Free energy of superfluid ^{3}He †

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We present a systematic scheme for calculating the free energy of superfluid Fermi liquids by an asymptotic expansion in the small parameter T_c/T_F. We use this scheme to evaluate the strong-coupling corrections to the free energy of superfluid ^{3}He. We show that the leading corrections can be expressed in terms of the normal-state quasiparticle scattering amplitude, and discuss the strong-coupling results using the *s-p* approximation for the scattering amplitude.

I. INTRODUCTION AND SUMMARY

In this paper we present a microscopic theory of the free energy of a neutral Fermi system with pairing and apply the theory to the superfluid phases of ^{3}He, assuming, as now seems almost certain, that these phases are characterized by spin triplet pairing with $l=1$.[1,2] We consider only spatially uniform systems in the absence of external fields, but initially specify no particular type of pairing. Hence, although we are primarily concerned with superfluid ^{3}He, the general theory developed in Sec. II of this paper should be relevant to other systems with pairing, such as atomic nuclei (allowing for finite-size effects as in Ref. 3) and perhaps the interiors of neutron stars. The basis of our theory is an asymptotic expansion of the free-energy difference $\Delta\Omega$ between the paired and normal states in powers of T_c/T_F, the ratio of the transition temperature to the Fermi temperature $T_F = p_F^2/2m^*k_B$. The leading term in this expansion is the free energy of the BCS pairing theory[4]; as is by now well known, this theory can account neither for the quantitative thermodynamic properties of superfluid ^{3}He, nor even for the stability of a phase with the particular magnetic properties of ^{3}He-*A*. This failure of the BCS theory is at first sight quite surprising: $T_c/T_F \sim 10^{-3}$ for ^{3}He, and one is inclined to neglect all effects of higher order in T_c/T_F, as is successfully done in the theory of superconductors.[5] The inability of simple BCS theory to describe the pairing in ^{3}He has two causes. The first is the strength of the residual interactions between ^{3}He quasiparticles. These interactions, as, for example, measured by the normal-state Landau parameters,[6] are an order of magnitude stronger in ^{3}He than in typical superconducting metals. The second reason, first emphasized by Mermin and Stare,[7] is that for $l>0$ pairing even small corrections to the BCS free energy can have significant physical consequences, because the free-energy differences in BCS theory between different states of the same l can be much smaller than the characteristic pairing energy itself.

In the early 1960s several authors pointed out that the BCS free energy represents the leading term in an expansion in powers of T_c/T_F, but seem to have mistakenly concluded that the first corrections are smaller by two powers of T_c/T_F [3,8]; the nature of the leading corrections was pointed out by Anderson and Brinkman,[9] who realized that these corrections could explain the stability of ^{3}He-*A*. We find that the most important corrections to the BCS approximation are smaller by one power of T_c/T_F and can be calculated from the two-particle scattering amplitude for quasiparticles on the Fermi surface in the normal state. Our result unifies the microscopic theory of the superfluid free energy and the microscopic theories of the static and transport properties of both the normal Fermi liquid[10,11] and the superfluid phases[12,13]: The results of all these theories can be expressed in terms of the same quasiparticle scattering amplitude.

In Sec. III we use our theory to calculate the free energy of ^{3}He in the neighborhood of the transition temperature in terms of the scattering amplitude, and in Sec. IV we evaluate the resulting expressions in the s- and p-wave scattering approximation to the scattering amplitude (s-p approximation). The phenomenological Ginzburg-Landau functional provides a convenient bridge between these results and the experimental thermodynamic properties of ^{3}He near T_c. The $l=1$ functional, in the notation of Mermin and Stare,[7] is

$$\Delta\Omega[A] = \tfrac{1}{3}\alpha(T)\,\mathrm{Tr}AA^\dagger + \beta_1\,|\mathrm{Tr}AA^T|^2 + \beta_2\,(\mathrm{Tr}AA^\dagger)^2 + \beta_3\,\mathrm{Tr}(AA^T)(AA^T)^* + \beta_4\,\mathrm{Tr}(AA^\dagger)^2 + \beta_5\,\mathrm{Tr}(AA^\dagger)(AA^\dagger)^*\,, \tag{1.1}$$

where the order parameter A_{ij} is an arbitrary 3×3 complex matrix related to the energy-gap matrix $\Delta_{\alpha\beta}(\hat{k})$ by

$$\Delta_{\alpha\beta}(\hat{k}) = \vec{\Delta}(\hat{k})\circ(i\vec{\sigma}\sigma_y)_{\alpha\beta}\,, \tag{1.2}$$

$$\Delta_i(\hat{k}) = \sum_{j=1}^{3} A_{ij}(\hat{k})_j\,, \tag{1.3}$$

and the coefficient $\alpha(T)$ has the form $(T/T_c-1)\alpha'$. In the BCS theory the Ginzburg-Landau parameters are

$$\begin{aligned} &\alpha^{\mathrm{BCS}}(T) = N(0)(T/T_c-1)\,,\\ &-\beta_5^{\mathrm{BCS}} = \beta_4^{\mathrm{BCS}} = \beta_3^{\mathrm{BCS}} = \beta_2^{\mathrm{BCS}} = -2\beta_1^{\mathrm{BCS}}\,,\\ &\beta_1^{\mathrm{BCS}} = -N(0)(1/\pi k_B T_c)^2\{\tfrac{1}{30}[\tfrac{7}{8}\zeta(3)]\}\,, \end{aligned} \tag{1.4}$$

with $N(0) = k_F^3/2\pi^2 v_F p_F$, the single-spin quasiparticle density of states at the Fermi surface. Following Mermin and Stare we call any theory in which the ratios of the β_i are as in (1.4) "weak coupling," and all other theories "strong coupling." In all weak-coupling theories $\Delta\Omega$ is minimized by the "isotropic" state first studied by Balian and Werthamer.[14] By studying the stationary points of the free-energy functional for general values of the β_i, Mermin and Stare[15] and Barton and Moore[16] have shown that (assuming $l=1$) the B phase of ^{3}He must be the isotropic state, the A phase must be the axial state, and the A_1 phase in a magnetic field must be the state obtained from the axial state by setting either $\Delta_{\uparrow\uparrow}$ or $\Delta_{\downarrow\downarrow}$ equal to zero. Hence, a successful microscopic theory must give strong-coupling corrections to the BCS theory such that, at pressures above the polycritical pressure, the β_i lie in a region of parameter space where the axial state is known to be stable. A microscopic theory must also fit the observed specific-heat discontinuities at T_c, which yield more detailed information on the β_i. To discuss the corrections to the BCS theory it is convenient to introduce reduced Ginzburg-Landau parameters $\overline{\beta}_i$ and $\Delta\overline{\beta}_i$ defined by

$$\begin{aligned} &\overline{\beta}_i = \left(\frac{N(0)}{\alpha'}\right)^2 \frac{\beta_i}{|\beta_1^{\mathrm{BCS}}|}\,,\\ &\Delta\overline{\beta}_i = \overline{\beta}_i - \overline{\beta}_i^{\mathrm{BCS}}\,, \end{aligned} \tag{1.5}$$

in terms of which the specific-heat discontinuities at T_c are

$$\begin{aligned} &\frac{\Delta c_A}{c_N} = 1.19\,\frac{2}{\overline{\beta}_2+\overline{\beta}_4+\overline{\beta}_5}\,,\\ &\frac{\Delta c_{A_1}}{c_N} = 0.594\,\frac{4}{\overline{\beta}_2+\overline{\beta}_4}\,,\\ &\frac{\Delta c_B}{c_N} = 1.43\,\frac{5}{3(\overline{\beta}_1+\overline{\beta}_2)+\overline{\beta}_3+\overline{\beta}_4+\overline{\beta}_5}\,. \end{aligned} \tag{1.6}$$

The specific-heat discontinuities have been accurately measured only at the melting curve, where Halperin *et al.*[17] found $\Delta\overline{\beta}_2+\Delta\overline{\beta}_4 = -0.70\pm 0.13$ and $\Delta\overline{\beta}_5 = -0.1\pm 0.13$. The most striking feature of these experimental results is the small size of $\Delta\overline{\beta}_5$ relative to $\Delta\overline{\beta}_2+\Delta\overline{\beta}_4$; all available theories fail on this point! The s-p approximation of Sec. IV gives $\Delta\overline{\beta}_2+\Delta\overline{\beta}_4 = -0.74$, in accidental agreement with experiment, but has $\Delta\overline{\beta}_5 = -0.90$. In spin-fluctuation theories[18-20] the over-all scale of the $\overline{\beta}_i$ is an adjustable parameter, but these theories fare even worse than the s-p approximation on the ratio $\Delta\overline{\beta}_5/(\Delta\overline{\beta}_2+\Delta\overline{\beta}_4)$: Brinkman, Serene, and Anderson (BSA)[18] find 2.0 for this ratio, while Tewordt, Fay, Dörre, and Einzel[20] obtain 2.13, compared to the experimental value of 0.14.

In our opinion, the disagreement between these theories and experiment is not surprising. Our general results in Sec. III demonstrate that the strong-coupling corrections depend sensitively on the detailed structure in the scattering amplitude for quasiparticles; only a reasonably good approximation to the scattering amplitude can give reliable strong-coupling corrections. A crude check for the quality of an approximation is provided by normal-state properties such as the Landau parameters and the low-temperature transport coefficients, which can all be expressed in terms of the quasiparticle scattering amplitude.[21] The spin-fluctuation model is based on a scattering amplitude which yields normal-state properties in significant disagreement with experiment and, therefore, cannot be expected to give quantitatively correct results for superfluid ^{3}He. The properties of the normal state at low pressures are represented adequately by the scattering amplitude in the s-p approximation,[22] but at high pressures the s-p approximation does not yield quantitative agreement with measured transport properties of normal ^{3}He,[13] and we cannot expect reliable quantitative results for the superfluid properties at high pressures. In the high-pressure region an adequate approximation to the scattering amplitude is lacking. We present the results in s-p approximation for the following reasons: The normal-state data indicate that the s-p scattering amplitude is probably a good approximation at low pressures, and we want to stimulate thermodynamic measurements in superfluid ^{3}He at low pressures to check this presumption. Secondly, we want to demonstrate that at higher pressures the s-p results, although (as expected) not quantitatively correct, are in reasonable qualitative agreement with experiments in superfluid ^{3}He.

In Fig. 1 we show the pressure dependence in the s-p approximation of the specific-heat discontinuities for the isotropic and axial states; this is equivalent to giving the free energies, since the two are related by

$$\Delta\Omega = -\tfrac{1}{3}\pi^2(\Delta c/c_N)N(0)k_B^2(T-T_c)^2 . \qquad (1.7)$$

At high pressures the calculated corrections stabilize the axial state relative to the isotropic state. At the melting curve the combination $\Delta\bar{\beta}_2 + \Delta\bar{\beta}_4 + \Delta\bar{\beta}_5$, which determines the specific-heat jump at the A transition, is a factor of 2 too large. The magnitude of this error lies in the expected range; the normal-state viscosity indicates roughly the same inaccuracy of the s-p scattering amplitude.[13] The calculated free energies cross at 9 bar, while the measured polycritical pressure is 21.5 bar, so the s-p approximation overestimates the strong-coupling corrections to the axial-isotropic free-energy difference even at relatively low pressures.

In Table I we give theoretical results for the $\Delta\bar{\beta}_i$ and the specific-heat discontinuities at zero pressure, where the s-p approximation should be most accurate. The s-p approximation parametrizes the scattering amplitude in terms of the Landau parameters, and we have followed the usual procedure[21] of setting $F_l^{s,a}=0$ for $l \geq 2$. To obtain the results in Table I we took $F_0^s = 10.07$, $F_0^a = -0.67$, and $F_1^s = 6.04$, the experimentally determined values given by Wheatley.[2] To indicate the sensitivity of the results to the remaining parameter F_1^a, we give results for two different choices of F_1^a: In the first row of Table I we used $F_1^a = -0.68$, the value obtained from the forward scattering sum rule on the scattering amplitude; in the second row we used $F_1^a = 0$, a choice consistent with the experimental value -0.15 ± 0.3 obtained by Corruccini, Osheroff, Lee, and Richardson.[23] [For other values of the Landau parameters, the $\Delta\bar{\beta}_i$ are easily obtained from Eq. (4.5).] The calculated strong-coupling corrections are small at zero pressure, of order 10%, but specif-

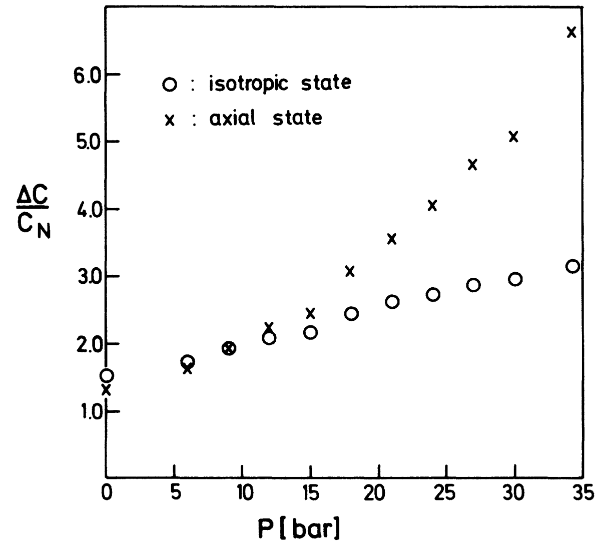

FIG. 1. Pressure dependence of the specific-heat discontinuities in the s-p approximation. The results are obtained from the Landau parameters F_0^s, F_1^s, F_0^a of Ref. 2; F_1^a is calculated from the forward scattering sum rule.

ic-heat measurements comparable in precision to those of Halperin *et al.*[17] could check the expected accuracy of the s-p approximation. Concerning the uncertainties at high pressures, our optimistic view is that a systematic study of the strong-coupling corrections, together with the transport coefficients in both the normal and superfluid phases, should allow us to learn much more about the normal-state quasiparticle scattering amplitude and to uncover its presently unknown structure.

