Critical properties of two tensor models with application to the percolation problem

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Two models having p-dimensional Cartesian tensor order parameters are introduced. In the first the tensor is constrained to be symmetric and traceless, and in the second it is constrained to be diagonal and traceless. The three-dimensional form of the first can be used to describe the isotropic to nematic phase transition in liquid crystals. The second model is a continuum generalization of the Ashkin-Teller-Potts model, which describes the percolation problem when $p = 1$. Both models have cubic invariants which according to Landau mean-field theory, give rise to first-order phase transitions. These models are studied near four dimensions when the cubic invariant is small using the ϵ expansion. A new fixed point, stable in $6 - \epsilon'$ dimensions, is located and its properties studied. The percolation exponents to second order in $\epsilon' = 6 - d$ follow from perturbations about this fixed point for the $p = 1$ Ashkin-Teller-Potts model and are $\eta = -(1/21)\epsilon - (206/3^3 7^3)\epsilon^2$ and $1/\nu = 2 - (5/21)\epsilon - (23/7^3 3^3 2^2)\epsilon^2$.

I. INTRODUCTION

According to the hypothesis of universality,[1] critical exponents should depend only on the symmetry of the order parameter, the dimensionality of space, symmetries of the Hamiltonian, and not on numerical values of interaction strengths. The renormalization group[2-5] provides an impressive theoretical justification for this hypothesis. Indeed critical exponents have been calculated for a wide variety of Hamiltonians for n-component vector order parameters.[6-8] Order parameters of other than vector symmetry are possible. In this paper, we consider two closely related models with nonvector order parameters which find their motivation, respectively, in the nematic-to-isotropic phase transitions in liquid crystals[9,10] and in the Ashkin-Teller-Potts[11,12] (ATP) model. In both models, the fundamental variables are p-dimensional symmetric-traceless tensors. In the first, all $n = \frac{1}{2}(p+2)(p-1)$ independent tensor components are allowed to fluctuate. We call this the Q model; it describes fluctuations in a nematic liquid crystal[9,10] when $p = 3$. In the second, the off-diagonal components are constrained to be zero leaving $n' = p - 1$ independent components. This model is a continuum generalization[13] of the p-state[14,15] ATP model,[12] and describes the percolation problem[14,15] when analytically continued to $p = 1$. A preliminary report of the Q-model results appears elsewhere.[16] The three-dimensional version of the ATP model has already received some attention in the literature.[12,17]

The fundamental difference between p-dimensional vector and tensor order parameters is that the latter have more invariants with respect to the p-dimensional rotation group than the former. Thus both vectors and tensors have one quadratic invariant, but tensors have one cubic invariant instead of zero, and two quartic invariants instead of one. Hence, the symmetry of the tensor order parameter alters the symmetry of the Hamiltonian in much the same way that externally imposed conditions alter the symmetry of Hamiltonians with vector order parameters. In particular, the Q and ATP models with no cubic terms will be very similar to the vector model in a hypercubic environment.[18-20]

Landau mean-field theory, as it is usually applied, implies a first-order transition whenever there is a cubic invariant.[21,10] Thus on the basis of Landau theory, one would expect phase transitions in the Q and ATP models to be first order. On the other hand, it is known that in two dimensions, the ATP model exhibits a continuous transition for $p \leq 4$.[22,23] Furthermore, series expansions for the three-state Potts model in three dimensions predict a continuous[24] or nearly continuous[25] phase transition. The situation for the Q model is similarly ambiguous. Monte Carlo calculations[26] on the rotationally invariant Maier-Saupe model[27] ($p = 3$ Q model) yield a latent heat for the nematic-to-isotropic transition substantially lower than predicted by mean-field theory. Experiments[28] also yield a lower latent heat than predicted by mean-field theory on the Maier-Saupe model though this may be due to interactions present in a real system that are not included in the model.[29-31]

The work presented here was initiated in the hope that the renormalization group and the ϵ-expansion might shed some light on these contra-

dictions. Our program has met with only partial success. Using the renormalization group, we have not been able to provide a global picture of the behavior of the Q and ATP models nor a definitive understanding of why the transitions in these models are more continuous than expected. We have, however, been able to analyze properties of these models in the vicinity of easily located fixed points and to obtain some new information which may provide the key to an eventual more complete understanding of their global behavior. Our principal results fall into two categories:

(a) Both the Q and ATP models have stable fixed points in $6-\epsilon'$ dimensions for $p<p_c=4+O(\sqrt{\epsilon'})$ and $\frac{10}{3}+O(\sqrt{\epsilon'})$, respectively. For larger values of p, the fixed points become stable in $6+\epsilon'$ dimensions as in the usual ϕ^3 field theory.[32] We call these the Q^3 fixed points to emphasize that they are determined solely by the cubic invariants to lowest order in ϵ'. Their stability in less than six dimensions is a new feature of the Q and ATP models which results from the internal degress of freedom of symmetric traceless tensors. The interpretation of the Q^3 fixed points is open to some question. It appears, as discussed further in Sec. V, that different interpretations are needed for $p<2$ and $p>2$. For $p<2$, it is almost certain that the Q^3 fixed point for the $p=1$ ATP model gives the critical exponents which we calculate to second order in ϵ' for the percolation problem.[15] For $p>2$, the situation is less clear. It seems unlikely that the Q^3 fixed point has anything to do with the continuous transitions in two dimensions in the ATP model. We conclude this because, on the basis of the ϵ expansion, it appears virtually certain that the region of existence of the Q^3 fixed point when extrapolated to two dimensions could not include all p between 2 and 4. Other interpretations for the Q^3 fixed point for $p>2$ are (i) it describes metastable behavior, (ii) it is completely inaccessible for physical systems and is merely a reflection of a singularity that occurs at $p=2$, (iii) it describes the critical behavior of some unknown physical system. As yet no connection between the Q^3 fixed point for the Q model for any value of p and a physically interesting problem has been established. In analogy with the ATP model, however, it is possible that there is some problem which corresponds to the $p=1$ Q^3 fixed point and that the region $2<p<p_c$ is open to the same interpretation as for $2<p<p_c$ for the ATP model.

(b) If the coefficient t of the cubic invariant is zero, the fixed-point structure in $4-\epsilon$ dimensions of both the Q and ATP models is similar to that of a ferromagnet in a hypercubic environment.[19] All of these fixed points are unstable with respect to t. The interpretation of this instability depends on the existence of a stable physical Q^3 fixed point. If there is no stable Q^3 fixed point, as certainly must be the case for large enough p in all dimensions, the instability represents a crossover[33] to first-order behavior for nonzero t. In this case, the usual Landau mean-field theory[21,10] provides an adequate description of the first-order transition for large initial values of t. If on the other hand, t is small, pretransitional effects are important even though the transition is first order. In this case, linearization in t near the $t=0$ fixed points provides a description of the transition. This treatment can in principle determine the effect of critical fluctuations on the continuous isotropic to biaxial transition in mixtures of rodlike and platelike molecules discussed by Alben[34] using mean-field theory. If there is a stable physical Q^3 fixed point, the instability in t could represent a crossover to this fixed point. Alternatively, the Q^3 fixed point could be inaccessible to systems whose potentials flow to the most stable $t=0$ fixed point when $t=0$. In this case, the instability in t would still lead to a first-order transition. A final possibility is that the accessibility of the Q^3 fixed point depends on the sign of t. We have been unable to determine which alternative is correct, though our prejudice at the moment is that the instability with respect to t for $p>2$ leads to a first-order transition.

The outline of this paper is as follows. Section I is the introduction. Section II presents the model and treats the mean-field theory. Section III discusses diagrammatic rules peculiar to the Q and ATP models and deals with the ϵ expansion for the $t=0$ fixed points in the isotropic state. Section IV treats the equation of state of these fixed points for states with uniaxial symmetry. Section V analyzes the Q^3 fixed points for short-range forces, calculates the percolation exponents to second order in ϵ', and discusses briefly the Q^3 fixed points for long-range forces. Finally, Sec. VI is a brief summary with some comments on the possibility of any of the ϵ expansion fixed points corresponding to the continuous transitions for the ATP model for $p=3$ and 4 in two dimensions.

