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Renormalization-group and mode-coupling theories of critical dynamics

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We discuss renormalization-group and mode-coupling theories of critical dynamics for several models. We show that to lowest order in $\epsilon = d_c - d$, where for $d > d_c$ conventional theory is valid, the renormalization-group differential equations are related by a simple transformation to those obtained from mode-coupling theory. Furthermore, the values of the dynamical fixed-point parameters essentially coincide with the critical amplitudes of the diverging transport coefficients, as determined by mode-coupling theory. In addition, we present a correct renormalization-group treatment for dynamical systems with more than one time scale, as for the binary liquid. We stress the fact that there are two distinct classes of critical dynamics, as illustrated by the results for the models studied here. We also discuss the instability of the conventional fixed point (time-dependent Ginzburg-Landau like) against mode-coupling perturbations for $d < d_c$, with particular emphasis on a model for superfluid helium.

I. INTRODUCTION

In this paper we present renormalization-group and mode-coupling analyses of several models of critical dynamics as defined by appropriate Fokker-Planck stochastic equations. Our results are valid to lowest order in $\epsilon = d_c - d$, where the conventional theory of transport anomalies is valid for dimensionality $d > d_c$. These models have been studied previously to this order in ϵ, both by renormalization-group analyses[1,2] and more recently via mode coupling theory.[3] Thus many of our results are not new in that they are contained either explicitly or implicitly in the works of Halperin, Hohenberg, and Siggia,[1] and Ma and Mazenko,[2] as well as to some extent in our earlier mode-coupling calculations.[3] However, our aim here is to clarify certain issues that have not yet been adequately discussed in existing literature. First, we wish to show the intimate relationship that exists between the renormalization-group and mode-coupling approaches. In particular, we show here that to order ϵ they yield identical results for the basic renormalization-group differential equations that determine the fixed-point values and dynamical critical exponents. More precisely, a simple scale transformation transforms the mode-coupling equations into the corresponding renormalization-group equations. In particular, the values of the dynamic fixed points are essentially given by the amplitudes of the diverging transport coefficients as determined by mode-coupling theory. Second, we attempt to clarify the previous renormalization-group work on the binary liquid, both as regards the proper treatment of a system with two distinct time scales as well as in the different role played by the "bare" transport coefficients in this case. Finally, we explicitly examine the problem of the relative stability of the conventional [time-dependent Ginzburg-Landau (TDGL)-like] and nonconventional fixed points with respect to the mode coupling and certain other perturbations. This is of particular interest with respect to the question of the correct fixed point for superfluid helium. In fact, some authors[4] have argued recently that such a system has a TDGL-like fixed point[5] for $d < d_c$, a result which is contrary to that of Halperin[1] *et al.* and of our present analysis. We argue here that this TDGL fixed point is in fact unstable with respect to the mode-coupling perturbation and hence that it is not the correct fixed point for superfluid He. This result is implicitly contained in the work of Halperin *et al.*[1]

The structure of our paper is as follows: in Sec. II we summarize some formal aspects of the dynamic renormalization group. In particular, we discuss the dynamical analog of the set of eigenoperators and eigenvalues (crossover exponents) which is of fundamental significance in the static renormalization group.[6,7] We also briefly discuss the formal relationship between the renormalization-group and mode-coupling theories. In Secs. III–VI we give a renormalization-group analysis of Fokker-Planck stochastic models for the critical dynamics of the Heisenberg ferromagnet, superfluid helium and planar ferromagnets, and binary liquids. Two models for helium are discussed, one a simplified symmetric model given in Sec. IV and the other a more satisfactory asymmetric model given in Sec. VI. In all these cases we find that the conventional fixed point is unstable with respect to the mode-coupling perturbation for $d < d_c$. In Sec. IV and V we discuss, respectively, the stability of the TDGL (conventional) fixed point for superfluid helium and the notion of asymptotic closure of long wave fluctuations in critical dynamics. We present some concluding remarks in Sec. VII. Finally we present in Appendix

A a simple method of eliminating to order ϵ the "rapidly varying" degrees of freedom in the Fokker-Planck stochastic equation.

II. DYNAMIC RENORMALIZATION GROUP

In this section we give a brief review of the formalism of the dynamic renormalization group[1,2,8] within the context of a stochastic equation for the probability distribution function $P(\{a\},t)$ of a set of gross variables[9] $\{a_i(\vec{k})\}$ with wave number k. We denote this set by the symbol $\{a\}$. For convenience we will limit our discussion to the case when the stochastic equation is of the form

$$\frac{\partial P}{\partial t}=\mathcal{L}P, \tag{2.1}$$

which is of the type studied in the text. To lowest order in ϵ the renormalized stochastic equation maintains this simple form; in higher order in ϵ, memory effects would have to be included. Although our discussion will be limited here to the form [Eq. (2.1)], a general discussion of an exact dynamic renormalization group which includes memory effects will be given elsewhere.[10] The stochastic operator $\mathcal{L}$ in Eq. (2.1) can be written generally as

$$\mathcal{L}=\sum_{ij}\sum_{\vec{k}}\frac{\partial}{\partial a_i(\vec{k})}k^{p_{ij}}L_{ij}\left(\frac{\partial}{\partial a_j^*(\vec{k})}+\frac{\partial\Phi}{\partial a_j^*(\vec{k})}\right)-\sum_i\sum_{\vec{k}}\frac{\partial}{\partial a_i(\vec{k})}v_{i\vec{k}}(\{a\}), \tag{2.2}$$

where p_{ij} is two or zero depending on whether or not conservation laws exist for the variables $a_i(\vec{k})$ and $a_j(\vec{k})$, L_{ij} is a "bare" Onsager kinetic coefficient, Φ is the free energy associated with $\{a\}$ divided by k_BT, and $v_{i\vec{k}}$ is the "instantaneous velocity" of $a_i(\vec{k})$. The so-called "streaming term" in $v_{i\vec{k}}$ in Eq. (2.2) describes, for the examples studied here, the coupling between modes and is proportional to a coupling constant λ_i. Thus the stochastic operator $\mathcal{L}(\{\mu\})$ is a function of a set of parameters $\{\mu\}$ which includes the bare Onsager coefficients $\{L_{ij}\}$, the mode coupling coefficients $\{\lambda_i\}$, and any thermodynamic variables such as temperature and magnetic field which occur in the free energy Φ. These latter variables must be adjusted to their critical-point values for the system to be at criticality.

The dynamical renormalization group can be discussed in a manner similar to that for statics. We first eliminate the "short wavelength" fluctuations which occur in Eq. (2.1) by suitably integrating over all Fourier components $a_i(\vec{k})$ with $b^{-1}\Lambda<k<\Lambda$, where Λ is the upper cutoff in the sum over $\vec{k}$ in Eq. (2.1) and $b\gtrsim 1$. Next, we apply the scale transformation

$$\vec{k}\to b\vec{k},\quad a_i(\vec{k})\to b^{x_i}a_i(b\vec{k}),\quad t\to b^z t, \tag{2.3}$$

where $x_i=1-\frac{1}{2}\eta$ when a_i is the order parameter. The above scale transformation of time is appropriate only when there is a single relevant time scale, as for the Heisenberg ferromagnet. More generally, as is discussed in the text, one must consider a set of dynamical exponents $\{z_i\}$ in order to scale the set of time scales. The two operations above (elimination of short wavelengths and scale transformation) can be considered to constitute a dynamical renormalization-group transformation on the set of *scaled* variables which parametrize $\mathcal{L}$. The relation of these scaled variables to the original physical variables $\{\mu\}$ can be obtained by applying the renormalization transformation to Eq. (2.2). This yields a relation between the original stochastic operator $\mathcal{L}^1$ and the new stochastic operator $\mathcal{L}^2$ of the form

$$\mathcal{L}^2(\{b^{x_i}a_i(b\vec{k})\},\xi,V,\{L_{ij}^2\},\{\lambda_i^2\})=b^{-z}\mathcal{L}^1(\{a_i(\vec{k})\},\xi/b,V/b^d,\{b^{z-p_{ij}-x_i-x_j}L_{ij}^2\},\{b^{z-\mu_i-x_i}\lambda_i^2\}), \tag{2.4}$$

where μ_i is the exponent which characterizes the transformation of $v_{i\vec{k}}$, i.e.;

$$v_{i,\vec{q}}^2(\{b^{x_j}a_j(b\vec{k})\},\xi,V,\lambda_i^2)=b^{-\mu_i}v_{i,b\vec{q}}^1(\{a_j(\vec{k})\},\xi/b,V/b^d,\lambda_i^2). \tag{2.5}$$

In the above the superscripts 1 and 2 denote the quantities before and after elimination of short wavelength fluctuations, respectively, V is the system volume, and ξ is the correlation length. We suppress all other thermodynamic variables. Thus upon iterating the renormalization transformation l times we can identify

$$\lambda_{i,l}=\lambda_i^1,\quad \lambda_{i,l+1}=b^{z-\mu_i-x_i}\lambda_i^2$$

and

$$L_{ij,l}=L_{ij}^1,\quad L_{ij,l+1}=b^{z-p_{ij}-x_i-x_j}L_{ij}^2,$$

in an obvious notation. These constitute the appropriate scale transformations. The renormalization-group equations may then be taken to be

$$\mu_{l+1}=R_b\mu_l, \tag{2.6}$$

where R_b represents the renormalization-group operation described above. A fixed point μ^* is given by the solution of

$$\mu^* = R_b \mu^* . \tag{2.7}$$

In order to find a fixed-point solution the system must be at criticality and the exponents η and z must take certain values.

It is often convenient to give a differential formulation of Eq. (2.6). For this purpose we introduce the variable $k = k_0 b^l$ for some constant k_0 and choose $(b-1)$ to be infinitesimal. We also define the scaled variables $L_{ij}(k) = L_{ij,\,l}$ and $\hat{\lambda}_i(k) = \lambda_{i,\,l}$, so that $\{\mu_l\} \to \{\hat{\mu}(k)\}$, and transform Eq. (2.6) into the differential form

$$k \frac{d\hat{\mu}(k)}{dk} = -G\hat{\mu}(k) \tag{2.8}$$

or, in the autonomous form

$$\frac{d\hat{\mu}}{d\tau} = G\hat{\mu}, \tag{2.9}$$

where $\tau = -\ln k$ and where G is the infinitesimal generator of the group transformation. Equation (2.9) is the dynamical analog (ignoring memory effects) of Wilson's differential form of the renormalization-group equation for the static Hamiltonian.[7] The dynamical fixed point corresponds to $G\hat{\mu}^* = 0$.

