Bicritical and tetracritical points in anisotropic antiferromagnetic systems

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Renormalization-group techniques developed to analyze bicritical and tetracritical points, specifically in n-component antiferromagnetic systems, are presented in detail. The treatment yields a scaling description of the critical behavior of anisotropic antiferromagnets in both parallel and skew, uniform and staggered magnetic fields, in particular, the bicritical, spin-flop transition is discussed. For $n \leq 3$ it is described by a stable, isotropic, Heisenberg-like fixed point. However for $n \geq 4$ a new *biconical* fixed point, with irrational ϵ-expansion coefficients, becomes stable and describes tetracritical behavior. Special attention is given to the singular shape of the (T, H) phase boundaries for both *iso*tropic and *aniso*tropic antiferromagnets.

I. INTRODUCTION

Successes gained over the last decade in the experimental and theoretical study of phenomena in the vicinity of critical points,[1,2] particularly the advent of the renormalization-group ϵ-expansion approach,[3,4] have more recently given one the courage to attempt the serious study of *multicritical points*. The simplest general characterization of a multicritical point may, perhaps, be given by first considering a system exhibiting a λ *line:* that is, a line of critical points, $T_c(g)$, which is generated by some "nonordering" field g applied to the system (e.g., a pressure, stress, magnetic field, etc.). A nonordering field alters nonuniversal critical parameters, like critical-point energies, specific heat, and spontaneous order amplitudes, but does *not* change the basic nature of the critical point so that, in particular, universal quantities such as the critical exponents do not vary with g.[5] Well-known examples are the shift of Curie points under applied pressure, the depression of the λ point in ^{4}He on dilution with ^{3}He, and, of particular concern in this paper, the shift in the Néel point of an anisotropic antiferromagnet by a uniform magnetic field. It is frequently observed in both real materials and model systems, that the invariance of the asymptotic critical behavior along a λ line extends only over a finite range of g and is terminated abruptly at some special value g_0. At this *multicritical point* ($T=T_0$, $g=g_0$), distinct, new critical exponents occur and, in general, beyond this value quite new phenomena arise. ["Beyond" should more generally be interpreted as "in the vicinity of the end point g_0 of the λ line in the (T, g) plane."] In what seems to be the simplest situation, the λ line $T_c(g)$ is merely continued by a single line $T_T(g)$ across which the transition becomes first order. However Griffiths[6] pointed out that if the (T, g) space was enlarged by adjoining the basic ordering field, say h, then for many model systems analyzed phenomenologically the line $T_T(g)$ when viewed in the full (T, g, h) space, is seen to be a line of *triple points* at the join of three first-order surfaces. These surfaces individually terminate in three distinct critical or λ lines [one being the original $T_c(g)$ line] which, in turn, then meet and terminate at the multicritical point. Since three λ lines are confluent, Griffiths dubbed[6] the point a *tricritical point.*

This terminology appears apt and has been adopted by most subsequent workers: But the name is perhaps slightly unfortunate, in that it suggests a unique linear ordering of possible "higher-order" multicritical points. Subsequent (and, indeed, even earlier) studies have shown that the conceivable thermodynamic geometries of multicritical points embedded in larger thermodynamic spaces can be very complex.[7-12] Indeed it seems likely that the full classification of multicritical points will, like the classification of knots, remain an esoteric and largely unsolved problem for some time. For the present it thus seems reasonable to proceed in a more frankly *ad hoc* fashion and investigate various multicritical points as they come to hand in significant contexts.

A step in this direction was taken by Liu and Fisher (Appendix of Ref. 8), who presented a phenomenological analysis of the multicritical points resulting from the *competition between two distinct types of ordering*. Liu and Fisher were principally concerned with ^{4}He, where the competition is between "diagonal" or "crystalline" ordering and "off-diagonal" or "superfluid" ordering as a function of the pressure $p \propto g$. However they utilized the analogy with anisotropic antiferromagnetic spin systems, where the corresponding competition between "parallel" or "Ising-like" and "perpendicular" or "XY-like" ordering takes place as a function of the magnetic field, $H_{\parallel} \propto g$, parallel to the axis of anisotropy. This situation is the focus of the research reported below.

The phenomenological analysis shows that for a certain range of the quartic parameters entering the phenomenological expansion of the free energy one may, below the multicritical point, encounter three distinct ordered phases. Adopting magnetic language these are, *first,* a pure parallel ordered phase, with critical temperature $T_a(g) \equiv T_c^{\parallel}(H_{\parallel})$, for transition to the disordered paramagnetic phase (see Fig. 1); *second,* a pure perpendicularly ordered phase, with critical temperature $T_b(g) \equiv T_c^{\perp}(H_{\parallel})$ (see Fig. 2); and *third,* an "intermediate" or "doubly ordered" phase (*not* shown in Fig. 1) separated from the pure parallel and pure perpendicular phases by two further λ lines $T_{ac}(g)$ and $T_{bc}(g)$. In helium the new intermediate phase would be a "supersolid."[13] The *four* λ lines then meet together at the multicritical point (T_0, g_0), which Liu and Fisher[8] accordingly termed a *tetracritical point.*[14]

On the other hand, for the second range of the quartic free-energy parameters, the phenomenological theory yields *no* intermediate phase but, rather, a first-order transition between the two purely ordered, parallel and perpendicular, phases along a "flop line" $T_{\varphi}(g) = T_{\varphi}(H_{\parallel})$, [or $g \propto H_{\parallel} = H_{\varphi}(T)$]. The topology is as shown in Fig. 1. In the case of the anisotropic antiferromagnet the flop line $T_{\varphi}(H_{\parallel})$ corresponds simply to the spin-flop transition, from antiferromagnetic parallel ordering to antiferromagnetic perpendicular ordering (as indicated schematically in Fig. 1), which was predicted many years ago by Néel.[15] Since the flop line meets with just *two* λ lines, namely, $T_a(g) = T_c^{\parallel}(H_{\parallel})$ and $T_b(g) = T_c^{\perp}(H_{\parallel})$, at the spin-flop multicritical point (T_b, H_b), this point was termed a *bicritical point* by Fisher and Nelson.[16] The prefix *bi* may also be regarded as indicating that this type of multicritical point results from the simplest form of competition between two distinct ordering mechanisms.

Although the experimental situation is by no means entirely transparent,[17-26] the essential correctness of the phase diagram shown in Fig. 1 as a description of real antiferromagnets with relatively small uniaxial anisotropy in carefully aligned fields seems fairly well established. Accepting this phase diagram, one may develop a scaling theory of bicritical points.[16] In addition to the underlying "modifying" or "deviating" field $H_{\parallel}$ it is natural and straightforward to introduce the corresponding ordering fields $h_a \propto H_{\parallel}^{\dagger}$ and $\vec{h}_b \propto \vec{H}_{\perp}^{\dagger}$ which, for an antiferromagnet, are parallel and

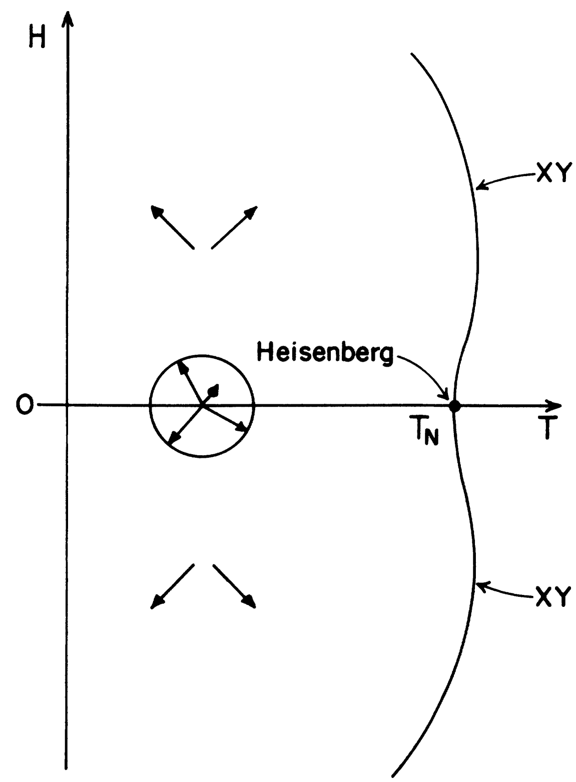

FIG. 2. Phase diagram for an antiferromagnet with *zero* anisotropy.

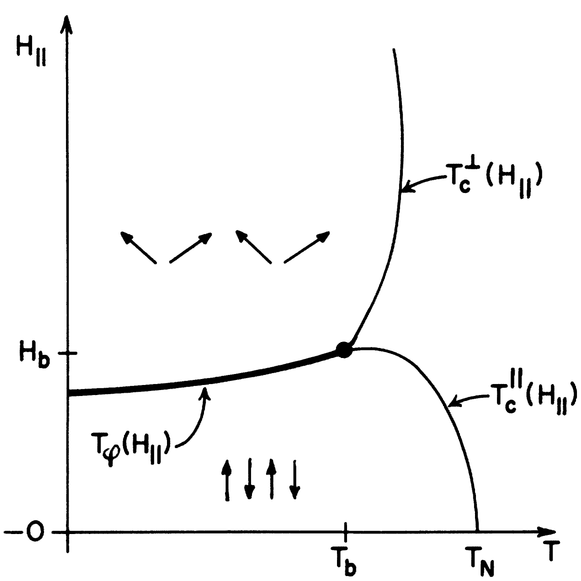

FIG. 1. Schematic phase diagram of an anisotropic antiferromagnet in a uniform magnetic field oriented parallel to the anisotropy axis.

perpendicular *staggered* fields which act oppositely on opposing sublattices. (Although these fields are normally physically inaccessible for antiferromagnets,[27] their response function may be studied by neutron diffraction. However, to our knowledge, such experiments have not yet been performed near a spin-flop point.) The scaling theory[16] leads to various predictions interrelating observable properties near the bicritical point. However to make these predictions more concrete and precise it is necessary to know the values of the basic bicritical exponents α, φ, $\Delta_\parallel$, and $\Delta_\perp$ entering the scaling formulation.[16] From a deeper theoretical viewpoint one also requires some assurance that the multicritical scaling behavior will be bicritical rather than tetracritical in nature.

Both these needs have been met by a recently announced[28] renormalization-group calculation employing the ϵ expansion.[3,4] In this paper we present the details of this calculation. We show, in particular, that for realistic Heisenberg spins with $n=3$ components, there is a stable bicritical fixed point while other fixed points, describing tetracritical scaling behavior, are unstable. Since diagonal order in helium may be described by $n_\parallel = 1$, and off-diagonal order by $n_\perp = 2$, so that $n = n_\parallel + n_\perp = 3$, this conclusion also indicates that a tetracritical point with an intermediate, supersolid phase should not occur in ^{4}He.[29] On the other hand, for n larger than a certain $n^\times$ depending on the dimensionality d, new, *biconical* tetracritical behavior is discovered. This may be relevant in certain experimental situations if the total number of ordering components satisfies $n \geqslant 4$. Additionally, for $n \gtrsim 11$ *decoupled*, tetracritical behavior sets in (see Ref. 16 and Sec. V below).

Second, our renormalization-group calculations demonstrate that the bicritical exponents should be isotropic or Heisenberg-like with $n = n_\parallel + n_\perp$. This serves to justify the explicit numerical exponent predictions made by Fisher and Nelson.[16] In particular, the fact that the crossover exponent $\phi = \varphi_H(n, d)$ exceeds unity for $n=2$ or 3 and $d<4$, leads to the conclusion that the two λ lines $T_c^\parallel(H_\parallel)$ and $T_c^\perp(H_\parallel)$, and the spin-flop line $T_\varphi(H_\parallel)$ should meet with a *common tangent* at the bicritical point. The details of this feature, in particular the significance of choosing the correct linear scaling axes to describe it, are discussed here in Sec. V E.[30]

The renormalization-group calculations also apply to a perfectly isotropic antiferromagnet in a small magnetic field H. Although perfect isotropy represents a strong idealization of any antiferromagnet, sufficiently precise realizations may exist to test the quite striking predictions which follow.[16] Within mean-field theory, imposition of the field H depresses the transition and the resultant initial temperature shift is quadratic in H. However, as explained in Sec. IV, our treatment predicts that the shift should vary rather as H^{ψ} with $\psi = 2/\phi \simeq 1.6$ (for $n=3$ isotropy). Furthermore the critical behavior in any nonzero field should reflect the reduced symmetry of $n-1$ components; i.e., it should be XY-like rather than Heisenberg-like and, concomitantly, the initial temperature is initially *raised* rather than lowered. (Of course in small fields this will be obscured by the usual crossover effects.)

In real systems alignment of the total, uniform external field $\vec{H} = (H_\parallel, \vec{H}_\perp)$ along the axis of magnetic anisotropy is often hard to achieve. Indeed, as discussed in Sec. VI, this may well be why a first-order spin-flop transition is sometimes not observed below T_b.[31] It is thus clearly of interest to include the perpendicular field components $\vec{H}_\perp$ in the calculation. As shown in Sec. VI, this is straightforward in principle but quite complicated and tedious in practice. Accordingly we have confined ourselves to a discussion of the transitions in the $(T, H_\parallel, \vec{H}_\perp)$ space from disorder to order, but have not discussed the transitions (including the spin-flop transition itself) which may take place within the already partially ordered phases.

The effects of finite ordering fields $H_\parallel^\dagger$ and $\vec{H}_\perp^\dagger$ is complicated by the fact that each is an ordering field *only* for its own phase: For the second phase it acts as a *non*ordering field which does not destroy the transition. Aspects of the resulting phase diagrams are discussed in Sec. VII.

The renormalization-group calculations themselves proceed in two stages. In the first, presented in Secs. II and III, a series of transformations and partial renormalizations are applied to the Hamiltonian to bring it into a form adapted to a detailed recursion-relation analysis. The basic procedures used were developed and applied previously in connection with a study of metamagnetic *tri*critical points.[32] The second stage, carried through in Sec. V, then consists of the more-or-less standard steps[3,4] of derivation of recursion relations, location of fixed points, and linearization to find exponents, all performed for a reduced Hamiltonian with quartic terms of lower symmetry than previously analyzed.[33] The most interesting theoretical points to emerge are, first, the existence of fixed points (the biconical fixed point) at which the exponents have ϵ-expansion coefficients which are nonrational algebraic functions of the number of components n (in contrast to the usual ratio of finite polynomials). In fact the exponents even vary nonanalytically for real n. A second feature of interest is the existence (for large n) of a stable fixed point describing two independent Hamiltonians with distinct critical exponents. As observed in Ref. 16 this essentially implies a breakdown of the usual, "total" scaling hypothesis.

II. SPIN-FLOP HAMILTONIAN

We first present the series of transformations needed to put an antiferromagnetic Hamiltonian into a form suitable for renormalization-group analysis. As mentioned, these transformations were developed for the discussion of metamagnets.[32] The present analysis is slightly more complicated, but the essential features are the same.