II. T_c/T_F EXPANSION OF $\Delta\Omega$: GENERAL RESULTS

In this and the following sections we calculate formally the free-energy difference $\Delta\Omega$ between the superfluid and normal states of a Fermi system by an asymptotic expansion in the small parameter T_c/T_F. The dominant term in this expansion is the weak-coupling free energy; it is of order $(T_c/T_F)^2$. The leading corrections to the weak-coupling result are of order $(T_c/T_F)^3$. The aim of this paper is a calculation of these leading corrections. Our main result is that the most

TABLE I. Ginzburg-Landau coefficients and specific-heat discontinuities for ^{3}He at zero pressure in the s–p approximation.

	$\Delta\bar{\beta}_1$	$\Delta\bar{\beta}_2$	$\Delta\bar{\beta}_3$	$\Delta\bar{\beta}_4$	$\Delta\bar{\beta}_5$	$\Delta c_B/c_N$	$\Delta c_A/c_N$	$\Delta c_{A_1}/c_N$
$F_1^a=-0.68$	-0.02	-0.01	-0.02	-0.06	-0.12	1.52	1.31	0.60
$F_1^a=0.0$	-0.04	-0.02	-0.03	-0.08	-0.10	1.55	1.32	0.61

important $(T_c/T_F)^3$ terms can be calculated from the two-particle scattering amplitude for *quasiparticles* on the Fermi surface in the normal state. The same scattering amplitude determines the transport coefficients in the Landau theory of Fermi liquids, so *in this sense* the free energy through order $(T_c/T_F)^3$ can be calculated from Fermi-liquid theory.

We present the derivation of these results in two parts. In the first part, comprising the remainder of this section, we rely only on general assumptions about the properties of the Green's functions and scattering amplitudes, such as the characteristic order of magnitude of these quantities and their characteristic energy and momentum scales. These considerations are sufficient to derive the T_c/T_F expansion for $\Delta\Omega$ and to show that the $(T_c/T_F)^3$ terms have the form just described.

The arguments in this part hold for both singlet and triplet pairing, for any angular momenta, and for all temperatures below T_c. In Sec. III, on the other hand, we concentrate on the contributions to $\Delta\Omega$ of fourth order in the order parameter and for triplet pairing. The principal results of Sec. III are Eqs. (3.15) and (3.30) which give explicitly the $(T_c/T_F)^3$ contributions to $\Delta\Omega$ for arbitrary odd l and to the $l=1$ Ginzburg-Landau parameters β_i, in terms of angular integrals of the scattering amplitude on the Fermi surface.

A. Formal preliminaries

In this section we give a brief introduction to the formalism used in our calculations. We begin from an expression for the grand canonical thermodynamic potential density Ω (hereafter simply called the free energy) as a stationary functional of the exact self-energy. This functional was first discussed for normal Fermi systems by Luttinger and Ward,[24] whose work was extended to the electron-phonon system by Eliashberg[25] and to general single-component superfluids by DeDominicis and Martin.[26] We use this functional both because its stationarity properties simplify our expansion and also because it can serve to systematically generate the strong-coupling corrections to response functions and to properties of spatially inhomogeneous systems.[27]

By using a 4×4 matrix representation for the Green's functions and self-energies, one can write the stationary free-energy functional Ω for translationally invariant systems in the compact form

$$\Omega[\hat{\Sigma}] = -\tfrac{1}{2}k_B T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}_4\{\hat{\Sigma}(\vec{k},\omega_n)\hat{G}(\vec{k},\omega_n) + \ln[-\hat{G}(\vec{k},\omega_n)^{-1}]\} + \Phi[\hat{G}]. \tag{2.1}$$

Here the self-energy $\hat{\Sigma}$ and Green's function $\hat{G}$ are 4×4 matrices constructed from the more familiar 2×2 spin-matrix self-energies and Green's functions,

$$\hat{\Sigma}(\vec{k},\omega_n) = \begin{pmatrix} \Sigma(\vec{k},\omega_n) & \Delta(\vec{k},\omega_n) \\ \Delta^\dagger(\vec{k},-\omega_n) & -\Sigma^T(-\vec{k},-\omega_n) \end{pmatrix},$$
$$\hat{G}(\vec{k},\omega_n) = \begin{pmatrix} G(\vec{k},\omega_n) & F(\vec{k},\omega_n) \\ F^\dagger(\vec{k},-\omega_n) & -G^T(-\vec{k},-\omega_n) \end{pmatrix}. \tag{2.2}$$

To avoid confusion we will denote the trace in the 4×4 space by Tr_4, and in the 2×2 space by Tr_2. $\Phi[\hat{G}]$ is a functional defined diagrammatically by the requirement that $2[\delta\Phi/\delta\hat{G}^T(\vec{k},\omega_n)]$ reproduces the formal skeleton-diagram expansion for $\hat{\Sigma}(\vec{k},\omega_n)$ as a functional of $\hat{G}$. Finally to make Ω a functional of $\hat{\Sigma}$ alone one fixes $\hat{G}$ in terms of $\hat{\Sigma}$ by the Dyson equation

$$\hat{G}^{-1}(\vec{k},\omega_n) = \hat{G}_0^{-1}(\vec{k},\omega_n) - \hat{\Sigma}(\vec{k},\omega_n), \tag{2.3}$$

where

$$\hat{G}_0(\vec{k},\omega_n) = \begin{pmatrix} G_0(\vec{k},\omega_n) & 0 \\ 0 & -G_0(-\vec{k},-\omega_n) \end{pmatrix},$$

and $G_0(\vec{k},\omega_n)$ is the Green's function for noninteracting fermions. The functional $\Omega[\hat{\Sigma}]$ has the following stationarity property:

$$\delta\Omega[\hat{\Sigma}]/\delta\hat{\Sigma}(\vec{k},\omega_n) = 0 \tag{2.4}$$

for any $\hat{\Sigma}$ satisfying the self-consistency equation

$$\delta\Phi[\hat{G}]/\delta\hat{G}(\vec{k},\omega_n) = \tfrac{1}{2}\hat{\Sigma}^T(\vec{k},\omega_n). \tag{2.5}$$

In this equation one treats all the elements of $\hat{G}$ and $\hat{\Sigma}$ as independent; our functional derivatives are defined by

$$\delta\Phi[\hat{G}] = k_B T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}_4\left(\frac{\delta\Phi}{\delta\hat{G}(\vec{k},\omega_n)}\,\delta\hat{G}^T(\vec{k},\omega_n)\right). \tag{2.6}$$

Equation (2.5) is an implicit equation for $\hat{\Sigma}$, since $\hat{G}$ on the left-hand side is defined by Eq. (2.3).

At one such stationary point $\Omega[\hat{\Sigma}]$ is equal to the true thermodynamic potential $\Omega(T,\mu)$, and the Green's functions have their physical values

$$G_{\alpha\delta}(\vec{k},\omega_n) = -\int_0^\beta d\tau\, e^{i\omega_n\tau}\langle T_\tau a_{\vec{k}\alpha}(\tau) a_{\vec{k}\delta}^\dagger(0)\rangle ,$$

$$G^T_{\alpha\delta}(-\vec{k},-\omega_n) = -\int_0^\beta d\tau\, e^{-i\omega_n\tau}\langle T_\tau a_{-\vec{k}\delta}(\tau) a_{-\vec{k}\alpha}^\dagger(0)\rangle ,$$

$$F_{\alpha\delta}(\vec{k},\omega_n) = -\int_0^\beta d\tau\, e^{i\omega_n\tau}\langle T_\tau a_{\vec{k}\alpha}(\tau) a_{-\vec{k}\delta}(0)\rangle ,$$

$$F^\dagger_{\alpha\delta}(\vec{k},\omega_n) = -\int_0^\beta d\tau\, e^{-i\omega_n\tau}\langle T_\tau a_{-\vec{k}\alpha}^\dagger(\tau) a_{\vec{k}\delta}^\dagger(0)\rangle . \tag{2.7}$$

In particular, at the physical stationary point, $\Delta(\vec{k},\omega_n)$ and $\Delta^\dagger(\vec{k},\omega_n)$ are matrix adjoints, and $\Sigma^T(\vec{k},\omega_n)$ is the transpose of $\Sigma(\vec{k},\omega_n)$:

$$\Delta^\dagger_{\alpha\beta}(\vec{k},\omega_n) = \Delta_{\beta\alpha}(\vec{k},\omega_n)^*, \quad \Sigma^T_{\alpha\beta}(\vec{k},\omega_n) = \Sigma_{\beta\alpha}(\vec{k},\omega_n). \tag{2.8}$$

In the normal state, above T_c, the pairing self-energy $\Delta(\vec{k},\omega_n)$ and Green's function $F(\vec{k},\omega_n)$ vanish. We now assume that for temperatures near T_c the normal state is correctly described by the microscopic version of Landau's Fermi-liquid theory, and we denote the corresponding self-consistent solution of (2.3) and (2.5) by $\hat{\Sigma}_N$, $\hat{G}_N$. Below T_c this normal solution should exist and represents a straightforward extrapolation of the Fermi-liquid theory. We take the properties of the normal state to be known and calculate only the difference $\Delta\Omega$ between the true free energy and the normal-state free energy $\Omega_N = \Omega[\hat{\Sigma}_N]$.

The natural variables for calculating the free-energy difference $\Delta\Omega$ are the "superfluid parts" of the self-energies and Green's functions, obtained by subtracting off the corresponding quantities for the normal state,

$$\hat{\Sigma}(\vec{k},\omega_n) - \hat{\Sigma}_N(\vec{k},\omega_n) = \begin{pmatrix} \Sigma(\vec{k},\omega_n) - \Sigma_N(\vec{k},\omega_n) & \Delta(\vec{k},\omega_n) \\ \Delta^\dagger(\vec{k},-\omega_n) & -\Sigma^T(-\vec{k},-\omega_n) + \Sigma^T_N(-\vec{k},-\omega_n) \end{pmatrix},$$
$$\hat{G}(\vec{k},\omega_n) - \hat{G}_N(\vec{k},\omega_n) = \begin{pmatrix} G(\vec{k},\omega_n) - G_N(\vec{k},\omega_n) & F(\vec{k},\omega_n) \\ F^\dagger(\vec{k},-\omega_n) & -G^T(-\vec{k},-\omega_n) + G^T_N(-\vec{k},-\omega_n) \end{pmatrix}. \tag{2.9}$$

When one takes $\hat{\Sigma}_N$ and $\hat{G}_N$ as known functions, $\Delta\Omega$ becomes a functional of the superfluid self-energy alone,

$$\Delta\Omega[\hat{\Sigma} - \hat{\Sigma}_N] = \Omega[\hat{\Sigma}] - \Omega[\hat{\Sigma}_N]. \tag{2.10}$$

Our scheme for calculating $\Delta\Omega$ will be to study the formal diagrammatic expansion of $\Phi[\hat{G}] - \Phi[\hat{G}_N]$ in powers of $\hat{G} - \hat{G}_N$.

Because $\Omega[\hat{\Sigma}]$ is stationary at $\hat{\Sigma}_N$, $\Delta\Omega$ cannot contain a term linear in $\hat{G} - \hat{G}_N$, and by using the relation

$$\delta\Phi[\hat{G}_N]/\delta\hat{G}(\vec{k},\omega_n) = \tfrac{1}{2}\hat{\Sigma}^T_N(\vec{k},\omega_n),$$

one sees easily from (2.1) that the linear term in $\Phi[\hat{G}] - \Phi[\hat{G}_N]$ is exactly cancelled by the linear term from the rest of the full functional $\Omega[\hat{\Sigma}]$. To simplify our discussion of the expansion of $\Phi[\hat{G}]$, we use this cancellation explicitly and omit the linear terms both in $\Phi[\hat{G}] - \Phi[\hat{G}_N]$ and in the other parts of the full free-energy functional. By this subtraction we obtain a new functional $\Delta\Phi$,

$$\Delta\Phi[\hat{G} - \hat{G}_N] = \Phi[\hat{G}] - \Phi[\hat{G}_N] - \tfrac{1}{2}k_B T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}_4\{\hat{\Sigma}_N(\vec{k},\omega_n)[\hat{G}(\vec{k},\omega_n) - \hat{G}_N(\vec{k},\omega_n)]\}, \tag{2.11}$$

in terms of which the stationarity condition for $\Delta\Omega$ becomes

$$\frac{\delta\Delta\Omega[\hat{\Sigma} - \hat{\Sigma}_N]}{\delta(\hat{\Sigma}(\vec{k},\omega_n) - \hat{\Sigma}_N(\vec{k},\omega_n))} = 0, \tag{2.12}$$

whenever

$$\frac{\delta\Delta\Phi[\hat{G} - \hat{G}_N]}{\delta(\hat{G}(\vec{k},\omega_n) - \hat{G}_N(\vec{k},\omega_n))} = \tfrac{1}{2}[\hat{\Sigma}^T(\vec{k},\omega_n) - \hat{\Sigma}^T_N(\vec{k},\omega_n)]. \tag{2.13}$$

Equation (2.13) serves to determine $\hat{\Sigma} - \hat{\Sigma}_N$ and $\hat{G} - \hat{G}_N$ self-consistently at the stationary points of $\Delta\Omega$. Hence any approximation to $\Delta\Phi$ generates an approximation to the free energy $\Delta\Omega$. We now can turn to constructing an expansion of $\Delta\Phi$ in powers of T_c/T_F.

B. T_c/T_F expansion of $\Delta\Phi$

The basic assumption we use to expand $\Delta\Phi$ is that k_BT_c is small compared to all relevant energies in the normal state. Our formal expansion parameter is therefore T_c/T_N, where k_BT_N is the smallest characteristic normal-state energy. An equivalent formulation of this assumption is that the superfluid coherence length $\xi_0 = \hbar v_F / 2\pi k_B T_c$ is large compared to ξ_N, the biggest relevant length in the normal state. The smallness of T_c/T_N, or equivalently of ξ_N/ξ_0, serves both to justify retaining only certain classes of diagrams in the diagrammatic expansion of $\Delta\Phi$, and to show that the incoherent parts of the Green's functions can be ignored in the diagrams retained. We expect that T_N is of the order of the Fermi temperature T_F, approximately 1 K in ^{3}He. A likely candidate for T_N is the spin-fluctuation temperature $(1-F_0^a)T_F \simeq T_F/4$, in which case T_c/T_N is still an excellent small parameter of order 10^{-3}. Since T_N cannot differ drastically from T_F, for notational simplicity we will express the asymptotic expansion in terms of the ratio T_c/T_F [equivalently $(k_F\xi_0)^{-1}$]. In this notation powers of T_F/T_N are absorbed into the coefficients of the asymptotic expansion.