We now digress to discuss the relationship between the renormalization-group and mode-coupling approaches.[3] The latter yields equations for the physical transport coefficients $L_{ij}(k)$, and the mode-coupling coefficients λ_i, for small values of the wave number k. These equations describe the effect of long-wavelength ($k \to 0$) fluctuations on the transport properties of the system. The renormalization group, on the other hand, yields equations for the *scaled* variables $\hat{L}_{ij}(k)$, and $\hat{\lambda}_i(k)$, in the limit $k \to \infty$, where k again describes the effects of long wavelength fluctuations, as $l \to \infty$. The relationship between these two approaches can be obtained by linking the scaled variables to the physical variables by suitable scale transformations which take into account the different limiting values of k which are involved in the two cases. Thus we introduce

$$\hat{L}_{ij}(k) = A_{ij} k^{z - z^c_{ij}} L_{ij}(k_m^2/k) \tag{2.10}$$

and

$$\hat{\lambda}_i(k) = A_i k^{z - w_i} \lambda_i , \tag{2.11}$$

where $z^c_{ij} = (p_{ij} + x_i + x_j)$ is the conventional dynamical critical exponent and $w_i = \mu_i + x_i$ is the scaling exponent of the mode-coupling term with λ_i. (Note that w_i in fact is the dynamic critical exponent for such systems as isotropic and planar ferromagnets and superfluid helium.) The arbitrary constants in Eqs. (2.10) and (2.11) arise from (a) the arbitrariness in choosing a scale for k and (b) the invariance of the renormalization-group equations (RGE) to multiplication of the members of $\{\hat{\mu}\}$ by some arbitrary constants. The different k dependences of $\hat{L}_{ij}$ and L_{ij} in Eqs. (2.10) and (2.10′) come from the fact that $\hat{L}_{ij}(k)$ is the transport coefficient after renormalization is repeated $\ln(k/k_0)/\ln b$ times, whereas $L_{ij}(k)$ contains the contributions of fluctuations whose wave numbers are *greater than* k. Substitution of Eqs. (2.10) and (2.11) into Eqs. (2.8) or (2.9) then yields differential equations for $L_{ij}(k)$ and λ_i. Note however, that since at criticality $L_{ij}(k)$ diverges for the models studied here as $k \to 0$ for $d < d_c$, one cannot ascribe fixed-point values to the L_{ij}. On the other hand, Eq. (2.10) can also be written

$$L_{ij}(k) = A_{ij}^{-1} k_m^{2(z^c_{ij} - z)} \hat{L}_{ij}(k_m^2/k) k^{z - z^c_{ij}} . \tag{2.10′}$$

This suggests that there is in general a simple relationship between the critical amplitudes of these diverging kinetic coefficients and the fixed-point values of $\hat{L}_{ij}$.

In fact, if we adjust arbitrary constants so as to obtain $A_{ij} = k_m^{2(z^c_{ij} - z)}$, we have as $k \to 0$

$$L_{ij}(k) \simeq \hat{L}^*_{ij} k^{z - z^c_{ij}} . \tag{2.12}$$

As we show in our discussion of the various models, the mode-coupling equations (MCE) derived previously[3] for the $\{L_{ij}, \lambda_i\}$ are equivalent (to order ϵ) to the RGE for $\{\hat{L}_{ij}, \hat{\lambda}_i\}$. Furthermore, the fixed-point values in general satisfy Eq. (2.12). This intimate relation between MC and RG approaches is to be expected, since in both cases one attempts to eliminate the short wavelength fluctuations in the dynamical equations of motion. In mode coupling in its original version, one does this in one step, whereas in the RG approach one integrates out these fluctuations in a series of small steps.

Returning to our discussion of the RGE, we now consider the concept of dynamic eigenoperators and eigenvalues (crossover exponents). Consider linearizing Eq. (2.9) for a small perturbation $\delta\hat{\mu}$ about the fixed-point value $\hat{\mu}^*$, i.e.; $\hat{\mu} = \hat{\mu}^* + \delta\hat{\mu}$. Then one obtains from Eq. (2.9) the eigenvalue problem

$$\mathcal{G}\delta\hat{\mu} = -y\delta\hat{\mu}, \tag{2.13}$$

where $d(\delta\hat{\mu})/d\tau = -y\delta\hat{\mu}$ and $\mathcal{G}$ is the linearized form of G. Thus in general one finds a set of dynamical eigenvalues $\{y_i\}$ which describe the response of the system to perturbations around the fixed point. In terms of the original parameter b we have $\delta\hat{\mu} = Ae^{y\tau} = Bb^{ly}$. Thus, as in statics, we can classify dynamical eigenoperators as relevant, marginal, or irrelevant, depending on whether y is greater than, equal to, or less than zero, respectively. These eigenvalues can be considered as crossover exponents, as in statics, which characterize the stability of a given fixed point with respect to a given perturbation. In Secs. II–VI we evaluate the fixed points, eigenoperators, and eigenvalues for the dynamical models mentioned earlier, to order $\epsilon = d_c - d$. Since[7] the static correlation function exponent $\eta = O(\epsilon^2)$, we take $\eta = 0$ in all of these calculations.

For completeness we note that in the case where there is only one dynamical exponent z rather than a set $\{z_i\}$ there exists a fixed-point stochastic operator $\mathcal{L}^*$ for the renormalized stochastic equation

$$\frac{\partial}{\partial t}\hat{P} = \hat{\mathcal{L}}\hat{P}, \tag{2.14}$$

where $\hat{P}$ is the scaled probability distribution function for the long wavelength fluctuations. This agrees with a proposal originally made by Kuramoto.[8] More generally, however, when there is more than one time scale, a fixed-point stochastic operator does *not* exist. However, there is still a fixed point $\hat{\mu}^*$ in the parameter space, which is obtained by requiring that the *form* of the stochastic equation remains invariant, as is discussed in more detail in subsequent sections. We now illustrate the various ideas summarized here by analyzing several models which have critical transport anomalies.

III. ISOTROPIC HEISENBERG FERROMAGNET

We begin by discussing a dynamical model for the Heisenberg ferromagnet in $6-\epsilon$ dimensions. A detailed renormalization-group analysis has been given elsewhere[2] to order ϵ. Our main interest here is to show the equivalence to order ϵ of the mode coupling[3] and renormalization-group treatments. We treat this model in more detail than for subsequent models, in order to illustrate the nature of our calculational scheme. This model is somewhat easier to discuss than the others, owing to the fact that it has only one kind of variable.

Our initial Fokker-Planck equation for the probability distribution function $P_0(\{S_{\vec{k}}\}, t)$ for the Fourier components $\{S_{\vec{k}}\}$ of the magnetization density $S(\vec{r})$, with

$$S_{\vec{k}} = V^{-1/2}\int d\vec{r}\, e^{-i\vec{k}\cdot\vec{r}} S(\vec{r}) \tag{3.1}$$

takes the form[11]

$$\frac{\partial}{\partial t}P_0(\{S_{\vec{k}}\}, t) = \mathcal{L}P_0(\{S_{\vec{k}}\}, t) \tag{3.2}$$

with

$$\mathcal{L} = -\sum_{\vec{q}\alpha}\frac{\partial}{\partial S^\alpha_{\vec{q}}}v^\alpha_{\vec{q}} + \sum_{\vec{q}\alpha} q^2 L_0 \frac{\partial}{\partial S^\alpha_{\vec{q}}}\left(\frac{\partial}{\partial S^\alpha_{-\vec{q}}} + \frac{\partial\Phi}{\partial S^\alpha_{-\vec{q}}}\right) \qquad (\alpha = x, y, z) \tag{3.3}$$

and

$$v^\alpha_{\vec{q}} = \frac{\lambda_0}{V^{1/2}}\sum_{\vec{k}}\sum_{\beta\gamma}\epsilon_{\alpha\beta\gamma}S^\beta_{\vec{q}-\vec{k}}\frac{\partial\Phi}{\partial S^\gamma_{-\vec{k}}}. \tag{3.4}$$

The various terms which appear in Eq. (3.3) are as follows: $\Phi(\{S_{\vec{k}}\})$ is the free energy divided by k_BT, $\epsilon_{\alpha\beta\gamma}$ is the three-dimensional Levi-Civita tensor, L_0 is the bare Onsager kinetic coefficient for spin diffusion, and λ_0 is the mode-coupling constant. As usual, the sum over k is from zero to an upper cutoff Λ. The mode-coupling term corresponds to a precession of spins in the local magnetic field and its effect is negligible for $d > 6$.[2,11-13] Thus the conventional theory of critical dynamics, which ascribes all the critical-dynamical anomaly to the thermodynamic driving force $\partial\Phi/\partial S^\alpha_{-\vec{q}}$ (with the Onsager coefficient remaining finite) is valid for $d > 6$. For $d < 6$ the mode-coupling term is significant and its effect can be treated for small ϵ by the following renormalization-group approach. First, we eliminate the short-wavelength spin fluctuations (SW) by integrating Eq. (3.2) over those $\{S_{\vec{k}}\}$ with wave number $b^{-1}\Lambda < k < \Lambda$, with $b > 1$. To first order in ϵ we can drop the quartic term in $\{S_{\vec{k}}\}$ from Φ and retain terms up to second order in λ_0 as noted in Ref. 2. Thus, as is shown in Appendix A, this yields a Fokker-Planck equation for the reduced probability distribution function $P_1(\{S_{\vec{k}}\}, t)$, for the remaining long-wavelength fluctuations (LW), which takes the following form:

$$\frac{\partial}{\partial t}P_1(\{S_{\vec{k}}\}, t) = \mathcal{L}_e P_1(\{S_{\vec{k}}\}, t). \tag{3.5}$$