The Hamiltonian considered is that appropriate for the uniaxial anisotropic antiferromagnet with n-component spins $\vec{S}(\vec{R}) = [S_1(\vec{R}) \equiv S_\parallel(\vec{R}); \vec{S}_\perp(\vec{R})]$ at the sites $\vec{R}$ of a d-dimensional lattice, namely,

$$\mathcal{H}_{\rm int} = -\sum_{\vec{R}\vec{R}'} [J(\vec{R}-\vec{R}')\vec{S}(\vec{R})\cdot\vec{S}(\vec{R}') + D(\vec{R}-\vec{R}')S_\parallel(\vec{R})S_\parallel(\vec{R}')] - \sum_{\vec{R}} [H_\parallel S_\parallel(\vec{R}) + \vec{H}_\perp\cdot\vec{S}_\perp(\vec{R})] - \sum_{\vec{R}} e^{i\vec{k}_0\cdot\vec{R}}[\vec{H}^\dagger\cdot\vec{S}(\vec{R})]. \tag{2.1}$$

The isotropic exchange coupling $J(\vec{R})$ leads to antiferromagnetic ordering on two interpenetrating sublattices A and B (with superlattice reciprocal vector $\vec{k}_0$), while $D(\vec{R})$ represents an anisotropy energy tending to align the spins along the "easy" or "parallel" axis. The staggered or ordering field is $\vec{H}^\dagger = (H_\parallel^\dagger, \vec{H}_\perp^\dagger)$. As in previous renormalization-group work the spins are taken to be continuous in magnitude, and with each spin $\vec{S}(\vec{R})$ is associated an *isotropic* spin weighting factor[3,4]

$$e^{-W(\vec{S})} = e^{-|\vec{S}|^2/2 - f_4|\vec{S}|^4}. \tag{2.2}$$

The total effective Hamiltonian is thus

$$\bar{\mathcal{H}}(\vec{S}(\vec{R})) = -\mathcal{H}_{\rm int}/k_BT - \sum_{\vec{R}} W(\vec{S}(\vec{R})), \tag{2.3}$$

and the trace operation, needed to define the partition function, simply involves integrating each spin component from $-\infty$ to ∞.

Following the techniques introduced in Ref. 32, we decompose the spins according to which sublattice they populate. Defining sublattice δ functions by

$$\Delta_a(\vec{R}) = 1, \quad \Delta_b(\vec{R}) = 0 \quad \text{if } \vec{R}\subset A,$$
$$\Delta_a(\vec{R}) = 0, \quad \Delta_b(\vec{R}) = 1 \quad \text{if } \vec{R}\subset B, \tag{2.4}$$

we can then write

$$\vec{S}(\vec{R}) = \vec{S}_a(\vec{R})\Delta_a(\vec{R}) + \vec{S}_b(\vec{R})\Delta_b(\vec{R}). \tag{2.5}$$

If this decomposition is inserted in (2.1), we may write

$$\begin{aligned}\bar{\mathcal{H}} = \tfrac{1}{2}\sum_{\vec{R}\vec{R}'} \{ & K_{aa}(\vec{R}-\vec{R}')[\vec{S}_a(\vec{R})\cdot\vec{S}_a(\vec{R}') + \vec{S}_b(\vec{R})\cdot\vec{S}_b(\vec{R}')] + K_{ab}(\vec{R}-\vec{R}')[\vec{S}_a(\vec{R})\cdot\vec{S}_b(\vec{R}') + \vec{S}_b(\vec{R})\cdot\vec{S}_a(\vec{R}')] \\ & + E_{aa}(\vec{R}-\vec{R}')[S_a^\parallel(\vec{R})S_a^\parallel(\vec{R}') + S_b^\parallel(\vec{R})S_b^\parallel(\vec{R}')] + E_{ab}(\vec{R}-\vec{R}')[S_a^\parallel(\vec{R})S_b^\parallel(\vec{R}') + S_b^\parallel(\vec{R})S_a^\parallel(\vec{R}')]\} \\ & + L_\parallel\sum_{\vec{R}}[S_a^\parallel(\vec{R}) - S_b^\parallel(\vec{R})] + \vec{L}_\perp\cdot\sum_{\vec{R}}[\vec{S}_a^\perp(\vec{R}) - \vec{S}_b^\perp(\vec{R})] + \vec{L}^\dagger\cdot\sum_{\vec{R}}[\vec{S}_a(\vec{R}) + \vec{S}_b(\vec{R})] \\ & - \tfrac{1}{2}\sum_{\vec{R}}[|\vec{S}_a(\vec{R})|^2 + |\vec{S}_b(\vec{R})|^2] - f_4\sum_{\vec{R}}[|\vec{S}_a(R)|^4 + |\vec{S}_b(R)|^4],\end{aligned} \tag{2.6}$$

where the reduced fields are

$$\vec{L}_\perp = \vec{H}_\perp/k_BT, \quad L_\parallel = H_\parallel/k_BT, \quad \vec{L}^\dagger = \vec{H}^\dagger/k_BT, \tag{2.7}$$

and the reduced interactions are

$$K_{aa}(\vec{R}-\vec{R}') = K_{bb}(\vec{R}-\vec{R}') = J(\vec{R}-\vec{R}')\Delta_a(\vec{R})\Delta_a(\vec{R}')/k_BT,$$
$$K_{ab}(\vec{R}-\vec{R}') = J(\vec{R}-\vec{R}')\Delta_a(\vec{R})\Delta_b(\vec{R}')/k_BT, \tag{2.8}$$

and

$$E_{aa}(\vec{R}-\vec{R}') = E_{bb}(\vec{R}-\vec{R}') = D(\vec{R}-\vec{R}')\Delta_a(\vec{R})\Delta_a(\vec{R}')/k_BT,$$
$$E_{ab}(\vec{R}-\vec{R}') = D(\vec{R}-\vec{R}')\Delta_a(\vec{R})\Delta_b(\vec{R}')/k_BT. \tag{2.9}$$

For theoretical convenience, we have changed the sign of all spins lying on the B sublattice. This device converts the $K_{\alpha\beta}(\vec{R})$ and $E_{\alpha\beta}(\vec{R})$ into predominantly *ferro*magnetic interactions, and makes the fields $L_\parallel$ and $\vec{L}_\perp$ acts as though they were staggered fields on a ferromagnet. Similarly the ordering field $\vec{L}^\dagger$ becomes a uniform field.

In order to diagonalize the quadratic part of $\bar{\mathcal{H}}$ we define the transformed spin variables

$$\vec{s}_\pm(\vec{q}) = \tfrac{1}{2}\sum_{\vec{R}} e^{i\vec{q}\cdot\vec{R}}[\vec{S}_a(\vec{R})\Delta_a(\vec{R}) \pm \vec{S}_b(\vec{R})\Delta_b(\vec{R})], \tag{2.10}$$

where $\vec{q}$ runs over a half-sized Brillouin zone.[32]

The inverse transformation is

$$\vec{S}_a(\vec{R}) = N_a^{-1}\sum_{\vec{q}} e^{-i\vec{q}\cdot\vec{R}}[\vec{s}_+(\vec{q})+\vec{s}_-(\vec{q})], \quad \vec{R}\subset A$$
$$\vec{S}_b(\vec{R}) = N_a^{-1}\sum_{\vec{q}} e^{-i\vec{q}\cdot\vec{R}}[\vec{s}_+(\vec{q})-\vec{s}_-(\vec{q})], \quad \vec{R}\subset B. \tag{2.11}$$

If (2.8) is substituted into (2.6), and if we write $\vec{s}_\pm(\vec{q}) = [\vec{s}^{\,\parallel}_\pm(\vec{q}), \vec{s}^{\,\perp}_\pm(\vec{q})]$, the resulting expression can be decomposed into three pieces as

$$\bar{\mathcal{H}} = \bar{\mathcal{H}}_\parallel + \bar{\mathcal{H}}_\perp + \bar{\mathcal{H}}_\times , \tag{2.12}$$

where the part involving parallel spin components only is

$$\begin{aligned}\bar{\mathcal{H}}_\parallel = & -N_a^{-1}\sum_{\vec{q}}\{[1-K_+(\vec{q})-E_+(\vec{q})]s^\parallel_+(\vec{q})s^\parallel_+(-\vec{q})+[1-K_-(\vec{q})-E_-(\vec{q})]s^\parallel_-(\vec{q})s^\parallel_-(-\vec{q})\}+2L_\parallel s^\parallel_-(\vec{0})+2L_\parallel^\dagger s^\parallel_+(\vec{0})\\ & -2f_4N_a^{-3}\sum_{\vec{q}\vec{q}'\vec{q}''}[s^\parallel_+(\vec{q})s^\parallel_+(\vec{q}')s^\parallel_+(\vec{q}'')s^\parallel_+(-\vec{q}-\vec{q}'-\vec{q}'')+6s^\parallel_+(\vec{q})s^\parallel_+(\vec{q}')s^\parallel_-(\vec{q}'')s^\parallel_-(-\vec{q}-\vec{q}'-\vec{q}'')\\ & \qquad +s^\parallel_-(\vec{q})s^\parallel_-(\vec{q}')s^\parallel_-(\vec{q}'')s^\parallel_-(-\vec{q}-\vec{q}'-\vec{q}'')],\end{aligned} \tag{2.13}$$

while the corresponding perpendicular part is

$$\begin{aligned}\bar{\mathcal{H}}_\perp = & -N_a^{-1}\sum_{\vec{q}}\{[1-K_+(\vec{q})]\vec{s}^{\,\perp}_+(\vec{q})\cdot\vec{s}^{\,\perp}_+(-\vec{q})+[1-K_-(\vec{q})]\vec{s}^{\,\perp}_-(\vec{q})\cdot\vec{s}^{\,\perp}_-(-\vec{q})\}+2\vec{L}_\perp\cdot\vec{s}^{\,\perp}_-(\vec{0})+2\vec{L}_\perp^\dagger\cdot\vec{s}^{\,\perp}_+(\vec{0})\\ & -2f_4N_a^{-3}\sum_{\vec{q}\vec{q}'\vec{q}''}\{[\vec{s}^{\,\perp}_+(\vec{q})\cdot\vec{s}^{\,\perp}_+(\vec{q}')][\vec{s}^{\,\perp}_+(\vec{q}'')\cdot\vec{s}^{\,\perp}_+(-\vec{q}-\vec{q}'-\vec{q}'')]+2[\vec{s}^{\,\perp}_+(\vec{q})\cdot\vec{s}^{\,\perp}_+(\vec{q}')][\vec{s}^{\,\perp}_-(\vec{q})\cdot\vec{s}^{\,\perp}_-(-\vec{q}-\vec{q}'-\vec{q}'')]\\ & \qquad +4[\vec{s}^{\,\perp}_+(\vec{q})\cdot\vec{s}^{\,\perp}_-(\vec{q}')][\vec{s}^{\,\perp}_+(\vec{q}'')\cdot\vec{s}^{\,\perp}_-(-\vec{q}-\vec{q}'-\vec{q}'')]+[\vec{s}^{\,\perp}_-(\vec{q})\cdot\vec{s}^{\,\perp}_-(\vec{q}')][\vec{s}^{\,\perp}_-(\vec{q}'')\cdot\vec{s}^{\,\perp}_-(-\vec{q}-\vec{q}'-\vec{q}'')]\},\end{aligned} \tag{2.14}$$

and finally the purely fourth-order part with mixed spin components is

$$\begin{aligned}\bar{\mathcal{H}}_\times = 4f_4N_a^{-3}\sum_{\vec{q}\vec{q}'\vec{q}''} & \{[s^\parallel_+(\vec{q})s^\parallel_+(\vec{q}')+s^\parallel_-(\vec{q})s^\parallel_-(\vec{q}')][\vec{s}^{\,\perp}_+(\vec{q}'')\cdot\vec{s}^{\,\perp}_+(-\vec{q}-\vec{q}'-\vec{q}'')\\ & +\vec{s}^{\,\perp}_-(\vec{q}'')\vec{s}^{\,\perp}_-(-\vec{q}-\vec{q}'-\vec{q}'')]+4s^\parallel_+(\vec{q})s^\parallel_-(\vec{q}')[\vec{s}^{\,\perp}_+(\vec{q}')\vec{s}^{\,\perp}_-(-\vec{q}-\vec{q}'-\vec{q}'')]\},\end{aligned} \tag{2.15}$$

where

$$K_\pm(\vec{q}) = \sum_{\vec{R}} e^{i\vec{q}\cdot\vec{R}}[K_{aa}(\vec{R})\pm K_{ab}(\vec{R})], \tag{2.16}$$

$$E_\pm(\vec{q}) = \sum_{\vec{R}} e^{i\vec{q}\cdot\vec{R}}[E_{aa}(\vec{R})\pm \vec{E}_{ab}(\vec{R})]. \tag{2.17}$$

One may note that for a model with only nearest-neighbor interactions, K_{aa} vanishes identically.

