,, \quad (4.54)$$

where, to lowest order in ϵ, we identify ν in γ_N as $\omega/\omega(q,\xi)$. The correlation function can be obtained using the fluctuation-dissipation theorem (4.4) and can be written in the dynamical scaling

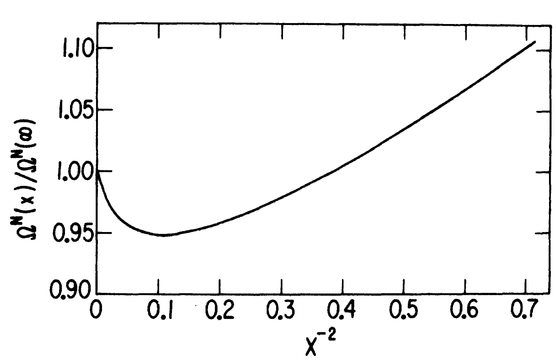

FIG. 7. Plot of $\Delta^N(x)=\Omega^N(x)/\Omega^N(\infty)$.

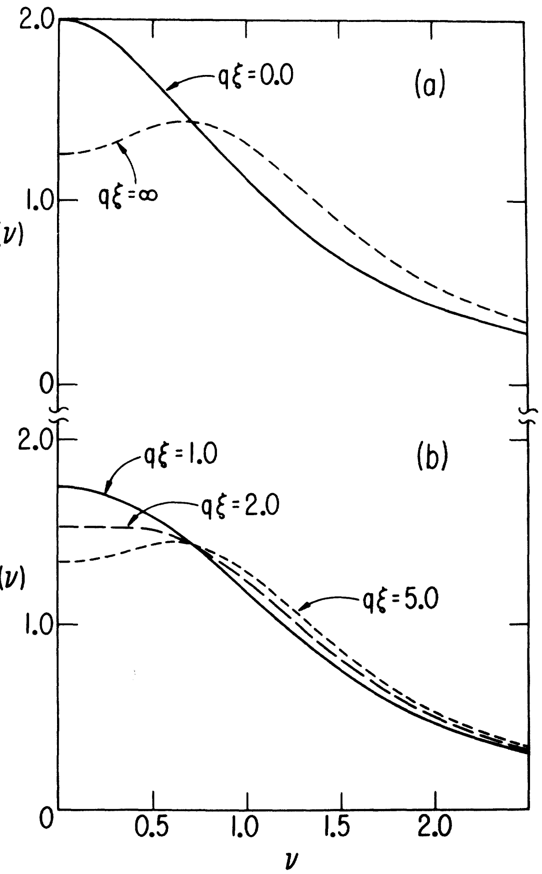

FIG. 8. Plot of the shape function $f_x^N(\nu)$ for $\epsilon=1$ (a) in the relaxational regime $x=0$ ($x=q\xi$) and in the critical regime $x=\infty$ and (b) for intermediate values of $x=1.0$, 2.0, and 5.0.

form (1.6). The behavior of the shape function $f_x^N(\nu)$ is shown in Fig. 8 for $\epsilon=1$. The shape function exhibits the following behavior[49]:

(i) In the relaxational regime, $x \ll 1$, $f_x^N(\nu)$ is a Lorentzian centered about $\nu=0$.

(ii) As x increases and we move toward the critical regime $x \gg 1$ we find that two fluctuation-induced peaks appear in the spectrum displaced symmetrically about the origin. For $\epsilon=1$ these two peaks first appear for $x\sim 2$.

(iii) The position of the peaks moves continuously outward from the origin until $x\sim 10$ where the position of the peaks attains a limiting value $\nu_M=\pm 0.65$.

(iv) The height of the peaks relative to the value of $f_x^N(0)$ increases as x increases reaching a maximum of 15% for $x\gg 1$.

(v) For $\epsilon \ll 1$ $f_x^N(\nu)$ is a Lorentzian centered about $\nu=0$ for all values of x. In the critical regime the two-peak structure first appears for $\epsilon \sim 0.45$.

These peaks we find are similar to those observed in the isotropic antiferromagnet $RbMnF_3$ by neutron scattering.[50] Peaks similar to those in Fig. 8 have been found previously by Wegner[51] for $T=T_N$ by numerically solving a set of coupled equations similar to (4.14), (4.16), and (4.17) in three dimensions. Wegner[52] derived these equations using a mode-coupling approach.