The effective stochastic operator

$$\mathcal{L}_e \equiv \langle\mathcal{L}\rangle_{\rm SW} - \langle\mathcal{L}'(\mathcal{L}^{(0)})^{-1}\mathcal{L}'\rangle_{\rm SW} \tag{3.6}$$

with

$$\mathcal{L}^{(0)} \equiv \sum_{\vec{q}}\sum_{\alpha} q^2 L_0 \frac{\partial}{\partial S^\alpha_{\vec{q}}}\left(\frac{\partial}{\partial S^\alpha_{-\vec{q}}} + \frac{\partial\Phi}{\partial S^\alpha_{-\vec{q}}}\right) \tag{3.7}$$

and

$$\mathcal{L}' = -\sum_{\vec{q}}\sum_{\alpha}\frac{\partial}{\partial S^\alpha_{\vec{q}}}\left[v^\alpha_{\vec{q}}(\{S_{\vec{k}}\}) - \langle v^\alpha_{\vec{q}}(\{S_{\vec{k}}\})\rangle_{\rm SW}\right]. \tag{3.8}$$

In the above $\langle O\rangle_{\rm SW}$ is the partial average of any operator O over the short wavelength spin fluctuations $\{S\}_{\rm SW}$ defined by

$$\langle O\rangle_{\rm SW}=\int d\{S\}_{\rm SW}\, O\, P^e_{\rm SW}(\{S\}), \tag{3.9}$$

where $P^e_{\rm SW}(\{S\})$ is the equilibrium distribution function for $\{S\}_{\rm SW}$. $\langle O\rangle_{\rm SW}$ is still an operator in that it acts on the long-wavelength spin fluctuations $\{S\}_{\rm LW}$. Also, note that both in our initial starting point [Eq. (3.2)] and in the derivation of Eq. (3.5), which is based on the Markoffian approximation, we have ignored memory effects. Although there exists an exact formal procedure for eliminating a part of the degrees of freedom that makes use of a projection operator, as noted in Sec. I, the present approximation suffices for our present purposes.

Next, we observe that the second term of Eq. (3.6) can be transformed by making use of the fact that

$$\sum_{\vec{k}\alpha}\frac{\partial}{\partial S^\alpha_{\vec{k}}}v^\alpha_{\vec{k}}\cdots=\sum_{\vec{k}\alpha}v^\alpha_{\vec{k}}\left(\frac{\partial}{\partial S^\alpha_{\vec{k}}}+\frac{\partial\Phi}{\partial S^\alpha_{\vec{k}}}\right)\cdots . \tag{3.10}$$

Thus this second term becomes

$$-\sum_{\vec{q}}^{\rm LW}\sum_{\alpha\beta}\frac{\partial}{\partial S^\alpha_{\vec{q}}}\langle\delta v^\alpha_{\vec{q}}(\mathcal{L}^{(O)})^{-1}\delta v^\beta_{-\vec{q}}\rangle_{\rm SW}\left(\frac{\partial}{\partial S^\beta_{-\vec{q}}}+\frac{\partial\Phi}{\partial S^\beta_{-\vec{q}}}\right), \tag{3.11}$$

where $\delta v^\alpha_{\vec{q}}\equiv v^\alpha_{\vec{q}}-\langle v^\alpha_{\vec{q}}\rangle_{\rm SW}$. In the present approximation the small q behavior of the expression in angular brackets in (3.11) is given by

$$\langle\delta v^\alpha_{\vec{q}}(\mathcal{L}^{(O)})^{-1}\delta v^\beta_{-\vec{q}}\rangle_{\rm SW}=-\delta_{\alpha\beta}q^2\frac{\lambda_0^2}{2L_0 d}\frac{1}{V}\sum_{\vec{k}}^{\rm SW}\frac{1}{k^2\chi_{\vec{k}}}\left(\frac{\partial\chi_{\vec{k}}}{\partial k}\right)^2, \tag{3.12}$$

where we have introduced the k-dependent susceptibility $\chi_{\vec{k}}$ through the approximation

$$\Phi(\{S\})=\tfrac{1}{2}\sum_{\vec{k}}\sum_{\alpha}\chi^{-1}_{\vec{k}}|S^\alpha_{\vec{k}}|^2. \tag{3.13}$$

We shall use this small q form hereafter. With these results we can therefore write

$$\mathcal{L}_e=-\sum_{\vec{q}\alpha}^{\rm LW}\frac{\partial}{\partial S^\alpha_{\vec{q}}}\langle v^\alpha_{\vec{q}}\rangle_{\rm SW}+\sum_{\vec{q}\alpha}^{\rm LW}q^2L_1'\frac{\partial}{\partial S^\alpha_{\vec{q}}}\left(\frac{\partial}{\partial S^\alpha_{-\vec{q}}}+\frac{\partial\Phi}{\partial S^\alpha_{-\vec{q}}}\right), \tag{3.14}$$

where the new Onsager kinetic coefficient is

$$L_1'=L_0+\frac{\lambda_0^2}{2L_0 d}\frac{1}{(2\pi)^d}\int^{\rm SW}dk\,\frac{1}{k^2\chi_{\vec{k}}}\left(\frac{\partial\chi_{\vec{k}}}{\partial k}\right)^2. \tag{3.15}$$

At criticality $\chi_{\vec{k}}=\chi_0 k^{-2}$ [$\eta=O(\epsilon^2)$], so that to lowest order in ϵ Eq. (3.15) reduces to

$$L_1'=L_1+(\lambda_0^2\chi_0/192\pi^3)(\ln b/L_0). \tag{3.16}$$

Next, we note that $\langle v^\alpha_{\vec{q}}\rangle_{\rm SW}$ is obtained from $v^\alpha_{\vec{q}}$ by setting all $\{S_{\vec{k}}\}_{\rm SW}$ equal to zero, so that we see from Eq. (3.15) that $\mathcal{L}_e$ takes essentially the same form as $\mathcal{L}$. This correspondence can be made more precise by making the scale transformation discussed in Sec. II (with $\eta=0$):

$$S_{\vec{k}}\to bS_{b\vec{k}},\quad t\to b^z t,\quad V\to b^d V. \tag{3.17}$$

The transformed stochastic equation at criticality is thus

$$\frac{\partial}{\partial t}P(\{S_{\vec{k}}\},t)=\mathcal{L}_1 P(\{S_{\vec{k}}\},t), \tag{3.18a}$$

where

$$\mathcal{L}_1=\mathcal{L}(b^{z-1-d/2}\lambda_0,\, b^{z-4}L_1')=\mathcal{L}(\lambda_1,L_1) \tag{3.18b}$$

and $\mathcal{L}(\lambda_0,L_0)$ is the original stochastic operator (3.3) at criticality. Equation (3.18) defines the renormalization transformation for the parameters λ and L of the stochastic equation. Written explicitly, this transformation from (λ_l,L_l) to (λ_{l+1},L_{l+1}) is, with $\Lambda=1$ hereafter,

$$\lambda_{l+1}=b^{z-4+\epsilon/2}\lambda_l, \tag{3.19a}$$

$$L_{l+1}=b^{z-4}\left(L_l+\frac{\chi_0\lambda_l^2}{192\pi^3}\frac{\ln b}{L_l}\right). \tag{3.19b}$$

These are the RGE obtained earlier by Ma and Mazenko.[2] They also define the renormalization transformation for the stochastic operator $\mathcal{L}_l\equiv\mathcal{L}(\lambda_l,L_l)$.

As noted in Sec. II, it is convenient to rewrite Eq. (3.19) in a differential form by choosing $b-1=\delta>0$ infinitesimal and introducing the wave number k by $k=k_0b^l$. Thus we find the following differential form of the RGE for $\hat\lambda(k)\equiv\lambda_l$ and $\hat L(k)\equiv L_l$:

$$k\frac{d}{dk}\hat\lambda(k)=(z-4+\tfrac{1}{2}\epsilon)\hat\lambda(k), \tag{3.20a}$$

$$k\frac{d}{dk}\hat L(k)=(z-4)\hat L(k)+\frac{\chi_0}{192\pi^3}\frac{\hat\lambda(k)^2}{\hat L(k)}. \tag{3.20b}$$

We now note that (3.15) in fact gives precisely the renormalization of transport coefficients of the mode-coupling theory.[3] In fact, if we denote by $L(k)$ the transport coefficient in which fluctuations with wave numbers greater than k are included as renormalization contributions, we see that L_1' is $L(b^{-1}\Lambda)$ and L_0 is $L(\Lambda)$. If we write Λ as k and choose $b-1$ to be infinitesimal, we recover the following differential form of the MCE[14] at criticality:

$$\frac{d}{dk}L(k)=-\frac{\chi_0\lambda_0^2}{192\pi^3}\frac{1}{k^{1+\epsilon}L(k)}. \tag{3.21}$$

Note that here we have retained the ϵ dependence which arises from the volume integral over $\vec{k}$, as this is a crucial contribution. As noted before, there is a simple relationship between the solutions of Eqs. (3.20) and (3.21), namely;

$$\hat{\lambda}(k) = Ak^{z-4+\epsilon/2}\lambda_0, \quad (3.22a)$$

$$\hat{L}(k) = (Ak_m^{\epsilon})k^{z-4} L(k_m^2/k), \quad (3.22b)$$

or

$$L(k) = A^{-1}k_m^{8-\epsilon-2z}k^{z-4}\hat{L}(k_m^2/k). \quad (3.22b')$$

The arbitrariness in the constants A and k_m has been discussed earlier.

The conventional ($\hat{\lambda}^* = 0$) and nonconventional ($\hat{\lambda}^* \neq 0$) fixed-point solutions of (3.20) as well as the dynamic critical exponents, eigenoperators, and eigenvalues are given in Tables I and II. Note that the eigenvalue $y_\lambda = \frac{1}{2}\epsilon$ for the conventional fixed-point signals its instability against the mode-coupling term for $d < 6$, i.e., the nonconventional fixed point is stable in that case, the conclusions also reached by Ma and Mazenko.[2]

We now consider the MCE [Eq. (3.21)]; its asymptotic solution for small k is

$$L(k) \simeq (\chi_0/96\pi^3)^{1/2}\lambda_0 k^{-\epsilon/2}. \quad (3.23)$$

According to Eq. (3.22a), $\hat{\lambda}(k) = \lambda^* = A\lambda_0$ near the nonconventional fixed point. Then, with the choice $\lambda^* = \lambda_0$ or $A = 1$, Eq. (3.23) becomes

$$L(k) \simeq L^* k^{-\epsilon/2}, \quad (3.24)$$

which also follows from Eq. (3.22b') be replacing $\hat{L}$ by its fixed-point value L^*. This leads to the observation made in Sec. II that the critical amplitudes of the diverging transport coefficients are just the fixed-point values of these transport coefficients in the RGE.