Next, we make small-momentum expansions of the Fourier-transformed interactions entering (2.13) and (2.14), by writing

$$\begin{aligned}1-K_+(\vec{q}) &= (T-T_0^\perp)/T+(j_0^\perp a^2/k_BT)q^2+\cdots,\\ 1-K_-(\vec{q}) &= (T-T_-^\perp)/T+(j_-^\perp a^2/k_BT)q^2+\cdots\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}1-K_+(\vec{q})-E_+(\vec{q}) &= (T-T_0^\parallel)/T+(j_0^\parallel a^2/k_BT)q^2+\cdots,\\ 1-K_-(\vec{q})-E_-(\vec{q}) &= (T-T_-^\parallel)/T+(j_-^\parallel a^2/k_BT)q^2+\cdots,\end{aligned} \tag{2.19}$$

where a is, say, the nearest-neighbor lattice spacing. It is then convenient to rescale the spins to fix the coefficients of the q^2 terms at unity by writing

$$\begin{aligned}s^\parallel_+(\vec{q}) &= (k_BT/2j_0^\parallel a^{2+d})^{1/2}\sigma_{1,\vec{q}},\\ s^\parallel_-(\vec{q}) &= (k_BT/2|j_-^\parallel|a^{2+d})^{1/2}\sigma_{2,\vec{q}},\\ \vec{s}^{\,\perp}_+(\vec{q}) &= (k_BT/2j_0^\perp a^{2+d})^{1/2}\vec{s}_{1,\vec{q}},\\ \vec{s}^{\,\perp}_-(\vec{q}) &= (k_BT/2|j_-^\perp|a^{2+d})^{1/2}\vec{s}_{2,\vec{q}}.\end{aligned} \tag{2.20}$$

We have denoted parallel spin components by σ and the perpendicular vectors by $\vec{s}$ for notational clarity. With a positive uniaxial anisotropy $D(\vec{R})$ it is easy to show that the inequalities

$$|j_0^\parallel| > |j_0^\perp| \quad \text{and} \quad |j_-^\parallel| > |j_-^\perp| \tag{2.21}$$

hold. Denoting $N_a^{-1}a^{-d}\sum_{\vec{q}}$ by $\int_{\vec{q}}$, we find that these transformations reduce the three contributions to the total Hamiltonian $\bar{\mathcal{H}}$ to the forms

$$\bar{\mathcal{H}}_{\parallel} = -\tfrac{1}{2}\int_{\vec{q}} (r_1^{\parallel}+e_1^{\parallel}q^2)\sigma_1\sigma_1 - \tfrac{1}{2}\int_{\vec{q}} (r_2^{\parallel}+e_2^{\parallel}q^2)\sigma_2\sigma_2 + h_{\parallel}\sigma_{2,\vec{0}} + h_{\parallel}^{\dagger}\sigma_{1,\vec{0}}$$
$$-\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}(u_{11}\sigma_1\sigma_1\sigma_1\sigma_1 + u_{12}\sigma_1\sigma_1\sigma_2\sigma_2 + u_{22}\sigma_2\sigma_2\sigma_2\sigma_2), \tag{2.22}$$

which involves only parallel or σ spins,

$$\bar{\mathcal{H}}_{\perp} = -\tfrac{1}{2}\int_{\vec{q}} (r_1^{\perp}+e_1^{\perp}q^2)\vec{s}_1\cdot\vec{s}_1 - \tfrac{1}{2}\int_{\vec{q}} (r_2^{\perp}+e_2^{\perp}q^2)\vec{s}_2\cdot\vec{s}_2 + \vec{h}_{\perp}\cdot\vec{s}_{2,\vec{0}} + \vec{h}_{\perp}^{\dagger}\cdot\vec{s}_{1,\vec{0}}$$
$$-\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}[v_{11}(\vec{s}_1\cdot\vec{s}_1)(\vec{s}_1\cdot\vec{s}_1) + v_{12}(\vec{s}_1\cdot\vec{s}_1)(\vec{s}_2\cdot\vec{s}_2) + \bar{v}_{12}(\vec{s}_1\cdot\vec{s}_2)(\vec{s}_1\cdot\vec{s}_2) + v_{22}(\vec{s}_2\cdot\vec{s}_2)(\vec{s}_2\cdot\vec{s}_2)], \tag{2.23}$$

which involves only perpendicular or $\vec{s}$ spins, and

$$\bar{\mathcal{H}}_{\times} = -\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}[w_{11}\sigma_1\sigma_1(\vec{s}_1\cdot\vec{s}_1) + w_{12}\sigma_1\sigma_1(\vec{s}_2\cdot\vec{s}_2) + w_{21}\sigma_2\sigma_2(\vec{s}_1\cdot\vec{s}_1) + w_{12}\sigma_1\sigma_2(\vec{s}_1\cdot\vec{s}_2) + w_{22}\sigma_2\sigma_2(\vec{s}_2\cdot\vec{s}_2)]. \tag{2.24}$$

In these expressions we have neglected terms of order q^4, which will be irrelevant, and have deleted the usual momentum-conserving subscripts on the spins. The basic temperature variables are then

$$\bar{r}_1^{\parallel} = k_B(T-T_0^{\parallel})/a^2 j_0^{\parallel}, \quad \bar{r}_2^{\parallel} = k_B(T-T_-^{\parallel})/a^2|j_-^{\parallel}|,$$
$$\bar{r}_1^{\perp} = k_B(T-T_0^{\perp})/a^2 j_0^{\perp}, \quad \bar{r}_2^{\perp} = k_B(T-T_-^{\perp})/a^2|j_-^{\perp}|, \tag{2.25}$$

while the reduced fields become

$$h_{\parallel} = (2k_BT/a^{2+d}|j_-^{\parallel}|)^{1/2}L, \quad h_{\parallel}^{\dagger} = (2k_BT/a^{2+d}j_0^{\parallel})^{1/2}L^{+},$$
$$\vec{h}_{\perp} = (2k_BT/a^{2+d}|j_-^{\perp}|)^{1/2}\vec{L}_{\perp}, \quad \vec{h}_{\perp} = (2k_BT/a^{2+d}j_0^{\perp})^{1/2}\vec{L}_{\perp}^{+}, \tag{2.26}$$

and the quartic amplitudes are

$$u_{11} = \tfrac{1}{2}f_4k_B^2T^2/a^{4-d}j_0^{\parallel 2}, \quad u_{12} = 3f_4k_B^2T^2/a^{4-d}j_0^{\parallel}|j_-^{\parallel}|,$$
$$u_{22} = \tfrac{1}{2}f_4k_B^2T^2/a^{4-d}j_-^{\parallel 2}, \tag{2.27}$$

$$v_{11} = \tfrac{1}{2}f_4k_B^2T^2/a^{4-d}j_0^{\perp 2}, \quad v_{12} = f_4k_B^2T^2/a^{4-d}j_0^{\perp}|j_-^{\perp}|,$$
$$\bar{v}_{12} = 2f_4k_B^2T^2/a^{4-d}j_0^{\perp}|j_-^{\perp}|, \quad v_{22} = \tfrac{1}{2}f_4k_B^2T^2/a^{4-d}j_-^{\perp 2} \tag{2.28}$$

$$w_{11} = f_4k_B^2T^2/a^{4-d}j_+^{\parallel}j_0^{\perp}, \quad w_{12} = f_4k_B^2T^2/a^{4-d}|j_-^{\parallel}|j_0^{\perp},$$
$$w_{21} = f_4k_B^2T^2/a^{4-d}j_0^{\parallel}|j_-^{\perp}|, \tag{2.29}$$
$$\bar{w}_{12} = 4f_4k_B^2T^2/a^{4-d}(j_0^{\parallel}|j_-^{\parallel}|j_0^{\perp}|j_-^{\perp}|)^{1/2},$$
$$w_{22} = f_4k_B^2T^2/a^{4-d}|j_-^{\parallel}||j_-^{\perp}|.$$

Finally the momentum factors are

$$e_1^{\parallel} = e_1^{\perp} = 1, \quad e_2^{\parallel} = \pm 1 = \mathrm{sgn}(j_-^{\parallel}), \quad e_2^{\perp} = \pm 1 = \mathrm{sgn}(j_-^{\perp}). \tag{2.30}$$

The signum functions are needed here to account for the possibility that $j_-^{\parallel}$ and $j_-^{\perp}$ might be negative; negative values for $e_2^{\parallel}$ and $e_2^{\perp}$ need not concern us since we will find that these terms are strongly irrelevant variables, going rapidly to zero as we repeat the renormalization procedure. (The same phenomenon was observed in Ref. 1.) It can be seen from (2.16)–(2.19) that $j_0^{\parallel}$ and $j_0^{\perp}$ will be positive for antiferromagnetic interactions. We note also that the inequalities $T_0^{\parallel} > T_-^{\parallel}$ and $T_0^{\perp} > T_-^{\perp}$ hold; from these it follows that we have

$$\bar{r}_1^{\parallel} < \bar{r}_2^{\parallel}, \quad \bar{r}_1^{\perp} < \bar{r}_2^{\perp}, \tag{2.31}$$

in the critical region. These inequalities are important because they will eventually allow us to integrate the spin variables $\sigma_{2,\vec{q}}$ and $\vec{s}_{2,\vec{q}}$ out of the problem.

III. HAMILTONIAN TRANSFORMATIONS FOR A PARALLEL FIELD

In this section we analyze the Hamiltonian of Sec. II for the simpler case where only the uniform field $H_{\parallel}$ acts. All other perpendicular and staggered fields are supposed to vanish. A schematic drawing of the anticipated phase diagram in the $(T, H_{\parallel})$ plane is shown in Fig. 1. As explained, the object of the renormalization analysis is to obtain concrete numerical predictions for the exponents describing critical behavior: (i) on the "parallel" critical line in fields below H_b; (ii) on the "perpendicular" critical line above H_b; and (iii) at the *bicritical* or spin-flop point (T_b, H_b) itself. In this section we present an account of preliminary renormalization-group procedures which simplify the Hamiltonian $\bar{\mathcal{H}}$ and allow a standard renormalization-group analysis to be made in Sec. V. The procedures developed here also allow a discussion of the special case of zero anisotropy, which is presented in Sec. IV.

On writing the Hamiltonian (2.22)—(2.24) symbolically in real space, we obtain

$$\bar{\mathcal{H}} = \int d\vec{R}[\tfrac{1}{2}\bar{r}_1^{\parallel}\sigma_1^2 + \tfrac{1}{2}e_1^{\parallel}(\vec{\nabla}\sigma_1)^2 + \tfrac{1}{2}\bar{r}_2^{\parallel}\sigma_2^2 + \tfrac{1}{2}e_2^{\parallel}(\vec{\nabla}\sigma_2)^2 + \tfrac{1}{2}\bar{r}_1^{\perp}|\vec{s}_1|^2 + \tfrac{1}{2}e_1^{\perp}(\vec{\nabla}\vec{s}_1)^2 + \tfrac{1}{2}\bar{r}_2^{\perp}|\vec{s}_2|^2 + \tfrac{1}{2}e_2^{\perp}(\vec{\nabla}\vec{s}_2)^2$$
$$+ h_{\parallel}\sigma_2 + u_{11}\sigma_1^4 + u_{12}\sigma_1^2\sigma_2^2 + u_{22}\sigma_2^4$$
$$+ v_{11}|\vec{s}_1|^4 + v_{12}|\vec{s}_1|^2|\vec{s}_2|^2 + \bar{v}_{12}(\vec{s}_1\cdot\vec{s}_2)^2 + v_{22}|\vec{s}_2|^4 + w_{11}\sigma_1^2|\vec{s}_1|^2 + w_{12}\sigma_1^2|\vec{s}_2|^2 + w_{21}\sigma_2^2|\vec{s}_1|^2$$
$$+ \bar{w}_{12}\sigma_1\sigma_2(\vec{s}_1\cdot\vec{s}_2) + w_{22}\sigma_2^2|\vec{s}_2|^2]. \quad (3.1)$$

The σ and $\vec{s}$ variables here denote, of course, the Fourier transforms back into real space of the variables appearing in (2.22)–(2.24).

We now shift the σ_2 spin variable to eliminate the linear field term, that is we make the replacement

$$\sigma_2 \Rightarrow \sigma_2 + M, \quad (3.2)$$

and obtain

$$\bar{\mathcal{H}} = -\int d\vec{R}[\tfrac{1}{2}r_1^{\parallel}\sigma_1^2 + \tfrac{1}{2}e_1^{\parallel}(\vec{\nabla}\sigma_1)^2 + \tfrac{1}{2}r_2^{\parallel}\sigma_2^2 + \tfrac{1}{2}e_2^{\parallel}(\vec{\nabla}\sigma_2)^2 + \tfrac{1}{2}r_1^{\perp}|\vec{s}_1|^2 + \tfrac{1}{2}e_1^{\perp}(\vec{\nabla}\vec{s}_1)^2 + \tfrac{1}{2}r_2^{\perp}|\vec{s}_2|^2 + \tfrac{1}{2}e_2^{\perp}(\vec{\nabla}\vec{s}_2)^2$$
$$+ 2u_{12}M\sigma_2\sigma_1^2 + 4u_{22}M\sigma_2^3 + 2w_{21}M\sigma_2(\vec{s}_1\cdot\vec{s}_1) + \bar{w}_{12}M\sigma_1(\vec{s}_1\cdot\vec{s}_2) + 2w_{12}M\sigma_2(\vec{s}_2\cdot\vec{s}_2)$$
$$+ u_{11}\sigma_1^4 + u_{12}\sigma_1^2\sigma_2^2 + u_{22}\sigma_2^4 + v_{11}|\vec{s}_1|^4 + v_{12}|\vec{s}_1|^2|\vec{s}_2|^2 + \bar{v}_{12}(\vec{s}_1\cdot\vec{s}_2)^2 + v_{22}|\vec{s}_2|^4$$
$$+ w_{11}\sigma_1^2|\vec{s}_1|^2 + w_{12}\sigma_1^2|\vec{s}_2|^2 + w_{21}\sigma_2^2|\vec{s}_1|^2 + \bar{w}_{12}\sigma_1\sigma_2(\vec{s}_1\cdot\vec{s}_2) + w_{22}\sigma_2^2|\vec{s}_2|^2], \quad (3.3)$$

where the displaced temperature variables are

$$r_1^{\parallel} = \bar{r}_1^{\parallel} + 2u_{12}M^2, \quad r_2^{\parallel} = \bar{r}_2^{\parallel} + 12u_{22}M^2,$$
$$r_1^{\perp} = \bar{r}_1^{\perp} + 2w_{21}M^2, \quad r_2^{\perp} = \bar{r}_2^{\perp} + 2w_{22}M^2, \quad (3.4)$$

while $M = M(h_{\parallel}, T)$ is chosen to satisfy the relation

$$\bar{r}_2^{\parallel}M + 4u_{22}M^3 = h_{\parallel}. \quad (3.5)$$

The definitions (2.25) indicate that the inequalities (2.31) will for small field $h_{\parallel}$ (and hence, small M) apply also to the unbarred r_i parameters defined in (3.4). Consequently, the two parameters $r_2^{\parallel}$ and $r_2^{\perp}$ will diverge indefinitely under iteration of the usual renormalization-group procedures.[34]

With this in mind we follow Ref. 32 and introduce *distinct* renormalization-group spin revealing factors[32] $\hat{c}_{\parallel}^{+}$, $\hat{c}_{\perp}^{+}$, $\hat{c}_{\parallel}^{-}$, and $\hat{c}_{\perp}^{-}$, where $\hat{c}_{\parallel}^{+}$ and $\hat{c}_{\perp}^{+}$ are chosen, as usual, to keep $e_1^{\parallel}$ and $e_1^{\perp}$ constant but $\hat{c}_{\parallel}^{-}$ and $\hat{c}_{\perp}^{-}$ are chosen to keep $r_2^{\parallel}$ and $r_2^{\perp}$ constant (and thus prevent them diverging). As in Ref. 1, this rescaling device causes many of the variables in the Hamiltonian to become strongly irrelevant, so converging rapidly to zero as the renormalizations progress. The surviving terms are found to be

$$\bar{\mathcal{H}} = -\int d\vec{R}[\tfrac{1}{2}r_1^{\parallel}\sigma_1^2 + \tfrac{1}{2}e_1^{\parallel}(\vec{\nabla}\sigma_1)^2 + \tfrac{1}{2}r_2^{\parallel}\sigma_2^2 + \tfrac{1}{2}r_1^{\perp}|\vec{s}_1|^2$$
$$+ \tfrac{1}{2}e_1^{\perp}(\vec{\nabla}\vec{s}_1)^2 + \tfrac{1}{2}r_2|\vec{s}_2|^2 + 2u_{12}M\sigma_2\sigma_1^2 + w_{21}M\sigma_2|\vec{s}_1|^2$$
$$+ \bar{w}_{12}\sigma_1(\vec{s}_1\cdot\vec{s}_2) + u_{11}\sigma_1^4 + w_{11}\sigma_1^2|\vec{s}_1|^2 + v_{11}|\vec{s}_1|^4]. \quad (3.6)$$