II. MODEL AND MEAN-FIELD THEORY

The model we consider has the reduced Hamiltonian

$$\mathcal{H}=\int d^dx\left[\tfrac{1}{4}(rQ_{ij}Q_{ij}+\nabla_kQ_{ij}\nabla_kQ_{ij})-tQ_{ij}Q_{jk}Q_{ki}+u(Q_{ij}Q_{ij})^2+H_{ij}Q_{ij}+vQ_{ij}Q_{jk}Q_{ki}Q_{ii}\right]. \tag{2.1}$$

Here d is the dimension of space and $Q_{ij}=Q_{ij}(\vec{x})$ is traceless, symmetric, and of dimension p. The dependence of Q on the position coordinates $\vec{x}$, is suppressed for notational convenience, and the Einstein convention with respect to repeated indices is understood. To complete the definition of the model we define the average of a quantity A, which is a functional of Q as the functional integral over Q:

$$\langle A\rangle = \int DQAe^{\mathcal{H}} \Big/ \int DQ\, e^{\mathcal{H}}. \tag{2.2}$$

The solution of this model for general H is cumbersome. We will consider only the case for which H does not change the symmetry of the spontaneous symmetry breaking state. Also we do not consider the other type of gradient term $\nabla_i Q_{ik}\nabla_j Q_{jk}$. This is analogous to the dipole term in the vector model. It couples the rotational properties of the space variables to the rotational properties of Q.

The connection between the Q and ATP models and the vector model can be seen most clearly by introducing an appropriate decomposition of Q:

$$Q_{ij}=\sum_\alpha A_\alpha a^\alpha_{ij}+\sum_{\alpha>\beta} B_{\alpha\beta}b^{\alpha\beta}_{ij},$$

$$a^\alpha_{ij}=\delta_{ij}\left(\frac{p-\alpha}{p-\alpha+1}\right)^{1/2}\times\begin{cases}0 & \text{if } i<\alpha,\\ 1 & \text{if } i=\alpha,\\ -1/(p-\alpha) & \text{if } i>\alpha,\end{cases} \tag{2.3}$$

$$b^{\alpha\beta}_{ij}=\tfrac{1}{2}\sqrt{2}\,(\delta_{i\alpha}\delta_{j\beta}+\delta_{i\beta}\delta_{j\alpha}).$$

There are $p-1$ a's and $\frac{1}{2}p(p-1)$ b's. The a's and b's form a basis set for the traceless symmetric $p\times p$ matrices. The normalization is chosen such that

$$Q_{ij}Q_{ij}=\sum_\alpha A^2_\alpha+\sum_{\alpha>\beta}B^2_{\alpha\beta}. \tag{2.4}$$

If the coefficients of the $\mathrm{Tr}Q^3$ and $\mathrm{Tr}Q^4$ terms in Eq. (2.1) were zero, the Q model would be isomorphic to the n-dimensional vector model with $n=\frac{1}{2}(p-1)(p+2)$. Since $B_{\alpha\beta}=0$ for the ATP model, the ATP model would be isomorphic to the n'-dimensional vector model with $n'=p-1$. The $\mathrm{Tr}Q^3$ and $\mathrm{Tr}Q^4$ terms however have a more complicated representation in this basis set. This is a reflection of the fact that they are invarient with respect to a p-dimensional rotation group instead of a n- or n'-dimensional group.

The above decomposition is the logical generalization of that used by Freiser[35] and Alben[34] for three-dimensional traceless tensors. The connection is easily established by setting $B_{\alpha\beta}=0$ and $A_1=A\cos\theta$ and $A_2=A\sin\theta$, then

$$Q_{ij}=A\begin{bmatrix}(\frac{2}{3})^{1/2}\cos\theta & 0 & 0\\ 0 & -\frac{1}{2}(\frac{2}{3})^{1/2}\cos\theta+\frac{1}{2}\sqrt{2}\,\sin\theta & 0\\ 0 & 0 & -\frac{1}{2}(\frac{2}{3})^{1/2}\cos\theta-\frac{1}{2}\sqrt{2}\,\sin\theta\end{bmatrix}. \tag{2.5}$$

When $\theta=\frac{2}{3}n\pi$, Q_{ij} is uniaxial; otherwise Q_{ij} is biaxial. In particular, if $\theta=\frac{1}{6}\pi$, Q_{ij} is purely biaxial. For the p-dimensional case, the A_α's can similarly be expressed in terms of an over-all amplitude and $p-2$ independent angles.[36]

The connection between the model of Eq. (2.1) and the ATP model may be established as follows. The p-state ATP[11,12] model consists of $(p-1)$-dimensional vectors $\vec{A}(\vec{x})$ on lattice sites $\vec{x}$, that are constrained to be at one of the vertices of a $(p-1)$-dimensional unit simplex. Thus, the vectors $\vec{A}$ must be at one of the vertices of an equilateral triangle inscribed in a unit circle in the three state Potts model and at one of the vertices of a tetrahedron inscribed in a unit sphere in the 4-state Ashkin-Teller model. The Hamiltonian for the p-state Potts model is

$$H=-J\sum_{\langle\vec{x},\vec{x}'\rangle}\vec{A}(\vec{x})\cdot\vec{A}(\vec{x}'), \tag{2.6}$$

where $\langle\vec{x},\vec{x}'\rangle$ signifies a sum over the nearest-neighbor pairs. It is easy to verify that if $v=0$, Eq. (2.1) for diagonal traceless tensors has minima for A_α such that the corresponding vector points to one of the vertices of a p-dimensional simplex. Thus by letting r, t, and u tend infinity in the appropriate way, it is possible to restrict the A_α such that the corresponding vector can point only to the vertices of a p-dimensional simplex. This is the same technique used by Wilson[3] to regain the Ising model from a continuum model and by Golner[13] to obtain the three-state Potts model from a continuum model. Thus Eq. (2.1) is the correct continuum generalization of the p-state Potts model described by Eq. (2.6). After one iteration in a renormalization scheme, $\mathrm{Tr}Q^4$ terms are always generated. We therefore allow for $v\neq 0$ from the beginning.

A. Simple cases

A few cases for which the structure of the $\mathrm{Tr}Q^3$ and $\mathrm{Tr}Q^4$ terms is simple can be identified. First,

for the case $p=2$, $\mathrm{Tr}Q^3=0$ and $\mathrm{Tr}Q^4=\frac{1}{2}(\mathrm{Tr}Q^2)^2$. The Q and Potts models for the case $p=2$ are completely isomorphic to the superfluid and Ising models, respectively. The Q model and ATP models for the case $p=3$ correspond to the de Gennes liquid crystal model[9] and to the usual Potts[11] model, respectively. For this case, the relation $\mathrm{Tr}Q^4=\frac{1}{2}(\mathrm{Tr}Q^2)^2$ is not obvious but is true. For this reason, the liquid crystal model and the Potts model have only one type of quartic interaction.

The ATP model with $t=0$ is of special interest in the limit $p\to\infty$. In this limit the constraint on the trace of Q becomes unimportant. The ATP model becomes isomorphic to the vector model with a hypercubic perturbation.[18,19] The analogy between these models will be discussed further in Secs. III and IV.

B. Mean-field theory

To study the mean-field theory for the Q and ATP models, we regard Eq. (2.1) as a free energy and Q_{ij} as the average value of the order parameter. As usual, we assume r to vary linearly with temperature: $r=a(T-T^*)$, where T^* is not necessarily the transition temperature T_c. In equilibrium in the mean field, Q assumes the value that minimizes Eq. (2.1). Since the minimum is independent of $\vec{x}$, the rotational independence of Eq. (2.1) allows Q to be taken as diagonal. Therefore, in the mean-field theory, there is no difference between the Q and ATP model.