The starting point for our considerations is the diagrammatic expansion of $\Delta\Phi[\hat{G}-\hat{G}_N]$. Each diagram contains two or more Green's-function lines representing elements of the matrix $\hat{G}-\hat{G}_N$; all the $\hat{G}-\hat{G}_N$ lines terminate in normal-state vertex functions represented by open circles. Some $\Delta\Phi$ diagrams of low order in $\hat{G}-\hat{G}_N$ are shown in Fig. 2, along with our diagrammatic conventions for the components of $\hat{G}-\hat{G}_N$. In our diagrams we do not distinguish between G and G^T, even though we took them as independent in the stationarity conditions (2.5) and (2.13). This is permissible if we modify the counting factors for the diagrams appropriately, since at the physical stationary points G^T is the transpose of G.

In order to convert the diagrammatic expansion of $\Delta\Phi$ into an asymptotic expansion in T_c/T_F, we need assumptions about the elements of a diagram, the normal-state vertices and the superfluid parts of the Green's functions. An m-point normal vertex function $\Gamma^{(m)}$ has $m/2$ incoming lines and $m/2$ outgoing lines because particle number is conserved in the normal state, and it depends on $m-1$ independent momenta $\vec{k}_i$ and frequencies $\omega_{n_i} = (2n_i+1)\pi k_B T$. Our assumption is that the characteristic scales for this momentum and frequency dependence are set by k_F and $k_B T_F$, respectively. Consequently, the order of magnitude of a vertex $\Gamma^{(m)}$ can be estimated by dimensional analysis which leads to

$$\Gamma^{(m)} \sim (k_F)^{-3(m/2-1)} k_B T_F . \tag{2.14}$$

This estimate requires, in particular, that $\Gamma^{(m)}$ contain no factor T_F/T_c. We are not aware of any relevant mechanism leading to such a strong temperature dependence of the normal-state vertices. We point out, however, that our assumptions exclude all critical phenomena from our considerations. Our theory is therefore limited to the temperature region where critical fluctuations are negligible; for ^{3}He this does not seriously restrict its range of applicability. Our assumptions about the normal-state Green's functions are taken from Fermi-liquid theory. G_N consists of quasiparticle and incoherent parts,

$$G_N(\vec{k}, \omega_n) = G_N(\vec{k}, \omega_n)_{\rm qp} + G_N(\vec{k}, \omega_n)_{\rm inc} , \tag{2.15}$$

where the quasiparticle Green's function is

$$G_{N\alpha\beta}(\vec{k}, \omega_n)_{\rm qp} = \frac{1}{Z}\, \frac{1}{i\omega_n - \hbar v_F (k-k_F)}\, \delta_{\alpha\beta} , \tag{2.16}$$

and the incoherent part $G_N(\vec{k}, \omega_n)_{\rm inc}$ is assumed to be of order $(k_B T_F)^{-1}$ and to vary with k and ω_n on the scales k_F and $k_B T_F$, respectively.

In contrast to the normal-state vertices and Green's function, which are *input parameters* in our theory, the superfluid Green's functions F, $F^\dagger$, and $G-G_N$ must be *calculated* from Eqs. (2.3) and (2.13). This we will do by a self-consistent procedure. Starting from an ansatz for $\hat{\Sigma}-\hat{\Sigma}_N$, the superfluid part of the self-energy, we will identify all the contributions to $\Delta\Phi$ through order $(T_c/T_F)^3$. Knowing $\Delta\Phi$ one can calculate $\hat{\Sigma}-\hat{\Sigma}_N$ from (2.13) to show that our ansatz is self-consistent. Fortunately, the ansatz only needs to be satisfied to leading order in T_c/T_F. Higher-order corrections first contribute to $\Delta\Omega$ in order $(T_c/T_F)^4$, as a con-

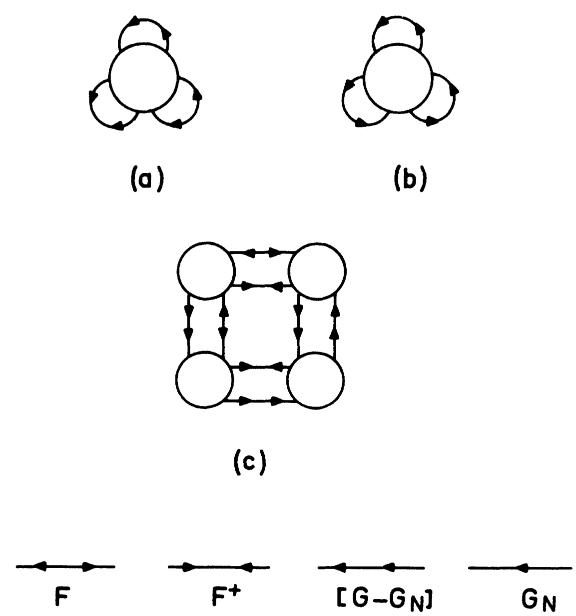

FIG. 2. Some diagrams for $\Delta\Phi[\hat{G}-\hat{G}_N]$, along with our diagrammatic conventions for the Green's-function lines. The open circles represent normal-state vertex functions.

sequence of the stationarity property of $\Omega[\hat{\Sigma}]$. Our ansatz is that $\Sigma(\vec{k}, \omega_n) = \Sigma_N(\vec{k}, \omega_n)$, and $\Delta(\vec{k}, \omega_n)$ is of order $k_B T_c$ and varies with k and ω_n on the scale of k_F and $k_B T_F$, respectively. These assumptions suffice to determine the order in T_c/T_F of any diagram for $\Delta\Phi$. The order-of-magnitude arguments which we give in the remainder of this section are independent of any specific type of paired state, so for simplicity we will use the $l=0$ Green's functions and suppress all spin indices.

Combining our assumptions about G_N, our ansatz for $\hat{\Sigma} - \hat{\Sigma}_N$, and the Dyson equation (2.3), we find

$$F(\vec{k}, \omega_n) = F_{\rm qp}(\vec{k}, \omega_n) + F_{\rm inc}(\vec{k}, \omega_n),$$
$$G(\vec{k}, \omega_n) - G_N(\vec{k}, \omega_n) = [G(\vec{k}, \omega_n) - G_N(\vec{k}, \omega_n)]_{\rm qp} + [G(\vec{k}, \omega_n) - G_N(\vec{k}, \omega_n)]_{\rm inc}, \quad (2.17)$$

where the quasiparticle parts, familiar from superconductivity theory, are given by

$$F_{\rm qp}(\vec{k}, \omega_n) = -\frac{1}{Z}\,\frac{\Delta}{\omega_n^2 + \xi_k^2 + |\Delta|^2},$$
$$[G(\vec{k}, \omega_n) - G_N(\vec{k}, \omega_n)]_{\rm qp} = -\frac{1}{Z}\,\frac{|\Delta|^2}{(i\omega_n - \xi_k)(\omega_n^2 + \xi_k^2 + |\Delta|^2)}. \quad (2.18)$$

Δ is the renormalized "energy gap,"

$$\Delta = (1/Z)\Delta(k_F, \pi T), \quad (2.19)$$

which we will use as the order parameter, and $\xi_k = \hbar v_F(k - k_F)$. For both $|\omega_n|$ and $|\xi_k|$ in the range $\lesssim k_B T_c$, $F_{\rm qp}$ and $(G - G_N)_{\rm qp}$ are of the same order,

$$F_{\rm qp}(\vec{k}, \omega_n) \sim [G(\vec{k}, \omega_n) - G_N(\vec{k}, \omega_n)]_{\rm qp} \sim (1/k_B T_F)(T_F/T_c). \quad (2.20a)$$

In this "low-energy range" of frequencies and momenta, the incoherent parts of the Green's functions are smaller than the quasiparticle parts by one power of T_c/T_F:

$$F_{\rm inc}(\vec{k}, \omega_n) \sim [G(\vec{k}, \omega_n) - G_N(\vec{k}, \omega_n)]_{\rm inc} \sim (1/k_B T_F) \times 1. \quad (2.20b)$$

As the frequency increases or the momentum moves away from the Fermi surface, the distinction between quasiparticle and incoherent Green's functions diminishes until, for $|\omega_n| \sim k_B T_F$ or $|\xi_k| \sim k_B T_F$, both parts are of the same order. In this "high-energy range" the order of magnitude of the Green's functions is given by

$$F_{\rm qp}(\vec{k}, \omega_n) \sim F_{\rm inc}(\vec{k}, \omega_n) \sim (1/k_B T_F)(T_c/T_F),$$
$$[G(\vec{k}, \omega_n) - G_N(\vec{k}, \omega_n)]_{\rm qp} \sim [G(\vec{k}, \omega_n) - G_N(\vec{k}, \omega_n)]_{\rm inc} \sim (1/k_B T_F)(T_c/T_F)^2. \quad (2.21)$$

Equations (2.14), (2.20), and (2.21) represent the basic order-of-magnitude estimates for classifying $\Delta\Phi$ diagrams with respect to their order in T_c/T_F. Factors of T_c/T_F in $\Delta\Phi$ diagrams arise explicitly from the superfluid Green's functions, as discussed above, and from frequency sums and momentum integrals involving these Green's functions. The "most dangerous" negative powers of T_c/T_F come from the quasiparticle Green's functions in the low-energy range. In this case, however, one must consider in detail the integration and summation restrictions necessary to keep the ξ and ω_n arguments of the Green's functions in the range $\lesssim k_B T_c$. For an order-of-magnitude estimate of these restrictions we use

$$k_B T \sum_{|\omega_n| \lesssim k_B T_c} \sim k_B T_F (T_c/T_F),$$
$$\int_{|k - k_F| \lesssim \xi_0^{-1}} d^3k \sim k_F^3 (T_c/T_F). \quad (2.22)$$

Hence, the restrictions lead to additional positive powers of T_c/T_F. Our T_c/T_F classifications are relative to the characteristic normal-state energy density $(k_B T_F) k_F^3$ which we obtain as a common factor for any diagram, collecting the prefactors T_F^{-1}, T_F, k_F^3 in the estimates (2.20), (2.21), (2.22), using (2.14) for the order of magnitude of the normal parts of a diagram, and estimating high-energy sums or integrals by $T\sum_n \sim T_F$ and $\int d^3k \sim k_F^3$.

Whereas the negative powers only depend on the number of quasiparticle lines in a diagram, the compensating positive powers depend in addition on the topology of a diagram, and so require the more detailed discussion which follows. For convenience, we group the diagrams by number of superfluid lines and discuss each group separately, starting with the easiest one to analyze.

1. Diagrams with four superfluid lines

We first discuss the contributions to $\Delta\Phi$ from the quasiparticle Green's functions in the low-energy range. According to the estimate (2.20), four quasiparticle lines in this energy range carry a factor $(T_c/T_F)^{-4}$. If the four arguments in these Green's functions are independent, not related by energy and momentum conservation, one obtains in addition a phase-space factor $(T_c/T_F)^4$ from restricting the four momentum integrations to the region within ξ_0^{-1} of the Fermi surface, and a further factor $(T_c/T_F)^4$ from restricting the four frequency summations to the range $\lesssim k_B T_c$. Such a diagram therefore contributes to $\Delta\Phi$ in order $(T_c/T_F)^4$ and will be neglected here.

Thus the only diagrams with four superfluid lines which are of interest here are those which separate by cutting the four superfluid lines, since in this case one of the four frequencies is fixed by energy conservation. This implies that only three of the frequencies have to be restricted to the range $\lesssim k_B T_c$; the fourth one is then automatically in this range. Such diagrams therefore contribute to $\Delta\Phi$ in order $(T_c/T_F)^3$. All diagrams of this type are shown in Figs. 3(f), 3(g), and 3(h). Note that momentum conservation, in contrast to energy conservation, does *not* reduce the order in T_c/T_F; if we restrict three momenta to the region near k_F, the fourth one given by momentum conservation does not automatically lie in this region. One therefore needs a further constraint on the three independent momenta which gives an additional factor T_c/T_F,[28] and a total momentum phase-space factor $(T_c/T_F)^4$, just as in the case of four independent momenta.

We will now show that the incoherent and the high-energy parts of the superfluid Green's functions do not contribute up to order $(T_c/T_F)^3$. This statement is obvious for the incoherent part in the low-energy range, since these Green's functions are at least one power in T_c/T_F smaller than the quasiparticle Green's functions. The discussion of the high-energy parts is slightly more complicated. Suppose that $m<4$ Green's functions are in the low-energy range and $4-m$ Green's functions have arguments at high energies. According to (2.20) and (2.21), these Green's functions carry at least a factor $(T_c/T_F)^{4-2m}$. The summation and integration restrictions necessary to keep m Green's functions in the low-energy range give a factor $(T_c/T_F)^{2m}$, so that the total contribution to $\Delta\Phi$ is of order $(T_c/T_F)^4$ and will be neglected.

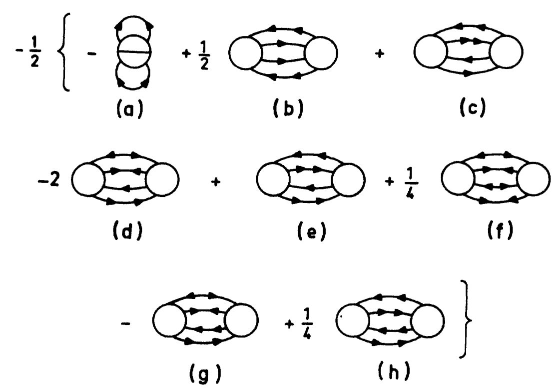

FIG. 3. Diagrams which contribute to $\Delta\Phi$ through order $(T_c/T_F)^3$.