Since gradients of the spin variables σ_2 and $\vec{s}_2$ no longer appear, we may integrate these variables completely out of the problem, and thereby obtain the reduced Hamiltonian

$$\bar{\mathcal{H}}_{\rm red} = -\int d\vec{R}[\tfrac{1}{2}r_{\parallel}\sigma^2 + \tfrac{1}{2}(\vec{\nabla}\sigma)^2 + \tfrac{1}{2}r_{\perp}|\vec{s}|^2 + \tfrac{1}{2}(\vec{\nabla}\vec{s})^2$$
$$+ u\sigma^4 + 2w\sigma^2|\vec{s}|^2 + v|\vec{s}|^4]. \quad (3.7)$$

In this expression we have deleted superfluous subscripts, lowered the remaining superscripts, and set

$$u = u_{11} - 8u_{12}^2M^2/r_2^{\parallel},$$
$$w = \tfrac{1}{2}w_{11} - 4w_{21}u_{12}M^2r_2^{\parallel} - \bar{w}_{12}^2M^2/4r_2^{\parallel}, \quad (3.8)$$
$$v = v_{11} - 2w_{21}^2M^2/r_2^{\parallel}.$$

Reference back to (3.4) and (2.25)–(2.29) shows the basic $(T, H_{\parallel})$ variation to be of the form

$$r_{\parallel} \approx a_{\parallel}(T - T_0^{\parallel}) + 12a_0^{\parallel}h_{\parallel}^2, \quad r_{\perp} \approx a_{\perp}(T - T_0) + 4a_0^{\perp}h_{\parallel}^2, \quad (3.9)$$

where $a_{\parallel}$, $a_0^{\parallel}$, $a_{\perp}$, and $a_0^{\perp}$ are positive constants, to lowest order in the anisotropy given by

$$a_{\parallel} \approx k_B/a^2j_0^{\parallel}, \quad a_{\perp} \approx k_B/a^2j_0^{\perp},$$
$$a_0^{\parallel} \approx a_0^{\perp} \approx \tfrac{1}{2}f_4k_B^2T^2/a^{4-d}|j_{-}^{\parallel}|j_0^{\perp}\bar{r}_2^{\parallel 2}. \quad (3.10)$$

In these expressions we have assumed that the field $H_{\parallel}$, and hence M, is small. Under these conditions the quartic parameters u, v, and w in

(3.8) are also positive. We take the uniaxial anisotropy small enough so that the complete range of flop transitions is swept out as we vary $H_\parallel$. For any nonzero anisotropy, we have $T_0^\parallel > T_0^\perp$, with equality holding for zero anisotropy. We make no attempt to answer the global questions involved in considering large anisotropies and consequent large fields $H_\parallel$. Indeed, under such circumstances one may well find that a tricritical point intervenes on the λ line $T_c(H_\parallel)$ before a spin-flop point is reached.

The Hamiltonian (2.1) has now been simplified sufficiently so that a detailed investigation of fixed points and critical exponents is feasible.

IV. FIELD BEHAVIOR FOR ZERO ANISOTROPY

It is straightforward and instructive to treat a fully isotropic antiferromagnet with no anisotropy, i.e., $D(R)\equiv 0$, under the influence of a uniform field $H\equiv H_\parallel$.[30] When $D(\vec{R})=0$, we find that the parameters in (3.9) satisfy

$$a_\parallel = a_\perp \quad \text{and} \quad T_0^\parallel = T_0^\perp. \tag{4.1}$$

Thus the temperature variations of $r_\parallel$ and $r_\perp$ in the reduced Hamiltonian (3.7) are identical. However coefficients of the field dependence necessarily remain unequal. This leads to an interesting effect not, apparently, noticed before.

In zero external field, the quartic couplings u, v, and w clearly become equal and the Hamiltonian exhibits full n-fold rotational symmetry. [This may be checked from (2.27)–(2.29) explicitly via (2.16)–(2.19), which yield $j_0^\perp = j_0^\parallel$.] One thus expects the usual Heisenberg-like critical exponents corresponding to an n-dimensional isotropic order parameter when $H_\parallel$ vanishes.

When $H_\parallel$ is nonzero, however, the degeneracy of $r_\parallel$ and $r_\perp$ is split, since $r_\parallel$ increases more rapidly with field than $r_\perp$. The analysis of quadratically anisotropic spin systems made by Fisher and Pfeuty[34] can now be applied. Under renormalization-group iteration the system crosses over and exhibits critical behavior characteristic of an $(n-1)$-dimensional order parameter with spin ordering perpendicular to the field axes. It must be remembered, however, that the crossover exponent $\varphi(n)$ normally considered[34,35] corresponds, by (3.9), to a variation of the variable $H_\parallel^2$. Thus the effective crossover exponent appropriate to the variable $H_\parallel$ in scaling expressions like[36]

$$\chi^\dagger(T,H_\parallel)\approx t^{-\gamma(n)}X_n(H_\parallel/t^{\tilde\phi(n)}), \tag{4.2}$$

with $t=(T-T_c)/T_c$, is given by

$$\tilde\phi(n)=\tfrac{1}{2}\varphi(n). \tag{4.3}$$

For an antiferromagnet with zero anisotropy, we hence expect a phase diagram of the sort shown in Fig. 2. The crossover index φ has been determined most accurately for three dimensions by series analysis[36] with the results $\phi(3)\simeq 1.25$ and $\phi(2)\simeq 1.175$. Hence, $\tilde\phi$ should be less than unity. Now the extended crossover scaling hypothesis[9] embodied in (4.2) implies that the phase boundary should be given asymptotically by

$$H_\parallel/t^{\tilde\phi}=\dot x, \tag{4.4}$$

where $\dot x$ is a constant, so that as $H_\parallel\to 0$ we have

$$T_c(H_\parallel)-T_c(0)\approx \dot c H_\parallel^{\tilde\psi}-AH_\parallel^2, \tag{4.5}$$

with

$$\tilde\psi=1/\tilde\phi, \quad \dot c=\dot x^{-\tilde\psi}.$$

The term $-AH_\parallel^2$, with A positive, represents the usual depression of T_c by a field, which is present even in mean-field theory; it enters here[30] as the leading nonlinear correction to the scaling field t. We expect $\dot c$ to be *positive* since it corresponds simply to the *increase* in T_c found when the anisotropy is present solely in the quadratic spin couplings.[36,37] The result (4.5) leads to the rather surprising bow-shaped critical line illustrated in Fig. 2. For a sufficiently isotropic, real antiferromagnetic material, the predictions

$$\tilde\psi\simeq 1.60\ (n=3), \quad 1.70\ (n=2) \tag{4.6}$$

should be testable experimentally.

V. RENORMALIZATION-GROUP ANALYSIS FOR A PARALLEL FIELD

A. Recursion relations

We now present a detailed renormalization-group analysis of the Hamiltonian (3.7), for nonzero anisotropy $D(R)$. For small $H_\parallel$, it follows from (3.9) and the inequality $T_0^\parallel > T_0^\perp$ that the temperature parameter r becomes negative before $r_\perp$ does, when T is reduced. Thus the system ultimately displays standard Ising-like critical behavior.[34] (This is, of course, also the case for zero field.) For fields sufficiently large compared to $(T_0^\parallel - T_0^\perp)^{1/2}$, however, the reverse situation evidently occurs. Then $(n-1)$-isotropic (i.e., "perpendicular" or "planar") critical behavior is instead realized.[34] Below the transition this changeover as $H_\parallel$ increases leads to the spin-flop transition.

A new analysis is needed in the bicritical region where $r_\parallel$ and $r_\perp$ are of comparable magnitude. Recursion relations to first order in $\epsilon=4-d$, for a d-dimensional system, can be constructed directly from the general expressions given by Fisher and Pfeuty.[34] However their relations were derived from Wilson's approximate recursion formula. Although this is, in fact, exact[4] to order ϵ, it nevertheless seems useful to reconstruct the relations using the exact momentum-integration method of Wilson.[4]

Accordingly, consider the generalized Hamiltonian for n-component spins,

$$\bar{\mathcal{H}} = -\frac{1}{2}\int d\vec{R}\left(\sum_{i=1}^{n}[r_i s_i^2 + (\vec{\nabla} s_i)^2] + 2\sum_{i,j=1}^{n} u_{ij} s_i^2 s_j^2\right). \quad (5.1)$$

On transforming to momentum space, the resulting momentum integrals may, for simplicity, be taken to run over a spherical zone (although this is not essential). As in Ref. 4, we assume the nonquadratic parts of the Hamiltonian are small, and calculate recursion relations by perturbation theory. A new, renormalized Hamiltonian $\bar{\mathcal{H}}'$ is generated from $\bar{\mathcal{H}}$ by choosing a rescaling factor $b > 1$ and integrating out all spin variables such that $b\vec{q}$ lies outside the original Brillouin zone.

When these standard techniques[4] are applied to (5.1), we obtain the recursion relations

$$r_i' = b^2\left(r_i + 8u_{ii}A(r_i) + 4\sum_j u_{ij}A(r_i)\right), \quad (5.2)$$

$$u_{ij}' = b^{\epsilon}\Big(u_{ij} - 8u_{ij}u_{ii}B(r_i, r_i) - 16u_{ij}^2 B(r_i, r_j) - 8u_{ij}u_{jj}B(r_j, r_j) - 4\sum_m u_{im}u_{mj}B(r_m, r_m)\Big), \quad (5.3)$$

which are valid to $O(\epsilon)$. The diagrammatic integrals arising here are

$$A(r) = \int_q^{>} (r+q^2)^{-1},$$

$$B(r, r') = \int_q^{>} (r+q^2)^{-1}(r'+q^2)^{-1}, \quad (5.4)$$

where the symbol $\int_q^{>}$ denotes a d-dimensional momentum integration over the shell $\Lambda b^{-1} < |\vec{q}| < \Lambda$. Apart from a few inessential modifications, these recursion formulas are identical to those found by Fisher and Pfeuty.[34]

The recursion relations (5.2) and (5.3) can be used to treat a model involving $n_\parallel$-component spins $\vec{\sigma}$ interacting with $n_\perp$-component spins $\vec{s}$, through the Hamiltonian

$$\bar{\mathcal{H}} = -\frac{1}{2}\int d\vec{R}\,[r_\parallel|\vec{\sigma}|^2 + (\vec{\nabla}\vec{\sigma})^2 + r_\perp|\vec{s}|^2 + (\vec{\nabla}\vec{s})^2 + u|\vec{\sigma}|^4 + 2w|\vec{\sigma}|^2|\vec{s}|^2 + v|\vec{s}|^4]. \quad (5.5)$$

For $n_\parallel = 1$ and $n_\perp = n-1$ this model reduces to the spin-flop Hamiltonian (3.7). For $n_\parallel = n_\perp$ it has been studied by Brézin *et al.*[33] but their results do not include ours. To order ϵ we obtain finally the recursion relations

$$r_\parallel' = b^2[r_\parallel + 4(n_\parallel+2)fu + 2n_\perp fw - 4(n_\parallel+2)gur_\parallel - 2n_\perp gwr_\perp], \quad (5.6)$$

$$r_\perp' = b^2[r_\perp + 4(n_\perp+2)fv + 2n_\parallel fw - 4(n_\perp+2)gvr_\perp - 2n_\parallel gr_\parallel], \quad (5.7)$$

$$u' = b^{\epsilon}[u - 4(n_\parallel+8)gu^2 - 4n_\perp gw^2], \quad (5.8)$$

$$v' = b^{\epsilon}[v - 4(n_\perp+8)gv^2 - 4n_\parallel gw^2], \quad (5.9)$$

$$w' = b^{\epsilon}w[1 - 16gw - 4(n_\parallel+2)u - 4(n_\perp+2)v], \quad (5.10)$$

where the functions

$$f(b) = \Lambda^2(1-b^{-2})/8\pi^2 \quad\text{and}\quad g(b) = \ln b/8\pi^2\Lambda^{\epsilon} \quad (5.11)$$

arise from the Feynman integrals over the outer momentum shell evaluated in the limit $d \to 4$ (i.e., $\epsilon = 0$).

B. Decoupled fixed points

For any value of n (> 0) the last three recursion relations above determine *six* fixed points. Four of these have $w^* = 0$ and hence represent *decoupled* Hamiltonians with *independent* fluctuations in the σ and s variables. Indeed these Hamiltonians satisfy the mean-field criterion for a tetracritical point,[8] namely,

$$(w^*)^2 < u^* v^*. \quad (5.12)$$

Thus they correspond to *tetra*critical rather than *bi*critical behavior as discussed further below. That these Hamiltonians will describe tetracritical points is also indicated by the decoupling. On defining

$$\bar{\epsilon} = 8\pi^2\Lambda^{\epsilon}\epsilon, \quad (5.13)$$

these fixed points are (a) $u^* = v^* = 0$, the trivial, always unstable Gaussian-Gaussian point; (b) $u^* = \bar{\epsilon}/4(n_\parallel+8)$, $v^* = 0$, an $n_\parallel$-Heisenberg-Gaussian point; (c) $u^* = 0$, $v^* = \bar{\epsilon}/4(n_\perp+8)$, a Gaussian-$n_\perp$-Heisenberg point; and finally (d) $u^* = \bar{\epsilon}/4(n_\parallel+8)$, $v^* = \bar{\epsilon}/4(n_\perp+8)$, a decoupled $n_\parallel$-Heisenberg-$n_\perp$-Heisenberg point.

The flows associated with these fixed points in the $w = 0$ plane are shown schematically in Fig. 3. The various crossover exponents associated with these in-plane flows are all of order ϵ. Calculating the renormalization-group eigenvalues corresponding to perturbations which take the system *out* of the $w = 0$ plane leads to the eigenvalues

$$\lambda_{(a)} = \epsilon, \quad \lambda_{(b)} = 6\epsilon/(n_\parallel+8),$$

$$\lambda_{(c)} = 6\epsilon/(n_\perp + 8), \quad \lambda_{(d)} = \epsilon\,\frac{32 - 2n_\| - 2n_\perp - n_\| n_\perp}{(n_\| + 8)(n_\perp + 8)}, \tag{5.14}$$

where we have written the b-dependent renormalization eigenvalues $\Lambda_{(\alpha)}(b)$ as $\Lambda_{(\alpha)} = b^{\lambda(\alpha)}$. The fixed points (a), (b), and (c) are evidently unstable to w-type perturbations for all $n_\|$ and $n_\perp > -8$.