In this paper, we will consider only equations of state for uniaxial systems. To do this, we single out the uniaxial component of Q_{ij} from Eq. (2.3),

$$Q_{ij}=Ma_{ij}^1+\sum_\alpha \beta_\alpha b_{ij}^{\alpha 1}+\hat{Q}_{ij}, \tag{2.7}$$

where $M=A_1$, $\beta_\alpha=B_{\alpha 1}$, and $\hat{Q}_{ij}$ has zeroes in the first row and column and a $(p-1)$-dimensional traceless tensor in rows and columns 2 through p. In a uniaxial system, β_α and $\hat{Q}_{ij}$ are zero in equilibrium. Inserting Eq. (2.7) into Eq. (2.1), we obtain the free energy density $F=F_u+F_b$, where

$$F_u=\tfrac{1}{4}rM^2-(p-2)\tilde{t}M^3+(u+\tilde{v})M^4 \tag{2.8}$$

and

$$F_b=C_1\,\mathrm{Tr}\hat{Q}^2+C_2\sum_\alpha \beta_\alpha\beta_\alpha+O(Q^3,\beta^3), \tag{2.9}$$

where $\tilde{t}=t/[p(p-1)]^{1/2}$, $\tilde{v}=[(p^2-3p+3)/p(p-1)]v$, and C_1 and C_2 are complicated functions of M. F_u is the mean-field free energy for a purely uniaxial system and is plotted in Fig. 1 for various values of r for $(p-2)\tilde{t}>0$. An evaluation of the mean-field free energy for the discrete ATP model [Eq. (2.6)] with $\langle\vec{A}(\vec{x})\rangle=M\vec{e}_x$, where $\vec{e}_x$ is a unit vector[15,34] along one of the multihedral directions yields Eq. (2.8) with $t=(1/6\Omega)[p(p-1)]^{1/2}T$ and $(u+\tilde{v})=(1/12\Omega)(p^2-3p+3)T$, where Ω is the volume of a unit cell.

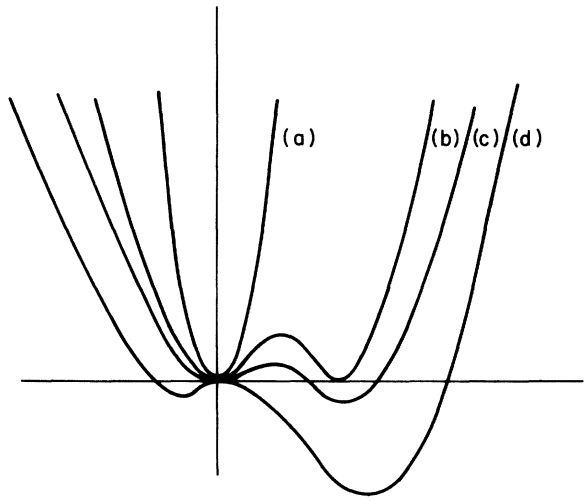

FIG. 1. Mean-field free energy [Eq. (2.8)] as a function of order parameter M. In curve (a) $r>(p-2)^2\tilde{t}^2/(u+\tilde{v})=r_c$, (b) $r=r_c$, (c) $0<r<r_c$, (d) $r<0$. If $(p-2)t<0$, the above curve must be reflected about the $M=0$ axis. Then if M is constrained to be non-negative, a second-order transition occurs when $r=0$.

Minimization of F_u with respect to M yields

$$\tfrac{1}{2}r-3(p-2)\tilde{t}M+4(u+\tilde{v})M^2=0. \tag{2.10}$$

If $t=0$, Eqs. (2.8) and (2.10) predict a second-order transition[21,10] with $M\sim(T-T^*)^{1/2}$. If $(p-2)\tilde{t}>0$ (<0) a first-order transition occurs to a state with positive (negative) M when the secondary minimum in Fig. 1 reaches zero. This occurs at a temperature $T_c>T^*$ which satisfies

$$r=(p-2)^2\tilde{t}^2/(u+\tilde{v}) \tag{2.11}$$

and

$$M(T_c)=\tfrac{1}{2}(p-2)\tilde{t}/(u+\tilde{v}). \tag{2.12}$$

An interesting situation occurs when p becomes less than 2. In this case, $(p-2)t$ becomes negative for the mean-field free energy for the ATP Hamiltonian [Eq. (2.6)]. Thus, the free energy is minimized for negative M. On the other hand, the ground state of Eq. (2.6) clearly has M positive. For the percolation problem (ATP model $p=1$), the order parameter is the probability, which must be non-negative, of being in the infinite cluster.[14] We are thus led to the conclusion that M is constrained to be positive for $p<2$ even though $(p-2)t$ is negative.[15] In this case, *a second-order transition occurs* at $T=T^*$ with

$$M=\tfrac{1}{6}\{[\,|\tilde{t}(p-2)|\,]^{-1}\}(T^*-T). \tag{2.13}$$

In other words, $\beta=1$. γ and ν retain their usual values of 1 and $\frac{1}{2}$. These exponents are in agreement with exact calculations for the percolation problem on a Bethe lattice[37] and lend strong support to our interpretation that M must remain positive for $p<2$. The fixed point in $6-\epsilon'$ dimensions for $p<2$ discussed in Sec. V is related to this constrained transition, and allows us to obtain percolation exponents.

In order for the uniaxial state to be stable, C_1 in Eq. (2.9) must be positive. Using Eq. (2.10), we find

$$C_1=[2(3-p)/(p-1)]vM^2+\tfrac{3}{2}t[p/(p-1)]^{1/2}M. \tag{2.14}$$

Since $tM>0$ (even for $p<2$), this implies the uniaxial state is stable when $v=0$. If $p>3$, the uniaxial state is at least locally stable if $v<0$; if $v>0$, C_1 will become negative for sufficiently large $|v|$ yielding and unstable uniaxial state. We also find $C_2=0$. This is expected since the principal axis of Q can point anywhere in the Q model. ($\beta_\alpha=0$ in the ATP model.)

III. ISOTROPIC STATE

In this section, we will apply the ϵ expansion in the standard way to the calculation of critical exponents for the Q and ATP models.[3-5] The initial Hamiltonian is restricted to variables with wavenumber of less than a cutoff which we set equal to unity. Degrees of freedom with $b^{-1}<q<1$, where $b>1$ are removed. Then space and dynamical variables are rescaled via $q\to bq$ and $Q\to\zeta Q$, where ζ is chosen so that the coefficient of $q^2Q_{ij}Q_{ij}$ remains unchanged. This prescription yields recursion relations for the potentials r, t, u, and v. Recursion relations for u and v to second order in u and v determine four fixed points to order $\epsilon=4-d$. To this order, the exponent η is zero and will be ignored. At each fixed point, linearized recursion relations then determine the correlation length exponent ν and crossover exponents for the t, u, and v fields. To first order in ϵ, t is a scaling field at each of the four fixed points with fixed-point-dependent crossover exponent $\phi_t\equiv\lambda_t\nu$. At fixed points where u and/or v are zero, u and v are scaling fields with exponents $\phi_u\equiv\lambda_u\nu$ and $\phi_v\equiv\lambda_v\nu$. At fixed points where both u and v are nonzero, linear combinations of u and v are scaling fields with exponents $\phi_1\equiv\lambda_1\nu$ and $\phi_2\equiv\lambda_2\nu$. The exponent ϕ_t is always positive indicating that a small deviation of t from its critical surface will always lead to a first-order transition, or possibly to the Q^3 fixed point, or some as yet unidentified fixed point. The most stable fixed point will have both u and v exponents negative.

In the noninteracting case ($t=u=v=0$) bilinear averages of the Q's can be calculated exactly using Eq. (2.2). In the Q model,

$$\langle Q_{ij}(\vec{q})Q_{kl}(-\vec{q})\rangle = (r+q^2)^{-1}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-(2/p)\delta_{ij}\delta_{kl}), \tag{3.1}$$

where

$$Q_{ij}(\vec{x})=\sum_q e^{i\vec{q}\cdot\vec{x}}Q_{ij}(\vec{q}).$$

The Feynman-graph perturbation theory for this model is similar to that of the well-known ϕ^4-field-theory model.[38] The topological structure of the graphs can be represented in exactly the same way if the correspondences in Fig. 2 are made. The cubic (three point) verticies are structureless. Although the topological structure of the graphs is identical to that of the ϕ^4-theory graphs, the p dependence of a graph can not be deduced from simple structural considerations such as the number of loops. The p dependence must be calculated from Eq. (3.1).