Finally, we note that in this example there is a fixed-point stochastic operator $\mathcal{L}^* \equiv \mathcal{L}(\lambda^*, L^*)$ which is the dynamical analog of the fixed-point Hamiltonian in equilibrium critical phenomena. This, however, is true only when critical dynamics is governed by a single dynamic critical exponent z.

IV. SYMMETRIC SUPERFLUID HELIUM MODEL AND PLANAR FERROMAGNET

First we define the models by writing down the stochastic operator $\mathcal{L}$. For helium

$$\mathcal{L} = L_0 \int d\vec{r} \frac{\delta}{\delta\psi(\vec{r})}\left(\frac{\delta}{\delta\bar{\psi}} + \frac{\delta\Phi}{\delta\bar{\psi}}\right) + \text{c.c.}$$
$$- \zeta_0 \int d\vec{r} \frac{\delta}{\delta S(\vec{r})} \nabla^2 \left(\frac{\delta}{\delta S(\vec{r})} + \frac{\delta\Phi}{\delta S(\vec{r})}\right)$$
$$- i\lambda \int d\vec{r} \left(\frac{\delta}{\delta\psi(\vec{r})}\psi(\vec{r})\frac{\delta\Phi}{\delta S(\vec{r})} - \frac{\delta}{\delta S(\vec{r})}\psi(\vec{r})\frac{\delta\Phi}{\delta\psi(\vec{r})}\right)$$
$$+ \text{c.c.}, \quad (4.1a)$$

where $\psi(\vec{r})$ and $S(\vec{r})$ are the local-order parameter and the local entropy density fluctuation, and

$$\Phi = \frac{1}{2}\sum_{\vec{k}} \left(\frac{|\psi_{\vec{k}}|^2}{\chi_{\vec{k}}} + \frac{|S_{\vec{k}}|^2}{C_p(\vec{k})}\right). \quad (4.1b)$$

$\bar{\psi}$ is the complex conjugate of ψ and c.c. denotes the complex conjugate of the term that precedes. At criticality we set $\chi_{\vec{k}} = k^{-\gamma/\nu}$ and $C_p(\vec{k}) = k^{-\alpha/\nu}$ where α and ν are the usual specific-heat (C_p) and correlation-length critical exponents. The stochastic operator for the planar ferromagnet is obtained from Eqs. (4.1) by identifying $\psi, \bar{\psi}$, and S as the components of spin density $(S_x + iS_y)/\sqrt{2}$, $(S_x - iS_y)/\sqrt{2}$, and S_z, respectively, and setting $C_p(\vec{k}) = 1$, where L_0 is now real. Here we should point out that the present model for superfluid helium is not quite satisfactory from the renormalization-group point of view since Φ already contains critical anomalies.[15] We will discuss a more satisfactory asymmetric model of superfluid helium in Sec. VI.

These systems can be treated in the same way as the ferromagnet so that our discussion will be rather brief. For superfluid helium the MCE for the thermal conductivity $\zeta(\vec{k})$ and the Onsager kinetic coefficient for the order parameter relaxation rate $L(\vec{k})$ (which is generally complex) are, for the dimensionality $d = 4 - \epsilon$ ($\epsilon > 0$),

$$\frac{d}{dk}L(k) = -\frac{k^{-1-\epsilon+\alpha/\nu}g^2}{L(k) + k^{\alpha/\nu}\zeta(k)}, \quad (4.2a)$$

$$\frac{d}{dk}\zeta(k) = -\frac{k^{-1-\epsilon}g^2}{L(k) + \bar{L}(k)}. \quad (4.2b)$$

The symbol $\bar{L}$ denotes the complex conjugate of L and g is the appropriate mode-coupling coefficient between the order parameter and entropy fluctuations given by $g^2 = \lambda^2/8\pi^2$. The corresponding RGE for the scalar quantities $\hat{\zeta}(k)$, $\hat{L}(k)$, and $\hat{g}(k)$ are

$$k\frac{d}{dk}\hat{L}(k) = (z-2)\hat{L}(k) + \frac{\hat{g}(k)^2}{\hat{L}(k) + \hat{\zeta}(k)}, \quad (4.3a)$$

$$k\frac{d}{dk}\hat{\zeta}(k) = \left(z - 2 - \frac{\alpha}{\nu}\right)\hat{\zeta}(k) + \frac{\hat{g}(k)^2}{\hat{L}(k) + \hat{\bar{L}}(k)}, \quad (4.3b)$$

$$k\frac{d}{dk}\hat{g}(k) = \left(z - \frac{d}{2} - \frac{\alpha}{2\nu}\right)\hat{g}(k). \quad (4.3c)$$

The solutions of these two sets of equations are related by

$$\hat{g}(k) = Ak^{z-d/2-\alpha/2\nu}g, \quad (4.4a)$$

$$\hat{L}(k) = Ak_m^{\epsilon-\alpha/\nu}k^{z-2}L(k_m^2/k), \quad (4.4b)$$

$$\hat{\zeta}(k) = Ak_m^{\epsilon+\alpha/\nu}k^{z-2-\alpha/\nu}\zeta(k_m^2/k). \quad (4.4c)$$

TABLE I. This gives the fixed-point values and dynamical critical exponents (z) to order $\epsilon = d_c - d$ for the models studied in this paper. For given fixed-point values of transport coefficients there are two fixed-point values $\pm\lambda^*$ and $\pm g^*$ which are mirror images to each other as first pointed out in Ref. 2. In the bottom row we have also set $C^* = 1$ and $g^* = \lambda^*/(8\pi^2)^{1/2}$. The conventional fixed-point behavior for the asymmetric model of superfluid helium has been studied in Ref. 5 although we do not reproduce the results here.

Model	Conventional fixed point: Fixed-point values	Conventional fixed point: Dynamic critical exponents	Nonconventional fixed point: Fixed-point values	Nonconventional fixed point: Dynamic critical exponents
1. Heisenberg ferromagnet	$L^* \neq 0$ $\hat{\lambda}^* = 0$	$z = 4$	$L^* = (\chi_0/96\pi^3\epsilon)^{1/2}\lvert\lambda^*\rvert$ $\lambda^* \neq 0$	$z = 4 - \frac{1}{2}\epsilon$
2. Symmetric model of superfluid helium	$\hat{g}^* = 0$ $\hat{L}^*, \hat{\zeta}^* \neq 0$	$z_L = 2$ $z_\zeta = 2 + \alpha/\nu$	$L^* = (5/3\epsilon)^{1/2}\lvert g^*\rvert$ $\zeta^* = \frac{1}{2}(5/3\epsilon)^{1/2}\lvert g^*\rvert$ $g^* \neq 0$	$z = 2 - \frac{2}{5}\epsilon$
3. Planar ferromagnet	$\hat{g}^* = 0$ $\hat{L}^*, \hat{\zeta}^* \neq 0$	$z = 2$	$L^* = (2/\epsilon)^{1/2}\lvert g^*\rvert$ $\zeta^* = (2/\epsilon)^{1/2}\lvert g^*\rvert$ $g^* \neq 0$	$z = \frac{1}{2}d = 2 - \frac{1}{2}\epsilon$
4. Binary liquid	$\hat{g}^* = 0$ $\eta^*, \zeta^* \neq 0$	$z_\zeta = 4$ $z_\eta = 2$	$g^{*2}/\zeta^*\eta^* = \frac{24}{19}\epsilon$ $g^* \neq 0$	$z_\zeta = 4 - \frac{18}{19}\epsilon$ $z_\eta = 2 - \frac{1}{19}\epsilon$
5. Asymmetric model of superfluid helium			$L^* = \lvert g\rvert(2.44/\epsilon)^{1/2} \times (1 + \frac{0.480}{0.732}i)$ $\zeta^* = \lvert g\rvert(5/6 \times 0.732\epsilon)^{1/2}$	$z = 2 - \frac{2}{5}\epsilon$

TABLE II. This gives the eigenoperators and eigenvalues to order ϵ for the linearized renormalization-group equations for the models considered in this paper. The symbols R, M, and I denote relevant, marginal, and irrelevant eigenoperators, respectively, for $\epsilon > 0$. Note that in all cases the conventional fixed point is unstable with respect to the mode-coupling perturbation (y_λ, $y_g > 0$) for $\epsilon > 0$.

Model	Conventional fixed point: Eigenoperator	Eigenvalue	Relevance	Nonconventional fixed point: Eigenoperator	Eigenvalue	Relevance
1. Heisenberg	$\delta\hat{L}$	$y_L = 0$	M	$\delta\hat{L}$	$y_L = -\epsilon$	I
ferromagnet	$\delta\hat{\lambda}$	$y_\lambda = \frac{1}{2}\epsilon$	R	$\delta\hat{\lambda}$	$y_\lambda = 0$	M
2. Symmetric superfluid	$\delta\hat{L}$	$y_L = 0$	M	$\mathrm{Im}\,\delta\hat{L} = \delta\hat{L} - \delta\bar{\hat{L}}$	$y_{\mathrm{Im}L} = -\frac{2}{3}\epsilon$	I
helium	$\delta\hat{\zeta}$	$y_\zeta = 0$	M	$\delta\hat{\zeta} + \frac{1}{16}(1 \mp \sqrt{73})(\delta\hat{L} + \delta\bar{\hat{L}})$	$y_\pm = \frac{1}{30}\epsilon(-19 \pm \sqrt{73})$	I
model	$\delta\hat{g}$	$y_g = \epsilon/2$	R	$\delta\hat{g}$	$y_g = 0$	M
3. Planar	$\delta\hat{L}$	$y_L = 0$	M	$\delta\hat{L} + \delta\hat{\zeta}$	$y_+ = -\epsilon$	I
ferromagnet	$\delta\hat{\zeta}$	$y_\zeta = 0$	M	$\delta\hat{L} - 2\delta\hat{\zeta}$	$y_- = -\frac{1}{4}\epsilon$	I
	$\delta\hat{g}$	$y_g = \frac{1}{2}\epsilon$	R	$\delta\hat{g}$	$y_g = 0$	M
4. Binary	$\delta\hat{\eta}$	$y_\eta = 0$	M	$\delta\hat{f} = \delta(\hat{g}^2/\hat{\eta}\hat{\zeta})$	$y_f = -\epsilon$	I
liquid	$\delta\hat{\zeta}$	$y_\zeta = 0$	M			
	$\delta\hat{g}$	$y_g = \frac{1}{2}\epsilon$	R	$\delta\hat{g}$	$y_g = 0$	M

The fixed-point solutions and corresponding eigenoperators and eigenvalues are listed in Tables I and II. We note the following points: For the nonconventional fixed point $[\hat{g}(k)\to g^*\neq 0]$ g^* is a free parameter as in the Heisenberg ferromagnet. If we choose $g^*=g$ or $A=1$ in Eq. (4.4a) then the critical amplitudes of the diverging transport coefficients are related (as in Sec. II) to the fixed-point values, i.e., $L(k)=L^*k^{-2\epsilon/5}$ and $\xi(k)=\xi^*k^{-3\epsilon/5}$, where we have used $\alpha/\nu=\frac{1}{5}\epsilon$.

