Equivalence of two exactly soluble models for tricritical points

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An exactly soluble n-component continuous spin model with tricritical points is shown to be equivalent, in the limit $n \to \infty$, to a one-component model with anharmonic long-range interactions.

In a recent paper[1] we studied an exactly soluble model for tricritical points. The Hamiltonian of the model reads

$$H_N^{\rm SC}(\vec{X}) = \frac{D^2}{8N^3}\left(\sum_{i=1}^N X_i^2\right)^4 + \frac{B}{4N}\left(\sum_{i=1}^N X_i^2\right)^2 + \frac{A}{2}\sum_{i=1}^N X_i^2 - C\sum_{\langle i,j\rangle} X_i X_j . \quad (1)$$

We were able to draw the complete phase diagram in the B-Δ-T space; here $\Delta = 2dC$ and d is the dimensionality of the lattice. In the B-Δ-T space we found a line of tricritical points.

At the same time, Emery[2] proposed and solved an n-component vector model which also exhibits tricritical behavior. This model is defined by

$$H_N = \sum_{j=1}^N nV(n^{-1}\vec{S}_j^2) - \sum_{i,j} J_{ij}(\vec{S}_i\cdot\vec{S}_j) - \sum_{j=1}^N \vec{S}_j\cdot\vec{h}, \quad (2)$$

where $\vec{S}_k$ is a vector in R^n.

The aim of this note is to demonstrate that in the limit $N, n\to\infty$ the model defined by (2) has the same free energy as the model defined by

$$H_N^{\rm SC}(\vec{X}) = NV\left(N^{-1}\sum_{i=1}^N X_i^2\right) - \sum_{i,j} J_{ij}X_iX_j - h\sum_{i=1}^N X_i \quad (3)$$

for $N\to\infty$. [Clearly (1) is a special case of (3).] To establish the equivalence between (2) and (3) we calculate the free energy corresponding to (3). We suppose that

$$V(t) \geq -V_0 > -\infty , \quad (4)$$

$$\lim_{t\to\infty} t^{-2}V(t) > 0 , \quad (5)$$

$$J_{ij} = J(|\vec{r}_i - \vec{r}_j|) \geq 0 , \quad J(0) = 0 , \quad (6)$$

and, following Kac and Thompson,[3] write

$$Z_N^{\rm SC} = \int_{R^N} d^N X\, e^{-\beta H_N(\vec{X})} = \sqrt{N}\int_0^\infty d\zeta\, \zeta^{N-1} e^{-\beta N V(\zeta^2)} Q_N^{\rm SP}(\zeta^2) . \quad (7)$$

We made the transformation $X_i \to \zeta X_i$ to obtain this last expression. $Q_N^{\rm SP}(\zeta^2)$ is then the partition function, without the renormalization factor A_n defined in Ref. 3, of a spherical model with $\zeta^2 J_{ij}$ and ζh as parameters. The free energy

$$-\beta F^{\rm SP}(\zeta^2) = \lim_{N\to\infty} N^{-1}\ln Q_N^{\rm SP}(\zeta^2) \quad (8)$$

is well known[3,4] and using the Laplace method[5] we can write

$$Z_N^{\rm SC} \underset{N\to\infty}{\sim} \exp\left(N \max_{0\le\zeta^2\le\infty} [-\beta V(\zeta^2) + \ln\zeta - \beta F^{\rm SP}(\zeta^2)]\right) \quad (9)$$

so that the free energy per spin of (3) is given by

$$-\beta\psi^{\rm SC}(\beta,h) = \max_{0\le\zeta^2\le\infty}[-\beta V(\zeta^2) + \ln\zeta - \beta F^{\rm SP}(\zeta^2)] . \quad (10)$$

The maximum in (10) can be evaluated with the aid of the derivative with respect to ζ^2; we then find a set of self-consistency equations determining the free energy of (3):

$$-\beta\psi^{\rm SC}(\beta,h) = -\beta V(\zeta^2) + \beta\zeta^2 t - \tfrac{1}{2}f_d(t) + \frac{\beta h^2}{4[t-\hat{J}(0)]} + \tfrac{1}{2}\ln 2\pi\beta^{-1} , \quad (11)$$

$$t(\zeta^2) = \left.\frac{\partial V(z)}{\partial z}\right|_{z=\zeta^2} , \quad (12)$$

$$\beta\zeta^2 = \tfrac{1}{2}f_{d,t}(t) + \frac{\beta h^2}{4[t-\hat{J}(0)]^2} , \quad (13)$$

where

$$f_d(t) = (2\pi)^{-d}\int_0^{2\pi} d^d\omega \ln[t - \hat{J}(\vec{\omega})] , \quad (14)$$

$$f_{d,t}(t) \equiv \frac{\partial}{\partial t}f_d(t) , \quad (15)$$

and

$$\hat{J}(\vec{\omega}) = \sum_{\vec{l}} J(\vec{l}) \cos\vec{\omega}\cdot\vec{l} . \quad (16)$$

This set of self-consistency equations is identical to that found by Emery[2] for the free energy of the model defined by (2) for $N, n \to \infty$. Consequently both models are equivalent.

The model defined by (3) has another interesting property. Provided that $V(z)$ is a convex function of z, for $z \geq 0$, and there is a phase transition, the critical temperature is given by

$$\beta_C^{SC} = \zeta_0^{-2} \tfrac{1}{2} f_{d,\,t} t(\zeta_0^2), \tag{17}$$

where ζ_0^2 is the solution of

$$t(\zeta_0^2) = \hat{J}(0). \tag{18}$$

Now (17) is also the critical temperature of a spherical model with $\sum X_i^2 = N\zeta_0^2$. Moreover, for $h=0$ and $\beta \geq \beta_C$ it can be seen that the free energy of (3) and the free energy of this spherical model are identical:

$$\psi^{SC}(\beta, h=0) = F^{SP}\left(\beta, h=0; \sum X_i^2 = N\zeta_0^2\right). \tag{19}$$

For $\beta \geq \beta_C$ the behavior of the model (3) is then that of the spherical model provided that $V(z)$ is a convex function on the positive real axis.

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[1]St. Sarbach and T. Schneider, Z. Phys. B 20, 399 (1975).
[2]V. J. Emery, Phys. Rev. B 11, 3397 (1975).
[3]M. Kac and C. J. Thompson, Phys. Norv. 5, 163 (1971).
[4]G. S. Joyce, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 2, pp. 375–442.
[5]See, e.g., A. Erdelyi, *Asymptotic Expansions* (Dover, London, 1956), pp. 36–38.