For the ATP model

$$\langle Q_{ii}(\vec{q})Q_{jj}(-\vec{q})\rangle=[2/(r+q^2)](\delta_{ij}-1/p). \tag{3.2}$$

The structure of the Potts model graphs is the

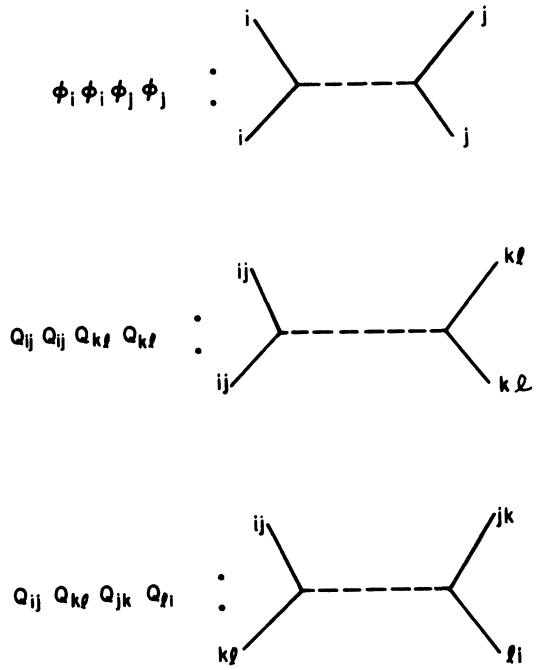

FIG. 2. Correspondence between the structure of the four-point vertex of the ϕ^4 model and the structures of the four-point verticies of the Q model.

same as that of the Q model in the case of the u verticies. Both the v and t verticies are structureless.

A. Q model

The recursion relations for the Q model are

$$r' = b^2\{r + 4K_4[\tfrac{1}{2}(1-b^{-2}) - r\ln b][2(p^2+p+2)u + 2(3+2p-6/p)v]\}, \tag{3.3}$$

$$t' = b^{1+\epsilon/2}\{1 - 24K_4 \ln b[4u + (p+3-12/p)v]\}t, \tag{3.4}$$

$$u' = b^{\epsilon}[u - 32K_4 \ln b(au^2 + guv + dv^2)], \tag{3.5a}$$

$$v' = b^{\epsilon}[v - 32K_4 \ln b(6uv + c^{-1}v^2)], \tag{3.5b}$$

where

$$K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(\tfrac{1}{2}d)$$

and

$$a = \tfrac{1}{4}(p^2+p+14) = \tfrac{1}{2}(n+8), \quad g = \tfrac{3}{2}+p-3/p, \quad c = (\tfrac{9}{4}+\tfrac{1}{2}p-9/p)^{-1}, \quad d = \tfrac{3}{4}+\tfrac{9}{2}1/p^2. \tag{3.6}$$

Equations (3.5) have four fixed points. They are (a) Gaussian

$$u_G^* = v_G^* = 0. \tag{3.7}$$

(b) Heisenberg

$$u_H^* = \epsilon/16(n+8)K_4, \quad v_H^* = 0. \tag{3.8}$$

(c) and (d) mixed-plus and mixed-minus

$$u_\pm^* = (\epsilon/64K_4)(a - 6gc + 36dc^2)^{-1}(1 + 12dc^2 - gc \pm \Delta), \tag{3.9}$$

$$v_\pm^* = c(\epsilon/32K_4 - 6u_\pm^*),$$

where

$$\Delta = (c/|c|)[(gc-1)^2 + 4dc^2(6-a)]^{1/2}, \tag{3.10}$$

where it is understood that the positive sign of the square root is to be taken. c^{-1} passes through zero from negative to positive at $p = 2.55$. The plus and minus fixed points map out two trajectories as a function of p. The factor of $c/|c|$ in Eq. (3.10) insures that fixed points on a given trajectory will have the same labeling for $c^{-1} = 0+$ and $c^{-1} = 0^-$. (Note that the value of u_H^* differs from that usually quoted[4,7] by a factor of $\frac{1}{4}$. This is due to our different normalization of the Q^2 term in the initial Hamiltonian. Critical exponents for this fixed point will of course be the same as usual.)

The stability of the fixed points (a)–(d) are determined by the λ exponents obtained by linearizing Eqs. (3.5) about their fixed-point values. We find

$$\lambda_u^G = \lambda_v^G = \epsilon, \tag{3.11a}$$

$$\lambda_u^H = -\epsilon, \quad \lambda_v^H = [(n-4)/(n+8)]\epsilon, \tag{3.11b}$$

$$\lambda_1^\pm = -\epsilon,$$

$$\lambda_2^\pm = 3\epsilon - 32K_4[(2a+6)u_\pm^* + (b+2c^{-1})v_\pm^*]. \tag{3.11c}$$

We are now in a position to discuss the position and stability of the various fixed points as a function of p. First note that for $n<4$ ($p<2.702$), the Heisenberg fixed point is the most stable. For $2<p<2.702$, the mixed-minus fixed point lies in the upper right-hand quadrant in the u-v plane, and the mixed-plus fixed point lies in the lower right-hand quadrant. Both mixed fixed points lie to the left of the Heisenberg point until $p \sim 2.6$ when the mixed-plus point moves to the right of the Heisenberg point. At $n=4$, the mixed-minus and Heisenberg points points merge and become marginally relevant (i.e., $\lambda_v^H = \lambda_2^- = 0$). This is most easily seen by noting that $a=6$ and $\Delta = gc - 1$ at $n=4$. A similar situation occurs for the hypercubic model[19,20] at $n=4$. For $n>4$, the mixed-minus fixed point moves into the lower half plane and becomes the most stable (λ_1^- and $\lambda_2^- < 0$). Finally, at $p = 3.62$ ($n = 7.38$), Δ becomes zero and the two mixed fixed points merge together. For $p > 3.62$, Δ is imaginary and the mixed fixed points become physically inaccessible. This presumably implies a first-order phase transition[33] even if $t=0$. Of special interest are the physical dimensions $p=2$ and $p=3$. As already noted $\mathrm{Tr}Q^4 = \frac{1}{2}(\mathrm{Tr}Q^2)^2$ in both these cases, so that one would expect physical quantities to depend only on $u + \frac{1}{2}v$. At $p=2$, the mixed-minus fixed point lies on the line $u + \frac{1}{2}v = u_H^* = \epsilon/160K_4$ and the mixed-plus point on the line $u + \frac{1}{2}v = 0$. Similarly at $p=3$, the mixed-minus fixed point lies on the line $u + \frac{1}{2}v = u_H^* = \epsilon/16 \times 13K_4$ (with $u > u_H$ and $v<0$) and the mixed-plus fixed point lies along the line $u + \frac{1}{2}v = 0$. The flow lines for $p=3$ (Fig. 2) indicate that a point in the upper half plane will flow to infinity asymptotically along the line $u + \frac{1}{2}v = u_H^*$. A point in the appropriate region of the lower half plane will flow to the stable mixed-minus fixed point with $u_-^* + \frac{1}{2}v_-^* = u_H^*$.

The exponents ν and ϕ_t can easily be obtained for all of the above fixed points from Eqs. (3.3) and (3.4)

$$\nu^{-1} = 2 - 8K_4u^*(p^2+p+2) - 8K_4v^*(3+2p-6/p), \tag{3.12}$$

$$\phi_t = \nu[1 + \tfrac{1}{2}\epsilon - 96u^* - 24(p+3-12/p)v^*]. \tag{3.13}$$

At the Heisenberg fixed point, v^* is zero and Eq. (3.12) reduces to the familiar result[4] for an n-component vector: $\nu^{-1} = 2 - [(n+2)/(n+8)]\epsilon$. As discussed in Sec. II, this result is expected. At $p=2$ and $p=3$, ν depends only on $u^* + \frac{1}{2}v^*$ as pre-

dicted. Thus, for $p=3$, ν has the same value if v starts off negative and flows to the stable mixed-minus fixed point or if v starts off positive and flows to infinity along $u+\frac{1}{2}v=u_H^*$. Hence, at $p=2$ and $p=3$, ν has the same value as Heisenberg systems with $n=2$ and 5: $\nu^{-1}=2-[(n+2)/(n+8)]\epsilon$. t is undefined at $p=2$ since $\mathrm{Tr}Q^3=0$. At $p=3$, ϕ_t depends only on $u+\frac{1}{2}v$ and has the value $\nu(1-\frac{1}{26}\epsilon)$.