Next, we discuss the conventional fixed-point solution $\hat{g}^*=0$ in some detail, as it provides a simple example of the problem posed by the existence of two time scales. First observe that the fixed-point condition $d\hat{L}/dk=d\hat{\xi}/dk=0$ cannot be satisfied in this case when both L^* and ξ^* are nonzero. Second, the consistent treatment of this case requires an introduction of two dynamic critical exponents, z_L and z_ξ, and corresponding characteristic frequencies ω_L and ω_ξ. The stochastic operator $\mathcal{L}$ is also split up into $\mathcal{L}_L(L_0)$ and $\mathcal{L}_\xi(\xi_0)$, where we can ignore the mode-coupling term here. We now introduce two dimensionless times $t_L\equiv\omega_L t$ and $t_\xi\equiv\omega_\xi t$ which we treat as independent variables in the stochastic equation, i.e.;

$$\left(\omega_L\frac{\partial}{\partial t_L}+\omega_\xi\frac{\partial}{\partial t_\xi}\right)P=(\mathcal{L}_L+\mathcal{L}_\xi)P. \tag{4.5}$$

The scale transformation of time [Eq. (2.3)] must now be generalized to include the following two scale changes of t_L and t_ξ, rather than a single scale change of t:

$$t_L\to b^{z_L}t,\quad t_\xi\to b^{z_\xi}t. \tag{4.6}$$

After applying the renormalization transformation the new stochastic equation which corresponds to Eq. (3.18) is

$$\left(\omega_L\frac{\partial}{\partial t_L}+b^{z_L-z_\xi}\omega_\xi\frac{\partial}{\partial t_\xi}\right)P$$
$$=[\mathcal{L}_L(b^{z_L-2}L_0)+b^{z_L-z_\xi}\mathcal{L}_\xi(b^{z_\xi-2-\alpha/\nu}\xi_0)]P. \tag{4.7}$$

The RGE which correspond to Eq. (4.3) are

$$k\frac{d}{dk}\hat{L}(k)=(z_L-2)\hat{L}(k), \tag{4.7a}$$

$$k\frac{d}{dk}\hat{\xi}(k)=\left(z_\xi-2-\frac{\alpha}{\nu}\right)\hat{\xi}(k). \tag{4.8b}$$

The fixed-point condition together with nonzero $\hat{L}^*$ and $\hat{\xi}^*$ leads to the conventional dynamic critical exponents $z_L=2$ and $z_\xi=2+\alpha/\nu$. As is clear from Table II, this fixed point is unstable against the mode-coupling perturbation g for $d<4$. Finally, we note that in this example there is no single fixed-point stochastic operator $\mathcal{L}^*$. However, Eq. (4.7) can be decoupled into two fixed-point stochastic equations

$$\omega_L\frac{\partial}{\partial t_L}P=\mathcal{L}_L(L^*)P, \tag{4.9a}$$

$$\omega_\xi\frac{\partial}{\partial t_\xi}P=\mathcal{L}_\xi(\xi^*)P. \tag{4.9b}$$

This case is rather trivial, though, since the order-parameter relaxation and the heat transport are completely independent here. However, an analogous situation exists for the binary liquid except that there are two interacting processes, as we discuss in Sec. V.

The treatment of the planar ferromagnet is completely parallel to that of the above model for superfluid helium. Both the MCE and the RGE for the planar ferromagnet can be obtained from those for helium by letting $\alpha\to 0$ in Eqs. (4.2) and (4.3), respectively, and noting that $L(k)$ is real for the planar ferromagnet. The results that one obtains in this case are summarized in Tables I and II. The mode-coupling coefficient g^2 in this case is given by $(k_BT)^2/8\pi^2\chi_\parallel$, which is twice the value of the corresponding g^2 in Ref. 3.

As a final note we comment on some recent "microscopic" studies by several different authors[4] of the critical dynamics of the single-component Bose fluid in $4-\epsilon$ dimensions. All of these studies found the same fixed point as that appropriate for the TDGL model.[5] To lowest order in ϵ this is the same as the conventional fixed point. However, all of these calculations are based on formulations in which only the order parameter is involved; couplings to other slowly varying modes such as the entropy density are not included. Our result indicating the instability of the conventional fixed point against such a mode-coupling perturbation for $\epsilon>0$ would suggest that the parameter space considered by these authors is simply too small to allow for the possiblity of non-TDGL fixed points. Although we have not carried out our calculation to $O(\epsilon^2)$, the following fixed-point behavior seems quite plausible for a single-component Bose fluid: For $\epsilon<0$ the Gaussian conventional fixed point is stable. For $\epsilon>0$ the non-Gaussian nonconventional fixed point is the most stable one. The non-Gaussian conventional (TDGL) fixed point is unstable against g, whereas the Gaussian nonconventional fixed point is unstable against the coefficient u of the quartic term in Φ. The Gaussian conventional fixed point is unstable against both u and g. In the above, Gaussian and non-Gaussian refer to the static fixed point, and conventional and nonconventional to the remaining dynamic fixed point.

V. BINARY LIQUID

We define our model by the following stochastic operator $\mathcal{L}$:

$$\mathcal{L} = -\zeta_0 \int d\vec{r} \frac{\delta}{\delta c(\vec{r})} \nabla^2 \left(\frac{\delta}{\delta c(\vec{r})} + \frac{\delta \Phi}{\delta c(\vec{r})} \right) - \eta_0 \int d\vec{r} \frac{\delta}{\delta \vec{u}(\vec{r})} \cdot (\nabla^2 - \nabla\nabla) \cdot \left(\frac{\delta}{\delta \vec{u}(\vec{r})} + \vec{u}(\vec{r}) \right) + \lambda \int d\vec{r} \left[\frac{\delta}{\delta c(\vec{r})} \nabla(c\vec{u}) - \frac{\delta}{\delta \vec{u}(\vec{r})} \cdot \left(\frac{\delta \Phi}{\delta c(\vec{r})} \nabla c(\vec{r}) \right)_\perp \right], \tag{5.1a}$$

where c is the local concentration fluctuation, $\vec{u}(r)$ is the transverse local velocity, and $(\;)_\perp$ denotes taking transverse components. Here also

$$\Phi = \frac{1}{2} \sum_{\vec{k}} \frac{|c_{\vec{k}}|^2}{\chi_{\vec{k}}} \tag{5.1b}$$

with $\chi_{\vec{k}} = k^{-\gamma/\nu}$ and

$$\langle \vec{u}_{\vec{k}} \vec{u}_{-\vec{k}} \rangle = 1 - \vec{k}\vec{k}/k^2. \tag{5.1c}$$

As mentioned earlier a correct analysis of this case requires an introduction of two time scales, $t_\zeta = \omega_\zeta t$ and $t_\eta = \omega_\eta t$, with corresponding dynamic critical exponents z_ζ and z_η for the diffusion and viscous relaxation, respectively. This point was not explicitly discussed in the previous RG treatment[1] of the binary liquid. After a renormalization transformation the stochastic equation at criticality takes the form

$$\left(\omega_\zeta \frac{\partial}{\partial t_\zeta} + b^{z_\zeta - z_\eta} \omega_\eta \frac{\partial}{\partial t_\eta} \right) P = [\mathcal{L}_\zeta(\zeta_{l+1}) + b^{z_\zeta - z_\eta} \mathcal{L}_\eta(\eta_{l+1}) + b^{(z_\zeta - z_\eta)/2} \mathcal{L}'(g_{l+1})] P, \tag{5.2}$$

where ζ_l, η_l, and g_l denote the concentration conductivity, the shear viscosity, and the mode-coupling coefficient, respectively, with $g_0^2 = \lambda^2/8\pi^2$. Likewise $\mathcal{L}_\zeta$, $\mathcal{L}_\eta$, and $\mathcal{L}'$ are the stochastic operators that describe the diffusion, viscous relaxation, and the coupling between these two, respectively. Although the scale invariance of the process at the fixed point $\zeta_{l+1} = \zeta_l = \zeta^*$, etc., is not apparent in Eq. (5.2), this can be seen more readily by eliminating either u or c by the method of Appendix A. Thus, up to the self-consistent second-order treatment in $\mathcal{L}'$, we have

$$\mathcal{L}_{\zeta,\text{eff}} = \mathcal{L}_\zeta - \langle \mathcal{L}'(1/\mathcal{L}_\eta)\mathcal{L}' \rangle_{\vec{u}},$$

$$b^z \mathcal{L}_{\eta,\text{eff}} = b^z \mathcal{L}_\eta - b^z \langle \mathcal{L}'(1/\mathcal{L}_\zeta)\mathcal{L}' \rangle_c,$$

where $z \equiv z_\zeta - z_\eta$ and $\langle \cdots \rangle_{\vec{u}}$ and $\langle \cdots \rangle_c$ denote partial averages over equilibrium distributions $P_e(\{u\})$ and $P_e(\{c\})$, respectively. The scale invariance for $\{c\}$ and $\{u\}$ follow as in the noninteracting case discussed in Sec. IV.