D. Calculation of $C_M(q,\omega)$

We can treat $Q_M(q,\omega)$ in much the same manner as $Q_N(q,\omega)$. We present only the results here (after exponentiating as in the Appendix):

$$\tilde{\Gamma}_M(q,\omega)=\Gamma_M q^2[1+Q_M(q,\omega)] = \frac{\Gamma_M \Lambda^{\epsilon/2} q^{2-\epsilon/2} x^{\epsilon/2}(1-\frac{3}{8}\epsilon)}{(1+x^2)^{\epsilon/4}(1-\frac{1}{2}i\nu)^{\epsilon/4}} W_M(x,\nu), \tag{4.55}$$

$$W_M(x,\nu)=1+\tfrac{1}{4}\epsilon\big[4-x^{-4}(\{[\beta-(1-\alpha)]^2-x^4\} \times \ln R_2-(1-\alpha)\beta\ln R_1)\big], \tag{4.56}$$

$$R_1(x,\nu)=\frac{1}{(1+x^2)^2(1-\frac{1}{2}i\nu)^2}\,\frac{W'(x,\nu)}{Y(x,\nu)S(x,\nu)}, \tag{4.57}$$

$$W'(x,\nu)=2(\beta+\alpha-1)/(\Delta-2), \tag{4.58}$$

$$Y(x,\nu)=2\alpha(\beta+1-\alpha)/(\Delta-2\alpha^2), \tag{4.59}$$

$$S(x,\nu)=\frac{(1+x^2)(\beta+1+x^2)+\alpha(x^2-1)}{\alpha[\beta-x^2+1-\alpha]}, \tag{4.60}$$

$$\Delta(x,\nu)=\beta^2+2\alpha-\beta^2[1+4(\alpha/\beta^2)]^{1/2}, \tag{4.61}$$

$$\alpha=1-i\nu(1+x^2), \tag{4.62}$$

$$\beta=-[x^4+2x^2(1+\alpha)+(1-\alpha)^2]^{1/2}, \tag{4.63}$$

$$R_2(x,\nu)=(1+x^2)(1-\tfrac{1}{2}i\nu). \tag{4.64}$$

Our definition of ν is still given as above (4.23). We can define a characteristic frequency for the magnetization as in (4.43). We easily find

$$\bar{\omega}_M(q,\xi)\chi_M(q,\xi)=\tilde{\Gamma}_M(q,0) = \Gamma_M\Lambda^{\epsilon/2}q^{2-\epsilon/2}(1+x^{-2})^{-\epsilon/4}(1-\tfrac{3}{8}\epsilon) \times\{1+\epsilon[1-x^{-2}\ln(1+x^2)]\} \tag{4.65}$$

or

$$\bar{\omega}_M(q,\xi)=q^{2-\epsilon/2}\Omega_M(x), \tag{4.66}$$

$$\Omega_M(x)=\Omega_M(\infty)\Delta_M(x), \tag{4.67}$$

where

$$\Omega_M(\infty)=\Gamma_M r\Lambda^{\epsilon/2}(1+\tfrac{5}{8}\epsilon) \tag{4.68}$$

and

$$\Delta_M(x) = (1+x^{-2})^{-\epsilon/4}(1+\epsilon)^{-1}\{1+\epsilon[1-x^{-2}\ln(1+x^2)]\} \tag{4.69}$$

starts at 1 for $x \to \infty$ and decreases monatonically to zero as x decreases in qualitative agreement with the results of Joukoff-Piette and Resibois.[46]

It is easy to extract the physical transport coefficient $\tilde{\Gamma}_M(\xi)$ from (4.65), we obtain

$$\tilde{\Gamma}_M(\xi) \equiv \lim_{q\to 0}\lim_{\omega\to 0}\frac{1}{q^2}\tilde{\Gamma}_M(q,\omega) = \Gamma_M(\Lambda\xi)^{\epsilon/2}(1-\tfrac{3}{8}\epsilon). \tag{4.70}$$

We see that the transport coefficient diverges as $T \to T_N$. Our result is in agreement with that of HHS[53] to $O(\epsilon)$.