Fixed point (d), however, is only unstable when

$$n_\| n_\perp + 2(n_\| + n_\perp) < 32 + O(\epsilon). \tag{5.15}$$

If this inequality is reversed, the fixed point becomes completely stable and terminates the critical surface flows. Since the system will then spontaneously break into essentially independent $n_\|$-Heisenberg and $n_\perp$-Heisenberg subsystems, a single scaling function cannot properly describe the asymptotic free energy when $n_\| \neq n_\perp$. Evidently this spontaneous decoupling provides one mechanism within the renormalization-group framework for the breakdown of scaling. Setting $n_\| = 1$ and $n_\perp = n - 1$, we see that this breakdown can occur only for $n > 11 + O(\epsilon)$. As such it is probably hard to realize in real physical systems.

C. Bicritical, Heisenberg fixed point

The two remaining fixed points lie at nonzero w. The first is the well-known isotropic n-Heisenberg fixed point[34,35] located at

$$u^* = w^* = v^* = \bar{\epsilon}/4(n_\| + n_\perp + 8), \tag{5.16}$$

with

$$r_\| = r_\perp = -\frac{\bar{\epsilon}(n_\| + n_\perp + 2)}{2(n_\| + n_\perp + 8)}. \tag{5.17}$$

As the interaction parameters u, v, and w at this fixed point satisfy the mean-field theory criterion for *bi*critical behavior[8] [namely, the converse of (5.12)], we conclude that this fixed point describes a spin-flop or bicritical point. Linearizing about this fixed point in (u, v, w) space, we find the three eigenvalues

$$\lambda_{(H)1} = -\epsilon, \quad \lambda_{(H)2} = -8\epsilon/(n_\| + n_\perp + 8),$$
$$\lambda_{(H)3} = -(4 - n_\| - n_\perp)\epsilon/(n_\| + n_\perp + 8), \tag{5.18}$$

correct to order ϵ.[38] This fixed point is fully stable and hence determines the critical behavior for

$$n_\| + n_\perp \leq 4 + O(\epsilon). \tag{5.19}$$

The range of $n_\|$ and $n_\perp$ values not covered by the inequalities (5.15) and (5.19) is the domain of stability of the sixth fixed point, which we will call "biconical" for reasons to be explained. The domains of stability to order ϵ for the Heisenberg, biconical, and decoupled fixed points are shown in Fig. 5. The critical behavior along the line $n_\| = n_\perp = \frac{1}{2}n$ was analyzed by Brézin *et al.*[33] These domains of stability are modified somewhat when the calculations are extended to higher orders in ϵ. For example by using the eigenvalue $\lambda_{(H)3}$ obtained to $O(\epsilon^3)$ by Ketley and Wallace,[39] we find that the Heisenberg fixed point is stable for

$$n_\| + n_\perp = n < n^\times(d) = 4 - 2\epsilon + c^\times \epsilon^2 + O(\epsilon^3), \tag{5.20}$$

where

$$c^\times = \tfrac{5}{12}[6\zeta(3) - 1]. \tag{5.21}$$

To estimate this oscillating series at $d = 3$, the diagonal Padé approximant

$$n^\times(d) \simeq (4 + 3.176\epsilon)/(1 + 1.294\epsilon) \tag{5.22}$$

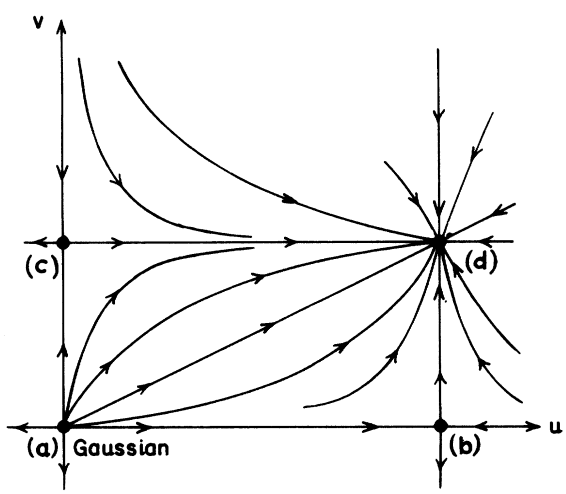

FIG. 3. Hamiltonian flows and fixed points in the $w=0$ plane.

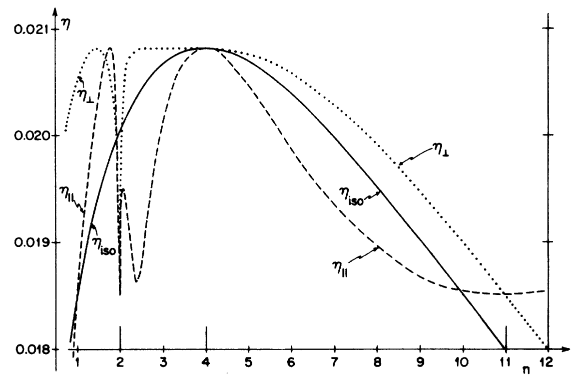

FIG. 4. Plot of the biconical exponents $\eta_\|$ and $\eta_\perp$ to $O(\epsilon^2)$ as a function of n. The isotropic exponent η to the same order is shown for comparison. We have set $\epsilon = 1$ for numerical evaluation.

may be formed; this yields $n^{\times}(3) \simeq 3.128$. Thus, in three dimensions, we still expect the Heisenberg fixed point to dominate for $n \leq 3$.

D. Tetracritical, biconical fixed point

We will analyze the remaining biconical fixed point in detail only for the uniaxial case $n_{\parallel} = 1$, $n_{\perp} = n - 1$, considered in the earlier sections. For $n^{\times}(\epsilon) < n < 11 + O(\epsilon)$ this fixed point determines the critical behavior. Its location to order ϵ is found to be given by

$$w^* = \bar{\epsilon}x/8, \quad u^* = \{1 + [1 - 9(n-1)x^2]^{1/2}\}\bar{\epsilon}/72 \,,$$
$$v^* = \{1 + [1 - (n+7)x^2]^{1/2}\}\bar{\epsilon}/8(n+7) \tag{5.23}$$

and

$$r_{\parallel}^* = -[12fu^* + 4(n-1)fw^*]/(b^{-2} - 1) \,,$$
$$r_{\perp}^* = -[4(n+1)fv^* + 2fw^*]/(b^{-2} - 1) \,, \tag{5.24}$$

where $x = x(n)$ is the real root of the cubic equation

$$9(4n^2 + 29n + 88)x^3 - 6(2n^2 + 28n + 179)x^2 + (n^2 + 5n + 472)x + 6(n - 11) = 0 \,. \tag{5.25}$$

Although the appropriate root of this equation is rational at $n = 11$ ($x = 0$), $n = 4$ ($x = \frac{1}{6}$), $n = 2$ ($x = \frac{1}{3}$), and at $n = 1$ ($x = \frac{10}{33}$) and -1 ($x = \frac{2}{3}$), the root is an irrational function of n. Specifically, for $n = 5$, we find

$$x(5) = [82 - (a + c\sqrt{82})^{1/3} - (a - c\sqrt{82})^{1/3}]/333 \,,$$

with

$$a = 18728, \quad c = 1998 \,. \tag{5.26}$$

Furthermore as a function of n the root exhibits a two-thirds root cusp at $n = 2$ described by

$$x(n) = \tfrac{1}{3} - \tfrac{1}{9}(6)^{-1/3}\Delta n^{2/3} - \tfrac{7}{162}\Delta n + \cdots, \quad \Delta n = n - 2 \,. \tag{5.27}$$

The renormalization-group eigenvalues needed for the various exponents can now be computed by linearizing the recursion relations (5.6)–(5.10). To order ϵ this yields

$$\lambda_{(B)1} = 2 + \tfrac{1}{2}(-3u^* - (n+1)v^* + \{[3u^* - (n+1)v^*]^2 + 4(n-1)w^{*2}\}^{1/2}) \,,$$
$$\lambda_{(B)2} = 2 + \tfrac{1}{2}(-3u^* - (n+1)v^* + \{[3u^* - (n+1)v^*]^2 + 4(n-1)w^{*2}\}^{1/2}) \,. \tag{5.28}$$

From these relations the biconical thermodynamic exponents can be calculated using the standard expression[4]

$$2 - \alpha = d\nu = d/\lambda_1 \,, \tag{5.29}$$

while the crossover exponent is given by[4]

$$\phi = \lambda_2/\lambda_1 \,. \tag{5.30}$$

In general, we must allow for distinct exponents $\eta_{\parallel}$ and $\eta_{\perp}$ governing the decay of order parallel and perpendicular to the anisotropy axis. The two gap exponents $\Delta_{\parallel}$ and $\Delta_{\perp}$ entering the free-energy scaling relation[16,30] for the biconical tetracritical point are related to $\eta_{\parallel}$ and $\eta_{\perp}$ by

$$\Delta_{\parallel} = \tfrac{1}{2}(d + 2 - \eta_{\parallel})\nu, \quad \Delta_{\perp} = \tfrac{1}{2}(d + 2 - \eta_{\perp})\nu \,, \tag{5.31}$$

and, similarly, we have the susceptibility exponents

$$\gamma_{\parallel} = (2 - \eta_{\parallel})\nu, \quad \gamma_{\perp} = (2 - \eta_{\perp})\nu \,. \tag{5.32}$$

Since we have

$$\eta_{\parallel}, \eta_{\perp} = O(\epsilon^2) \,, \tag{5.33}$$

it is clear that $\gamma_{\parallel}$ and $\gamma_{\perp}$ are the same to order ϵ and may, for brevity, be denoted $\gamma_B(n)$ (as in Ref. 28, where this distinction was not explicitly made).

The biconical exponents $\gamma_B(n)$ and $\phi_B(n)$ to order ϵ evaluated at $\epsilon = 1$ are listed in Table I together with the corresponding truncated Heisenberg exponents. (A graph of these results has been presented in Ref. 28.) Note that the biconical fixed point merges with the Heisenberg fixed point at $n = 4 + O(\epsilon)$, which is why the values coincide for $n = 4$. Similarly the biconical fixed point merges with the Ising/($n_{\perp} = 10$) decoupled fixed pint at $n = 11$ so that,

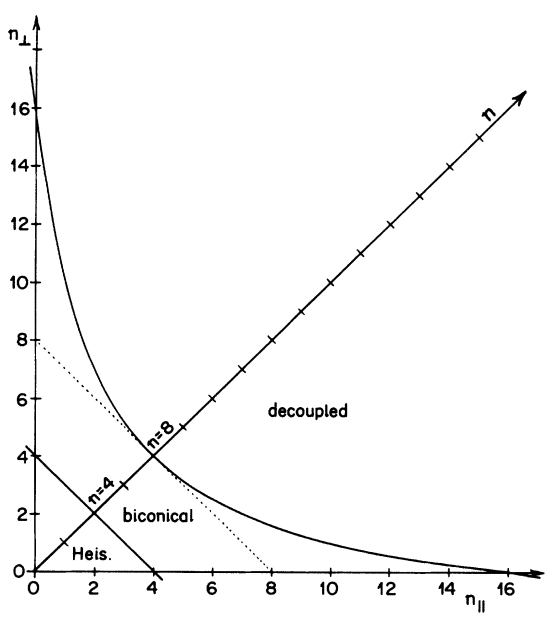

FIG. 5. Domains of stability of the decoupled, Heisenberg, and biconical fixed points.

TABLE I. Biconical and Heisenberg exponents evaluated to order ϵ at $\epsilon=1$.

n	$x(n)$	Biconical $\gamma_B(n)$	$\phi_B(n)$	Heisenberg $\gamma_H(n)$	$\phi_H(n)$
1	0.303 03	1.1667	1.0720	1.1667	1.0555
2	0.3333	1.1667	1.1667	1.2000	1.1000
3	0.230 20	1.2230	1.1761	1.2273	1.1364
4	0.166 67	1.2500	1.1667	1.2500	1.1667
5	0.120 53	1.2673	1.1551	1.2692	1.1923
6	0.086 01	1.2805	1.1470	1.2857	1.2143
7	0.059 66	1.2921	1.1438	1.3000	1.2333
9	0.023 12	1.3136	1.1504	1.3235	1.2647
10	0.010 31	1.3238	1.1578	1.3333	1.2777
11	0	1.3333	1.1667	1.3421	1.2895
13	−0.015 21	1.3505	1.1858	1.3571	1.3095
15	−0.024 55	1.3651	1.2048	1.3698	1.3261

to order ϵ, we have $\gamma_B(11)=\gamma_H(10)$.

Although the values of γ are numerically close for the biconical and Heisenberg fixed points, the values of ϕ differ significantly; this might enable these fixed points to be distinguished experimentally (or in numerical calculation and simulations). It may be remarked that as a result of the singular variation of $x(n)$, given in (5.27), the exponent $\gamma_B(n)$ displays a cubic cusp at $n=2$ (see the figure in Ref. 28).

It is in fact possible to determine the tetracritical exponents $\eta_\parallel$ and $\eta_\perp$ to leading order by straightforward techniques.[4] Thus one discovers that the biconical fixed point does *not*, in fact, have a single, isotropic exponent η. The inequality of the fixed-point values $r_\parallel^*$ and $r_\perp^*$, as evidenced by (5.24), leads to distinct exponents $\eta_\parallel$ and $\eta_\perp$. To order ϵ^2 these are given by

$$\eta_\parallel=8[3u^{*2}+(n-1)w^{*2}]+O(\epsilon^3)\ , \tag{5.34}$$

$$\eta_\perp=8[(n+1)v^{*2}+w^{*2}]+O(\epsilon^3)\ , \tag{5.35}$$

where only the fixed-point values (5.28) of u^*, w^*, and v^* to $O(\epsilon)$ are needed.

A plot of $\eta_\parallel$ and $\eta_\perp$ to $O(\epsilon^2)$ as a function of n is presented in Fig. 4; the isotropic exponent η to the same order is shown for comparison. As in the case of $\gamma_B(n)$ and $\phi_B(n)$, the exponents $\eta_\parallel(n)$ and $\eta_\perp(n)$ exhibit cusps at $n=2$. Note that, in parts of the region $1\leq n\leq 4$, both $\eta_\parallel$ and $\eta_\perp$ become greater than the isotropic η, even though the isotropic or Heisenberg fixed point dominates the critical behavior in this region; this violates the folklore that "the largest η wins."

In the region of biconical stability, $4<n+O(\epsilon)<11$, it is not hard to show that $0\leq w^*\leq\bar\epsilon/(n+8)$, while u^* and v^* exceed $\bar\epsilon/(n+8)$. Accordingly the fixed point satisfies the mean-field criterion[7] (5.12) for *tetra*criticality. Thus, a new phase with *both* parallel and perpendicular ordering simultaneously present is expected to appear below T_b in place of the usual spin-flop line. In confirmation, recall that the equation of state to order ϵ^0 is always given by the phenomenological theory. One does not expect that the corrections of order ϵ and higher would alter such qualitative features of the thermodynamic behavior corresponding to the fixed point. [As already mentioned, the condition for bicriticality,[8] namely, $(w^*)^2\geq u^*v^*$, is satisfied at the Heisenberg fixed point.]