The RGE for $d = 4 - \epsilon$ are

$$k \frac{d}{dk} \hat{\zeta}(k) = (z_\zeta - 4)\hat{\zeta}(k) + \frac{3}{4} \frac{\hat{g}(k)^2}{\hat{\eta}(k)}, \tag{5.3a}$$

$$k \frac{d}{dk} \hat{\eta}(k) = (z_\eta - 2)\hat{\eta}(k) + \frac{1}{24} \frac{\hat{g}(k)^2}{\hat{\zeta}(k)}, \tag{5.3b}$$

$$k \frac{d}{dk} \hat{g}(k) = \left(\frac{z_\zeta + z_\eta}{2} - 3 + \frac{\epsilon}{2} \right) \hat{g}(k). \tag{5.3c}$$

The corresponding MCE are

$$k \frac{d}{dk} \zeta(k) = -\frac{3}{4} \frac{g^2}{k^\epsilon \eta(k)}, \tag{5.4a}$$

$$k \frac{d}{dk} \eta(k) = -\frac{1}{24} \frac{g^2}{k^\epsilon \zeta(k)}. \tag{5.4b}$$

The general relationships between the solutions of Eqs. (5.3) and (5.4) are

$$\hat{g}(k) = A k^{(z_\zeta + z_\eta)/2 - 3 + \epsilon/2} g, \tag{5.5a}$$

$$\hat{\zeta}(k) = k_m^\epsilon A B k^{z_\zeta - 4} \zeta(k_m^2/k), \tag{5.5b}$$

$$\hat{\eta}(k) = k_m^\epsilon A B^{-1} k^{z_\eta - 2} \eta(k_m^2/k). \tag{5.5c}$$

If we compare these relationships with analogous relationships such as (4.4) we see that in this case there is an extra free parameter as well as two exponents z_ζ and z_η. The fixed-point solution of Eqs. (5.3) requires that $d\hat{\zeta}/dk = d\hat{\eta}/dk = d\hat{\lambda}/dk = 0$. This is in contrast to the earlier treatment by Halperin *et al.*[1] based on a single z whose equations do not have a solution in which all three derivatives are simlutaneously zero. For the nonconventional fixed point ($g^* \neq 0$) Eqs. (5.3) yields

$$z_\zeta = 4 - \tfrac{18}{19}\epsilon, \tag{5.6a}$$

$$z_\eta = 2 - \tfrac{1}{19}\epsilon, \tag{5.6b}$$

and

$$g^{*2}/\zeta^* \eta^* = f = \tfrac{24}{19}\epsilon, \tag{5.7}$$

which agree with results of Halperin *et al.*[1] At the conventional fixed point ($g^* = 0$) we have $z_\zeta = 4$ and $z_\eta = 2$. As can be seen from Table II and as is generally true for the models studied here, the conventional fixed point is unstable against the mode-coupling coefficient $\hat{g}$ for $d < 4$. We note as in the previous example that there is no single fixed-point operator for the binary liquid.

For the nonconventional case, with $g^* = Ag$, specifying g^* is not sufficient to determine ζ^* and η^*, in contrast to the other models studied here. Rather, only the product $\zeta^*\eta^*$ is fixed to be $(19/24\epsilon)(g^*)^2$. Likewise the choice $A = 1$ is not sufficient to determine uniquely the critical amplitudes of $\zeta(k)$ and $\eta(k)$. This fact is related to the existence of the following "constants of the motion" of Eqs. (5.3) and (5.4) for the values of z_ζ and z_η given in Eqs. (5.6):

$$\hat{\rho} \equiv \hat{\zeta}(k)/\hat{\eta}(k)^{18}, \tag{5.8a}$$

$$\rho \equiv \zeta(k)/\eta(k)^{18}. \tag{5.8b}$$

Namely, $\hat{\rho}$ and ρ are independent of k. The fixed-point values (critical amplitudes) depend on $\hat{\rho}$ as well as g^* (ρ as well as g). That is, we have

$$\zeta^* = \hat{\rho}^{1/19}[(g^*)^2/f]^{18/19}, \tag{5.9a}$$

$$\eta^* = \hat{\rho}^{-1/19}(g^{*2}/f)^{1/19}, \tag{5.9b}$$

and

$$\zeta(k) = \rho^{1/19}(g^2/f)^{18/19}k^{-18\epsilon/19}, \tag{5.10a}$$

$$\eta(k) = \rho^{-1/19}(g^2/f)^{1/19}k^{-\epsilon/19}. \tag{5.10b}$$

The critical amplitudes in Eqs. (5.10) are again given by the fixed-point values ζ^* and η^* if in addition to choosing[16] $g^* = g$ we also require that $\hat{\rho} = \rho$. The free parameter B then becomes equal to $k_m^{17\epsilon/19}$. The appearance of these constants of motion in this problem reflects the fundamentally different character of the critical dynamics of the binary liquid as compared to that of the other systems studied here. To make this even more transparent we note that we can also write ρ as $\rho = \zeta(\Lambda)/\eta(\Lambda)^{18}$, where $\zeta(\Lambda)$ and $\eta(\Lambda)$ are the "bare" transport coefficients which are determined by the short-wavelength microscopic motion of the system. Similarly, we have $\hat{\rho} = \zeta_0/\eta_0^{18}$, where ζ_0 and η_0 are the initial transport coefficients, that is, the transport coefficients which appear in the stochastic equation before any renormalization is performed. Therefore, in contrast to the earlier examples considered, for which the fixed-point values and critical amplitudes were completely determined by specifying the coupling constants, which are to some extent presumably universal, for the binary liquid the fixed-point values and critical amplitudes are not universal, since microscopic details enter through the ratios ρ and $\hat{\rho}$.

The above result is in a certain sense not surprising. Namely, some years ago one of us introduced the concept of "asymptotic closure of long-wavelength fluctuations" in critical dynamics, for the class of second-order phase transitions in which there is spontaneous breakdown of continuous symmetry.[13] According to this hypothesis, the nonlinear coupling among just the long-wavelength fluctuations completely determines the critical dynamics. The examples considered in Secs. III and IV belong to this class, whereas the binary liquid does not. Namely, in the binary liquid, while the product of the critical amplitudes $\zeta^*\eta^*$ of the thermal conductivity and the shear viscosity contains only g^*, the strength of coupling between long-wavelength fluctuations, the ratio ζ^*/η^* involves $\rho = \zeta(\Lambda)/\eta(\Lambda)^{18}$ which cannot be determined by the MCE or the RGE themselves but depends on the "initial data" that involve short-wavelength fluctuations. Therefore, in this latter case one would naturally expect that the short-wavelength fluctuations should somehow influence its critical dynamics. The results obtained above provide concrete manifestations of this concept.

VI. ASYMMETRIC MODEL OF SUPERFLUID HELIUM

In this section we describe a mode-coupling calculation for a more realistic model of superfluid helium than that considered in Sec. IV. The model is very similar to the one considered by Halperin, Hohenberg, and Siggia[1] and is described by a stochastic equation whose stochastic operator takes the same form as (4.1a) except that Φ now assumes the following analytic form:

$$\Phi = \int d r \{\tfrac{1}{2} r_0 |\psi(\vec{r})|^2 + \tfrac{1}{2}|\vec{\nabla}\psi(\vec{r})|^2 + \tilde{u}_0 |\psi(\vec{r})|^4 + [S(\vec{r})^2/2C_0] + v_0 S(\vec{r})|\psi(\vec{r})|^2\}, \tag{6.1}$$

where r_0, $\tilde{u}_0$, C_0, and v_0 are analytic functions of thermodynamic variables. As before, we limit ourselves to criticality.

First we are concerned with the static part of the problem where we derive Φ_q obtained by eliminating from Φ the short-wavelength fluctuations $\{\psi_{\vec{k}}\}$, $\{\bar{\psi}_{\vec{k}}\}$, and $\{S_{\vec{k}}\}$ with $q < k < 1$:

$$e^{-\Phi_q} = \int^{>q} e^{-\Phi}, \tag{6.2}$$

where the symbol $\int^{>q}$ stands for the integrations over short-wavelength fluctuations. The ϵ-expansion calculation can be readily performed by first dividing Φ into the following three parts:

$$\Phi = \Phi^{<q} + \Phi^{>q} + \Phi', \tag{6.3}$$

where $\Phi^{<q}$ ($\Phi^{q>}$) is obtained from Φ by omitting all the fluctuations with wave numbers greater (smaller) then q, and Φ' is the remainder containing both short- and long-wavelength fluctuations. Then we treat Φ' as a perturbation anticipating that $\tilde{u}_0 \sim O(\epsilon)$ and $v_0 \sim O(\epsilon^{1/2})$ and obtain aside from an unimportant constant,

$$\Phi_q = \Phi^{<q} + \langle\Phi'\rangle_q - \tfrac{1}{2}\langle(\Phi')^2\rangle_q^c + (1/3!)\langle(\Phi')^3\rangle_q^c + \cdots, \tag{6.4}$$

where $\langle\cdots\rangle_q$ is the average with the weight $e^{-\Phi^{>q}}$ and $\langle\cdots\rangle_q^c$ denotes the cumulant average. For small ϵ, Φ_q takes the form:

$$\Phi_q = \frac{1}{2}\sum_k^{<q}[r(q)+k^2]|\psi_{\vec{k}}|^2 + \frac{1}{V}\tilde{u}(q)\sum_{\{\vec{k}_i\}}^{<q}\bar{\psi}_{\vec{k}_1}\psi_{\vec{k}_2}\bar{\psi}_{\vec{k}_3}\psi_{\vec{k}_1+\vec{k}_3-\vec{k}_2} + \frac{1}{2C(q)}\sum_k^{<q}|S_{\vec{k}}|^2 + \frac{1}{V^{1/2}}v(q)\sum_{\{\vec{k}_i\}}^{<q}S_{\vec{k}_1}\bar{\psi}_{\vec{k}_2-\vec{k}_1}\psi_{\vec{k}_2}+\cdots, \tag{6.5}$$

where $\sum^{<q}$ denotes a summation over wave numbers smaller than q. Here, retaining only the lowest order corrections in ϵ, we find with $K_4 = 1/8\pi^2$

$$r(q) = r_0 + 8K_4u_0\int_q^\Lambda 2k^{1-\epsilon}dk, \tag{6.6a}$$

$$\frac{1}{C(q)} = \frac{1}{C_0} - 4K_4v_0^2\int_q^\Lambda \frac{dk}{k^{1+\epsilon}}, \tag{6.6b}$$

$$v(q) = v_0 - 4K_4v_0(4u_0+v_0^2C_0)\int_q^\Lambda \frac{dk}{k^{1+\epsilon}} \tag{6.6c}$$

and similarly for $\tilde{u}_q$ where

$$u_0 \equiv \tilde{u}_0 - \tfrac{1}{2}v_0^2C_0. \tag{6.7}$$

u_0 is the coefficient of $|\psi(\vec{r})|^4$ of Φ when $\{S(\vec{r})\}$ is eliminated.[5] Equation (6.6a) determines the critical temperature by requiring

$$\lim_{q\to 0} r(q) = 0,$$

and will not be needed subsequently.