In evaluating the shape function for the magnetization it is convenient to use the same characteristic frequency as for the staggered magnetization. We can write

$$\tilde{\Gamma}_M(q,\omega) = \omega(q,\xi)\chi_M\gamma_M(x,\nu), \tag{4.71}$$

where $\omega(q,\xi)$ is given by (4.50), so that

$$G_M(q,\omega) = \chi_M(q)\gamma_M(x,\nu)/[-i\nu+\gamma_M(x,\nu)] \tag{4.72}$$

and

$$\gamma_M(x,\nu) = \frac{1}{3}\frac{(1-\frac{3}{8}\epsilon)(1-2\ln\frac{4}{3})W_M(x,\nu)}{(1-\frac{1}{2}i\nu)^{\epsilon/4}(1+x^{-2})}. \tag{4.73}$$

In arriving at (4.73) we have used the result $r\Gamma_M/\Gamma_N = 1/3$.[54] We can extract the correlation function using the fluctuation-dissipation theorem (4.4) and write it in the dynamical scaling form

$$C_M(q,\omega) = [\chi_M(q)/\omega(q,\xi)]f_x^M(\nu). \tag{4.74}$$

We plot $f_x^M(\nu)$ in Fig. 9 for $\epsilon = 1$. The shape function is essentially Lorentzian for all values of x showing the characteristic hydrodynamic pole at $\nu = 0$ as $x \to 0$. This behavior is in qualitative agreement with the experiments of Tucciarone *et al.*[55]

E. Universal ratios

Because of the fixed point structure of the RNG the parameters Γ_N, $\Gamma_M r$, and λ_{MN} are interelated (only one is independent). We note then that certain ratios are universal. In particular the ratio of the characteristic frequencies [as defined by (4.43)] are universal

$$\frac{\bar{\omega}_M(q,\xi)}{\omega_N(q,\xi)} = \frac{\Omega_M(\infty)}{\Omega_N(\infty)}\frac{\Delta_M(x)}{\Delta_N(x)} \tag{4.75}$$

and

$$\Omega_M(\infty)/\Omega_N(\infty) = a^*[1+\tfrac{1}{8}\epsilon(3+4\ln 4-6\ln 3)]. \tag{4.76}$$

HHS have defined the ratios, in our notation,

$$R_N = \Gamma_N(\xi)\xi^{d/2}r^{-1/2}/\chi_N\lambda_{MN}, \tag{4.77}$$

$$R_M = \Gamma_M(\xi)/\lambda_{MN}\xi^{\epsilon/2}r^{-1/2}, \tag{4.78}$$

$$R_{N/M} = R_N/R_M. \tag{4.79}$$

Using our results for $\Gamma_N(\xi)$ and $\Gamma_M(\xi)$ we obtain the results

$$R_N = (af)^{-1/2}(1-\tfrac{3}{4}\epsilon\ln\tfrac{4}{3}), \tag{4.80}$$

$$R_M = (a/f)^{1/2}(1-\tfrac{3}{8}\epsilon), \tag{4.81}$$

$$R_{N/M} = (1/a)[1+\tfrac{3}{8}\epsilon(1-2\ln\tfrac{4}{3})]. \tag{4.82}$$

We can consistently only extract the leading term in an ϵ expansion for these terms. In $R_{N/M}$, for example, the $O(\epsilon)$ correction to $a = \frac{1}{3}$ is of the same order as the $\frac{3}{8}(1-2\ln\frac{4}{3})$ term. Since we