For the diagrams with four superfluid lines we have now achieved the important result that the quasiparticle parts of the Green's functions are sufficient to calculate the free energy up to order $(T_c/T_F)^3$. To demonstrate the further simplifications which are possible we will next work out in detail the diagram with four off-diagonal Green's functions, Fig. 3(f). The contribution of this diagram to $\Delta\Phi$ is given by

$$-\tfrac{1}{2}(\tfrac{1}{4})(k_BT)^3\sum_{\omega_{n_1}}\sum_{\omega_{n_2}}\sum_{\omega_{n_3}}\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\int\frac{d^3k_3}{(2\pi)^3}\,|\Gamma^{(4)}(\vec{k}_1,\omega_{n_1},\vec{k}_2,\omega_{n_2};\vec{k}_3,\omega_{n_3},\vec{k}_1+\vec{k}_2-\vec{k}_3,\omega_{n_1}+\omega_{n_2}-\omega_{n_3})|^2$$
$$\times F^\dagger_{qp}(\vec{k}_1,\omega_{n_1})F^\dagger_{qp}(\vec{k}_2,\omega_{n_2})F_{qp}(\vec{k}_3,\omega_{n_3})F_{qp}(\vec{k}_1+\vec{k}_2-\vec{k}_3,\omega_{n_1}+\omega_{n_2}-\omega_{n_3})\,. \quad (2.23)$$

In contrast to the product of quasiparticle Green's functions in (2.23), which is strongly peaked for values of all four frequency arguments less than k_BT_c and all four momentum arguments within ξ_0^{-1} of the Fermi surface, the normal vertex $\Gamma^{(4)}$ varies with frequency on the scale of k_BT_F and with momentum on the scale of k_F. Thus to leading order in T_c/T_F, $\Gamma^{(4)}$ can be replaced in (2.23) by its zero-frequency limit, with $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ on the Fermi surface:

$$\Gamma^{(4)}(\vec{k}_1,\omega_{n_1},\vec{k}_2,\omega_{n_2};\vec{k}_3,\omega_{n_3},\vec{k}_1+\vec{k}_2-\vec{k}_3,\omega_{n_1}+\omega_{n_2}-\omega_{n_3})$$
$$\Rightarrow\Gamma^{(4)}(k_F\hat{k}_1,0,k_F\hat{k}_2,0;k_F\hat{k}_3,0,k_F(\hat{k}_1+\hat{k}_2-\hat{k}_3),0)\,. \quad (2.24)$$

The resulting function of the angular variables is independent of temperature to leading order in T_c/T_F, and, when all four momenta are on the Fermi surface, is proportional to the conventional dimensionless quasiparticle scattering amplitude $T(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)$,[11]

$$T(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)=\frac{k_F^3}{\pi^2 v_F p_F Z^2}\Gamma^{(4)}(k_F\hat{k}_1,0,k_F\hat{k}_2,0;k_F\hat{k}_3,0,k_F\hat{k}_4,0)\,. \quad (2.25)$$

(This choice of variables for T is highly redundant, but will be temporarily convenient; because all four quasiparticles are on the Fermi surface, T depends on only two independent variables.)

The smallness of T_c/T_F allows one further simplification: To lowest order, the four energy variables ξ in (2.23) are independent, so (2.23) is unchanged to order $(T_c/T_F)^3$ if the quasiparticle Green's functions are replaced by the following distributions of equivalent weight concentrated at the Fermi surface:

$$\frac{1}{Z}\,\frac{-\Delta}{\omega_n^2+\xi_k^2+|\Delta|^2}\Rightarrow\frac{\pi}{Z}\,\delta(\xi_k)\frac{-\Delta}{(\omega_n^2+|\Delta|^2)^{1/2}}\,. \quad (2.26)$$

Similarly in the diagrams with $(G-G_N)_{\rm qp}$ lines we can make the substitution

$$-\frac{1}{Z}\,\frac{|\Delta|^2}{(i\omega_n-\xi_k)(\omega_n^2+\xi_k^2+|\Delta|^2)}\Rightarrow-\frac{i\pi}{Z}\,\delta(\xi_k)\left(\frac{\omega_n}{(\omega_n^2+|\Delta|^2)^{1/2}}-\frac{\omega_n}{|\omega_n|}\right). \quad (2.27)$$

This enables us to perform the three k integrals in (2.23), leaving

$$-\tfrac{1}{2}(v_Fp_F)k_F^3(\tfrac{1}{4})\pi^2\left(\frac{k_BT_c}{2v_Fp_F}\right)^3\int\frac{d\Omega_1}{4\pi}\int\frac{d\Omega_2}{4\pi}\int\frac{d\Omega_3}{4\pi}\,\delta(|\hat{k}_1+\hat{k}_2-\hat{k}_3|-1)[T(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_1+\hat{k}_2-\hat{k}_3)]^2$$
$$\times\left(\frac{T}{T_c}\right)^3\sum_{n_1}\sum_{n_2}\sum_{n_3}\frac{\Delta^*}{(\omega_{n_1}^2+|\Delta|^2)^{1/2}}\,\frac{\Delta^*}{(\omega_{n_2}^2+|\Delta|^2)^{1/2}}$$
$$\times\frac{\Delta}{(\omega_{n_3}^2+|\Delta|^2)^{1/2}}\,\frac{\Delta}{[(\omega_{n_1}+\omega_{n_2}-\omega_{n_3})^2+|\Delta|^2]^{1/2}}\,, \quad (2.28)$$

which is explicitly of order $(T_c/T_F)^3$ and has the form we wanted: an angular integral of a quadratic function of the two-particle scattering amplitude for quasiparticles on the Fermi surface.

It is easy to check that the important steps leading to (2.28), in particular replacing $\Gamma^{(4)}$ by T and replacing the quasiparticle Green's functions by distributions on the Fermi surface, depend on properties shared by F, $F^\dagger$, and $G-G_N$ and can be done for any l. Hence a completely parallel treatment works for all the diagrams of order $(T_c/T_F)^3$ with four superfluid Green's-function lines. For $l\neq 0$ the nontrivial spin and angular dependences of $\Delta_{\alpha\beta}(\hat{k})$ only complicate the remaining spin sums and angular integrals.

2. *Diagrams with more than four superfluid lines*

We will show that all these diagrams contribute to the free energy in higher order than $(T_c/T_F)^3$. Obviously the most dangerous contributions come from quasiparticle lines. Arguments identical to those given before show that diagrams with m superfluid lines contribute in order $(T_c/T_F)^m$ if they involve m independent k integrations and ω_n summations. In the worst case $(m=5)$, energy conservation can reduce this order by one power of T_c/T_F to order $(T_c/T_F)^4$. Ring diagrams of the type shown in Fig. 2(c) are also of order $(T_c/T_F)^4$, no matter how many superfluid lines are involved. For circulating momentum q and circulating frequency ω_m of order ξ_0^{-1} and k_BT_c, respectively, each bubble in such a diagram contributes in order 1. The restriction on the circulating momentum gives a factor $(T_c/T_F)^3$ and the restriction on the circulating frequency gives a factor T_c/T_F. These diagrams therefore contribute to $\Delta\Phi$ in order $(T_c/T_F)^4$ and can be consistently neglected.

3. *Diagrams with three superfluid lines*

The two diagrams of this type are shown in Figs. 2(a) and 2(b). Each of these diagrams consists of three superfluid Green's-function lines connected by a normal six-point vertex $\Gamma^{(6)}$. Counting powers of T_c/T_F originating from the superfluid lines, k integrations and ω_n summations, one finds that the "superfluid part" in these diagrams gives a factor $(T_c/T_F)^3$. We will show that the vertex $\Gamma^{(6)}$ introduces further powers in T_c/T_F unless it has the form of two four-point vertices connected by a single normal Green's function. Therefore, to order $(T_c/T_F)^3$, we will again be able to express the free energy in terms of quasiparticle scattering amplitudes.

We first note that to order $(T_c/T_F)^3$ we need to keep only the quasiparticle part of $G-G_N$. Furthermore, since $\Gamma^{(6)}(\vec{k},\omega_n;\cdots)$ varies with momentum on the scale of k_F, we can again replace $(G-G_N)_{\rm qp}$ by the distribution (2.27),

$$k_BT\sum_n\int\frac{d^3k}{(2\pi k_F)^3}[G(\vec{k},\omega_n)-G_N(\vec{k},\omega_n)]_{\rm qp}\Gamma^{(6)}(\vec{k},\omega_n;\cdots)$$
$$\simeq-\frac{i}{Z}\,\frac{k_BT}{2\pi v_Fp_F}\sum_n\int\frac{d\Omega}{4\pi}\left(\frac{\omega_n}{(\omega_n^2+|\Delta|^2)^{1/2}}-\frac{\omega_n}{|\omega_n|}\right)\Gamma^{(6)}(k_F\hat{k},\omega_n;\cdots)\,. \quad (2.29)$$

It is now important to observe that the frequency sum in (2.29) vanishes if one neglects the ω_n dependence of $\Gamma^{(6)}$. We therefore can replace $\Gamma^{(6)}(\vec{k}, \omega_n; \cdots)$ in (2.29) by the difference

$$\Gamma^{(6)}(\vec{k},\omega_n;\cdots) - \Gamma^{(6)}(\vec{k},\omega_0;\cdots),$$

where ω_0 denotes here and in the following some fixed frequency of order $k_B T_C$. Crudely speaking, this means that only the "frequency derivative" of $\Gamma^{(6)}$ contributes to the free-energy difference. The constant part of $\Gamma^{(6)}$, which is unknown, fortunately drops out of our calculation.

To proceed further we consider the diagrammatic expansion for the difference

$$\Gamma^{(6)}(\vec{k},\omega_n;\cdots) - \Gamma^{(6)}(\vec{k},\omega_0;\cdots),$$

which is obtained from the expansion for $\Gamma^{(6)}(\vec{k}, \omega_n; \cdots)$ in the following way. First, in each diagram for $\Gamma^{(6)}$, one replaces in all possible ways one of the Green's functions which carry the external frequency ω_n by the difference $G_N(\ldots, \omega_n+\omega_m) - G_N(\ldots, \omega_0+\omega_m)$, and sets ω_n equal to ω_0 everywhere else in the diagram. Next, one replaces in all possible ways two Green's functions carrying ω_n by the difference and sets $\omega_n = \omega_0$ elsewhere, and so on. We now concentrate on the differences of Green's functions occurring in these diagrams. The high-energy part and the incoherent part of $G_N(\ldots, \omega_n+\omega_m) - G_N(\ldots, \omega_0+\omega_m)$ are of order $(\omega_n-\omega_0)/(k_B T_F)^2$. Since ω_n and ω_0 are both $\lesssim k_B T_c$, these parts lead to an additional factor T_c/T_F and hence can be neglected. From the remaining quasiparticle part we find, after performing the ξ integration, a factor $-i\pi[\mathrm{sgn}(\omega_n+\omega_m) - \mathrm{sgn}(\omega_0+\omega_m)]$. This factor vanishes unless ω_m is of order $k_B T_c$, and thus leads to an additional factor of T_c/T_F in any diagram for $\Gamma^{(6)}$ which contains an internal frequency sum over ω_m. The only diagrams for $\Gamma^{(6)}$ which do not produce an extra factor of T_c/T_F are those shown in Fig. 4; they consist of two four-point vertices connected by a single quasiparticle Green's-function line whose frequency is fixed by energy conservation and thus not summed over. Only when inserted in such a line does the difference of Green's functions not produce an additional factor T_c/T_F. At most one such line can occur in any diagram, since the skeleton diagrams for $\Delta\Phi$ contain no self-energy insertions. Summing all these diagrams is equivalent to replacing the open circles in Fig. 4 by the full four-point vertices. To leading order in T_c/T_F these vertices can again be taken at zero frequency and with all external momenta on the Fermi surface.

The resulting diagrams for $\Delta\Phi$ which contribute in order $(T_c/T_F)^3$ are shown in Figs. 3(d) and 3(e). Only the quasiparticle parts of the superfluid Green's functions and of the explicit normal Green's function have to be included in this order, and, as before, the quasiparticle Green's functions can all be replaced by distributions concentrated on the Fermi surface. For the normal Green's functions the appropriate substitution is

$$\frac{1}{Z}\frac{1}{i\omega_n - \xi_k} \Rightarrow -\frac{i\pi}{Z}\,\delta(\xi_k)\,\frac{\omega_n}{|\omega_n|}\,. \tag{2.30}$$

As an example we give the contribution from diagram 3(d):

$$\begin{aligned}\tfrac{1}{2}(v_F p_F)k_F^3(2)\pi^2\left(\frac{k_B T_c}{2v_F p_F}\right)^3 &\int\frac{d\Omega_1}{4\pi}\int\frac{d\Omega_2}{4\pi}\int\frac{d\Omega_3}{4\pi}\,\delta(|\hat{k}_1+\hat{k}_2-\hat{k}_3|-1)\,T(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_1+\hat{k}_2-\hat{k}_3)\\ &\times T(\hat{k}_3,-\hat{k}_2;\hat{k}_1,-\hat{k}_1-\hat{k}_2+\hat{k}_3)\\ &\times\left(\frac{T}{T_c}\right)^3\sum_{n_1}\sum_{n_2}\sum_{n_3}\left(\frac{\omega_{n_1}}{(\omega_{n_1}^2+|\Delta|^2)^{1/2}}-\frac{\omega_{n_1}}{|\omega_{n_1}|}\right)\\ &\times\frac{\Delta^*}{(\omega_{n_2}^2+|\Delta|^2)^{1/2}}\,\frac{\Delta}{[(\omega_{n_1}+\omega_{n_2}-\omega_{n_3})^2+|\Delta^2|]^{1/2}}\,\frac{\omega_{n_3}}{|\omega_{n_3}|}\,.\end{aligned}\tag{2.31}$$

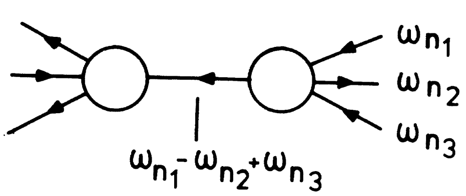

FIG. 4. A diagram for $\Gamma^{(6)}$ with an internal line whose frequency is fixed by energy conservation.

Notice that the product of quasiparticle scattering amplitudes in the integrand of (2.31) cannot be reduced to the simple product T^2 as in (2.28), because one incoming quasiparticle and one outgoing quasiparticle are interchanged between the two amplitudes. The same more complicated product of scattering amplitudes occurs in Fig. 3(e) also. In any case, we have now shown that the $(T_c/T_F)^3$ terms from diagrams with three superfluid lines can be expressed in terms of quasiparticle quantities.