Because of the symmetry implied by the unequal values of $r_\parallel^*$ and $r_\perp^*$ and by the values of u^*, v^*, and w^*, the spins will tend to lie on an easy double cone with axis parallel to the original easy axis, and with a conical angle θ determined by n via the fixed-point values. Specifically we find

$$\tan^2\theta(n)=\frac{r_\perp^*u^*-r_\parallel^*w^*}{r_\parallel^*v^*-r_\perp^*w^*}\ . \tag{5.36}$$

In interpreting this formula, however, it must be recalled that the σ and s spins represent distinct rescalings of the original parallel and perpendicular spin components, unless one has $j_0^\perp=j_0^\parallel$ in (2.18) and (2.19). This equality will hold when the anisotropy is of single-ion type [i.e., $D(\vec{R})=0$ when $\vec{R}\neq 0$]. Even in this case, however, some differential rescaling may take place through subsequent renormalizations. Nevertheless the biconical nature of the predominant spin fluctuations near the tetracritical point should be detectable in scattering experiments (once such a point is found) and, in any case, justifies the name given to the fixed point. The formula (5.31) has no meaning for $n<4$, since the length of the spin components in the (now unstable) biconical phase can be imaginary.

E. Scaling and scaling fields

Once one has identified a renormalization-group fixed point corresponding to a particular critical or multicritical point, scaling of the free energy and correlations in the vicinity of the multicritical point follows by the usual renormalization-group arguments.[4] The only issue of special significance to be discussed, however, concerns the identification of the appropriate linear scaling fields. For a bicritical point occurring in zero field, symmetry dictates that the linear scaling fields are $t=(T-T_b)/T_b$, and $H_\parallel$, $\vec{H}_\perp$, $H_\parallel^\dagger$, and $\vec{H}_\perp^\dagger$. In the presence of a finite field $H_\parallel$ in the vicinity of the bicritical field H_b, symmetry still shows that $H_\perp$, $H_\parallel^\dagger$, and $\vec{H}_\perp^\dagger$ are scaling fields. However, the fields t and $\Delta H_\parallel=H_\parallel-H_b$ must, in general, mix to produce new linear scaling fields $\tilde{t}$ and g. The need to define the modifying (or deviating) field g by

$$g = \Delta H_{\parallel}/k_B T_b - p_g t = h_{\parallel} - p_g t \qquad (5.37)$$

follows from simple geometric considerations of the bicritical phase diagram. Thus, as explained in Refs. 16 and 30, the mixing parameter p_g must equal the slope of the tangent to the first-order spin-flop line $T_\phi(h_\parallel)$ or T_b in the $(h_\parallel, t)$ or $(H_\parallel, k_B T)$ planes. Conversely a calculation of p_g dictates the slope of the spin-flop line. (Note we are assuming that the spins and their magnetic moments are dimensionless.) Since the crossover exponent ϕ, which then enters in the scaling combination $g/\tilde{t}^{\phi}$, exceeds unity, it is actually *sufficient* for *asymptotic* scaling purposes to replace $\tilde{t}$ simply by t (as was done in Ref. 16). However the *approach* to asymptotic scaling will, in fact, be more rapid in terms of the linear scaling field

$$\tilde{t} = t + q_t h_\parallel \ , \qquad (5.38)$$

where $-q_t$ is evidently the reciprocal slope of the $\tilde{t} = 0$ axis in the $(H_\parallel, k_B T)$ plane.

Within the renormalization group the mixing parameters p_g and q_t can in principle be derived from the two fixed-point eigenvectors corresponding to the $r_\parallel$ and $r_\perp$ parameters in the linearized form of the recursion relations (5.6) and (5.7). However, in general, this would entail knowledge of the full course of the nonlinear renormalizations leading to the fixed-point vicinity. This knowledge cannot be gained, but an estimate of the mixing parameters within the ϵ expansion may be obtained by assuming $H_\parallel$ is small and utilizing (3.9) and (3.10). When this is done we obtain[30]

$$p_g \simeq [1 - (j_0^{\perp}/j_0^{\parallel})] k_B (T_b - T_{-}^{\parallel})^2 / 16 f_4 H_b T_b \ , \qquad (5.39)$$

$$q_t \simeq 8(n+2) f_4 H_b T_b / n k_B (T - T_{-}^{\parallel})^2 \ , \qquad (5.40)$$

where the anisotropy has been estimated in terms of H_b and T_b. We note first that both q_t and p_g are positive (the latter being in accord with experimental observation). (Second, as is to be expected, both p_g and q_t vanish as $H_b \to 0$.)

Various predictions resulting from scaling at the bicritical point were discussed explicitly in Ref. 16 under the assumption, justified in detail above, that for $n = 2$ or 3 the bicritical exponents are just those appropriate to the corresponding isotropic spin Hamiltonian. The best estimates for the exponents are then those derived from analysis of series expansions.[36,37] We will not repeat the discussion here except to point out that utilization of the scaling field $\tilde{t}$ in place of t leads to a refined prediction for the location of the $n_\parallel = 1$ and $n_\perp = n - 1$ phase boundaries $H_c^{\pm}(T)$, in the vicinity of the bicritical point.[30] Specifically the scaled relation $g/\tilde{t}^{\phi} = \pm w_{\pm}$ yields

$$H_c^{\pm}(t)/k_B T \approx p_g t \pm w_{\pm}(t + g_g h_\parallel)^{\phi}$$
$$\approx p_g t \pm w_{\pm}' t^{\phi} + w_{\pm}'' t^{2\phi - 1} + O(t^{3\phi-2}) \ , \qquad (5.41)$$

where

$$w_{\pm}' = w_{\pm}(1 + p_g q_t)^{\phi}, \quad w_{\pm}'' = \phi w_{\pm}^2 (1 + p_g q_t)^{2\phi - 1} \ . \qquad (5.42)$$

Note that the $t^{2\phi-1}$ correction term in (5.41) is relatively singular and introduces a stronger asymmetry into the two branches than implied merely by the differences between w_+ and w_-. For these reasons it is probably preferable to make experimental fits to the first part of (5.41) rather than to the expanded version.[30]

As mentioned in Ref. 16, this prediction implies that the λ lines $H_c^+(T)$ and $H_c^-(T)$ should meet at the bicritical point H_b with a *common tangent* which is also the tangent to the first-order spin-flop line beneath T_b. This prediction is in contrast to the mean-field result, where the three lines meet at distinct angles. The relative amplitudes w_+/w_- should be a universal parameter; however, its evaluation requires further calculation.[30] One may anticipate, nonetheless, that it will exceed unity on the basis of the observation that for fixed nearest-neighbor coupling J the critical temperatures of the Heisenberg, XY, and Ising models are ordered according to

$$(T_c^{XY} - T_c^H)/(T_c^I - T_c^H) < 1 \ . \qquad (5.43)$$

The ratio of critical-temperature differences here should be a measure of $(w_-/w_+)^{1/\phi}$. These conclusions seem to be in reasonable accord with currently available measurements which, however, are not of as high a precision as desirable.[30] (It must also be remembered that $\phi \simeq 1.25$ is quite close to unity, so that as in historical observations of inverse ferromagnetic susceptibility plots, which approach the axis tangentially as t^γ with $\gamma \simeq 1.2$–1.4, the tangency may not be at all obvious to the unaided eye.)

VI. UNIFORM BUT SKEW EXTERNAL FIELD

A. Skew fields and the ordered phases

We now turn to the experimentally interesting case, in which the uniform external field is skew, i.e., applied at some nonzero angle, to the anisotropy axis. In practice it is hard experimentally to avoid some misalignment resulting in the imposition of a skew field. Indeed there have been suggestions that the transition between the antiferromagnetic and flopped states with the field *along* the anisotropy axis is continuous and not first order, as expected theoretically. It has even been asserted that there is no transition at all,[31] i.e., that the magnetization and other variables change rapidly but continuously as $H_\parallel$ is increased below T_b. One possible reason for the failure to observe the anticipated first-order transition is just the misalignment of the external field with respect to the easy axis of the crystal. Thus a mean-field-theory analysis[26,40] indicates that a true first-order transition should be observed only if the angle between the easy axis and the field is less than some temperature-dependent critical angle determined by the effective anisotropy and exchange fields. Although this conclusion is probably valid far from T_b its status close to the bicritical point is not yet clear. In any event the critical angle is expected to vanish as T_b is approached from below.

To complicate matters further, in the often studied antiferromagnet $MnCl_2 \cdot 4H_2O$, the easy axis is not along the crystallographic c axis (along which the field is usually applied[20-22,30]), but is displaced by an angle of about[23] 7°. We note, however, that a recent study,[24] with careful alignment of the field, has shown that the spin-flop transition below T_b is almost certainly first order in this material just as expected theoretically.

There is a further theoretical possibility for the nonobservation of a first-order transition in certain materials. When there is an additional anisotropic interaction of cubic symmetry and of the appropriate sign it is possible [even though such interactions are technically "irrelevant" at the bicritical point when $n < n^\times(d)$] that the spin-flop point appears thermodynamically to be tetracritical in nature even for $n < n^\times(d)$. In such a case, there should in fact again be two *further* λ lines separating the antiferromagnetic and flopped phases from the extra, doubly ordered, intermediate phase.[8] Such a possibility has been investigated using Feynman-graph ϵ-expansion techniques by Aharony and Bruce[41] and seems likely to be of particular relevance to displacive transitions. Aharony and Bruce do indeed find tetracritical thermodynamic behavior. However it should be noted that these two extra λ lines approach one another very rapidly as $T \to T_b$ so that, in fact, within the asymptotic scaling regime the point of confluence still appears to be *bi*critical.

In this paper we do not explore the possible ordered phases or the transitions between them. Rather we restrict ourselves to the bicritical region in the disordered phase. Although the effects of a misalignment of the field are less dramatic they are interesting and significant. In particular, misalignment may make it hard to verify the tangency of the λ lines at T_b in the $(T, H_\parallel)$ plane and to derive the crossover exponent ϕ that way from (5.36).

B. Transformation of the Hamiltonian

We consider the system described by the reduced Hamiltonian $\bar{\mathcal{H}}(\sigma_1, \sigma_2; \vec{s}_1, \vec{s}_2)$ of (3.1) in an external uniform field $\vec{h} = (h_\parallel, \vec{h}_\perp)$, where $\vec{h}_\perp$ is the component of the field perpendicular to the anisotropy axis. On shifting both the σ_2 and $\vec{s}_2$ variables by writing $\sigma_2 \to \sigma_2 + M_\parallel$ and $\vec{s}_2 \to \vec{s}_2 + M_\perp$, and choosing $M_\parallel$ and $\vec{M}_\perp$ to eliminate the linear field terms, the Hamiltonian can be written

$$\bar{\mathcal{H}} = \bar{\mathcal{H}}_0 + \bar{\mathcal{H}}_1 , \tag{6.1}$$

where $\bar{\mathcal{H}}_0$ is the Hamiltonian given in (3.3) except that the coefficients of the quadratic terms are replaced by

$$\begin{aligned} r_1^\parallel &= \bar{r}_1^\parallel + 2u_{12}M_\parallel^2 + 2w_{12}\vec{M}_\perp^2 , \\ r_2^\parallel &= \bar{r}_2^\parallel + 12u_{22}M_\parallel^2 + 2w_{22}\vec{M}_\perp^2 , \\ r_1^\perp &= \bar{r}_1^\perp + 2w_{21}M_\parallel^2 + 2v_{12}\vec{M}_\perp^2 , \\ r_2^\perp &= \bar{r}_2^\perp + 2w_{22}M_\parallel^2 + 4v_{22}\vec{M}_\perp^2 . \end{aligned} \tag{6.2}$$

In these relations $M_\parallel$ and $\vec{M}_\perp$, the components of the magnetization parallel to and perpendicular to the anisotropy axis, are give, for small $\vec{h}$, by

$$M_\parallel(r_2^\parallel + 4u_{22}M_\parallel^2) = h_\parallel , \quad r_2^\perp \vec{M}_\perp = \vec{h}_\perp . \tag{6.3}$$

The second part of the Hamiltonian is

$$\begin{aligned} \bar{\mathcal{H}}_1 = -\int d\vec{R}\, [& \bar{v}_{12}(\vec{M}_\perp \cdot \vec{s}_1)^2 + 4v_{22}(\vec{M}_\perp \cdot \vec{s}_2)^2 + 4w_{22}M_\parallel(M_\perp \cdot \vec{s}_2)\sigma_2 \\ & + \bar{w}_{12}M_\parallel(\vec{M}_\perp \cdot \vec{s}_1)\sigma_1 + 4v_{22}(\vec{M}_\perp \cdot \vec{s}_2)|\vec{s}_2|^2 + 2v_{12}(M_\perp \cdot \vec{s}_2)|\vec{s}_1|^2 \\ & + 2\bar{v}_{12}(\vec{M}_\perp \cdot \vec{s}_1)(\vec{s}_1\vec{s}_2) + 2w_{12}(\vec{M}_\perp \cdot \vec{s}_2)\sigma_1^2 + 2w_{22}(\vec{M}_\perp \cdot \vec{s}_2)\sigma_2^2 + \bar{w}_{12}(\vec{M}_\perp \cdot \vec{s}_1)\sigma_1\sigma_2] , \end{aligned} \tag{6.4}$$

Although this expression contains quadratic and cubic terms in the spin variables, no terms of third order in σ_1 or in $\vec{s}_1$ alone are developed by the new spin shifts. This is important since such terms would be relevant under the renormalization group discussed in Sec. III. Consequently, we expect that the Hamiltonian (6.1) will have the same stable fixed points for a range of values of $h_\parallel$ and $\vec{h}_\perp$.

The new complication which does appear in (6.4) is the bilinear coupling between those spin components $\vec{s}$, parallel to the anisotropy axis, and those σ, along the transverse field. This necessitates a preliminary diagonalization of the quadratic terms. Physically it corresponds to a canting of the spins which undergo ordering.