B. ATP model

The recursion relations for the ATP model are

$$r'=b^2\{r+16K_4[\tfrac{1}{2}(1-b^{-2})-r\ln b]\,[(p+1)u+3(1-1/p)v]\}, \tag{3.14}$$

$$t'=b^{1+\epsilon/2}\{1-24K_4\ln b[2u+3(1-2/p)v]\}t, \tag{3.15}$$

$$u'=b^{\epsilon}\{u-16K_4\ln b[(p+7)u^2+6(1-1/p)uv+(9/p^2)v^2]\}, \tag{3.16a}$$

$$v'=b^{\epsilon}\{v-16K_4\ln b[12uv+9(1-2/p)v^2]\}. \tag{3.16b}$$

As for the Q model, there are four fixed points determined by Eqs. (3.16). They are

(a) Gaussian

$$u_G^*=v_G^*=0. \tag{3.17}$$

(b) Heisenberg

$$u_H^*=\epsilon/16(n'+8)K_4,\quad v_H^*=0. \tag{3.18}$$

(c) and (d) Mixed-plus and mixed-minus (see note after summary)

$$u_\pm^*=(\epsilon/32K_4b')[a'\pm(a'^2-\tfrac{4}{9}b')^{1/2}], \tag{3.19}$$

$$v_\pm^*=\{1/36[1-(2/p)]K_4\}(\tfrac{1}{4}\epsilon-48K_4u_\pm^*),$$

where

$$a'=(p-2)^2-\tfrac{2}{3}(p-1)(p-2)+\tfrac{8}{3},$$
$$b'=(p+7)(p-2)^2-8(p-1)(p-2)+16. \tag{3.20}$$

The λ exponents follow from linearizing Eqs. (3.16),

$$\lambda_u^G=\lambda_v^G=\epsilon \tag{3.21a}$$

$$\lambda_u^H=-\epsilon,\quad \lambda_v^H=[(n'-4)/(n'+8)]\epsilon, \tag{3.21b}$$

and

$$\lambda_1^\pm=-\epsilon,$$
$$\lambda_2^\pm=3\epsilon-32K_4[(p+13)u_\pm^*+(12-21/p)v_\pm^*], \tag{3.21c}$$

where $n'=p-1$.

We now analyze the position and stability of these fixed points. As might be expected, the fixed point structure for the ATP model is very similar to that for the hypercubic model for all values of p. For $p<5$ ($n'<4$), the mixed-plus fixed point is in the lower-right quadrant and the mixed-minus in the upper-right-hand quadrant. The Heisenberg fixed point is the most stable. At $p=5$, the mixed-plus and Heisenberg fixed points merge and become marginally stable. For $p>5$, the mixed-plus fixed point moves into the upper half plane and becomes the most stable fixed point. This is analogous to the cubic fixed point passing through the Heisenberg fixed point at $n=4$ and becoming the most stable. As $p\to\infty$, $u_+^*\to\epsilon/48pK_4$, $v_+^*\to\epsilon/144K_4$ and $u_-^*\to 0$, $v_-^*\to\epsilon/144K_4$. Thus, in this limit, the mixed-plus fixed point becomes identical to the cubic fixed point and the mixed-minus fixed point becomes identical to the Ising fixed point of the hypercubic model.[19] (Remember the factor of $\frac{1}{4}$ resulting from different normalization.) This is expected since the traceless constraint becomes unimportant at $p=\infty$. At $p=2$ and $p=3$ the mixed-plus fixed point lies on the line $u+\frac{1}{2}v=0$ and the mixed-minus fixed point on the line $u+\frac{1}{2}v=u_H^*$.

ν and ϕ_t follow from Eqs. (3.14) and (3.15):

$$\nu^{-1}=2-16K_4(p+1)u^*-48K_4(1-1/p)v^*, \tag{3.22}$$

$$\phi_t=\nu[1+\tfrac{1}{2}\epsilon-48K_4u^*-72K_4(1-2/p)v^*]. \tag{3.23}$$

For the Heisenberg fixed point, $\nu^{-1}=2-\epsilon[(n'+2)/(n'+8)]$ as expected. For $p=2$ and 3, ν^{-1} depends only on $u^*+\frac{1}{2}v^*$, and both the Heisenberg and mixed-minus fixed points yield the same numerical value for ν^{-1}. ϕ_t depends only on $u^*+\frac{1}{2}v^*$ at $p=3$.

IV. ORDERED STATES WITH UNIAXIAL SYMMETRY

In this section, we will calculate the equation of state for uniaxial states of the Q model to first order in ϵ[39] and in the infinite p limit,[40] to first order in a perturbation series in t. The restriction to uniaxial states requires $v<0$. Thus, we will be able to determine the equation of state associated with the stable mixed-minus fixed point between $n=4$ and $n=7.38$. For $2<n<4$, the equation of state reduces to the usual Heisenberg equation.[39] In the large-p limit, we predict a first-order transition for negative v.[18] Presumably, there is also a first-order transition for positive v in the large-p limit. We will not consider this case. The development presented in this section also applies to the ATP model. For $n'\le 4$, the Heisenberg fixed point is stable with a Heisenberg equation of state. For $n'>4$, the stable fixed point has v positive and does not have a uniaxial ground state. We will, therefore, not consider the ATP model explicitly. We note,

however, that the ground state for $v>0$ probably corresponds to matrices with entries of equal magnitude but alternating sign along the diagonal. The equation of state should reduce to the hypercubic equation[41] in the large-p limit.

In the uniaxial state, only A_1 [Eq. (2.3)] is nonzero. We, therefore, write A_1 as a constant plus a fluctuating field with zero average

$$A_1(\vec{x}) = M + L(\vec{x}), \tag{4.1}$$

and define fluctuating nonuniaxial components β_α and $\hat{Q}_{ij}$ as in Eq. (2.7). Using this relation the Hamiltonian of Eq. (2.1) can be written[37]

$$\mathcal{H} = \int (H_0 + H_1 + H_2 + H_3 + H_4)\, d^d x. \tag{4.2}$$

The terms H_1, H_2, H_3, and H_4 are regarded as perturbations. The unperturbed Hamiltonian H_0 is given by

$$H_0 = \tfrac{1}{4}[r_L L^2 + (\vec{\nabla}\cdot L)^2] + \tfrac{1}{4}(r_T \beta_i \beta_i + \nabla_j \beta_i \nabla_j \beta_i) + \tfrac{1}{4}(r_R \hat{Q}_{ij}\hat{Q}_{ij} + \nabla_i \hat{Q}_{ij} \nabla_i \hat{Q}_{ij}). \tag{4.3}$$

The quantities r_L, r_T, and r_R are exact self-energies. They are given by

$$\int d^d\vec{x} \langle L(\vec{x}) L(0)\rangle = 2r_L^{-1}, \tag{4.4}$$

$$\int d^d\vec{x} \langle \beta_i(\vec{x})\beta_j(0)\rangle = 2r_R^{-1}\delta_{ij}, \tag{4.5}$$

$$\int d^d\vec{x} \langle \hat{Q}_{ij}(\vec{x})\hat{Q}_{kl}(0)\rangle = \begin{cases} \{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \\ -[2/(p-1)]\delta_{ij}\delta_{kl}\} r_R^{-1}, \\ \delta_{ij}\delta_{kl}[\delta_{ik} - 1/(p-1)] 2r_R^{-1}. \end{cases} \tag{4.6}$$

The first form on the right-hand side of Eq. (4.7) is used for the Q model, the second for the ATP model. Of course the β_i are exactly zero in the ATP model.