4. *Diagrams with two superfluid lines*

The two diagrams with two superfluid lines are shown in Figs. 5(a) and 5(b). Diagram 5(b) contributes to the free energy in order $(T_c/T_F)^3$ and can be analyzed with the same methods as used before. Diagram 5(a), however, demands special considerations; it is the only diagram which contributes in order $(T_c/T_F)^2$. We will call this the "weak-coupling diagram", because to leading order in T_c/T_F it generates the well-known BCS expression for the free-energy functional, and to all orders in T_c/T_F it generates a weak-coupling Ginzburg-Landau functional. The weak-coupling diagram furthermore determines the transition temperature, since it is the only diagram which contributes in second order in the order parameter. The vertex function in this diagram is irreducible in the particle-particle channel because $\delta\Phi/\delta\hat{G}$ contains only skeleton $\hat{\Sigma}$ diagrams; we indicate this by a line through the vertex, separating the incoming and outgoing lines. This particle-particle irreducible vertex plays the role of the pairing interaction in simple BCS theory, and will be denoted here by $V(\vec{k}_1, \omega_{n_1}; \vec{k}_2, \omega_{n_2})$ to emphasize this correspondence.

Using the rules (2.20) and (2.21) one finds easily that the leading contributions from this diagram are of order $(T_c/T_F)^2$. Contributions of this order come from both the low-energy and the high-energy parts of the F functions, which reflects the well-known fact that the weak-coupling theory depends on a high-energy cutoff. Our point of view is that neither the particle-particle irreducible interaction nor the high-energy parts of the Green's functions are known to any reliable accuracy. We therefore regard the transition temperature T_c as a parameter to be taken from experiment. The $(T_c/T_F)^2$ terms in the free energy are then given by the BCS free-energy functional, expressed in terms of the measured T_c and the measured density of states $N(0)$.

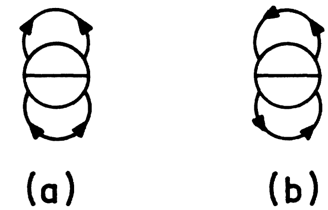

FIG. 5. Two diagrams with two superfluid Green's-function lines; (a) is the weak-coupling diagram, of order $(T_c/T_F)^2$, while (b) first contributes in order $(T_c/T_F)^3$.

The weak-coupling diagram also leads to additional contributions of order $(T_c/T_F)^3$, which come, for example, from the incoherent parts of the F functions at low energies. These corrections, which are usually neglected in strong-coupling theories for ^{3}He, cannot be analyzed with the methods of this section, but will be discussed in Sec. III.

The final diagram which we must discuss is shown in Fig. 5(b). It contains two $(G - G_N)$ lines connected by the particle-hole irreducible interaction $I(\vec{k}_1, \omega_{n_1}; \vec{k}_2, \omega_{n_2})$. Using (2.27) to evaluate the leading contribution from the quasiparticle parts of $(G - G_N)$, we find

$$-\tfrac{1}{2}(v_F p_F)k_F^3 \frac{1}{\pi^2}\left(\frac{k_B T_c}{2v_F p_F}\right)^2 \int \frac{d\Omega_1}{4\pi}\int\frac{d\Omega_2}{4\pi}\left(\frac{T}{T_c}\right)^2 \sum_{n_1}\sum_{n_2} I(k_F\hat{k}_1, \omega_{n_1}; k_F\hat{k}_2, \omega_{n_2}) \times\left(\frac{\omega_{n_1}}{(\omega_{n_1}^2 + |\Delta^2|)^{1/2}} - \frac{\omega_{n_1}}{|\omega_{n_1}|}\right)\left(\frac{\omega_{n_2}}{(\omega_{n_2}^2 + |\Delta|^2)^{1/2}} - \frac{\omega_{n_2}}{|\omega_{n_2}|}\right). \qquad (2.32)$$

Although (2.32) appears to be of order $(T_c/T_F)^2$, it is actually of higher order in T_c/T_F, because the frequency sums in (2.32) vanish if we ignore the dependence of the particle-hole irreducible interaction on ω_{n_1} or ω_{n_2}. To evaluate (2.32) we follow closely our previous treatment of the frequency dependence of $\Gamma^{(6)}$. We first replace $I(\vec{k}_1, \omega_{n_1}; \vec{k}_2, \omega_{n_2})$ in (2.32) by the difference

$$I(\vec{k}_1, \omega_{n_1}; \vec{k}_2, \omega_{n_2}) - I(\vec{k}_1, \omega_0; \vec{k}_2, \omega_{n_2}) - I(\vec{k}_1, \omega_{n_1}; \vec{k}_2, \omega_0) + I(\vec{k}_1, \omega_0; \vec{k}_2, \omega_0), \qquad (2.33)$$

and then calculate (2.33) from the diagrammatic expansion for the irreducible interaction I by the following procedure: In each diagram for I we replace in all possible ways one (and two, and three, ...) Green's functions carrying ω_{n_1} and also one (and two, and three, ...) Green's functions carrying ω_{n_2} by the differences $G_N(\ldots, \omega_{n_1} + \cdots) - G_N(\ldots, \omega_0 + \cdots)$ and $G_N(\ldots, \omega_{n_2} + \cdots) - G_N(\ldots, \omega_0 + \cdots)$, respectively; everywhere else we set the external frequencies ω_{n_1} and ω_{n_2} equal to ω_0. Hence, any diagram for (2.33) contains at least two insertions of Green's-function differences, one carrying ω_{n_1} and one carrying ω_{n_2}. As before, the quasiparticle part of an insertion produces a factor $[\mathrm{sgn}(\omega_{n_\alpha}+\omega_m)-\mathrm{sgn}(\omega_0+\omega_m)]$, which vanishes unless the internal frequency ω_m is of order k_BT_c. Each summation over an internal frequency carried by an insertion therefore introduces a factor T_c/T_F. The leading term of (2.33), of order T_c/T_F, comes from diagrams in which the insertions carry a single internal frequency. Such diagrams must contain exactly two insertions and must be reducible by cutting the two insertion lines. Two distinct classes of I diagrams generate diagrams for (2.33) with these properties: the diagrams reducible in the particle-particle channel and those reducible in the crossing particle-hole channel. The two corresponding diagrams for $\Delta\Phi$ of order $(T_c/T_F)^3$ are shown in Figs. 3(b) and 3(c). To leading order in T_c/T_F the full four-point vertices in these diagrams can be taken at fixed (zero) external frequencies and with all momenta on the Fermi surface, and therefore can again be replaced by quasiparticle scattering amplitudes.

It is easy to verify that the high-energy and incoherent parts of the normal Green's functions can be neglected in the insertions. These parts are of order $(\omega_{n_1}-\omega_0)/(k_BT_F)^2$ or $(\omega_{n_2}-\omega_0)/(k_BT_F)^2$ and would introduce further powers in T_c/T_F since ω_{n_1}, ω_{n_2}, and ω_0 are of order k_BT_c. Finally we note that to leading order in T_c/T_F the difference of normal Green's functions in diagrams 3(b) and 3(c) can be replaced by the distribution

$$-(i\pi/Z)\delta(\xi_k)[\mathrm{sgn}(\omega_{n_\alpha}+\omega_m)-\mathrm{sgn}(\omega_0+\omega_m)]. \tag{2.34}$$

The final results for $\Delta\Phi$ are independent of the arbitrarily chosen (in the sense that any frequency $\lesssim k_BT_c$ would work as well) subtraction point ω_0. The subtraction in (2.34) only guarantees absolute convergence of the frequency sums in $\Delta\Phi$ and can be neglected if these sums are performed in the appropriate order.

We have now found all the diagrams which contribute to $\Delta\Phi$ through order $(T_c/T_F)^3$. These diagrams are shown in Fig. 3 with their correct combinatorial coefficients and signs. All but the weak-coupling diagram are of order $(T_c/T_F)^3$ and can be evaluated by replacing the Green's functions by distributions on the Fermi surface and the four-point vertex functions by quasiparticle scattering amplitudes. The weak-coupling diagram in general requires more careful treatment, since it is of order $(T_c/T_F)^2$ and contains significant contributions from the high-energy range of frequency and momentum.

To end this section we will verify our ansatz for the superfluid self-energy, on which we based the preceding analysis. We first consider the free-energy functional $\Delta\Omega_{\mathrm{wc}}[\hat{\Sigma}-\hat{\Sigma}_N]$ obtained by keeping only the weak-coupling diagram $\Delta\Phi_{\mathrm{wc}}$ in the free-energy expression (2.1). Since $\Delta\Phi_{\mathrm{wc}}$ is a functional only of $F(\vec{k},\omega_n)$ and $F^\dagger(\vec{k},\omega_n)$, Eq. (2.13) implies that the ansatz $\Sigma=\Sigma_N$ is satisfied exactly at the minimum point $(\hat{\Sigma}-\hat{\Sigma}_N)_{\mathrm{wc}}$ of $\Delta\Omega_{\mathrm{wc}}$. The weak-coupling gap equation,

$$\Delta_{\mathrm{wc}}(\vec{k},\omega_n)=k_BT\sum_{n_1}\int\frac{d^3k_1}{(2\pi)^3}V(\vec{k},\omega_n;\vec{k}_1,\omega_{n_1})\times F_{\mathrm{wc}}(\vec{k}_1,\omega_{n_1}), \tag{2.35}$$

furthermore shows that $\Delta_{\mathrm{wc}}(\vec{k},\omega_n)$ is of order k_BT_c and varies with $\vec{k}$ and ω_n on the scales k_F and k_BT_F, respectively, since the pairing interaction V varies on these characteristic scales. Hence, our ansatz is fulfilled in the weak-coupling approximation and therefore in lowest order in T_c/T_F. To find the leading corrections we have to add to $\Delta\Omega_{\mathrm{wc}}$ the $\Delta\Phi$ diagrams of order $(T_c/T_F)^3$. We see from (2.13) that at the minimum of this new functional there will be nonvanishing contributions to $\Sigma-\Sigma_N$ from diagrams 3(b–e, g, h) and corrections to the off-diagonal self-energy Δ from diagrams 3(d, f, g). However, these strong-coupling corrections to Σ_N and Δ_{wc} are of order $(T_c/T_F)\Sigma_N$ and $(T_c/T_F)\Delta_{\mathrm{wc}}$, respectively, and hence give $(T_c/T_F)^4$ contributions in all the $\Delta\Phi$ diagrams which are already of order $(T_c/T_F)^3$. To discuss the effect of the strong-coupling corrections to the self-energies on $\Delta\Omega_{\mathrm{wc}}$ we use the stationarity of $\Delta\Omega_{\mathrm{wc}}$ at $(\hat{\Sigma}-\hat{\Sigma}_N)_{\mathrm{wc}}$; this implies that strong-coupling corrections to $\Delta\Omega_{\mathrm{wc}}$ first enter in second order in $(\hat{\Sigma}-\hat{\Sigma}_{\mathrm{wc}})$ and therefore lead to free-energy corrections of order $(T_c/T_F)^4$.

III. GINZBURG-LANDAU FREE ENERGY

In Sec. II we showed that to calculate the free energy through order $(T_c/T_F)^3$, one can put $\Sigma(\vec{k},\omega_n)=\Sigma_N(\vec{k},\omega_n)$ in the free-energy functional, which then reduces to a functional $\Delta\Omega[\Delta(\vec{k},\omega_n)]$ of the pairing self-energy matrix $\Delta(\vec{k},\omega_n)$ alone. We now use this functional to calculate the free energy in the Ginzburg-Landau region near T_c, or more precisely, to calculate the free energy in order $(T-T_c)^2$. To make contact with the phenomenological Ginzburg-Landau free-energy functionals, such as (1.1), we must relate the self-energy matrix $\Delta(\vec{k},\omega_n)$ and the phenomenological order-parameter matrix $\Delta(\hat{k})$. For this purpose we note that in order $(T-T_c)^2$ we can restrict the domain of the functional $\Delta\Omega[\Delta(\vec{k},\omega_n)]$ to the linear space of solutions of the gap equation at T_c,

$$\Delta_{\alpha\beta}(\vec{k}_1,\omega_{n_1}) = -k_B T_c \sum_{n_2} \int \frac{d^3k_2}{(2\pi)^3} V(\vec{k}_1,\omega_{n_1};\vec{k}_2,\omega_{n_2})_{\alpha\beta;\gamma\delta} \times |G_N(\vec{k}_2,\omega_{n_2})|^2 \Delta_{\gamma\delta}(\vec{k}_2,\omega_{n_2}). \tag{3.1}$$

For a system invariant under rotation of spin and orbital coordinates, the solutions of (3.1) have the form

$$\Delta_{\alpha\beta}(\vec{k},\omega_n) = Z\Delta_{\alpha\beta}(\hat{k})\,\theta(|\vec{k}|,n), \tag{3.2}$$

where Z is given by

$$Z = 1 - \mathrm{Im}[\Sigma_N(k_F,\pi T_c)/\pi T_c]. \tag{3.3}$$

$\theta(|\vec{k}|,n)$, which we normalize by $\theta(k_F,0)=1$, carries the $|\vec{k}|$ and ω_n dependence of the gap function. It varies with $|\vec{k}|$ and n on the scales k_F and T_F/T_c and can, in principle, be calculated from the linear gap equation (3.1). Here we consider $\theta(|\vec{k}|,n)$ to be known and fixed and identify the open amplitudes $\Delta_{\alpha\beta}(\hat{k})$ with the phenomenological order-parameter matrix. With $\theta(|\vec{k}|,n)$ fixed, $\Delta\Omega$ becomes a functional of the order parameter $\Delta_{\alpha\beta}(\hat{k})$ alone. Expanding $\Delta\Omega$ through order Δ^4 then leads to the Ginzburg-Landau free-energy functional discussed in the following:

We first consider $\Delta\Omega_{wc}$, the weak-coupling part of the free energy. Inserting (3.2) into $\Delta\Omega_{wc}[\Delta(\vec{k},\omega_n)]$ and using the gap equation (3.1), one obtains in order $(T-T_c)^2$

$$\Delta\Omega_{wc} = \tfrac{1}{2}\alpha \int \frac{d\Omega_k}{4\pi} \mathrm{Tr}_2[\Delta(\hat{k})\Delta^\dagger(\hat{k})] + \tfrac{15}{2}\beta_{wc} \int \frac{d\Omega_k}{4\pi} \mathrm{Tr}_2\{[\Delta(\hat{k})\Delta^\dagger(\hat{k})]^2\}, \tag{3.4}$$

where

$$\alpha = (T-T_c)\frac{\partial}{\partial T}\Big[k_B T \sum_n \int \frac{d^3k}{(2\pi)^3} |ZG_N(\vec{k},\omega_n)|^2 \theta^2(|\vec{k}|,n) + k_B T \sum_{n_1} \int \frac{d^3k_1}{(2\pi)^3} k_B T \sum_{n_2} \int \frac{d^3k_2}{(2\pi)^3} Z^{-2} V_l(|\vec{k}_1|,\omega_{n_1};|\vec{k}_2|,\omega_{n_2}) \times |ZG_N(\vec{k}_1,\omega_{n_1})|^2 |ZG_N(\vec{k}_2,\omega_{n_2})|^2 \theta(|\vec{k}_1|,n_1)\theta(|\vec{k}_2|,n_2)\Big]_{T=T_c},$$

$$\beta_{wc} = \tfrac{1}{30} k_B T_c \sum_n \int \frac{d^3k}{(2\pi)^3} |ZG_N(\vec{k},\omega_n)|^4 \theta^4(|\vec{k}|,n). \tag{3.5}$$