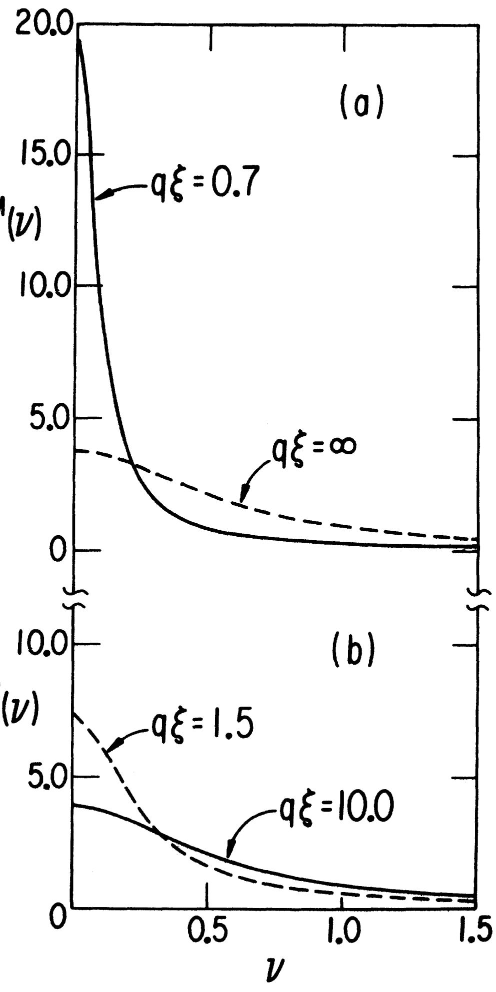

FIG. 9. Plot of the shape function $f_x^M(\nu)$ for $\epsilon = 1$ (a) in the hydrodynamic regime $x = 0.7$ ($x = q\xi$) and in the critical regime $x = \infty$ and (b) for intermediate values of $x = 1.5$ and $x = 10.0$.

have not calculated the first-order corrections to $a=\frac{1}{3}$ and $f=\epsilon/K_4$ we can only conclude,

$$R_N=(3/8\pi^2\epsilon)^{1/2}\ , \tag{4.83}$$

$$R_M=(1/24\pi^2\epsilon)^{1/2}\ , \tag{4.84}$$

and

$$R_{N/M}=3\quad . \tag{4.85}$$

in agreement with the results of HHS. Similarly we can only conclude that $\Omega_M(\infty)/\Omega_N(\infty)=\frac{1}{3}+O(\epsilon)$.

V. DISCUSSION

Our calculations to $O(\epsilon)$ for the isotropic antiferromagnet have led to nontrivial results. We have been able to carry out the complete calculation for T greater than or equal to T_N. We see that the three point vertex (which arises from the Poisson-bracket terms) leads to qualitative corrections to the TDGL model. We found a new dynamical fixed point corresponding to nonzero λ_{MN}. The TDGL fixed point ($\lambda_{MN}=0$) is unstable. A particularly interesting feature of our calculation is the establishment of fluctuation induced peaks in the shape function. This feature could not be established from general arguments.[56] Although peaks of this type have been observed in neutron scattering experiments,[50] we have not found the central peaks observed in these experiments. One possible resolution of this discrepancy is that we should include a coupling to the energy in our model. Halperin, Hohenberg, and Siggia point out that for magnetic systems, where $d=3$ and $N=3$ and the specific-heat index is believed to be negative, the energy coupling should change correction terms but not the leading singularities near the critical point. This does not rule out the possibility of energy conservation changing the shape functions near the critical point.

Since we have only carried out a calculation to first order in ϵ we must consider the effects of higher-order terms in ϵ. We know from ϵ-expansion calculations for static indices that second-order corrections can be significant and second-order results have given, in the static case, the "best" values for the indices[9] (that is, they represent the best truncation of the asymptotic series). It seems fairly evident, from the involved analysis of Sec. III, that any attempt to carry out the RNG to $O(\epsilon^2)$ will be extremely complicated. However, if we believe that the stable fixed-point structure we found to first order in ϵ persists to higher orders in ϵ we can avoid an explicit discussion of the RNG and use a generalization of Wilson's[1,37] Feynman-graph method to calculate to higher orders in ϵ. Halperin, Hohenberg, and Siggia have developed such a Feynman-graph approach.[57] While the value of z is not changed by higher-order terms, due to symmetry considerations, the corrections they find to the ratios defined by (4.77)–(4.79) due to $O(\epsilon^2)$ terms appear significant. The development of this technique from our point of view involves a detailed discussion of higher-order response functions which serve as dynamical generalizations of the n-point vertices treated by Wilson in his Feynman-graph method. We will discuss these points in detail elsewhere.

The next logical step in studying the antiferromagnet is to calculate the response and correlation functions below T_N. Even before we carry out this calculation, we can note, from continuity arguments, that the fluctuation induced peaks that we found at and above T_N will persist below T_N. The very interesting question is whether these peaks go continuously over into the usual spin waves as one goes from large x to small x. We are presently investigating this question.

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APPENDIX

Our exponentiations are guided by the simple guidelines: (a) Preserve the predictions of dynamical scaling as given by (4.31); (b) preserve the positivity of $\mathrm{Re}\Gamma(q,\omega)$ which is demanded by the positivity of the correlation function; (c) perform a minimal amount of exponentiation (i.e., exponentiate the simplest possible function).