The perturbation terms H_k, $k=1,\ldots,4$ are of order k in the fluctuation fields L, β_i, and $\hat{Q}_{ij}$. Their expressions are quite lengthy and we will not reproduce them here. Using Eqs. (4.3)–(4.6), we can proceed in the standard way to calculate the equation of state. We find

$$H/M^\delta + t/M\omega_t = f(\tau/M^{1/\beta}), \tag{4.7}$$

where τ is the reduced temperature variable $(T-T_c)/T_c$,

$$\delta = 3+\epsilon, \quad \beta = \nu(1-\tfrac{1}{2}\epsilon), \quad \omega_t = (1/\beta)\phi_t = \lambda_t(1+\tfrac{1}{2}\epsilon), \tag{4.8}$$

and

$$f(x) = 1 + x + l\epsilon\{6(3+x)\ln(3+x) + 2(p-1)(1+x)\ln(1+x) + (p^2-p-2)s(s+x)\ln(s+x) + 12x\ln 2 - 18(1+x)\ln 3 + (p^2-p-2)[xs^2(s-1)\ln(s-1) - s^2(1+x)\ln s]\}, \tag{4.9}$$

where

$$l\epsilon = 4K_4\{u^* + [(p^2-3p+3)/p(p-1)]v^*\}, \quad s = \frac{u^* + 3v^*}{u^* + [(p^2-3p+3)/p(p-1)]v^*}. \tag{4.10}$$

The standard normalization for $f(x)$ was used: $f(0)=1$ and $f(-1)=0$. Equations (4.8) and (4.10) are valid for both the Heisenberg and mixed-minus fixed points depending on which is the more stable. In particular, it is easy to verify that these equations reduce to the usual Heisenberg equation of state when $t=0$ and $u^*=u_H^*$, $v^*=0$. If $H=0$ but $t\neq 0$, Eq. (4.7) indicates that there will be a crossover away from the $t=0$ fixed point at $\tau=\tau_c\sim t^{1/\phi_t}$. M will undergo a change of order t^{1/ω_t} at this crossover. We are unable at the moment to determine whether this crossover is to a first-order transition or to the Q^3 fixed point. Though it seems likely that the crossover is to a first-order transition for $p>2$. It is certain, however, that there is some critical value of $p=p_c(d)$ such that for $p>p_c(d)$, the Q^3 fixed point does not exist. In this case, the crossover is to a first-order transition. The crossover behavior can then be compared with $\tau_c\sim t^2$ [Eq. (2.11)] and $M\sim t$ [Eq. (2.12)] predicted by mean-field theory.

Limit of large p. In this section we follow the method developed by Wallace[18] for treating the equation of state in the limit $n\to\infty$. In contrast to the method used to calculate the equation of state in the ϵ expansion this method is valid only when v may be regarded as a small perturbation. The behavior under the renormalization-group recursion relations is controlled by the unstable fixed point of Eq. (3.8). In the limit $p\to\infty$ there is a normal crossover behavior as a function of v.

In the case of the Q model we regard u as being of order $1/p^2$ so that the contribution of the quartic term to the free energy is of the same order as that of the quadratic term. If u is of order $1/p^2$, M is of order p. To insure that M remains of order p for nonzero v, we assign v to be of order $1/p^2$. As pointed out by Aharony,[19] this assign-

ment of the order of v is valid only if v can be regarded as a small perturbation. If the renormalization-group behavior were dominated by a fixed point with v^* not of order $1/p^2$ this approach would be invalid. However the analysis of Sec. II shows that there is no fixed point with $v^* \neq 0$ for large p. For the Q model this calculation of the equation of state is valid for either sign of v.

In the calculation below we obtain the equation of state to order $(1/p)^0$. We retain only the leading terms in p. Of the three self-energies only r_R is needed to this order. There is a contribution to r_R from a single loop diagram. We now follow Wallace step by step and obtain the equation of state

$$H/M^{\delta} - v/M^{\omega_v} + t/M^{\omega_t} = (\tau/M^2 + 1)^{\gamma}. \tag{4.11}$$

The critical exponents are

$$\delta = 3 + 2\epsilon/(2-\epsilon), \quad \gamma = 2/(2-\epsilon),$$

$$\omega_v = 2\epsilon/(2-\epsilon) = \phi_v/\beta,$$

$$\omega_t = (2+\epsilon)/(2-\epsilon) = \phi_t/\beta, \beta = 2.$$

For $v = 0$ this equation implies a first-order transition with $T_c \sim t^{2/\omega_t}$, since there is no Q^3 fixed point for $p = \infty$. This should be compared with the mean-field approximation result: $T_c \sim t^2$. If $t = 0$ and $v < 0$, there is also a first-order phase transition as for the vector model with hypercubic coupling. If $t = 0$ and $v > 0$, Eq. (4.11) predicts a negative value of r_R. This implies that the uniaxial state is not stable, Eq. (4.11) is invalid in this case.

The same type of analysis may be applied to the ATP model. For, the Potts model $u \sim 1/p$, $v \sim 1/p$, $t \sim 1/\sqrt{p}$, and $M^2 \sim p$. Exactly the same equation of state is derived. The ATP model, however, has a stable fixed point for $v^* > 0$. Therefore, the scaled equation of state is only valid for $v < 0$.

V. Q^3 FIXED POINT

In this section, we will show that there is a nontrivial fixed point for both the ATP and Q models that exists below six dimensions for a range of p. It is controlled by, and is stable with respect to, t.[42] We call this the Q^3 fixed point to emphasize that its existence depends on the presence of a cubic invariant. The stability in $6 - \epsilon'$ dimensions is a new feature of the tensor models that is not found in the usual ϕ^3 field theories. Near six dimensions u and v are driven to zero by the recursion relations. To simplify the calculation we set $u = v = 0$ as initial conditions. If u and v were actually zero the averages calculated from Eq. (2.2) would diverge. To avoid this conceptual problem we argue that an infinitesimal value of u removes this divergence.

As we shall see, it is difficult to determine unambiguously the range of p for which the Q^3 fixed point exists in $6 - \epsilon'$ dimensions. To zeroth order in ϵ', it exists for all $p < p_c$, where $p_c = \frac{10}{3}$ for the ATP and 4 for the Q model. These values are most likely reduced by a term proportional to $\sqrt{\epsilon'}$. For both models, different interpretations are needed for $p < 2$ and for $2 < p < p_c(\epsilon)$. For $p < 2$, the fixed point value of r is less than zero, i.e., the transition temperature is renormalized upward as is the case for the vector order-parameter model. The fixed-point reduced Hamiltonian has two minima as a function of $|Q|$. One shallow one at $|Q| \sim |r/t|$ and a deep one at $|Q| \sim |t/u|$ [cf. Fig. 1(d) with $r < 0$]. As discussed in Sec. I, the deep well corresponds to a situation in which a fraction of a state is preferentially occupied and is physically inaccessible. Thus the phase transition occurs when the order parameter falls into the shallower well. For $p > 2$, the fixed point value of r is greater than zero. The fixed-point Hamiltonian has a minima at $|Q| = 0$ and at $Q \sim |t/u|$ [c.f. Fig. 1(c)]. A first-order transition occurs if the order parameter falls into the deep minimum at $Q \sim |t/u|$. This corresponds to the usual Landau mean-field theory and presumably to the calculations around four dimensions presented in Secs. III and IV. It thus seems most likely that the Q^3 fixed point for $2 < p < p_c$ corresponds to critical fluctuations about the metastable minimum at $|Q| = 0$. Other possibilities are discussed in the Introduction. The conjecture of Alexander[43] that the minimum $|Q| \sim |t/u|$ is washed out by fluctuations among equivalent wells should also be mentioned. It should be emphasized that the problem of convergence discussed above is strictly a conceptual one. The calculations below are insensitive to the existence of the minimum at $Q \sim |t/u|$.

$p = 2$ is special to both the Q and ATP and corresponds, respectively, to the XY and Ising models. We would therefore, expect no nonclassical exponents to arise at any fixed point above four dimensions. It does in fact turn out that the critical exponents evaluated at the Q^3 fixed point for $p = 2$ for both models obtain the mean-field values.