Evaluating (3.5) to leading order in T_c/T_F one finds the BCS results,

$$\alpha = N(0)[(T-T_c)/T_c][1+O(T_c/T_F)],$$
$$\beta_{wc} = N(0)(1/\pi k_B T_c)^2[\tfrac{7}{240}\zeta(3)][1+O(T_c/T_F)]. \tag{3.6}$$

The terms of order T_c/T_F in α and β_{wc}, which give contributions to the free energy in order $(T_c/T_F)^3$, cannot be calculated with the methods of this paper; these terms depend on the frequency and energy structure of the pairing interaction and on corrections to the quasiparticle Green's functions (these are corrections to BCS theory treated in the strong-coupling theory of superconductivity). We believe, however, that these $(T_c/T_F)^3$ terms are negligible compared to the $(T_c/T_F)^3$ terms which come from the strong-coupling diagrams. To estimate these terms, we have evaluated them in the spin-fluctuation model of BSA, which gives

$$\alpha^{SF} = N(0)[(T-T_c)/T_c](1+0.017\delta),$$
$$\beta_{wc}^{SF} = N(0)(1/\pi k_B T_c)^2[\tfrac{7}{240}\zeta(3)](1-0.006\delta), \tag{3.7}$$

in the notation of BSA. Comparing the weak-coupling terms of order $(T_c/T_F)^3$ given by (3.7) with the strong-coupling terms calculated by BSA, one sees that the former are roughly a factor of 100 smaller and can be neglected. Since the spin-fluctuation model generally gives qualitatively correct results for ^{3}He, we believe that in a more rigorous calculation the T_c/T_F terms in (3.6) would remain small relative to the strong-coupling corrections. In any case, however, these terms are of secondary importance since they only give a multiplicative correction to the BCS free energy, independent of the specific state considered. This effect by itself can therefore never stabilize any state other than the equilibrium state in BCS theory.

To evaluate the strong-coupling Δ^4 corrections from Fig. 3 we need the quasiparticle Green's-function distributions to lowest order in Δ:

$$F_{\alpha\beta}(\vec{k},\omega_n) \Rightarrow -\delta(\xi_k)(\pi/Z)\Delta_{\alpha\beta}(\hat{k})(1/|\omega_n|),$$
$$[G_{\alpha\beta}(\vec{k},\omega_n) - G_{N\alpha\beta}(\vec{k},\omega_n)] \Rightarrow \delta(\xi_k)(i\pi/2Z)[\Delta(\hat{k})\Delta^\dagger(\hat{k})]_{\alpha\beta}\omega_n/|\omega_n|^3. \tag{3.8}$$

Through order Δ^4 only diagrams b, c, d, and f in Fig. 3 contribute to $\Delta\Phi$. From (3.8) we can calculate the frequency sums in these diagrams; the same frequency sums occur in spin-fluctuation theories, and have been evaluated by BSA:

diagrams b and c

$$(\pi k_B T_c)^4 \sum_{\omega_{n_1}} \sum_{\omega_{n_2}} \sum_{\omega_{n_3}} \frac{\omega_{n_1}}{|\omega_{n_1}|^3} \frac{\omega_{n_2}}{|\omega_{n_2}|^3} \frac{\omega_{n_3}}{|\omega_{n_3}|} \frac{\omega_{n_1}+\omega_{n_2}-\omega_{n_3}}{|\omega_{n_1}+\omega_{n_2}-\omega_{n_3}|} \simeq 6.8;$$

diagram d

$$(\pi k_B T_c)^4 \sum_{\omega_{n_1}} \sum_{\omega_{n_2}} \sum_{\omega_{n_3}} \frac{1}{|\omega_{n_1}|} \frac{1}{|\omega_{n_2}|} \frac{\omega_{n_3}}{|\omega_{n_3}|^3} \frac{\omega_{n_1}+\omega_{n_2}-\omega_{n_3}}{|\omega_{n_1}+\omega_{n_2}-\omega_{n_3}|} \simeq 10.1; \tag{3.9}$$

diagram f

$$(\pi k_B T_c)^4 \sum_{\omega_{n_1}} \sum_{\omega_{n_2}} \sum_{\omega_{n_3}} \frac{1}{|\omega_{n_1}|} \frac{1}{|\omega_{n_2}|} \frac{1}{|\omega_{n_3}|} \frac{1}{|\omega_{n_1}+\omega_{n_2}-\omega_{n_3}|} \simeq 30.1.$$

To perform the spin sums we decompose the scattering amplitude into spin-symmetric and spin-antisymmetric parts in the conventional way,

$$T_{\alpha\beta,\gamma\rho} = T^{(s)}\delta_{\alpha\rho}\delta_{\beta\gamma} + T^{(a)}(\vec{\sigma})_{\alpha\rho}\cdot(\vec{\sigma})_{\beta\gamma}, \tag{3.10}$$

and write the order parameter for triplet pairing in the form

$$\Delta_{\alpha\beta}(\hat{k}) = \vec{\Delta}(\hat{k})\cdot(\vec{\sigma}i\sigma_y)_{\alpha\beta}. \tag{3.11}$$

The scattering amplitude must be odd under exchange of particles in the final state,

$$T_{\alpha\beta,\gamma\rho}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4) = -T_{\beta\alpha,\gamma\rho}(\hat{k}_2,\hat{k}_1;\hat{k}_3,\hat{k}_4), \tag{3.12}$$

and combining (3.10) with (3.12) we find

$$\begin{aligned} T^{(s)}(\hat{k}_2,\hat{k}_1;\hat{k}_3,\hat{k}_4) &= -\tfrac{1}{2}[T^{(s)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4) + 3T^{(a)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)], \\ T^{(a)}(\hat{k}_2,\hat{k}_1;\hat{k}_3,\hat{k}_4) &= -\tfrac{1}{2}[T^{(s)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4) - T^{(a)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)]. \end{aligned} \tag{3.13}$$

This symmetry allows us to eliminate the product $T^{(s)}T^{(a)}$; for example, (3.13) implies

$$\begin{aligned} T^{(s)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)T^{(a)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4) &= \tfrac{1}{2}[T^{(s)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)]^2 + \tfrac{1}{2}[T^{(a)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)]^2 \\ &\quad - 2[T^{(a)}(\hat{k}_2,\hat{k}_1;\hat{k}_3,\hat{k}_4)]^2. \end{aligned} \tag{3.14}$$

Using (3.9), (3.11), and (3.14) we find the contributions to $\Delta\Phi$ from diagrams b, c, d, and f in Fig. 3:

$$\begin{aligned} \Delta\Phi_b = -\frac{N(0)}{(k_B T_c)^2}\frac{k_B T_c}{v_F p_F}\frac{6.8}{16} &\times \int\frac{d\Omega_1}{4\pi}\int\frac{d\Omega_2}{4\pi}\int\frac{d\Omega_3}{4\pi}\delta(|\hat{k}_1+\hat{k}_2-\hat{k}_3|-1) \\ &\times\{[T^{(s)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)]^2[|\vec{\Delta}(\hat{k}_1)|^2|\vec{\Delta}(\hat{k}_2)|^2 - |\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_2)|^2 + |\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}^*(\hat{k}_2)|^2] \\ &+[T^{(a)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)]^2[3|\vec{\Delta}(\hat{k}_1)|^2|\vec{\Delta}(\hat{k}_2)|^2 + 5|\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_2)|^2 - 5|\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}^*(\hat{k}_2)|^2]\}, \end{aligned}$$

$$\begin{aligned} \Delta\Phi_c = -\frac{N(0)}{(k_B T_c)^2}\frac{k_B T_c}{v_F p_F}\frac{6.8}{8} &\times \int\frac{d\Omega_1}{4\pi}\int\frac{d\Omega_2}{4\pi}\int\frac{d\Omega_3}{4\pi}\delta(|\hat{k}_1+\hat{k}_2-\hat{k}_3|-1) \\ &\times\{[T^{(s)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)]^2[|\vec{\Delta}(\hat{k}_1)|^2|\vec{\Delta}(\hat{k}_3)|^2 - |\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_3)|^2 + |\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}^*(\hat{k}_3)|^2] \\ &+[T^{(a)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)]^2[3|\vec{\Delta}(\hat{k}_1)|^2|\vec{\Delta}(\hat{k}_3)|^2 + |\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_3)|^2 - |\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}^*(\hat{k}_3)|^2]\}, \end{aligned}$$

$$\begin{aligned} \Delta\Phi_d = \frac{N(0)}{(k_B T_c)^2}\frac{k_B T_c}{v_F p_F}\frac{10.1}{2} &\times \int\frac{d\Omega_1}{4\pi}\int\frac{d\Omega_2}{4\pi}\int\frac{d\Omega_3}{4\pi}\delta(|\hat{k}_1+\hat{k}_2-\hat{k}_3|-1) \\ &\times([T^{(s)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)T^{(s)}(\hat{k}_3,-\hat{k}_2;\hat{k}_1,-\hat{k}_4) \\ &+T^{(a)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)T^{(a)}(\hat{k}_3,-\hat{k}_2;\hat{k}_1,-\hat{k}_4)]|\vec{\Delta}(\hat{k}_1)|^2[\vec{\Delta}(\hat{k}_4)\cdot\vec{\Delta}^*(\hat{k}_2)] \\ &+[T^{(s)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)T^{(a)}(\hat{k}_3,-\hat{k}_2;\hat{k}_1,-\hat{k}_4) + T^{(a)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)T^{(s)}(\hat{k}_3,-\hat{k}_2;\hat{k}_1,-\hat{k}_4)] \\ &\times\{[\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}^*(\hat{k}_2)][\vec{\Delta}^*(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_4)] - [\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_4)][\vec{\Delta}^*(\hat{k}_1)\cdot\vec{\Delta}^*(\hat{k}_2)]\}), \end{aligned}$$

$$\Delta\Phi_f = -\frac{N(0)}{(k_BT_c)^2}\frac{k_BT_c}{v_Fp_F}\frac{30.1}{8}$$
$$\times\int\frac{d\Omega_1}{4\pi}\int\frac{d\Omega_2}{4\pi}\int\frac{d\Omega_3}{4\pi}\delta(|\hat{k}_1+\hat{k}_2-\hat{k}_3|-1)$$
$$\times\left([T^{(s)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)]^2\{[\vec{\Delta}^*(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_3)][\vec{\Delta}^*(\hat{k}_2)\cdot\vec{\Delta}(\hat{k}_4)]\right.$$
$$-[\vec{\Delta}(\hat{k}_3)\cdot\vec{\Delta}(\hat{k}_4)][\vec{\Delta}^*(\hat{k}_1)\cdot\vec{\Delta}^*(\hat{k}_2)]+[\vec{\Delta}^*(\hat{k}_2)\cdot\vec{\Delta}(\hat{k}_3)][\vec{\Delta}^*(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_4)]\}$$
$$+[T^{(a)}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)]^2\{-5[\vec{\Delta}^*(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_3)][\vec{\Delta}^*(\hat{k}_2)\cdot\vec{\Delta}(\hat{k}_4)]+3[\vec{\Delta}^*(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_4)][\vec{\Delta}^*(\hat{k}_2)\cdot\vec{\Delta}(\hat{k}_3)]$$
$$\left.+5[\vec{\Delta}^*(\hat{k}_1)\cdot\vec{\Delta}^*(\hat{k}_2)][\vec{\Delta}(\hat{k}_3)\cdot\vec{\Delta}(\hat{k}_4)]\}\right), \tag{3.15}$$

where $\hat{k}_4=\hat{k}_1+\hat{k}_2-\hat{k}_3$. For unitary states, Eqs. (3.15) simplify appreciably because of the unitarity condition

$$|\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}(\hat{k}_2)|^2-|\vec{\Delta}(\hat{k}_1)\cdot\vec{\Delta}^*(\hat{k}_2)|^2 = [\vec{\Delta}(\hat{k}_1)\times\vec{\Delta}^*(\hat{k}_1)]\cdot[\vec{\Delta}(\hat{k}_2)\times\vec{\Delta}^*(\hat{k}_2)]=0.$$

Because all four quasiparticles are on the Fermi surface, the scattering amplitude in (3.15) depends on only two independent angles, which we take to be the angles introduced by Abrikosov and Khalatnikov[21,29]:

$$T(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)=T(\theta,\phi), \tag{3.16}$$

with θ the angle between $\hat{k}_1$ and $\hat{k}_2$ and ϕ the angle between the plane containing $\hat{k}_1$ and $\hat{k}_2$ and the plane containing $\hat{k}_3$ and $\hat{k}_4$;

$$\cos\theta=\hat{k}_1\cdot\hat{k}_2,\qquad \cos\phi=(\hat{k}_1\cdot\hat{k}_3-\hat{k}_2\cdot\hat{k}_3)/(1-\hat{k}_1\cdot\hat{k}_2). \tag{3.17}$$

We will next perform the integrations over all the angular variables in (3.15) except the two contained in the T matrices. This enables us to express the strong-coupling corrections in terms of weighted integrals over products of scattering amplitudes. To integrate over the δ functions in (3.15) we take the polar axis for $\hat{k}_3$ to be along $\hat{k}_1+\hat{k}_2$, and measure the azimuthal angle of $\hat{k}_3$ from the plane containing $\hat{k}_1$ and $\hat{k}_2$, so $d\Omega_3=d\cos\theta_3\,d\phi$. The argument of the δ function can be expressed in terms of these angles and the angle θ,

$$\delta(|\hat{k}_1+\hat{k}_2-\hat{k}_3|-1)=\delta(\cos\theta-2\cos\tfrac{1}{2}\theta\cos\theta_3+1). \tag{3.18}$$

