Renormalization-group methods for critical dynamics: II. Detailed analysis of the relaxational models

B. I. Halperin
Bell Laboratories, Murray Hill, New Jersey 07974

P. C. Hohenberg
Bell Laboratories, Murray Hill, New Jersey 07974
and Physik Department, Technische Universität München, 8046 Garching, West Germany

Shang-keng Ma*
Department of Physics, University of California, San Diego, La Jolla, California 92037
(Received 12 December 1975)

The relaxational models introduced in a previous treatment of critical dynamics are studied in detail using renormalization-group methods. The earlier results are justified by an analysis to all order in $\epsilon = 4 - d$, where d is the dimensionality. The diagrammatic formalism of the full dynamic renormalization group is presented, and applied to the earlier models. A generalization of Wilson's Feynman-graph expansion method is used to calculate the exponents to second order in ϵ. In model C, where a nonconserved order parameter is coupled to a conserved energy field, ambiguities were found in the earlier recursion-relation treatment for $2 < n < 4$, $d \approx 4$ (n is the number of components of the order parameter). These ambiguities are discussed in the present work, but are not fully resolved.

I. INTRODUCTION

In a previous paper[1] (hereafter referred to as I), the renormalization-group method for critical dynamics was applied to a number of purely relaxational models. Results were presented for dynamic critical exponents, and various universality classes were found depending on the dimensionality d, the number of components n of the order parameter, and the conservation laws in the system. The present paper details the renormalization-group analysis which led to the results of I, and in addition presents the Feynman-graph expansion of the critical exponents to order ϵ^2 for model C, in which the nonconserved order parameter couples to a conserved energy field. This expansion displays singular behavior in the limit where the transport coefficient λ_E for the energy vanishes, leading to ambiguities in the critical behavior for $2<n<4$, $d \to 4$, which we have not resolved. The principal results of our analysis are the proof of scaling relations to all orders in ϵ, and the determination of some of the boundaries of regions with different dynamical behavior for model C, near $d=4$. We also calculate a physical amplitude ratio of characteristic frequencies for the energy and the order parameter, to linear order in $\epsilon = 4-d$. In Sec. II recursion relations valid to lowest order in $\epsilon = 4-d$ are written down in a somewhat more general form than in I, and their fixed-point behavior is determined. Section III presents the diagrammatic formalism used to justify the recursion-relation analysis to all orders in ϵ. In Sec. IV the dynamic renormalization group is studied for case C, and the different regions are determined. It is shown that the first-order recursion relations studied in I and in Sec. II become invalid in the domain $2<n<4$, near $d=4$, where $\lambda_E \to 0$. Section V describes the dynamics of the energy field for the different models studied, while Sec. VI contains the second-order ϵ expansion for case C. Comparison with other renormalization-group treatments of relaxational models is made in Sec. VII.

II. RECURSION RELATIONS

In I we defined one-field models (cases A and B), and two-field models with conserved energy fields (C and D). We shall generalize the discussion slightly by introducing two-field models with *nonconserved* energy fields, which we call A′ and B′. The defining equations of A′ and B′ are the same as for C and D, respectively [Eqs. (I2.15)–(I2.27)], except that we replace $-\lambda_0^E \nabla^2$ by Γ_0^E, in (I2.26) and (I2.27).

Recursion relations valid to first order in ϵ for the static coefficients are the same for cases A′, B′, C, and D, and are given by Eqs. (I4.5)–(I4.8), except that (I4.8) contains several misprints. The correct form is

$$\gamma_{l+1} = b^{d-2a-a_E}[\gamma_l - 4(n+2)u_l\gamma_l B \ln b - 2n\gamma_l^3 C_l B \ln b]\,. \tag{2.1}$$

Recursion relations for the dynamic coefficients of cases A and C were given by Eqs. (I4.21), (I4.23),

and (I4.25). (We shall see below, however, that the equations for model C are valid to first order in ϵ only if $\mu_l = \lambda_l^E/\Gamma_l C_l$ is not zero.)