In developing an exponentiation procedure we deal with quantities of the form

$$\Gamma(q,\omega)=\Gamma(0,0)\left(1+\epsilon\sum_i\left[a_i(x,\nu)\ln b_i(x,\nu)\right]+\epsilon C(x,\nu)\right), \tag{A1}$$

where we have renormalized at $q=\omega=0$ such that

$$\epsilon\sum_i\left[a_i(0,0)\ln b_i(0,0)\right]+\epsilon C(0,0)=0 \tag{A2}$$

and $\Gamma(0,0)=\mathrm{const}\,\xi^{\epsilon/2}$, thus satisfying (4.32a). There will be divergences arising from the a_i terms in (A1) in the limits of large x or ν. We define C such that it has no divergences for large x and ν. Because our choice of renormalization point ($x=\nu=0$), we have a good description of the small x, ν behavior, but now we must treat the large x, ν behavior accurately. We can see how things go for large x and ν by allowing $x^2 \to \lambda x^2$ and $\nu \to \lambda\nu$ and expanding a_i and b_i in powers of λ:

$$a_i = \lambda^2 a_i^{(1)} x^2 i\nu + \lambda(a_i^{(2)} x^2 + a_i^{(3)} i\nu) + a_i^{(4)} + O(1/\lambda),$$

$$b_i = \lambda^2 b_i^{(1)} x^2 i\nu + \lambda(b_i^{(2)} x^2 + b_i^{(3)} i\nu) + b_i^{(4)} + O(1/\lambda).$$

There are several cases of interest:

(i) If $a_i^{(1)}$, $a_i^{(2)}$, or $a_i^{(3)}$ do not vanish, then $b_i^{(1)}$, $b_i^{(2)}$, $b_i^{(3)}$ must vanish and $b_i^{(4)} = 1$ or we have a breakdown in scaling. In this case we need not exponentiate and can include these terms in the $C(x,\nu)$ terms.

(ii) The main case of interest is where the $a_i^{(1)}$, $a_i^{(2)}$, and $a_i^{(3)}$ vanish. In this case we need to exponentiate and $\epsilon a_i^{(4)}$ will contribute to the exponent. We then need to choose that part of b_i to exponentiate. We will define this piece as b_i^0. Clearly b_i^0 should give the large x^2 and ν behavior correctly and we can assume the form

$$b_i^0 = A\,(b_i^{(1)} x^2 i\nu + b_i^{(2)} x^2 + b_i^{(3)} i\nu + B).$$

We include the factor A since we take a logarithm of b_i^0 and an overall scale factor can not change the large x, ν values of $\ln b_i^0$. We include B since it also can not change the large x, ν limits. We choose A and B by requiring that b_i^0 be separable in x and $i\nu$ (this is for convenience) and that $b_i^0(0,0) = 1$ (which makes it easy to maintain the original normalization at $x=\nu=0$). With the proper choice for A and B we can write

$$b_i^0(x,\nu) = [1 + (b_i^{(1)}/b_i^{(3)})x^2][1 + (b_i^{(1)}/b_i^{(2)})i\nu].$$

Here we must look at the explicit values of the b_i's and be sure that $b_i^0(x,\nu)$ has no zero, otherwise we may violate condition b and introduce spurious behavior in the shape function. If there is a zero then we must choose B in a different manner. In the cases treated thus far this situation has not developed. We can then write

$$\Gamma(q,\omega) = \Gamma(0,0)\left(1 + \epsilon\sum_i a_i^{(4)} \ln b_i^0(x,\nu) + \epsilon\sum_i [a_i(x,\nu)\ln b_i - a_i^{(4)} \ln b_i^0] + C\right),$$

which can be written to $O(\epsilon)$ as

$$\Gamma(q,\omega) = \Gamma(0,0)\left(1 + \epsilon\sum_i a_i^{(4)} \ln b_i^0(x,\nu)\right)\left[1 + \epsilon\left(C(x,\nu) + \sum_i [a_i(x,\nu)\ln b_i - a_i^{(4)}\ln b_i^0]\right)\right]$$

$$= \Gamma(0,0)\prod_i [b_i^0(x,\nu)]^{\epsilon a_i^{(4)}}\left[1 + \epsilon\left(C(x,\nu) + \sum_i [a_i(x,\nu)\ln b_i - a_i^{(4)}\ln b_i^0]\right)\right].$$

This form satisfies all of our requirements.[58]

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[38]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 248 (1972).