We use (3.18) to eliminate $\cos\theta_3$,

$$\int\frac{d\Omega_3}{4\pi}\delta(|\hat{k}_1+\hat{k}_2-\hat{k}_3|-1)=\int\frac{d\phi}{2\pi}\frac{1}{4\cos(\theta/2)}, \tag{3.19}$$

and then express the angular integrals over $\hat{k}_1$ and $\hat{k}_2$ in terms of the angle θ between $\hat{k}_1$ and $\hat{k}_2$, the direction $\hat{k}$ of $\hat{k}_1+\hat{k}_2$, and the azimuthal angle ψ around $\hat{k}$ of the plane containing $\hat{k}_1$ and $\hat{k}_2$. The angular integrals $\int(d\Omega_1/4\pi)\int(d\Omega_2/4\pi)$ in $\Delta\Phi_b$, $\Delta\Phi_c$, and $\Delta\Phi_f$ then take the form

$$\frac{1}{2}\int_0^1 d\cos(\theta/2)\int_0^{2\pi}\frac{d\phi}{2\pi}[T(\theta,\phi)]^2\int\frac{d\Omega_{\hat{k}}}{4\pi}\int_0^{2\pi}\frac{d\psi}{2\pi}f_\nu(\vec{\Delta}(\hat{k}_1),\vec{\Delta}(\hat{k}_2),\vec{\Delta}(\hat{k}_3),\vec{\Delta}(\hat{k}_4)), \tag{3.20}$$

with $\nu=b$, c, or f the diagram index; the integrals in $\Delta\Phi_d$ become

$$\frac{1}{2}\int_0^1 d\cos(\theta/2)\int_0^{2\pi}\frac{d\phi}{2\pi}T(\theta,\phi)T(\theta',\phi')\int\frac{d\Omega_{\hat{k}}}{4\pi}\int_0^{2\pi}\frac{d\psi}{2\pi}f_d(\vec{\Delta}(\hat{k}_1),\vec{\Delta}(\hat{k}_2),\vec{\Delta}(\hat{k}_4)), \tag{3.21}$$

where the angles θ' and ϕ' are related to θ and ϕ by

$$\cos\theta'=-\hat{k}_2\cdot\hat{k}_3=\cos\phi-\cos^2(\theta/2)(\cos\phi+1),$$
$$\cos\phi'=\frac{\hat{k}_1\cdot\hat{k}_3+\hat{k}_1\cdot\hat{k}_2}{1+\hat{k}_2\cdot\hat{k}_3} =\frac{3\cos^2(\theta/2)-1-[\cos^2(\theta/2)-1]\cos\phi}{\cos^2(\theta/2)+1+[\cos^2(\theta/2)-1]\cos\phi}. \tag{3.22}$$

The angles θ and ϕ fix the relative positions of the tetrad of vectors $\hat{k}_\alpha(\alpha=1-4)$, which for fixed (θ,ϕ) can be thought of as a rigid body whose orientation is given by $\hat{k}$ and ψ. In the body coordinate system the polar coordinates $(\overline{\theta}_\alpha,\overline{\phi}_\alpha)$ of $\hat{k}_\alpha$ are $(\overline{\theta}_1,\overline{\phi}_1)=(\theta/2,0)$, $(\overline{\theta}_2,\overline{\phi}_2)=(\theta/2,\pi)$, $(\overline{\theta}_3,\overline{\phi}_3)=(\theta/2,\phi)$, and $(\overline{\theta}_4,\overline{\phi}_4)=(\theta/2,\phi+\pi)$.

In order to carry out the integrations over the orientation of the rigid tetrad $(\hat{k}_1,\hat{k}_2,\hat{k}_3,\hat{k}_4)$ at fixed θ and ϕ, it is convenient to expand the order parameter in spherical harmonics. Each component $\Delta_i(\hat{k}_\alpha)$ of the spin vector $\vec{\Delta}(\hat{k}_\alpha)$ is a linear combination of spherical harmonics of a single l, and so can be written as

$$\Delta_l(\hat{k}_\alpha)=\sum_{m=-l}^{l} B_{lm}Y_{lm}(\hat{k}_\alpha). \qquad (3.23)$$

The $\hat{k}$ and ψ integrals in (3.20) and (3.21) then reduce to a sum of integrals of the type

$$\int \frac{d\Omega_k}{4\pi}\int_0^{2\pi}\frac{d\psi}{2\pi}\, Y_{lm_1}(\hat{k}_\alpha)Y_{lm_2}(\hat{k}_\beta)Y_{lm_3}(\hat{k}_\gamma)Y_{lm_4}(\hat{k}_\rho). \qquad (3.24)$$

To separate in the integrand of (3.24) the dependence on the fixed angles θ and ϕ from the dependence on $\hat{k}$ and ψ we use the transformation

$$Y_{lm}(\hat{k}_\alpha)=\sum_{m'=-l}^{l} D^{(l)}_{mm'}(R)^*Y_{lm'}(\overline{\theta}_\alpha,\overline{\phi}_\alpha), \qquad (3.25)$$

where R is the rotation which maps the coordinate system (k_x,k_y,k_z) into the body coordinate system, and $D^{(l)}_{mm'}(R)$ is the corresponding rotation matrix. This allows us to express (3.24) in terms of integrals over a product of four rotation matrices,

$$\int \frac{d\Omega_k}{4\pi}\int_0^{2\pi}\frac{d\psi}{2\pi}\, Y_{lm_1}(\hat{k}_\alpha)Y_{lm_2}(\hat{k}_\beta)Y_{lm_3}(\hat{k}_\gamma)Y_{lm_4}(\hat{k}_\rho)$$
$$=\sum_{m_1'\cdots m_4'} Y_{lm_1'}(\overline{\theta}_\alpha,\overline{\phi}_\alpha)Y_{lm_2'}(\overline{\theta}_\beta,\overline{\phi}_\beta)Y_{lm_3'}(\overline{\theta}_\gamma,\overline{\phi}_\gamma)Y_{lm_4'}(\overline{\theta}_\rho,\overline{\phi}_\rho)\int dR\, D^{(l)}_{m_1m_1'}(R)^*D^{(l)}_{m_2m_2'}(R)^*D^{(l)}_{m_3m_3'}(R)^*D^{(l)}_{m_4m_4'}(R)^*. \qquad (3.26)$$

The dependence of (3.24) on the scattering angles θ and ϕ is now contained in the product of four spherical harmonics whose arguments are determined by θ and ϕ alone. To evaluate the remaining integrals in (3.26) we employ two standard identities satisfied by rotation matrices,[30]

$$D^{(l)}_{m_1m_2}(R)^*D^{(l)}_{m_3m_4}(R)^* = \sum_{L=0}^{2l}\sum_{M,M'}\langle lm_1lm_3|LM\rangle D^{(L)}_{MM'}(R)^*\langle LM'|lm_2lm_4\rangle \qquad (3.27)$$

and

$$\int dR\, D^{(L_1)}_{M_1M_2}(R)^*D^{(L_2)}_{M_3M_4}(R)^*$$
$$=\langle L_1M_1L_2M_3|00\rangle\langle 00|L_1M_2L_2M_4\rangle$$
$$=(-1)^{M_1+M_2}(2L_1+1)^{-1}\delta_{L_1,L_2}\delta_{M_1,-M_3}\delta_{M_2,-M_4}, \qquad (3.28)$$

which together give for (3.26)

$$\sum_{L=0}^{2l}\frac{(-1)^{m_1+m_2}}{2L+1}\langle lm_1lm_2|Lm_1+m_2\rangle\langle lm_3lm_4|L-(m_1+m_2)\rangle\Phi_L^{(l)}(\theta,\phi),$$
$$\Phi_L^{(l)}(\theta,\phi)=\sum_{m_1'\cdots m_4'}(-1)^{m_1'+m_2'}\langle Lm_1'+m_2'|lm_1'lm_2'\rangle\langle L-m_1'-m_2'|lm_3'lm_4'\rangle Y_{lm_1'}(\overline{\theta}_\alpha,\overline{\phi}_\alpha)Y_{lm_2'}(\overline{\theta}_\beta,\overline{\phi}_\beta)Y_{lm_3'}(\overline{\theta}_\gamma,\overline{\phi}_\gamma)Y_{lm_4'}(\overline{\theta}_\rho,\overline{\phi}_\rho). \qquad (3.29)$$

Note that the Clebsch-Gordan coefficients in (3.29) require $m_1+m_2+m_3+m_4=0$ and $m_1'+m_2'+m_3'+m_4'=0$.

Taken together, Eqs. (3.15), (3.20)–(3.23), and (3.29) express the order $(T_c/T_F)^3$ strong-coupling contributions to the Δ^4 terms in the free-energy functional in terms of specified angular averages of the normal quasiparticle scattering amplitudes $T^{(s)}(\theta,\phi)$, $T^{(a)}(\theta,\phi)$.

For convenience we here described the order parameter by the coefficients B_{lm} defined in (3.23). To relate our results to phenomenological free-energy functionals, we need to express our products of four B_{lm}'s in terms of a conventional set of fourth-order invariants. This then yields the fourth-order coefficients as specific weighted angular integrals over quasiparticle scattering amplitudes. We have carried out this program in detail for the case $l=1$, using the fourth-order invariants introduced by Mermin and Stare, and obtained the following results:

$$\Delta\beta_i=\Delta\beta_i^{b+c}+\Delta\beta_i^d+\Delta\beta_i^f,\ (i=1,\ldots,5). \qquad (3.30)$$

$\Delta\beta_i^{b+c}$, $\Delta\beta_i^d$, and $\Delta\beta_i^f$ are the contributions to the fourth-order coefficients coming from diagrams b and c, d, and f, respectively;

$$\Delta\beta_i^{b+c}=-\eta\left(\frac{6.8}{16}\right)\langle W_i^{b+c}(\theta,\phi)[T^{(s)}(\theta,\phi)]^2 + V_i^{b+c}(\theta,\phi)[T^{(a)}(\theta,\phi)]^2\rangle,$$
$$\Delta\beta_i^d=-\eta\left(\frac{10.1}{4}\right)\langle W_i^d(\theta,\phi)[T^{(s)}(\theta,\phi)T^{(s)}(\theta',\phi') + T^{(a)}(\theta,\phi)T^{(a)}(\theta',\phi')] + V_i^d(\theta,\phi)[T^{(s)}(\theta,\phi)T^{(a)}(\theta',\phi') + T^{(a)}(\theta,\phi)T^{(s)}(\theta',\phi')]\rangle, \qquad (3.31)$$
$$\Delta\beta_i^f=-\eta\left(\frac{30.1}{16}\right)\langle W_i^f(\theta,\phi)[T^{(s)}(\theta,\phi)]^2 + V_i^f(\theta,\phi)[T^{(a)}(\theta,\phi)]^2\rangle.$$

η is a common factor,

$$\eta = N(0)\tfrac{1}{30}(k_B T_c v_F p_F)^{-1} = \frac{8\pi^2}{7\zeta(3)}\frac{k_B T_c}{v_F p_F}\left|\beta_1^{BCS}\right|, \tag{3.32}$$

and $\langle\ldots\rangle$ denotes the angular average $\int_0^1 d(\cos\theta/2) \times \int_0^{2\pi} d\phi/2\pi(\ldots)$. The weighting functions W_i^α and V_i^α are given in Table II.

Equation (3.31) allows a qualitative discussion of the combination $\beta_4+\beta_5-2\beta_1-\beta_3$ of fourth-order coefficients. This particular combination contains information on the stability of the axial state; a necessary condition for its stability is

$$\beta_4+\beta_5-2\beta_1-\beta_3 < -\tfrac{1}{2}(\beta_1+\beta_3) < 0. \tag{3.33}$$

In weak-coupling theory $\beta_4+\beta_5-2\beta_1-\beta_3=0$; the axial state is unstable. The strong-coupling corrections of order $(T_c/T_F)^3$ can be obtained from Eq. (3.31):

$$\beta_4+\beta_5-2\beta_1-\beta_3 = -\eta\frac{301}{4}\langle[(\cos^2\tfrac{1}{2}\theta+\sin^2\tfrac{1}{2}\theta\cos\phi)^2-\cos^2\theta] \times [T^{(a)}(\theta,\phi)]^2\rangle. \tag{3.34}$$

Notice that the weighting function in (3.34) has vanishing total weight,

$$\langle(\cos^2\tfrac{1}{2}\theta+\sin^2\tfrac{1}{2}\theta\cos\phi)^2-\cos^2\theta\rangle = 0. \tag{3.35}$$

Therefore, $\beta_4+\beta_5-2\beta_1-\beta_3$ depends sensitively on the detailed structure of the triplet scattering amplitude. The positive weight is located around $\theta=\frac{1}{2}\pi, \phi=0$, in the region of large-angle scattering of *particle-hole* pairs with small total momentum; the negative weight is found around $\theta=\pi, \phi=\frac{1}{2}\pi$ in the region of large-angle scattering of *particle-particle* pairs with small total momentum. The sign of the discriminant $\beta_4+\beta_5-2\beta_1-\beta_3$, therefore, depends on whether particle-hole or particle-particle scattering dominates. It is generally accepted that in ^{3}He the dominant scattering occurs in the triplet channel of particle-hole pairs with small total momentum. This implies that $\beta_4+\beta_5-2\beta_1-\beta_3$ is negative, and the axial state can be stable. However, Eq. (3.34) shows that because of the competing effects of particle-hole and particle-particle scattering, a quantitative calculation of $\beta_4+\beta_5-2\beta_1-\beta_3$ requires an accurate scattering amplitude in *both* the particle-hole and particle-particle channels.

TABLE II. Weighting functions for the $(T_c/T_F)^3$ corrections to the p-wave Ginzburg-Landau coefficients.