It is convenient to restate the recursion relations for the dynamic parameters of all the cases under consideration (where applicable) in a somewhat more general form:

$$1/\lambda_{l+1} = (1/\lambda_l) b^{d-2a+2-z}, \tag{2.2}$$

$$1/\Gamma_{l+1} = (1/\Gamma_l) b^{d-2a-z} + \gamma_l^2 I_1 + O(\epsilon^2), \tag{2.3}$$

$$1/\lambda_{l+1}^E = (1/\lambda_l^E) b^{d-2a_E+2-z}, \tag{2.4}$$

$$1/\Gamma_{l+1}^E = (1/\Gamma_l^E) b^{d-2a_E-z} + \gamma_l^2 J_1 + O(\epsilon^2), \tag{2.5}$$

where $\gamma_l^2 I_1$ and $\gamma_l^2 J_1$ are the contributions from the diagrams in Figs. 1(a) and 1(b) of I, written down in Eqs. (I3.27) and (I3.28). There are, however, a number of errors in (I3.27), so we reproduce the correct equation here:

$$\Sigma_a(k,\omega) = -4\gamma_0^2 \int \frac{d^d p}{(2\pi)^d} \left(\frac{C_0}{p^2+r_0}\right) \times \left(\frac{\Gamma_0(p^2+r_0) + (\lambda_0^E/C_0)(p-k)^2}{-i\omega + \Gamma_0(p^2+r_0) + (\lambda_0^E/C_0)(p-k)^2}\right). \tag{2.6}$$

We note that z is equal to $4+O(\epsilon)$ when ψ is conserved, and $z = 2+O(\epsilon)$ when ψ is not conserved, while $a_E = 2 + O(\epsilon)$. Since the exponent of b in Eq. (2.5) is $\gtrsim 2$ in all cases, the coefficient $1/\Gamma_l^E$ is strongly irrelevant in all cases. If $\gamma_0 \neq 0$, and the specific-heat exponent α is positive, then γ_l^2 approaches a finite fixed-point value of order ϵ. [We choose C_l to be $O(1)$.] We see that for any finite value of $1/\Gamma_0^E$, including $1/\Gamma_0^E = 0$, we have for large l,

$$1/\Gamma_l^E \to \text{const} \times \epsilon/\Gamma_l, \quad \text{case A}', \qquad \to \text{const} \times \epsilon/\lambda_l, \quad \text{case B}'. \tag{2.7}$$

If $\alpha < 0$, then $1/\Gamma_l^E \to 0$ for large l.

We may further note that I_1 vanishes in the limit $1/\Gamma_l^E \to 0$. Hence the second term on the right-hand side of (2.3) is of order ϵ^2, for large l, in model A′, and the recursion relation for $1/\Gamma_{l+1}$ is the same in that case as for model A, $\gamma_0 = 0$, to first order in ϵ. In fact the two models are identical in their critical behavior to all orders in ϵ. Indeed, since $1/\Gamma_l^E$ is irrelevant, we may set $1/\Gamma_0^E = 0$ to begin with. In this case the interaction between order-parameter fluctuations via the energy field is instantaneous in time, and the order-parameter response functions in model A′ are then exactly the same as for model A. Furthermore, when $1/\Gamma_0^E = 0$, we have for the energy response in case A′, the exact relation

$$\chi_E(k,\omega) = C_0 + D_1(k,\omega), \tag{2.8}$$

where $D_1(k,\omega)$ is the response function for the variable $\gamma_0 C_0 |\psi(r)|^2$ in case A. Equation (2.8) applies asymptotically for the critical behavior when $1/\Gamma_0^E$ is finite. The energy relaxation will be discussed further in Sec. V below.

In a similar manner we may show that cases B and B′ have identical critical behaviors. Thus, the addition of a *non-conserved* energy field makes no difference to the dynamic critical behavior. In case D, we find that $1/\lambda_l^E$ is a strongly irrelevant variable which approaches zero as $l \to \infty$. Again it is clear that we could have set $1/\lambda_0^E = 0$ in the first place, which gives an instantaneous interaction between fluctuations of the order parameter, so that the order-parameter correlation functions in model D are identical in the critical region to those in model B. The energy response functions are also identical for models B and D if we consider the limit $T \to T_c$, with k/κ fixed. If we let $k \to 0$ for any fixed finite value of κ, however, the energy will eventually change to a diffusive behavior, with

$$\omega_E(k) = \lambda_0 k^2/C \propto \kappa^{\tilde{\alpha}/\nu} k^2. \tag{2.9}$$

The crossover between the two behaviors occurs for k/κ of order κ/Λ.