The fixed-point value of t is of order $\epsilon^{1/2}$. The diagram contributing to the self-energy in order ϵ is shown in Fig. 4(a), those contributing to order ϵ^2 are those shown in Fig. 4(b). The critical exponent η is chosen such that the renormalization-group operation acts as the identity on the coefficient of the gradient term. Since the graph of Fig. 4(a) is not momentum independent, η will be of order ϵ. The equation expressing the condi-

tion that the gradient term not change is

$$1 = b^{-\eta}[1 - a_1 t^2 A'(b) - 2a_1^2 t^4 B'(b) - a_1 a_2 t^4 C'(b)]. \tag{5.1}$$

The recursion relation for the self-energy is

$$r' = b^{2-\eta}[r - a_1 t^2 A(b) - 2a_1^2 t^4 B(b) - a_1 a_2 t^4 C(b)]. \tag{5.2}$$

The recursion relation for t must be calculated to order $\epsilon^{5/2}$. The graph contributing to order $\epsilon^{3/2}$ is shown in Fig. 3(c), and those contributing to order $\epsilon^{5/2}$ are shown in Fig. 3(d). The recursion relation is

$$t' = b^{(1/2\epsilon - 3/2\eta)}[t + a_2 t^3 D(b) + 3a_1 a_2 t^5 E(b) + 3a_2^2 t^5 F(b) + a_3 t^5 G(b)]. \tag{5.3}$$

In these equations $a_1 = 2^4 3^2(1 - 2/p)$ $a_2 = 2^5 3^2(1 - 3/p)$, and $a_3 = 3^4 2^9(1 - 6/p + 10/p^2)$ for the ATP model and $a_1 = 2^2 3^2(p + 2 - 8/p)$, $a_2 = 2^2 3^2(p + 4 - 24/p)$ and $a_3 = 3^4 2^3(6p - 16 - 192/p + 640/p^2)$ for the Q model. The quantities $A(b)$, $B(b)$, $C(b)$, $A'(b)$, $B'(b)$, $C'(b)$, $D(b)$, $E(b)$, $F(b)$, and $G(b)$ are related to the momentum integrals associated with the graphs through the relations[44]

$$A(b) = I_1(q=0) = K_d[-2r \ln(b) + r^2 \tfrac{3}{2} b^2 - r\epsilon \ln^2(b)] + O(\epsilon^3), \tag{5.4}$$

$$B(b) = I_2(q=0) = K_6^2[-\tfrac{3}{4} b^2 - \tfrac{1}{2} \ln^2(b) - \tfrac{5}{12} \ln(b)]r + O(\epsilon^2), \tag{5.5}$$

$$C(b) = I_3(q=0) = K_6^2[-2 \ln^2(b) - 2 \ln(b)]r + O(\epsilon^2), \tag{5.6}$$

$$A'(b) = \left(\frac{\partial}{\partial q^2} I_1(q)\right)_{q=0} = K_d[-\tfrac{1}{3} \ln(b) + \tfrac{1}{9}\epsilon \ln(b) - \tfrac{1}{6}\epsilon \ln^2(b) + \tfrac{1}{3} r b^2] + O(\epsilon^2), \tag{5.7}$$

$$B'(b) = \left(\frac{\partial}{\partial q^2} I_2(q)\right)_{q=0} = K_6^2[-\tfrac{1}{12} b^2 + \tfrac{11}{108} \ln(b) + \tfrac{1}{18} \ln^2(b)] + O(\epsilon), \tag{5.8}$$

$$C'(b) = \left(\frac{\partial}{\partial q^2} I_3(q)\right)_{q=0} = K_6^2[-\tfrac{1}{3} \ln^2(b)] + O(\epsilon), \tag{5.9}$$

$$D(b) = I_4(q=0) = K_d[\ln(b) - \tfrac{3}{2} r b^2 + \tfrac{1}{2}\epsilon \ln^2(b)] + O(\epsilon^2), \tag{5.10}$$

$$E(b) = I_5(q=0) = K_6^2[\tfrac{1}{4} b^2 - \tfrac{11}{36} \ln(b) - \tfrac{1}{6} \ln^2(b)] + O(\epsilon), \tag{5.11}$$

$$F(b) = I_6(q=0) = K_6^2[\tfrac{1}{4} \ln(b) + \tfrac{1}{2} \ln^2(b)] + O(\epsilon), \tag{5.12}$$

$$G(b) = I_7(q=0) = K_6^2 \ln(b) + O(\epsilon). \tag{5.13}$$

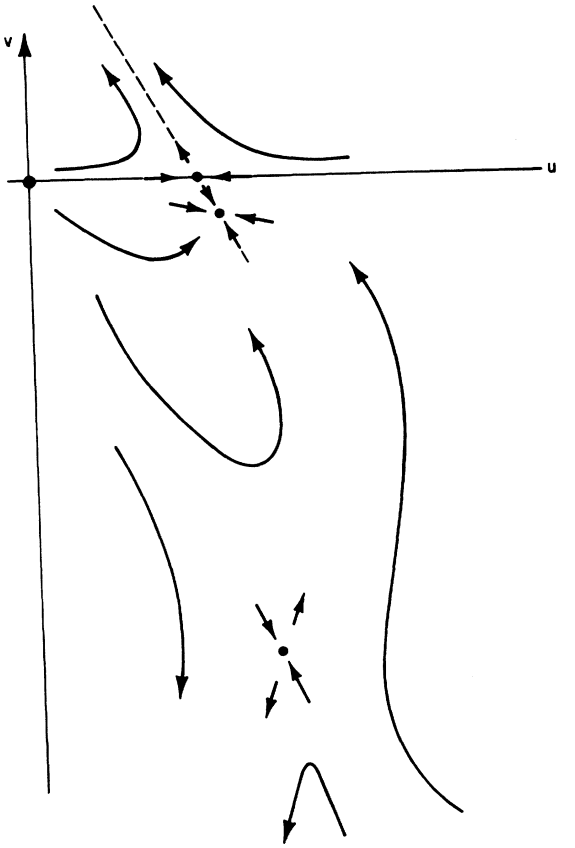

FIG. 3. Flow diagram for the case $p = 3$. Fixed points are shown as filled circles.

To find the graph associated with the integral I_n consider the graphs in Fig. 4 to be labeled from one to seven from left to right and from top to bottom. In writing Eqs. (5.4)–(5.13) we have retained only terms which diverge in the limit $b \to \infty$. Also, we have made use of the fact that the fixed-point values of r and t^2 are of order ϵ to evaluate the integrals to the order required to evaluate the critical exponents to order ϵ^2. The fixed point and the value of η are obtained by simultaneous solution of Eq. (5.1) and the equation obtained by setting $t = t'$ in Eq. (5.3). To check that all the b dependence cancels out it is necessary to know the fixed-point value of r to order ϵ. This comes from the nondivergent part of $A(b)$. The result is

$$r^* = \tfrac{1}{2} a_1 t^2 + O(\epsilon^2). \tag{5.14}$$

In this and all subsequent equations we absorb a factor of K_d into t^2. After checking that all the b dependence does cancel out we find the fixed point and the value of η

$$\eta = -\frac{1}{3}\left(\frac{a_1 \epsilon}{2a_2 - a_1}\right) - \frac{a_1}{54} \frac{(3a_2^2 + a_1 a_2 + 36a_3)\epsilon^2}{(2a_2 - a_1)^3}, \tag{5.15}$$

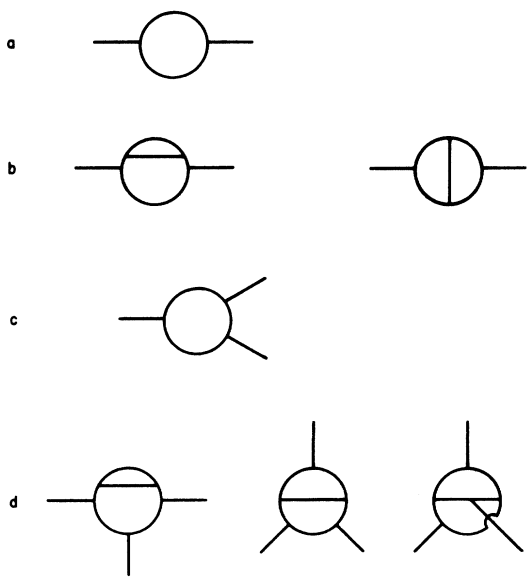

FIG. 4. (a) Self-energy diagram of order ϵ. (b) Self-energy diagrams of order ϵ^2. (c) Diagrams contributing to Eq. (5.3) to order $\epsilon^{3/2}$. (d) Diagrams contributing to Eq. (5.3) to order $\epsilon^{5/2}$.

$$t^{*2} = -\epsilon/(2a_2 - a_1) - (2a_2 - a_1)^{-3}(-\tfrac{5}{2}a_1a_2 + \tfrac{17}{18}a_1^2 + \tfrac{3}{2}a_2^2 + 2a_3)\epsilon^2. \tag{5.16}$$

The value of ν may be calculated from Eq. (5.2). The result is

$$1/\nu = 2 - \eta - [2a_1/(2a_2 - a_1)]\epsilon + [a_1/(2a_2 - a_1)^3](-\tfrac{49}{18}a_1^2 + a_2^2 + \tfrac{14}{3}a_1a_2 - 4a_3)\epsilon^2. \tag{5.17}$$

Note that both η and $(1/\nu - 2)$ are proportional to a_1 which is zero at $p = 2$ for both the Q and ATP models. Therefore, η and ν take on their classical values of 0 and $\frac{1}{2}$ at $p = 2$ as expected.