Equations (6.6b) and (6.6c) can be made self-consistent by replacing v_0, u_0, and C_0 in front of the integrals by $v(k)$, $u(k)$, and $C(k)$ as follows:

$$\frac{1}{C(q)} = \frac{1}{C_0} - 4K_4\int_q^\Lambda \frac{v(k)^2}{k^{1+\epsilon}}dk, \tag{6.8a}$$

$$v(q) = v_0 - 4K_4\int_q^\Lambda [4u(k)+v(k)^2C(k)]\frac{v(k)}{k^{1+\epsilon}}dk. \tag{6.8b}$$

In the differential form Eqs. (6.8) become

$$\frac{dC(k)}{dk} = -\frac{4K_4v(k)^2C(k)^2}{k^{1+\epsilon}}, \tag{6.9a}$$

$$\frac{dv(k)}{dk} = -\frac{4K_4v(k)}{k^{1+\epsilon}}[4u(k)+v(k)^2C(k)]. \tag{6.9b}$$

If we know that[17]

$$u(k) = u(0)k^\epsilon, \tag{6.10}$$

Eqs. (6.9) determine $v(k)$ and $C(k)$ self-consistently. We note that the calculational scheme quite similar to that described here has been also proposed by Rudnick[17] for the usual Wilson Hamiltonian.

We now go on to discuss the dynamic aspect. One could eliminate the short-wavelength fluctuations from the stochastic equation by the method illustrated in Sec. III and described in Ref. 10 in a general way and obtain the effective stochastic operator with the renormalized transport coefficients. Alternatively, one can use the fact that up to the first order in ϵ these renormalized transport coefficients at criticality are the same as the wave-number-dependent transport coefficients $L(q)$ and $\zeta(q)$. The calculations of the latter are described in Appendix B, and we find

$$\frac{dL(k)}{dk} = -\frac{4K_4}{k^{1+\epsilon}}\frac{[\lambda+iC(k)v(k)L(k)]^2}{2\zeta(k)+C(k)L(k)}, \tag{6.11a}$$

$$\frac{d\zeta(k)}{dk} = -\frac{K_4}{k^{1+\epsilon}}\frac{\lambda^2}{\mathrm{Re}L(k)}. \tag{6.11b}$$

Equations (6.9) and (6.11) together with Eq. (6.10) constitute the MCE for the present model.

Now, the RGE for the quantities $\hat{C}(k)$, etc.,[18] which corresponds to our $C(k)$, etc., have been recently obtained by Halperin, Hohenberg, and Siggia[1] which read up to $O(\epsilon)$ in the present notation as

$$k\frac{d}{dk}\hat{C}(k) = -\frac{\alpha}{\nu}\hat{C}(k) + 4K_4\hat{v}(k)^2\hat{C}(k)^2, \tag{6.12a}$$

$$k\frac{d}{dk}\hat{v}(k) = \left(\frac{\epsilon}{2}+\frac{\alpha}{2\nu} - 16K_4\hat{u}(k) - 4K_4\hat{C}(k)\hat{v}(k)^2\right)\hat{v}(k), \tag{6.12b}$$

$$k\frac{d}{dk}\hat{L}(k) = (z-2)\hat{L}(k) + 4K_4[\hat{\lambda}(k)+i\hat{C}(k)\hat{v}(k)\hat{L}(k)]^2 \times[2\hat{\zeta}(k)+\hat{C}(k)\hat{L}(k)]^{-1}, \tag{6.12c}$$

$$k\frac{d}{dk}\hat{\zeta}(k) = 2\left(z-d-\frac{\alpha}{\nu}\right)\hat{\zeta}(k) + \frac{K_4}{\mathrm{Re}\hat{L}(k)}\hat{\lambda}(k)^2, \tag{6.12d}$$

$$k\frac{d}{dk}\hat{\lambda}(k) = \left(z-\frac{d}{2}-\frac{\alpha}{2\nu}\right)\hat{\lambda}(k), \tag{6.12e}$$

where z is the dynamic critical exponent.

The solutions of Eqs. (6.12) are related to those of the MCE in the following way:

$$C(k) = A^{-1}k^{-\alpha/\nu}\hat{C}(k_m^2/k), \tag{6.13a}$$

$$v(k) = A^{1/2}k^{\epsilon/2+\alpha/2\nu}\hat{v}(k_m^2/k), \tag{6.13b}$$

$$L(k) = BA^{1/2}k^{z-2}\hat{L}(k_m^2/k), \tag{6.13c}$$

$$\zeta(k) = BA^{-1/2}k^{z-2-\alpha/\nu}\hat{\zeta}(k_m^2/k), \tag{6.13d}$$

$$\lambda = B\,k^{z-2+\epsilon/2-\alpha/2\nu}\hat{\lambda}(k_m^2/k), \tag{6.13e}$$

where A, B, and k_m are arbitrary positive numbers and we have chosen

$$u(k) = k^\epsilon\hat{u}(k_m^2/k), \tag{6.14}$$

Eq. (6.14) implies that $u(0)$ of Eq. (6.10) is nothing but the fixed-point value u^* of $\hat{u}(k)$ as $k\to\infty$.

The nontrivial fixed-point condition $k\,d\hat{\lambda}(k)/dk=0$ with $\hat{\lambda}\neq 0$ yields the well-known result[1]

$$z=2-\tfrac{1}{2}\epsilon+\alpha/2\nu \qquad (6.15)$$

and hence $\hat{\lambda}=B^{-1}\lambda$ is independent of k. We now choose $B=1$. If we further choose $A=1$ so that the critical amplitudes of $C(k)$ and $v(k)$ are given by the $k\to\infty$ limits C^* and v^* of $\hat{C}(k)$ and $\hat{v}(k)$, respectively, we again recover the results of Sec. IV, $L(k)=L^*k^{-2\epsilon/5}$ and $\zeta(k)=\zeta^*k^{-3\epsilon/5}$. Here, however, L^* and ζ^* are the fixed-point values of $\hat{L}(k)$ and $\hat{\zeta}(k)$ of Eqs. (6.12) as $k\to\infty$, which are obtained in Ref. 1 and are quoted in Table I, and differ from those of Sec. IV. Here again the critical amplitudes are uniquely determined once the mode-coupling coefficients λ and v and the specific heat, all of them the equilibrium properties, are fixed, quite in contrast to the binary liquid of Sec. V.

VII. CONCLUDING REMARKS

One of the main purposes of this paper was to display explicitly the close relationship between the renormalization-group and mode-coupling approaches to critical dynamics. Up to first order in ϵ we were able to exhibit their equivalence, and the relationship somewhat resembles that of the Schrödinger and Heisenberg representations in quantum mechanics. Beyond first order in ϵ the relationship may be more complicated primarily owing to the different attitudes toward dealing with the static aspects of the problem taken by those working in the two approaches. The mode-coupling approach regards whatever is known about the static aspects of the problem as input to the theory, whereas the renormalization-group approach tries to deal with both the static and dynamic aspects simultaneously.

Another point that emerges concerns the cases with multiple fixed points. The concept of fixed point is not apparent in the mode-coupling approach. However, the choice of a fixed point is made *a priori* in the mode-coupling approach when the starting model system is chosen. In fact in this approach one is always led to the stable fixed point of the RG as allowed in the space of parameters entering the chosen model, as one can see, e.g., in Eq. (2.10′). Here, in the limit of small k, $L_{ij}(k_m^2/k)$ approaches the value at its stable fixed point with the appropriate exponent z.

We may regard our RG treatment and the mode-coupling theory of a binary liquid as the two extreme representations of the same theory in the sense that in RG all the parameters g, ζ, and η attain finite fixed-point values, whereas in MC the concept of fixed point does not enter at all. In the same sense, the treatment of Halperin *et al.* is intermediate between the two extremes in that only ζ and the ratio $g^2/\zeta\eta$ attain finite fixed-point values.

Finally we remark that the two approaches described here are not limited to critical dynamics, and indeed a "mode-coupling"-type treatment of static critical phenomena was presented by Wilson himself.[17,19]

APPENDIX A

We describe here a simple way to eliminate the rapidly-varying degrees of freedom $\{B\}$ from the slowly-varying degrees of freedom $\{A\}$ in the stochastic equation for the entire system ($\{A\}$ plus $\{B\}$). This equation is of the form

$$\frac{\partial}{\partial t}P_{AB}(t)=\mathcal{L}P_{AB}(t) \qquad (A1)$$

with

$$\mathcal{L}=\mathcal{L}_A+\mathcal{L}_B+\mathcal{L}_{AB}, \qquad (A2)$$

where $\mathcal{L}_A$ and $\mathcal{L}_B$ describe the variables $\{A\}$ and $\{B\}$, respectively, and $\mathcal{L}_{AB}$ describes the interactions between them. We now eliminate the variables $\{B\}$ to second order in $\mathcal{L}_{AB}$. First we integrate Eq. (A1) over $\{B\}$ to obtain an equation for $P_A\equiv\int d\{B\}P_{AB}$,

$$\frac{\partial}{\partial t}P_A(t)=\mathcal{L}_A P_A(t)+\int d\{B\}\mathcal{L}_{AB}P_{AB}(t). \qquad (A3)$$

We now assume that $\{A\}$ and $\{B\}$ were uncoupled at some initial time $t=0$ and that $P_B(0)$ equals its thermal equilibrium P_B^e. Thus we only need to consider the first-order effect of $\mathcal{L}_{AB}$ on $P_{AB}(t)$ in (A3), where we choose t to be small enough that $P_A(t)$ remains practically unchanged (the Markoff approximation). To this order we then find from Eq. (A1)

$$P_{AB}(t)\cong\left(1+\int_0^t ds\,e^{s\mathcal{L}_0}\mathcal{L}_{AB}\right)P_B^e P_A(t) \qquad (A4)$$

with $\mathcal{L}_0=\mathcal{L}_A+\mathcal{L}_B$. We now substitute this into the second term on the right-hand side of (A3) to obtain finally

$$\frac{\partial}{\partial t}P_A(t)=\mathcal{L}_e P_A(t), \qquad (A5)$$

where the effective stochastic operator is given by

$$\mathcal{L}_e=\mathcal{L}_A-\langle\mathcal{L}_{AB}(1/\mathcal{L}_0)\mathcal{L}_{AB}\rangle_B \qquad (A6)$$

with

$$\langle O\rangle_B\equiv\int d\{B\}\,OP_B^e \qquad (A7)$$

for an arbitrary operator $O(\{A\},\{B\})$. In deriving Eqs. (A5) and (A6) we have assumed that t is still sufficiently large that $\langle \mathcal{L}_{AB} e^{s\mathcal{L}_0} \mathcal{L}_{AB}\rangle_B$ vanishes before the time $s=t$. We have also split up L so that $\langle \mathcal{L}_{AB}\rangle_B = 0$. That is, if $\langle \mathcal{L}_{AB}\rangle_B$ is not zero, then we define $\mathcal{L}_A + \langle \mathcal{L}_{AB}\rangle_B$ as $\mathcal{L}_A$ and $\mathcal{L}_{AB} - \langle \mathcal{L}_{AB}\rangle_B$ as $\mathcal{L}_{AB}$. In the text $\{A\}$ and $\{B\}$ correspond, respectively, to the long- and short-wavelength fluctuations.