III. DIAGRAMMATIC FORMALISM AND RENORMALIZATION GROUP TO ALL ORDERS IN ϵ

A. General formulation

In order to discuss the renormalization group properly, we shall first define a diagrammatic formalism. This formalism must be capable of describing equations of motion more complicated than the simple Markoffian forms (I2.11), (I2.25), and (I2.26), since these will arise in the intermediate stages of the renormalization group beyond first order in ϵ. We have chosen to use a formalism similar to that employed by Tucker and Halperin,[2] although a number of other methods may be used equally well—e.g., the formalism of Martin, Siggia, and Rose,[3] which was used to study planar models,[4] or a number of other methods.[5,6]

We shall begin by treating the single-field case appropriate to model A. The general equation of motion to be considered will be described by a "bare" propagator G_l and a set of vertices $U_l^{(2)}$, $U_l^{(4)}$, $U_l^{(6)}$, ... The propagator G_l is represented diagrammatically by a directed line, carrying a wave vector k, a spin index α, and a frequency ω, and is assigned the value

$$G_l = [(-i\omega/\Gamma_l) + r_l + k^2]^{-1}. \tag{3.1}$$

Note that G_l is the frequency Fourier transform of a retarded function $G_l(k,t)$, and the direction of the

arrow represents forward propagation in time.

The vertex of degree m, $U_l^{(m)}$, is represented diagrammatically by a dotted line with an even number m of distinct positions for attaching propagators, each of which may originate at the vertex (propagating forward in time from the given vertex), or may end at the vertex (having originated at some earlier time). The value of each vertex must be represented by a set of functions

$$U_l^{(m)} \equiv U^{(m)}_{\sigma_1\sigma_2\cdots\sigma_m}(k_1,\omega_1,k_2,\ldots,k_m,\omega_m) \times \delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\cdots\delta_{\alpha_{m-1}\alpha_m} \times (2\pi)^d\delta^d\left(\sum_{i=1}^m \sigma_i k_i\right)(2\pi)\delta\left(\sum_{i=1}^m \sigma_i\omega_i\right). \quad (3.2)$$

Here k_i and ω_i are the wave vector and frequency of the propagator attached to the ith site, and the index σ_i is equal to ± 1, according to whether the attached propagator originates ($\sigma_i = +1$) or ends ($\sigma_i = -1$) at the vertex. Note that the δ functions in (3.2) assure conservation of wave vector and frequency at the vertex.

B. Bare perturbation expansion

For the simple Markoffian starting equations of motion (I2.11), the only nonzero vertices are the "noise vertex"

$$U^{(2)}_{+-} = \Gamma_0^{-1}, \quad (3.3)$$

and the interaction vertices

$$U^{(4)}_{+---} = U^{(4)}_{-+--} = U^{(4)}_{--+-} = U^{(4)}_{---+} = u_0. \quad (3.4)$$

(These vertices are all independent of frequency and wave vector.) Furthermore the bare propagator is given by (3.1) with the values r_0 and Γ_0 equal to the corresponding constants in the equations of motion.

The frequency-dependent susceptibility $\chi_\psi(k,\omega)$ is given formally by the sum of all topologically distinct connected diagrams that can be formed with one incoming external line and one outgoing external line. All intermediate propagators are to be integrated with weight $d^dk'\,d\omega'/(2\pi)^{d+1}$, and summed over spin index, with frequencies running from $-\infty$ to ∞, and wave vectors running from 0 to Λ. The correlation function $C_\psi(k,\omega)$ is similarly given by the sum of all connected diagrams with two outgoing lines, carrying k,ω and $-k,-\omega$, respectively. We may define a self-energy $\Sigma(k,\omega)$ in the usual way as minus the sum of all irreducible contributions to χ_ψ, without the external legs, and we may write

$$G(k,\omega)^{-1} \equiv \chi_\psi(k,\omega)^{-1} = G_0(k,\omega)^{-1} + \Sigma(k,\omega). \quad (3.5)$$

Similarly, we may identify a *complete* (dressed) four-point interaction $\overline{U}^{(4)}_{+---}$, in terms of the sum of all connected diagrams with three incoming and one outgoing external lines, divided by the values of the external propagators.