C. Case $n = 2$

Because of the complexity of the Hamiltonian we will first analyze the XY or planar case $n=2$, when $\vec{s}_1$ and $\vec{s}_2$ are simply scalar fields. On rewriting the Hamiltonian (6.1) in the form

$$\bar{\mathcal{H}} = -\int dR \sum_{i,\mu} \left(\tfrac{1}{2}(\nabla s_\mu^i)^2 + \tfrac{1}{2} \sum_j s_\mu^i r_\mu^{ij} s_\mu^j + \cdots \right), \tag{6.5}$$

where $i = \parallel, \perp$ and $\mu = 1, 2$, the matrices r_μ^{ij} are seen to be

$$r_1^{ij} = \begin{bmatrix} r_1^{\parallel} & \bar{w}_{12} M_\parallel M_\perp \\ \bar{w}_{12} M_\parallel M_\perp & r_1^{\perp} + 2\bar{v}_{12} M_\perp^2 \end{bmatrix}, \tag{6.6}$$

$$r_2^{ij} = \begin{bmatrix} r_2^{\parallel} & 4w_{22} M_\parallel M_\perp \\ 4w_{22} M_\parallel M_\perp & r_2^{\perp} + 8v_{22} M_\perp^2 \end{bmatrix}, \tag{6.7}$$

where we may recall that the parameters $\bar{w}_{12}$, $\bar{v}_{12}$, etc., are defined in (2.28) and (2.29). The r matrices have eigenvalues $\lambda_1^\pm$ and $\lambda_2^\pm$ given explicitly by

$$\lambda_1^\pm = \tfrac{1}{2}\{r_1^\parallel + r_1^\perp + 2\bar{v}_{12}M_\perp^2 \pm [(r_1^\parallel - r_1^\perp - 2\bar{v}_{12}M_\perp^2)^2 + 4\bar{w}_{12}^2 M_\parallel^2 M_\perp^2]^{1/2}\}, \tag{6.8}$$

$$\lambda_2^\pm = \tfrac{1}{2}\{r_2^\parallel + r_2^\perp + 8v_{22}M_\perp^2 \pm [(r_2^\parallel - r_2^\perp - 8v_{22}M_\perp^2)^2 + 64w_{22}^2 M_\parallel^2 M_\perp^2]^{1/2}\}, \tag{6.9}$$

and corresponding eigenvectors $y_\mu^\pm$.

Denoting the linear combinations of spin variables which diagonalize the quadratic parts of the Hamiltonain by $\sigma_\mu^\pm$, we can now write

$$\bar{\mathcal{H}} = -\int d\vec{R} \sum_{\mu=1,2;\, j=\pm} [\tfrac{1}{2}(\nabla\sigma_\mu^j)^2 + \tfrac{1}{2}\lambda_\mu^j(\sigma_\mu^j)^2 + O(\sigma^3)], \tag{6.10}$$

where $O(\sigma^3)$ denotes all the terms cubic and quartic in the σ_μ^j arising from (6.1). We do not explicitly display these terms since there are very many of them, most of which turn out to be strongly irrelevant under the renormalization procedure used.

From (6.8) and (6.9) it is clear that for any value of $M_\perp$ the eigenvalues λ_2^j are larger than the λ_1^j; thus all terms in the Hamiltonian of the form $\sigma_2^k \sigma_1^l$ with $k > l$ are strongly irrelevant and decay to zero as in Sec. III. Provided $h_\perp \neq 0$ (or equivalently $m_\perp \neq 0$), we also have $\lambda_1^+ > \lambda_1^-$, so that again the majority of the remaining terms are irrelevant. Finally we are left with a new reduced Hamiltonian of the form

$$\bar{\mathcal{H}}_{\text{red}} = -\int d\vec{R}\, [\tfrac{1}{2}(\nabla\sigma_1^-)^2 + \tfrac{1}{2}\lambda_1^-(\sigma_1^-)^2 + \tfrac{1}{2}\lambda_1^+(\sigma_1^+)^2 + \tfrac{1}{2}\lambda_2^-(\sigma_2^-)^2 + \tfrac{1}{2}\lambda_2^+(\sigma_2^+)^2 + (\sigma_1^-)^2(v_1\sigma_1^+ + v_2\sigma_2^+) + u_1(\sigma_1^-)^4], \tag{6.11}$$

in which v_1, v_2, and u_1 are complicated functions of the initial parameters known only for small $\vec{h}$, whose precise form is not, in any case, very informative. Evidently we may now integrate out the σ_1^+ and $\sigma_2^\pm$ fields, to obtain simply

$$\mathcal{H} = -\int d\vec{R}\, [\tfrac{1}{2}(\nabla\sigma_1^-)^2 + \tfrac{1}{2}\lambda_1^-(\sigma_1^-)^2 + \bar{u}(\sigma_1^-)^4], \tag{6.12}$$

with

$$\bar{u} = u_1 - v_1^2/2\lambda_1^+ - v_2^2/2\lambda_2^+ . \tag{6.13}$$

This, of course, is just the $n=1$ Ising-like Hamiltonian, whose analysis is well known.

At least for small values of $m_\perp$ it is fairly easy to verify from (6.13) that $\bar{u}$ is positive; however, it is not so clear that $\bar{u}$ remains positive as $h_\perp$ is increased arbitrarily. (A negative $\bar{u}$, of course, indicates the possibility of tricritical behavior.[31,42]) Nevertheless, for our purposes we may assume this is the case since all phase boundaries close to the $h_\perp = 0$ bicritical point should remain continuous with no tricritical points in the immediate vicinity. However, the evaluation of $\bar{u}$ being still restricted to $h_\parallel$ small, means that our analysis does not completely exclude the possibility of confluent tricritical points.

Now, for a set of initial interaction parameters for which $\bar{u} > 0$, the Hamiltonian (6.12) has Ising-like critical behavior. We conclude that the phase boundary $T = T_c(h_\parallel, h_\perp)$ is Ising-like everywhere with $(n=1)$ critical exponents, except for a single $(n=2)$ point in the $h_\perp = 0$ plane, namely, the spin-flop bicritical point with XY-like exponents. For fixed $T < T_b$, the phase boundary should be a smooth curve in the $(h_\parallel, h_\perp)$ plane.

Very close to the spin-flop point, the analysis of Sec. V can be adapted to investigate the shape of the phase boundary in the vicinity of the bicritical point for the Hamiltonian (6.10). From the general discussion of anisotropy crossover,[34-37] the change in the critical temperature at fixed $h_\parallel = h_b$ (the bicritical value) when a small trans-

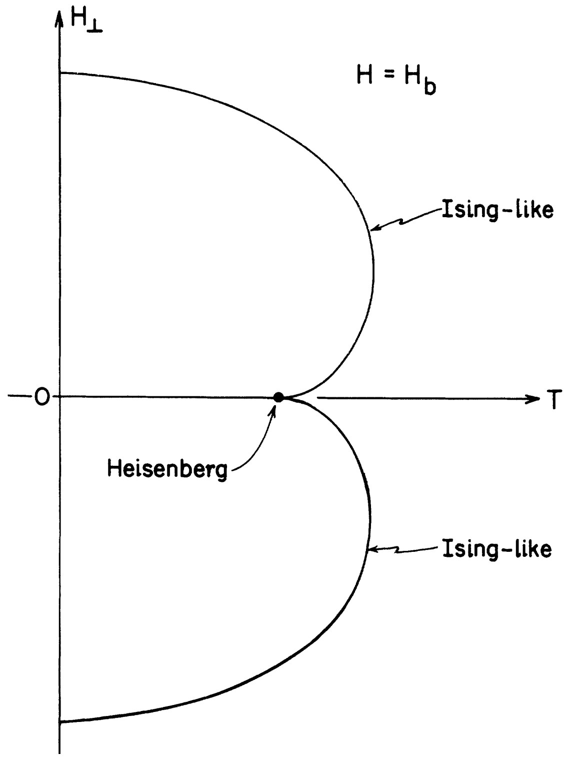

FIG. 6. Cross section of the critical surface with $H_{\parallel}=H_b$.

verse field is applied should vary as

$$T_c(h_\perp)-T_c(h_\perp=0)\sim(\lambda_1^+-\lambda_1^-)^{1/\phi}\ , \qquad (6.14)$$

where ϕ is the $n=2$ crossover exponent. Since we have $r_1^{\parallel}=r_1^{\perp}$ at the flop point, we easily find from (6.2), (6.3), (6.8), and (6.9) that

$$\lambda_1^+-\lambda_1^-\sim|M_\perp|\sim|h_\perp| \qquad (6.15)$$

as $h_\perp\to 0$. Since $\phi>1$ (as before) the phase boundary in the $h_\parallel=h_b$ plane is therefore tangent to the line $h_\perp=0$ as shown in Fig. 6. From Sec. V we know that, in the $(h_\parallel, T)$ plane, the phase boundary is tangent to the flop line $T_\phi(h_\parallel)$; by continuity, the $(T, h_\parallel, h_\perp)$ boundary surface has an isolated cusp-like singularity at which it is tangent to the flop line and from which it deviates as $|\vec{h}-\vec{h}_b|^{1/\phi}$, where $\vec{h}_b=(h_b, 0)$ is the value of the external field vector at the bicritical point.

D. General case

A similar analysis may be performed when the transverse spin variables $(\vec{s}_1, \vec{s}_2)$ have more than one component. However we must now distinguish between that transverse component parallel to $\vec{h}_\perp$ and the other $n-2$ perpendicular components. Since we have assumed isotropy in the transverse spins, we may, by rotational invariance, always suppose that $\vec{h}_\perp$ is directed along the first component of $\vec{s}_1$.

The same diagonalization as for the $n=2$ case may then be performed. After following the renormalization procedures of Sec. III and integrating out the irrelevant spin variables σ_2^i, etc., the Hamiltonian finally becomes

$$\mathcal{H}_{\rm red}=-\tfrac{1}{2}\int d\vec{R}\,[(\nabla\sigma_1^+)^2+(\nabla\sigma_1^-)^2+(\nabla\vec{s}_1^{\,\perp})^2+\lambda_1^+(\sigma_1^+)^2+\lambda_1^-(\sigma_1^-)^2+r_1^\perp|\vec{s}_1^{\,\perp}|^2+O(s^4)]\ , \qquad (6.16)$$

where now $\vec{s}_1^{\,\perp}$ denotes the $(n-2)$-component spin vector orthogonal to $\vec{h}_\perp$. We will again assume that the coefficients of the four spin interactions are all positive. Now this Hamiltonian has the same structure as the general quartic Hamiltonian (5.1) so that an identical analysis applies.

There are three possible relationships which may obtain between the quartic coefficients in (6.16), namely: (i) $\lambda_1^+=\lambda_1^-=r_1^\perp$, (ii) $\lambda_1^+>\lambda_1^-=r_1^\perp$, and (iii) $\lambda_1^+>\lambda_1^-\neq r_1^\perp$. By (6.8), the first case is possible only for $\vec{M}_\perp=0$ and $T=T_b(h_\parallel)$; this corresponds to the $\vec{h}_\perp=0$ bicritical point discussed in detail in Sec. V. One subset of case (ii) has also been discussed, namely, the phase boundary in the $M_\perp=0$ plane. The two possibilities in (iii) are $\lambda_1^-<r_1^\perp$, corresponding to Ising-like critical surfaces, and $\lambda_1^->r_1^\perp$, describing $n-2$ critical behavior. We must investigate for what ranges of initial parameters these two alternatives are realized. It is trivial to show that $\lambda_1^-\lessgtr r_1^\perp$ according to whether

$$[2\bar{v}_{12}(r_1^\parallel-r_1^\perp)-\bar{w}_{12}^2M_\parallel^2]M_\perp^2\lessgtr 0\ . \qquad (6.17)$$

Clearly, for $r_1^\parallel<r_1^\perp$, the left side is always negative. Furthermore, in the vicinity of the bicritical point we have $r_1^\parallel-r_1^\perp=O(M_\perp^2)$, so that it is again negative. Thus we conclude that the critical surface in the $(T, h_\parallel, h_\perp)$ space is Ising-like in the neighborhood of the bicritical point for $|\vec{h}_\perp|>0$.

In reaching this conclusion we have made the tacit assumption that the irrelevant variables do not, under renormalization-group iteration, cause a crossing of the renormalized coefficients $r_1^\perp$ and λ_1^-. If the initial values satisfy $\lambda_1^-(l=0)<r_1^\perp(l=0)$ we presume that this inequality is maintained for all l, in order that the effective Hamiltonian with all irrelevant variables neglected may be discussed in terms of the initial interaction parameters. As with other such global questions we have been unable to decide the point; it deserves further study.

A mean-field type of analysis suggests that, for a uniaxial antiferromagnet in which the j inequalities (2.21) hold, the left side of (6.17) is negative everywhere; it would follow that for $h_\perp \neq 0$, the critical surface would be Ising-like everywhere. However, from this analysis again one cannot exclude the possibility that the sign of the inequality is reversed by the irrelevant variables. Thus, at a finite distance from the flop point, there is the possibility of an $n-1$ critical line separating Ising-like and $n-2$ critical surfaces, although this seems unlikely to be realized in nature. Indeed an $n-1$ critical line would have to satisfy, for finite $M_\perp^2$, the condition

$$2\bar{v}_{12}(r_1^\parallel - r_1^\perp) = \bar{w}_{12}^2 M_\parallel^2 \; . \tag{6.18}$$

Some straightforward algebra using the initial coefficients (2.25)–(2.29) shows that, for reasonable values of these parameters, one cannot, with fixed $M_\parallel^2$, find a positive solution for $M_\perp^2$. The implications is again that an $n-1$ critical line is unlikely to exist.

The critical surface will have the same cusplike singularity at the flop point found for the $n=2$ case. Near the $n-1$ critical line in the $h_\perp = 0$ plane, we find from (6.17) that

$$r_1^\perp - \lambda_1^- \sim M_\perp^2 \sim h_\perp^2 \; , \tag{6.19}$$

for small $h_\perp$. Thus the effective crossover exponent is $\frac{1}{2}\phi$ and the critical surface comes in perpendicular to the $h_\perp = 0$ line at fixed T. The general appearance of the critical surface in the $(T, h_\parallel, h_\perp)$ plane is thus as sketched in Fig. 7. Similar considerations show that, if an $n-1$ critical line exists for finite $h_\perp$, the $n-2$ and Ising-like critical surfaces meet smoothly with a tangency exponent $\tilde{\psi}$ exceeding unity.

We see from this analysis that in an experiment designed to investigate the spin-flop bicritical point, very careful alignment of the field along the anisotropy axis is required. Otherwise, for $h_\perp \neq 0$ the critical surface is Ising-like with associated Ising-Heisenberg crossover effects (assuming that the flop point corresponds to the isotropic fixed point, as it should for real systems with $n \leq 3$). Thus, the measured exponents may be either Ising-like or lie between the Ising and Heisenberg values because of the crossover effects. Moreover, because of the cusplike singularity in the critical surface at the bicritical point, measurements of T_b will be sensitive to precise field alignment. However, the system provides a rich, experimentally accessible range of critical behavior in the $(T, h_\parallel, h_\perp)$ space even in the disordered phase, which (for $n=3$) displays Heisenberg, XY, and Ising-like critical behavior in different parts of the critical surface as shown in Fig. 7.