	$\alpha=f$	$\alpha=b+c$	$\alpha=d$
W_1^α	$-4x_1^2+x_2^2+x_3^2$	$-\frac{3}{2}x_1^2-3x_2^2+\frac{3}{2}$	0
V_1^α	$20x_1^2-5x_2^2-5x_3^2$	$\frac{15}{2}x_1^2+3x_2^2-\frac{7}{2}$	$3x_1x_3-x_2$
W_2^α	$-2x_1^2+8x_2^2-2x_3^2$	$\frac{1}{2}x_1^2+x_2^2+\frac{9}{2}$	$2x_1x_3-4x_2$
V_2^α	$2x_1^2-8x_2^2+2x_3^2$	$-\frac{21}{2}x_1^2-9x_2^2+\frac{43}{2}$	$-3x_1x_3+x_2$
W_3^α	$8x_1^2-2x_2^2-2x_3^2$	$3x_1^2+6x_2^2-3$	$-3x_1x_3+x_2$
V_3^α	$-8x_1^2+2x_2^2+2x_3^2$	$-3x_1^2+6x_2^2-1$	$-3x_1x_3+x_2$
W_4^α	$-2x_1^2-2x_2^2+8x_3^2$	$\frac{1}{2}x_1^2+x_2^2+\frac{9}{2}$	$-3x_1x_3+x_2$
V_4^α	$2x_1^2+2x_2^2-8x_3^2$	$\frac{19}{2}x_1^2+11x_2^2-\frac{37}{2}$	$2x_1x_3-4x_2$
W_5^α	$2x_1^2-3x_2^2-3x_3^2$	$-\frac{1}{2}x_1^2-x_2^2-\frac{9}{2}$	0
V_5^α	$-10x_1^2+15x_2^2+15x_3^2$	$\frac{5}{2}x_1^2+x_2^2+\frac{21}{2}$	$x_1x_3+3x_2$

$x_1=\cos\theta,\quad x_2=\cos^2(\theta/2)+\sin^2(\theta/2)\cos\phi,$

$x_3=\cos^2(\theta/2)-\sin^2(\theta/2)\cos\phi.$

IV. s-p APPROXIMATION

To obtain explicit results for the strong-coupling corrections to the free energy of superfluid ^{3}He, one must know the quasiparticle scattering amplitude in the normal state. Unfortunately this scattering amplitude cannot be calculated from first principles, since its structure is dominated by complicated many-body mechanisms[31]; neither is it fully determined by the available experimental information, since the measurable quantities only fix a few weighted averages of the scattering amplitude. The only possible way to proceed is to make an educated guess for the scattering amplitude and to check it against all available experimental information. Our investigation shows that the thermodynamic properties of superfluid ^{3}He can provide additional possibilities for such a check. In the present chapter we discuss the strong-coupling corrections to the fourth-order Ginzburg-Landau coefficients, using the s-p approximation for the scattering amplitude.

Any approximation for the scattering amplitude should be guided by three rigorous results. The first of these is the exchange antisymmetry of the scattering amplitude, Eq. (3.13), which in terms of the angles θ and ϕ becomes

$$T^{(s)}(\theta,\phi) = -\tfrac{1}{2}[T^{(s)}(\theta,\phi+\pi)+3T^{(a)}(\theta,\phi+\pi)],$$
$$T^{(a)}(\theta,\phi) = -\tfrac{1}{2}[T^{(s)}(\theta,\phi+\pi)-T^{(a)}(\theta,\phi+\pi)]. \tag{4.1}$$

The second is the relation between the forward scattering amplitude and the Landau parameters,[10]

$$T^{(\alpha)}(\theta,\phi=0) = \sum_l A_l^\alpha P_l(\cos\theta),$$
$$A_l^\alpha = F_l^\alpha/[1+F_l^\alpha/(2l+1)], \tag{4.2}$$

where $\alpha = s$ or a. Finally, the coefficients A_l^α satisfy the forward scattering sum rule,[32]

$$\sum_{l=0}^{\infty} (A_l^s + A_l^a) = 0 . \tag{4.3}$$

Conventionally, an approximation is checked against measured transport coefficients such as the viscosity, thermal conductivity, and spin-diffusion coefficient. Since the strong-coupling corrections are more sensitive to detailed structures in the scattering amplitude than are these transport coefficients, any acceptable approximation for our strong-coupling calculation should at least give values for the transport coefficients in reasonable agreement with experiment.

To our knowledge, no scattering amplitude for ^{3}He quasiparticles is available which satisfies these requirements in the whole pressure range of interest. The s-p approximation gives reasonably good normal-state transport coefficients at low pressures, as shown by Dy and Pethick,[22] but fails near melting pressure, according to the recent work of Pethick, Smith, and Bhattacharyya.[13]

The s-p approximation is based on the observation that, if one can neglect the scattering in all states in which the relative angular momentum of the incoming quasiparticles along their total momentum is larger than one, then Eqs. (4.1) and (4.2) uniquely determine the scattering amplitude in terms of the Landau parameters. In our calculation we took $F_l^\alpha = 0$ for $l \geq 2$, the approximation used by Dy and Pethick. The approximate scattering amplitude is then

$$\begin{aligned} T^{(s)}(\theta, \phi) = \tfrac{1}{4}[&(A_0^s - 3A_0^a) + (A_1^s - 3A_1^a)\cos\theta \\ &+ 3(A_0^s + A_0^a)\cos\phi + 3(A_1^s + A_1^a)\cos\theta\cos\phi] , \\ T^{(a)}(\theta, \phi) = \tfrac{1}{4}[&-(A_0^s - 3A_0^a) - (A_1^s - 3A_1^a)\cos\theta \\ &+ (A_0^s + A_0^a)\cos\phi + (A_1^s + A_1^a)\cos\theta\cos\phi] . \end{aligned} \tag{4.4}$$

Using this scattering amplitude, all the integrals in Eq. (3.31) can be performed analytically. As a result one obtains the strong-coupling corrections to the Ginzburg-Landau parameters as a quadratic form in the Landau parameters A_l^α,

$$\Delta\beta_i = \sum_{j=1}^{10} B_{ij} G_j , \tag{4.5}$$

where G_j denotes the ten second-order combinations of the four $l \leq 1$ Landau parameters A_l^α. The coefficient matrix B_{ij} is given in Table III.

The Landau parameters A_0^s, A_0^a, and A_1^s and the Fermi momentum k_F can be taken from experiment. If in addition one uses the forward scattering sum rule (4.3) to determine A_1^a, then the strong-coupling corrections in the s-p approximation are fully determined by T_c and by measured normal-state quantities, and can be compared with the results from specific-heat measurements in the superfluid state. This comparison together with a further discussion of the s-p results can be found in Sec. I. Unfortunately, accurate measurements of the specific heat have until now only been performed at melting pressure, where the s-p approximation is known to be inadequate and therefore cannot give reliable results for the strong-coupling effects. It is interesting, however, that both the viscosity in the normal state[13] and the specific-heat jump at the normal to A transition indicate that the s-p approximation overestimates the scattering strengh by roughly a factor of two. In the low-pressure region where the transport data are well represented by the s-p approximation, our theory cannot be checked because of the lack of experimental information on the thermodynamic properties of superfluid ^{3}He. We have presented the strong-coupling results in the s-p approximation with the hope that low-pressure specific-heat data will soon be available for a decisive test of our theory and of the s-p approximation.

To check the possibility of f-wave pairing, we have calculated in the s-p approximation the eleven fourth-order coefficients. The calculated correc-

TABLE III. $(T_c/T_F)^3$ corrections to the p-wave Ginzburg-Landau coefficients in the s–p approximation.

	$(A_0^s)^2$	$(A_0^a)^2$	$(A_1^s)^2$	$(A_1^a)^2$	$A_0^sA_0^a$	$A_1^sA_1^a$	$A_0^sA_1^s$	$A_0^sA_1^a$	$A_0^aA_1^s$	$A_0^aA_1^a$	
$\Delta\beta_1$	3	−45	−2	−48	33	38	0	−35	−28	83	$\times \frac{k_BT_c}{v_Fp_F} \lvert\beta_1^{\text{BCS}}\rvert$
$\Delta\beta_2$	−27	4	−19	−21	−16	−4	24	8	13	21	
$\Delta\beta_3$	−16	−16	−11	−12	−27	−19	22	22	22	21	
$\Delta\beta_4$	−14	−71	−7	−17	−37	−34	15	21	17	35	
$\Delta\beta_5$	11	−73	0	−59	48	59	−13	−28	−10	39	

tions stabilize none of the states suggested as possibly compatible with NMR experiments.[33]

To complete this section we will briefly comment on the spin-fluctuation model. In the framework developed here the spin-fluctuation model amounts to a special approximation for the scattering amplitude,

$$T^{(s)}(\theta,\phi)=\frac{3}{2}\frac{\bar{I}}{1-\bar{I}+\alpha\bar{I}\sin^2(\theta/2)\cos^2(\phi/2)}\,,$$

$$T^{(a)}(\theta,\phi)=-\frac{\bar{I}}{1-\bar{I}+\alpha\bar{I}\sin^2(\theta/2)\sin^2(\phi/2)}-\frac{1}{2}\frac{\bar{I}}{1-\bar{I}+\alpha\bar{I}\sin^2(\theta/2)\cos^2(\phi/2)}\,. \quad (4.6)$$

This approximation is at least incorrect in the forward scattering limit since it does not correctly reproduce all the known Landau parameters. For example, if $\bar{I}$ and α are chosen to fit the Landau parameters A_0^s and A_0^a, then the mass-enhancement parameter A_1^s is a factor of two too small. For this reason one cannot expect (and does not find) quantitatively correct results for the strong-coupling parameters. The ansatz (4.6) does, however, provide a link to previous calculations in the spin-fluctuation model and allows a critical discussion of the conventional approximations. This discussion can be found in Ref. 34.

V. DISCUSSION

Our main purpose in this paper was to present a systematic framework, based on an asymptotic expansion in T_c/T_F, for calculating the strong-coupling corrections to the BCS pairing free energy. We identified the leading strong-coupling corrections, of order $(T_c/T_F)^3$, and showed that these corrections can be expressed in terms of quasiparticle properties in the normal state. We then applied this scheme to the superfluid phases of ^{3}He and calculated the coefficients $\beta_1,\ldots,\beta_5$ in the $l=1$ Ginzburg-Landau free-energy functional. Our results show that these coefficients depend sensitively on the detailed structure of the quasiparticle scattering amplitude. Consequently, the adequacy of our framework cannot be assessed decisively by comparison with the available experimental data, because no reliable scattering amplitude is known for ^{3}He quasiparticles in the relevant pressure range. Since we have at present only theoretical support for our framework, we presented our arguments in more than usual detail, both to help the reader judge their validity, and also to make clear what a more accurate theory would involve. If our framework is adequate for superfluid ^{3}He, it can also be used to investigate the strong-coupling corrections to the magnetic properties, to the properties of inhomogeneous systems, and to transport properties in the superfluid state.

In our opinion the most critical point in our framework is the neglect of terms of order higher than $(T_c/T_F)^3$ in the asymptotic expansion of the free-energy functional. Due mainly to the decrease in T_c/T_F, the strong-coupling corrections become smaller with decreasing pressure, until at vapor pressure the free-energy terms of order $(T_c/T_F)^3$ give only 10% corrections to the weak-coupling $T_c/T_F)^2$ terms. One therefore expects very good convergence of the T_c/T_F expansion at low pressures, and it is likely that our $(T_c/T_F)^3$ terms describe well the small corrections to the weak-coupling theory. Near the melting pressure, however, the strong-coupling corrections are almost $\frac{1}{2}$ as large as the weak-coupling free energy, and this might cause some doubts about our neglect of higher-order terms in T_c/T_F. It is unfortunately impossible to accurately discuss the higher-order corrections for the purpose of checking our approximations. For this purpose we would need information about the high-energy parts of the Green's functions and about higher-order vertex functions in the whole energy range; none of these are known with any accuracy. The most dangerous diagram which we have neglected is shown in Fig. 6. It consists of four F functions connected by those parts of the normal eight-point vertex which cannot be decomposed into a simple product of two four-point vertices. This diagram contributes to the free-energy difference in order $(T_c/T_F)^4\log^4(T_F/T_c)$. It must be compared with diagram 3(f), of order $(T_c/T_F)^3$, which comes from those parts of the eight-point vertex which can be decomposed into a product of two four-point vertices. If we take $T_c/T_F\simeq 3\times10^{-3}$, then the additional "small" factor $(T_c/T_F)\log^4(T_F/T_c)$ is approximately 3, so diagram 6 appears to be comparable to those strictly of order $(T_c/T_F)^3$. We nevertheless expect the contribution of diagram 6 to be small compared to the contribution of diagram 3(f), because the frequency sums in

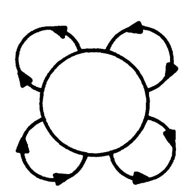

FIG. 6. A diagram of order $(T_c/T_F)^4\log^4(T_F/T_c)$. The open circle represents that part of the normal eight-point vertex function which cannot be decomposed into a product of two normal four-point vertex functions.

diagram 3(f) converge relatively slowly and contribute a factor 30. This factor has the same source as the factor $\log^4(T_F/T_c)$ in diagram 6 and must, therefore, be included for a consistent comparison. We then conclude that the contribution from diagram 6 is smaller than that from diagram 3(f) by approximately an order of magnitude, which seems to be consistent with the results that Tewordt, Fay, Dörre, and Einzel[20] obtained calculating diagram 6 in spin-fluctuation theory. This estimate is only valid, however, provided that the unknown eight-point vertex in Fig. 6 is not an order of magnitude greater than the product of two four-point vertices in Fig. 3(f).

In addition to the higher-order contributions generated by the diagrams we neglected, there are other higher-order contributions to the free energy which are naturally included in our framework. These contributions arise from solving (2.13) self-consistently for the energy gap and self-energy using our strong-coupling functional, and then evaluating the free-energy functional using the self-consistent solutions. The calculations in Secs. III and IV implicitly include these higher-order corrections: We calculated the Ginzburg-Landau parameters β_i through order $(T_c/T_F)^3$, but Δ^2 and the free energy are both $\sim 1/\beta_i$. Furthermore, it is easily seen that the strong-coupling corrections to the diagonal self-energy which are generated by our strong-coupling functional do not contribute to the free energy through fourth order in Δ; and, as pointed out in the beginning of Sec. III, all corrections to the $|\vec{k}|$ and ω_n dependence of the gap as determined by the linearized gap equation enter the free energy in higher order than $(T - T_c)^2$.

Our personal hope is that the diagrams of order $(T_c/T_F)^3$ give a reasonably accurate description of strong-coupling effects even at the melting curve of ^{3}He. Otherwise we cannot expect to find a reliable strong-coupling theory because of our lack of knowledge of the nonquasiparticle properties and of the high-order normal-state correlations which enter in the strong-coupling terms beyond $(T_c/T_F)^3$.

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