APPENDIX B

Let us consider a general critical dynamic model expressed in terms of the following stochastic equation for the probability distribution function $P(\underline{a}, t)$

$$\frac{\partial}{\partial t} P(\underline{a}, t) = \mathcal{L}(\underline{a})\, P(\underline{a}, t), \tag{B1}$$

where $\mathcal{L}(\underline{a})$ is a stochastic operator and $\underline{a}$ is a set of the gross variables arranged as a vector. Introduce a propagator matrix $\underline{G}(t)$ by

$$\underline{G}(t) \equiv \langle \underline{a} e^{t\mathcal{L}} \underline{a}^* \rangle \cdot \underline{\chi}^{-1} \tag{B2}$$

where $\underline{a}^*$ is the complex conjugate to $\underline{a}$,

$$\underline{\chi} \equiv \langle \underline{a}\underline{a}^* \rangle, \tag{B3}$$

and

$$\langle XOY \rangle \equiv \int d\underline{a}\, X(\underline{a})\, O Y(\underline{a})\, P_e(\underline{a}), \tag{B4}$$

and X and Y are arbitrary functions of $\underline{a}$. O is an operator acting on everything to the right and $P_e(\underline{a})$ is the equilibrium probability distribution function of the following form:

$$P_e(\underline{a}) = \mathfrak{N} e^{-\Phi(\underline{a})}. \tag{B5}$$

Next we introduce a Mori-type projection operator $\mathcal{P}$ through

$$\mathcal{P} X P_e \equiv \underline{a}^* \cdot \underline{\chi}^{-1} \cdot \langle \underline{a} X \rangle P_e + \langle X \rangle \tag{B6}$$

and use the identity[20]

$$\frac{d}{dt} e^{t\mathcal{L}} = e^{t\mathcal{L}} \mathcal{P}\mathcal{L} + \int_0^t ds\, e^{(t-s)\mathcal{L}}\, \mathcal{P}\mathcal{L} e^{s(1-\mathcal{P})\mathcal{L}} (1-\mathcal{P})\mathcal{L} + e^{t(1-\mathcal{P})\mathcal{L}} (1-\mathcal{P})\mathcal{L} \tag{B7}$$

for the time-displacement operator in Eq. (B2) to deduce

$$\frac{d}{dt} \underline{G}(t) = -\underline{\chi}^{-1} \cdot \underline{L}_0 \cdot \underline{G}(t) - \int_0^t ds\, \underline{\chi}^{-1} \cdot \underline{K}(s) \cdot \underline{G}(t-s) \tag{B8}$$

with

$$\underline{L}_0 \equiv -\langle \underline{a} \mathcal{L} \underline{a}^* \rangle, \tag{B9}$$

$$\underline{K}(t) \equiv \langle \underline{J} e^{t(1-\mathcal{P})\mathcal{L}} \underline{J}^\dagger \rangle \tag{B10}$$

where $\underline{J}$ and $\underline{J}^\dagger$ are the vector functions of $\underline{a}$ and $\underline{a}^*$ defined through

$$\int d\underline{a}\, J(\underline{a}) \cdots = \int d\underline{a} ([\mathcal{L}, \underline{a}] - \underline{L}_0 \cdot \underline{\chi}^{-1} \cdot \underline{a}) \cdots, \tag{B11a}$$

$$\underline{J}^\dagger(\underline{a}^*)\, P_e(\underline{a}) \equiv ([\mathcal{L}, \underline{a}^*] + \underline{a}^* \cdot \underline{\chi}^{-1} \cdot \underline{L}_0) P_e(\underline{a}). \tag{B11b}$$

Here we have used the fact that $\int d\mathrm{a} \cdots = \mathcal{L}P_e = 0$.

We now apply this result to the calculations of the wave-number-dependent transport coefficients $L(q)$ and $\zeta(q)$ which are expressed as

$$L(q) = L_0 + \int_0^\infty K_\psi(q,t)\, dt, \tag{B12a}$$

$$\zeta(q) = \zeta_0 + \int_0^\infty K_S(q,t)\, dt, \tag{B12b}$$

where

$$K_\psi(q,t) \equiv \langle J_\psi(q) e^{t(1-\mathcal{P})\mathcal{L}} J_\psi^\dagger(q) \rangle, \tag{B13a}$$

$$K_S(q,t) \equiv \langle J_S(q) e^{t(1-\mathcal{P})\mathcal{L}} J_S^\dagger(q) \rangle. \tag{B13b}$$

The fluxes $J_\psi(q)$, etc., can be obtained from Eqs. (B11) where it is enough to retain only the $O(\epsilon^{1/2})$ contributions, and we thus find

$$J_\psi(q) = \left(L_0 v_0 - \frac{i\lambda}{C_0}\right) \frac{1}{V^{1/2}} \sum_{\vec{k}} S_{\vec{k}-\vec{q}} \psi_{\vec{k}}, \tag{B14a}$$

$$J_\psi^\dagger(q) = -\left(L_0 v_0 - \frac{i\lambda}{C_0}\right) \frac{1}{V^{1/2}} \sum_{\vec{k}} S_{\vec{q}-\vec{k}} \psi_{\vec{k}}^*, \tag{B14b}$$

$$J_S(q) = i\lambda \frac{1}{V^{1/2}} \sum_{\vec{k}} (\vec{q}\cdot\vec{k} - \tfrac{1}{2} q^2) \psi_{\vec{k}}^* \psi_{\vec{k}-\vec{q}} = [\text{complex conjugate of } J_S^\dagger(q)]. \tag{B14c}$$

Note that the terms involving λ_0 and v_0 arise from the streaming and dissipative mode coupling, respectively, and the signs in front conform to the general rule found previously.[21] Up to $O(\epsilon)$ Eq. (B13) can be evaluated by noting that (a) the time displacement simply gives rise to independent free propagations of $S_{\vec{k}}$ and $\psi_{\vec{k}}$ such as

$$S_{\vec{k}}(t) \to e^{-(\zeta_0/C_0)k^2 t} S_{\vec{k}}, \tag{B15a}$$

$$\psi_{\vec{k}}(t) \to e^{-(L_0/\chi_k)t} \psi_{\vec{k}}(0), \tag{B15b}$$

$$\psi_{\vec{k}}^*(t) \to e^{-(L_0^*/\chi_k)t} \psi_{\vec{k}}^*(0), \tag{B15c}$$

and (b) P_e can be replaced by its Gaussian part, and we obtain

$$L(q) = L_0 + \left(\frac{\lambda}{C_0} + iL_0 v_0\right)^2 \int_{\vec{k}} \frac{4C_0 \chi_k}{L_0/\chi_k + (2\zeta_0/C_0)(\vec{q}-\vec{k})^2}, \tag{B16a}$$

$$\zeta(q) = \zeta_0 + \lambda^2 \int_{\vec{k}} q^{-2} [(2\vec{q}\cdot\vec{k})^2 - q^4] \frac{\chi_k \chi_{\vec{q}-\vec{k}}}{L_0/\chi_{\vec{q}-\vec{k}} + L_0^*/\chi_k}, \tag{B16b}$$

where

$$\int_{\vec{k}} \equiv \frac{1}{(2\pi)^d} \int d\vec{k}.$$

Using $\chi_k = k^{-2}$ at criticality we note that near four dimensions we can take the limit $q \to 0$ in the integrands of Eqs. (B16) and introduce the lower cutoff at $k = q$.[3,10] Furthermore, the second-order results [Eqs. (B16)] can be made self-consistent by replacing v_0, C_0, L_0, and ζ_0 in the second terms by $v(k)$, $C(k)$, $L(k)$, and $\zeta(k)$, respectively. That is,

$$L(q) = L_0 + \int_{\vec{k}}^{\mathrm{SW}} \left(\frac{\lambda}{C(k)} + iL(k)\,v(k)\right)^2 \times \frac{4C(k)}{k^4[L(k) + 2\zeta(k)/C(k)]}, \tag{B17a}$$

$$\zeta(q) = \zeta_0 + \lambda^2 \int_{\vec{k}}^{\mathrm{SW}} \frac{4(\hat{q}\cdot\hat{k})^2}{k^4\,\mathrm{Re}L(k)}, \tag{B17b}$$

with $\hat{q} = \vec{q}/q$, etc. These can then be converted to the following differential equations that determine $L(q)$ and $\zeta(q)$ where $C(k)$ and $v(k)$ are detained by solving Eq. (6.9):

$$\frac{dL(k)}{dk} = -\frac{4K_4}{k^{1+\epsilon}} \frac{[\lambda + iC(k)\,v(k)\,L(k)]^2}{2\zeta(k) + C(k)\,L(k)}, \tag{B18a}$$

$$\frac{d\zeta(k)}{dk} = -\frac{K_4}{k^{1+\epsilon}} \frac{\lambda^2}{\mathrm{Re}L(k)}. \tag{B18b}$$

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