[39]HHS have allowed the index η_M in (3.12c) to be a parameter in their model. The net effect is that z is determined by keeping λ_{NM} invariant rather than Γ_N and $\Gamma_M r$. The structure of the fixed point remains the same.

[40]K. Kawasaki, Prog. Theor. Phys. 39, 285 (1968).

[41]See Sec. V A in MM (Ref. 20.).

[42]See, for example, Eq. (4.4) in MM (Ref. 20).

[43]L. van Hove, Phys. Rev. 95, 1374 (1954).

[44]This discrepancy was resolved (private communication, B. I. Halperin) after this paper was accepted for publication.

[45]Note that our definitions of $\bar{t}$, τ, and $W_N(x,\nu)$ differ from those in Ref. 29. In particular $W_N(x,\nu)$, as defined here differs by an additive constant (independent of x and ν) from Eq. (5) in Ref. 29. These differences arise from a slightly different method of exponentiation used here. Since our final results for the shape function are essentially unchanged, except for a change in scale for ν [since $\omega(q,\xi)$ is different], we have support for the insensitivity of our results to the exponentiation procedure.

[46]An extensive program of investigation on this problem from the development of microscopic methods to detailed numerical analysis of the resulting equations has been carried out. Some of the relevant references are: P. Resibois and M. De Leener, Phys. Rev. 178, 806 (1969); P. Resibois and C. Piette, Phys. Rev. Lett. 24, 514 (1970); C. Joukoff-Piette and P. Resibois, Phys. Lett. A 42, 531 (1973); P. C. Hohenberg, M. De Leener, and P. Resibois, Physica 65, 505 (1973).

[47]D. L. Huber and D. A. Krueger, Phys. Rev. Lett. 24, 111 (1970).

[48]R. Nathans, F. Menzinger, and S. Pickart, J. Appl. Phys. 39, 1237 (1968); H. Lau, L. Corliss, A. Delapalme, J. Hastings, R. Nathans, and A. Tucciarone, Phys. Rev. Lett. 23, 1225 (1969).

[49]Note that the shape function calculated here and in Ref. 29 are in close qualitative agreement despite the use of a slightly different exponentiation procedure. (See Ref. 45.)

[50]A. Tucciarone, H. Y. Lau, L. M. Corliss, A. Delapalme, and J. W. Hastings, Phys. Rev. B 4, 3206 (1971).

[51]F. Wegner, Z. Phys. 218, 265 (1969).

[52]F. Wegner, Z. Phys. 216, 433 (1968).

[53]See Eq. (F. 15) of Ref. 26.

[54]See the comments in Sec. IV E.

[55]A. Tucciarone, J. M. Hastings, and L. M. Corliss, Phys. Rev. B 8, 1103 (1973).

[56]W. Marshall, in *Critical Phenomena*, edited by M. S. Green and J. V. Sengers, Natl. Bur. Stand. Mics. Pub. No. 273 (U.S. GPO, Washington, D.C., 1965).

[57]These authors use the elegant techniques due to P. C. Martin, E. D. Siggia, and H. A. Rose [Phys. Rev. A 8, 423 (1973)].

[58]There are two special cases worth mentioning where (a) $b_i(x,\nu) = b_i(x^2, x^2 f(\nu))$ or (b) $b_i(x,\nu) = b_i(\nu, \nu f(x^2))$. The large-$x$, $-\nu$ behavior is still validly represented by b_i^0, but the small-x (case a) or small-ν (case b) (behavior) is not of this form. A direct expansion of $b_i(x,\nu)$ would have $b_i^{(3)} = 0$ or $b_i^{(2)} = 0$, respectively. We can avoid this difficulty by working with a new frequeny $\bar{\nu} = x^2\nu/(a + x^2)$ (case a) or new $\bar{x} = x^2\nu/(a+\nu)$ (case b). The parameter a is determined from a small-x or small-ν expansion of $b_i(x,\nu)$. Case a arises in the ferromagnetic example. See M. Nolan and G. F. Mazenko (unpublished).