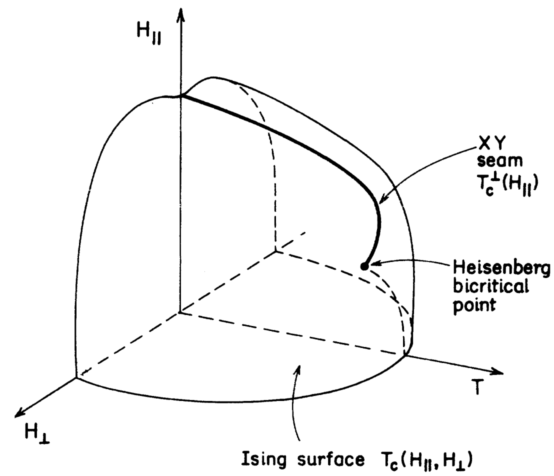

FIG. 7. Critical surface of an $n=3$ uniaxial antiferromagnet in the full $(T, H_\parallel, H_\perp)$ space.

VII. EFFECTS OF ORDERING FIELDS

In this concluding section we discuss the effects of ordering fields on the critical surfaces. In particular we carry out an analysis similar to that in Sec. V but with applied fields (i) $h_\parallel$ and $h_\parallel^\dagger$ and (ii) $h_\parallel$ and $h_\perp^\dagger$, and concentrate on the disordered or single-phase region. In case (i), we make the shifts $\sigma_2 \to \sigma_2 + M_\parallel$ and $\sigma_1 \to \sigma_1 + M_\parallel^\dagger$ and obtain

$$\tilde{\mathcal{H}} = \bar{\mathcal{H}}_0 + \bar{\mathcal{H}}_1, \tag{7.1}$$

where $\bar{\mathcal{H}}_0$ is again the Hamiltonian of (3.3) except that the coefficients of the quadratic terms are this time given by

$$\begin{aligned}
r_1^\parallel &= \bar{r}_1^\parallel + 12u_{11}M_\parallel^{\dagger 2} + 2u_{12}M_\parallel^2 , \\
r_2^\parallel &= \bar{r}_2^\parallel + 2u_{12}M_\parallel^{\dagger 2} + 12u_{22}M_\parallel^2 , \\
r_1^\perp &= \bar{r}_1^\perp + 2w_{11}M_\parallel^{\dagger 2} + 2w_{21}M_\parallel^2 , \\
r_2^\perp &= \bar{r}_2^\perp + 2w_{12}M_\parallel^{\dagger 2} + 2w_{22}M_\parallel^2 ,
\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}
\bar{\mathcal{H}}_1 = -\int d\vec{R}\, [\, & 4u_{12}M_\parallel M_\parallel^\dagger \sigma_1\sigma_2 + \bar{w}_{12}M_\parallel M_\parallel^\dagger \vec{s}_1 \cdot \vec{s}_2 \\
& + 4u_{11}M_\parallel^\dagger(\sigma_1)^3 + 2w_{11}M_\parallel^\dagger\sigma_1(\vec{s}_1\cdot\vec{s}_1) + 2w_{12}M_\parallel^\dagger\sigma_1(\vec{s}_2\cdot\vec{s}_2) \\
& + 2u_{12}M_\parallel^\dagger\sigma_1(\sigma_2)^2 + \bar{w}_{12}M_\parallel^\dagger\sigma_2(\vec{s}_1\cdot\vec{s}_2)] ,
\end{aligned} \tag{7.3}$$

with $M_\parallel$ and $M_\parallel^\dagger$ being defined as the solutions of

$$M_\parallel^\dagger[\bar{r}_1^\parallel + 4u_{11}(M_\parallel^\dagger)^2 + 2u_{12}M_\parallel^2] = h_\parallel^\dagger , \tag{7.4}$$

$$M_\parallel[\bar{r}_2^\parallel + 4u_{22}M_\parallel^2 + 2u_{12}(M_\parallel^\dagger)^2] = h_\parallel . \tag{7.5}$$

In $\tilde{\mathcal{H}}$ we now have bilinear couplings between the pairs (σ_1, σ_2) and $(\vec{s}_1, \vec{s}_2)$ so that a diagonalization similar to that of Sec. VI must be performed. We also note that in (7.3) there is a term cubic in σ_1 which is *relevant* under the renormalization group of Sec. III. Thus, we can reach a fixed point corresponding to a transition in the parallel component only if $\boldsymbol{M}_\parallel = 0$; this is, of course, just as expected. If we work in a region of the $(T, h_\parallel, h_\parallel^\perp)$ space in which $r_1^\parallel < r_1^\perp$ we will find no transition, exactly as in a conventional magnet in an ordering field. Conversely, if one is in a region where $r_1^\parallel > r_1^\perp$, there is the possibility of a transition. Now, according to the discussion of Sec. III, we can treat the σ_1 spins as irrelevant variables. Furthermore, there are no terms cubic in $\vec{s}_1$ (which would be relevant). Thus, as usual, there is only a single relevant field in the problem, namely, $r_1^\perp$. Even after diagonalization of the $\vec{s}_1, \vec{s}_2$ quadratic form to eliminate the cross term, inspection of (7.3) shows that no such relevant terms can develop. Thus there will be a continuous order transition to a state where the n − 1 transverse components order. A study of the eigenvalues shows that, as $H_\parallel^\dagger \to 0$ the phase boundary, at constant $H_\parallel$, meets the critical line in the $(T, H_\parallel)$ at right angles, as indicated in Fig. 8. Of course, this analysis assumes as before that, after integrating out the irrelevant spin components, the coefficients of the quartic term remain positive.

Should the quartic coupling term be driven *negative* by the imposition of a strong enough field $h_\parallel^\dagger$, the "balloon like" critical surface for $r_1^\parallel > r_1^\perp$ will terminate in a symmetric pair of *lines of tricritical points* (see Fig. 8). The transition surface bounding the (n − 1)-ordered state becomes first order in character on the other side of this tricritical line. A mean-field-theory calculation (for *fixed* length spins) by Khajehpour, Wang, and Kromhout[43] does, in fact, explicity produce these lines of tricritical points, which then terminate at the bicritical point.[43] A similar phase diagram was constructed by Chang *et al.*,[10] who applied a homogeneity hypothesis to a somewhat simpler model situation.

Renormalization-group arguments can be given to show that these lines of tricritical points are also present in the continuous spin model treated here and run into the bicritical points. Consider the Hamiltonian (3.1), but with an additional field term $h_\parallel^\dagger \sigma_1$ added. Suppose that this Hamiltonian is investigated by first renormalizing away the imposed *uniform* field $h_\parallel$, *without* shifting the σ_2 spin by $M_\parallel^\dagger$. One is then left with a Hamiltonian of the form (3.7), but with a field $h_\parallel^\dagger$ coupling to the σ spins. In the case n = 2, this Hamiltonian is identical to an effective Hamiltonian arising in a

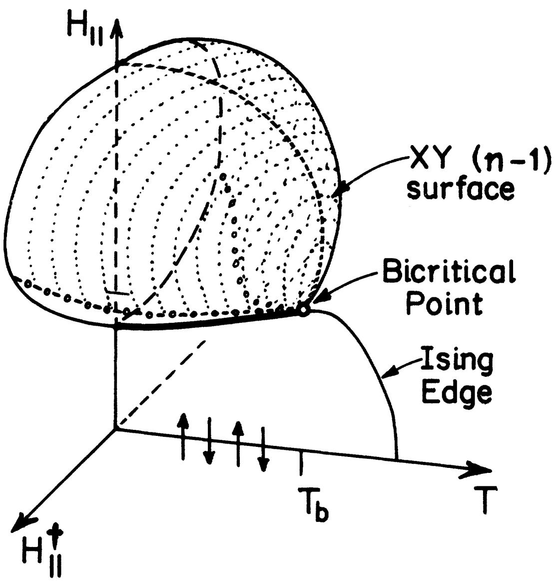

FIG. 8. Phase diagram of an $n = 3$ uniaxial antiferromagnet with both $\boldsymbol{H}_\parallel$ and $H_\parallel^\dagger$ applied. The *XY* transition surface is divided by a line (−o−o−) of tricritical points into regions of first order (clear) and continuous (stipled) transitions.

study of metamagnetic tricritical behavior.[32] The analysis presented there can be applied directly, provided that the anisotropy $g \equiv r_\parallel - r_\perp$ is positive and of order unity.[32] The result is that the Hamiltonian (3.7) *with* the field term $h_\parallel^\dagger \sigma$, is equivalent to an Ising-like Hamiltonian with a renormalized quadratic coupling term, *and* an effective quadratic coupling given by

$$u_{\rm eff} \approx u - A h_\parallel^{\dagger 2}, \tag{7.6}$$

where A depends weakly on the various coupling constants in (3.7).[32] Thus, tricritical points will occur at large values of g when the renormalized quadratic coupling is zero, and $h_\parallel^\dagger$ is strong enough to drive (7.6) through zero.

The analysis of Ref. 32 may readily be extended to the case of small g (with n = 2). One simply first iterates the recursion relations (5.6)–(5.10) for (3.7) until g is large. The field $h_\parallel^\dagger$ obeys the simple recursion relation $(h_\parallel^\dagger)' = b^{1+d/2} h_\parallel^\dagger$ during this process, and the difference g grows according to $g' = b^{\lambda_g} g$, where[34,35] $\lambda_g = 2 - \frac{1}{5}\epsilon$. If b is chosen so that $g' = O(1)$, the analysis of Ref. 32 can be carried out for the "partially renormalized" parameters g', $(h_\parallel^\dagger)'$, etc. Then (7.6) becomes

$$u_{\rm eff} \approx u' - A(h_\parallel^\dagger / g^{\Delta/\phi})^2, \tag{7.7}$$

where

$$\Delta/\phi \equiv \varphi_\parallel = \tfrac{3}{2} - \tfrac{1}{10}\epsilon + O(\epsilon^2). \tag{7.8}$$

Thus, the field in which the transition becomes tricritical goes to zero as $g^{\phi_\parallel}$ when g vanishes, and the lines of tricritical points terminate at the bicritical point as indicated by mean-field theory,[43] but with a geometry determined by the nonclassical exponent $\varphi_\parallel$. The analysis sketched above can be extended readily to general n, and the conclusions are the same. The combination $h_\parallel^\dagger/g^{\Delta/\phi}$ in (7.7) with Δ and ϕ given by their bicritical, i.e., isotropic n-Heisenberg values, is readily seen to be appropriate from the scaling formulation Ref. 16. The coefficient of ϵ in (7.8) is thus more generally $(n+2)/4(n+8)$ and the coefficients for ϵ^2 and ϵ^3 could also be quoted.[4]

An identical analysis may be performed for case (ii) with $h_\parallel$ and $h_\perp^\dagger$ nonzero. Shifts $M_\parallel$ and $\vec{M}_\perp^\dagger$ are performed on σ_2 and on one perpendicular component of $\vec{s}_1$. This leads to a term in the Hamiltonian of the form $4v_{11}(\vec{M}_\perp^\dagger\cdot\vec{s}_1)|\vec{s}_1|^2$ which is now relevant; thus no transition occurs in these spin components. However, there is an extra complication when $\vec{s}_1$ has two or more components, in that there is likely to be Gaussian critical behavior in $n-2$ of these below T_c when $h_\perp^\dagger=0$ and $h_\parallel \geq h_b$. Indeed, because we have assumed that the $n-1$ components of $\vec{s}_1$ and $\vec{s}_2$ are coupled isotropically one has a situation similar to the isotropic Heisenberg model below T_c, where a spin-wave analysis leads to a divergent longitudinal susceptibility as $H\to 0$.[44] We have not investigated this in the present context, but by considering only the region $h_\parallel < h_b$, where the parallel components order, we can easily see that there is a critical surface in the $(T, h_\parallel, h_\perp^\dagger)$ space with Ising-like exponents. Again, the balloonlike surface is expected to terminate in a line of tricritical points, in agreement with the results of mean-field theory.[43] The phase diagram in this space is thus shown in Fig. 9.

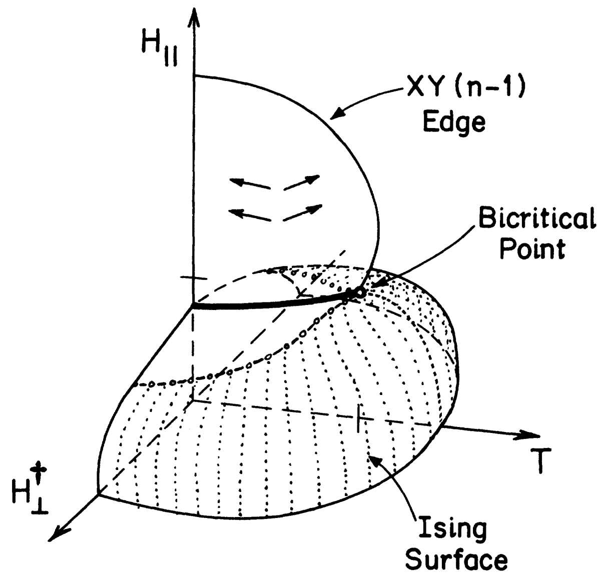

FIG. 9. Phase diagram of an $n=3$ uniaxial antiferromagnet in the presence of fields $H_\parallel$ and $H_\perp^\dagger$. Here, the *Ising* transition surface is divided by a line (–o –o –) of tricritical points into regions of first order (clear) and continuous (stipled) transitions.

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[12] Of striking experimental interest is the recent "polycritical point" discovered in the (p, T) diagram of liquid ^{3}He at $p_0 \simeq 22$ bar, $T_0 \simeq 2.4$ mK, by D. N. Paulsen, R. T. Johnson, and J. C. Wheatley, Phys. Rev. Lett. 30, 829 (1973), and D. N. Paulson, H. Kojima, and J. C. Wheatley, Phys. Rev. Lett. 32, 1098 (1974). Similarly, in the earlier discovery of the A transition point in ^{3}He along its melting line [D. D. Osheroff, R. C. Richardson, and D. M. Lee, Phys. Rev. Lett. 28, 885 (1972); W. J. Gully, D. D. Osheroff, D. T. Lawsen, R. C. Richardson, and D. M. Lee, Phys. Rev. A 8, 1633 (1973)] it is found that the transition "splits" in a magnetic field H, corresponding, so it is believed, to separate ordering of the "up" and "down" Fermi spheres: In the (H, T) plane the A point in zero field is thus also a multicritical point.

[13] The attractive term "supersolid" for a phase with both diagonal (solid) and off-diagonal (superfluid) ordering seems to have been coined independently by T. Tsuneto and H. Matsuda [*Proceedings of the 12th International Conference on Low Temperature Physics*, edited by E. Kanda (Keigaku, Kyoto, Japan, 1971), p. 145; Prog. Theor. Phys. Suppl. No. 46, 41 (1970)] and by W. J. Mullin [Phys. Rev. Lett. 26, 611 (1971)]. However, it has been objected to on the gronnds that such a supersolid would not be "more solid than diamond": But the meaning of the prefix "super" is "going beyond, transcending," as in "supernatural," "superscript," etc.

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