Evaluating Eqs. (5.15) and (5.17) for the ATP model at $p = 1$, and using the scaling equations $\gamma = \nu(2 - \eta)$ and $2\beta = d\nu - \gamma$, we obtain the percolation exponents

$$\begin{aligned} \eta &= -\tfrac{1}{21}\epsilon - (206/3^3 7^3)\epsilon^2, \\ 1/\nu &= 2 - \tfrac{5}{21}\epsilon - (23/7^3 3^3 2)\epsilon^2, \\ \gamma &= 1 + \tfrac{1}{7}\epsilon + (355/7^3 3^2 2^2)\epsilon^2, \\ \beta &= 1 - \tfrac{1}{7}\epsilon - (271/7^3 3^2 2^2)\epsilon^2. \end{aligned} \tag{5.18}$$

Substituting for $\epsilon = 1$ the results for 5 dimensions are found to be $\gamma = 1.17$ and $\beta = 0.835$. This can be compared with Kirkpatrick's Monte Carlo calculation[45] which yielded $\gamma = 1.3 \pm 0.1$ and $\beta = 0.66 \pm 0.05$. For both β and γ, the ϵ^2 term is of the right sign but not large enough to get agreement with the Monte Carlo result.

We will now discuss briefly the values of p for which t^{*2} is positive in $6 - \epsilon$ dimensions (i.e., the values of p for which the Q^3 fixed point exists). To first order in ϵ, t^{*2} is proportional to $\epsilon/(p_c^0 - p)$, where $p_c^0 = \frac{10}{3}$ for the ATP and $p_c^0 = 4$ for the Q model. Thus t^{*2} is positive as long as $p < p_c^0$. If $p > p_c^0$, t^{*2} is positive in $6 + \epsilon'$ dimensions. Note, however, that the ϵ expansion breaks down as p approaches p_c^0. The second-order term in Eq. (5.16) is proportional to $\epsilon^2/(p_c^0 - p)^3$ suggesting that in general

$$t^{*2} = (p_c^0 - p) f[\epsilon/(p_c^0 - p)^2], \tag{5.19}$$

where f is a function that reduces t^{*2} to Eq. (5.16) for small $\epsilon/(p_c^0 - p)^2$. A similar functional form for fixed-point potentials occurs for random n-component spin models[46] where $p_c^0 - p$ is replaced by $n - 1$. In this case, the exponents at $n = 1$ can be calculated directly[47] and are found to have corrections near four dimensions proportional to $\sqrt{\epsilon}$, in agreement with the scaling form of Eq. (5.19). In the case of the ATP and Q models, however, a direct calculation of t^{*2} at $p = p_c^0$ yields a negative (i.e., unphysical) value of t^{*2}. Therefore $f(x)$ changes sign for some $x_c \neq \infty$. We are now faced with a problem. If $f(x) > 0 (< 0)$ for $x < x_c$ $(> x_c)$, then there would be two disjoint regions for which $t^{*2} > 0$: (i) $p < p_c^0 - (\epsilon/x_c)^{1/2} = p_c < p_c^0$ and (ii) $p_c^0 < p < p_c^0 + (\epsilon/x_c)^{1/2}$. The second region however, is most likely an artifact since it was off limits to the first-order expansion in ϵ. In other words Eq. (5.19) is valid for $p < p_c$ only. For $p > p_c^0$, $t^{*2} = 0$. There is not much point in estimating p_c^0 to the order we have calculated. A single Pade' approximant which replaces $1 + x$ by $1/(1 - x)$ would be the only reasonable way to obtain p_c, and this is clearly not a good enough approximation. It thus appears that the Q^3 fixed point only exists for $p < p_c$, where p_c decreases with ϵ. The possibility that the fixed point exists for all $p < 4$ in two dimensions seems highly unlikely.

We conclude this section with the observation that the Q^3 fixed point also exists for long-range forces with potentials that die off as $r^{-d-\sigma}$ as long as $\sigma < \frac{1}{3}d$. In this case, the long-range (LR) exponents to lowest order in $\epsilon'' = 3\sigma - d$ are $\eta_{LR} = 2 - \sigma$ and $1/\nu_{LR} = \sigma - (a_1/a_2)\epsilon''$. An analysis identical to that of Sak[48] shows that the long-range fixed point is stable relative to the short-range fixed point as long as $\sigma < 2 - \eta$, where η is the short-range exponent. A curious situation arises for $p < 2$ in which case η is negative. In this case it would appear that the long-range fixed point is stable for $\sigma > 2$. However, in the region below six dimensions where the short-range fixed point

exists, the long-range fixed point only exists for $\sigma < 2$. Thus, below six dimensions, short-range behavior is unstable with respect to long-range behavior when η is negative below six dimensions whenever $\sigma < 2$. This is compared with the stronger condition that $\sigma < 2 - \eta$ when η is positive. The long-range Q^3 fixed points for the ATP and Q models have not yet been related to any physical problem.

VI. SUMMARY

In this paper, we have studied the critical properties of models having p-dimensional diagonal traceless (ATP model) and symmetric traceless (Q model) tensor order parameters using the ϵ expansion. These models differ from the more familiar vector model in that they have a cubic invariant t and two quartic invariants rather than a single quartic invariant. We have found two sets of nontrivial fixed points that can be treated using the ϵ expansion: one with *vanishing* t which gives nonclassical exponents below four dimensions; and one with a *nonvanishing* t which gives nonclassical exponents below six dimensions. The first set of fixed points is unstable with respect to t. This instability leads in most cases to a first-order transition describable by the usual Landau mean-field theory with a cubic invariant. The fixed point near six dimensions describes the percolation problem when the ATP model is analytically continued to $p = 1$.

We have been unable to make any definitive statement about the relationship between the continuous transition that is known to occur for the ATP model for $p < 3$ in two dimensions and the fixed-point structure obtainable from the ϵ expansions. It seems unlikely to us that these transitions have anything to do with the Q^3 fixed point in $6 - \epsilon'$ dimensions since the critical value of p for which this fixed point becomes nonexistent is less than or equal to $\frac{10}{3}$. One possibility is that the continuous transitions in two dimensions are described by fixed points in which the cubic potential t vanishes. In $4 - \epsilon$ dimensions, the fixed points with vanishing t are those discussed in Secs. III and IV. Thus, in $4 - \epsilon$ dimensions, the XY and the three-state ATP, and the Heisenberg and the four-state ATP model would belong to the same universality class and have the same critical properties. This would remain true down to a critical dimension d_c of order $3\frac{1}{3}$ where the fifth-order potentials (i.e., $\mathrm{Tr}Q^2$ and $\mathrm{Tr}Q^5$) become relevant. The fifth-order potential breaks the rotational invariance in spin space and distinguishes between the ATP and usual spin models. Thus below d_c, the ATP and spin-model fixed points would separate leading to different critical behavior for the two models. Different critical behavior is to be expected in two dimensions since the ATP models have long-range order in two dimensions whereas the two- and three-component spin models do not. The role of the fifth-order potential in distinguishing between the ATP and spin models is intriguing and is currently being investigated. We note in closing that it is entirely possible that it is impossible using the ϵ expansion to locate fixed points which, when followed to two dimensions become the fixed points describing the continuous transitions for $p = 3$ and 4.

While this paper was in the final stage of preparation two other papers[49,50] dealing with the ATP model were published. The first paper[49] by Zia and Wallace generalizes the ATP model in essentially the same way as we do in the present paper. They, however, factorized the fixed-point values for $u_\pm^*$ and $v_\pm^*$, Eqs. (3.19) and (3.20). Factorizing these equations we obtain

$$u_+^* = (\epsilon/48K_4)1/(p+2),$$

$$v_+^* = (\epsilon/144K_4)p/(p+2),$$

$$u_-^* = (\epsilon/48K_4)1/(p^2-7p+14),$$

$$v_-^* = (\epsilon/144K_4)p(p-5)/(p^2-7p+14),$$

in agreement with Ref. 49. Reference 50 derives the equation of state for the three state Potts model in agreement with Eqs. (4.9) and (4.10).

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