In drawing diagrams in terms of the bare vertices, we shall make use of a number of simplifying conventions. In accord with traditional usage, we shall draw the interaction vertex $U^{(4)}$ as a single point, rather than a dotted line with four distinct sites. Also we shall not indicate the direction of the propagators. This leads to a vast reduction in the number of diagrams to be drawn, at the price of introducing combinatorial factors whose verification will be left to the reader. As a further simplification we shall not indicate the locations of the noise vertices $U^{(2)}_{++}$. It should be noted that at most one noise vertex can occur on any propagating line, between two interaction vertices. Furthermore, the insertion of a noise vertex converts the value of a line from the bare propagator $G_0(k,\omega)$ to the bare "correlation function"

$$C_0(k,\omega) = 2G_0(k,\omega)\Gamma_0^{-1}G_0(-k,-\omega) = 2\omega^{-1}\,\mathrm{Im}G_0(k,\omega). \quad (3.6)$$

Thus the values of diagrams quoted in the text represent the sum over all possible ways noise vertices may be inserted in the lines. It can be shown that the evaluation of diagrams may be accomplished by using the rules of finite-temperature perturbation theory in quantum statistics (the Matsubara formulation), and letting $\hbar \to 0$ at the end of the calculation.[5,6]

C. Renormalized perturbation expansion

At the lth stage of the renormalization procedure, we define the constants r_l, Γ_l, and u_l as

$$r_l = \lim_{\omega,k\to 0}\{G_0^{-1}(k,\omega) + \Sigma(k,\omega)\}_l, \quad (3.7)$$

$$\frac{1}{\Gamma_l} = \lim_{\omega,k\to 0}\frac{i\partial}{\partial\omega}\{[G_0^{-1}(k,\omega) + \Sigma(k,\omega)]\}_l, \quad (3.8)$$

$$u_l = \lim_{\mathrm{all}\,\omega_i,k_i\to 0}\{\overline{U}^{(4)}_{+---}\}_l, \quad (3.9)$$

where the symbol $\{\ \}_l$ indicates that all intermediate wave vectors in the diagrams are to be integrated from Λ/b^l to Λ, and that the external frequencies and wave vectors, as well as the overall magnitude of G_0, Σ, and $U^{(m)}$ must then be rescaled according to the operation $(R_b^s)^l$ [see Eqs. (I4.3a) and (I4.3b)]. The rescaling parameter a is chosen such that

$$\lim_{k,\omega\to 0}\frac{\partial}{\partial k^2}\{G_0^{-1} + \Sigma\}_l = 1. \quad (3.10)$$

In general we define vertices $U_l^{(m)}$ from the diagrammatic expansions of the dressed m-point interaction according to

$$U^{(m)}_{l;\sigma_1\cdots\sigma_m} = \{\overline{U}^{(m)}_{\sigma_1\cdots\sigma_m}\}_l, \quad m = 4, 6, \ldots \tag{3.11}$$

$$\tfrac{1}{2}U^{(2)}_{l,+-} = \tfrac{1}{2}U^{(2)}_{l,-+} = \{G_0^{-1} + \Sigma\}_l - G_l^{-1}, \tag{3.12}$$

$$U^{(2)}_{l,++} = \{\overline{U}^{(2)}_{++}\}_l . \tag{3.13}$$

It may be shown that if just one of the indices σ_i is positive, and if all frequencies are set equal to zero, then the value of $U^{(m)}_{l;\sigma_1\cdots\sigma_m}(k_i\omega_i)$ is just equal to the usual static (renormalized) vertex $U_l^{(m)}(k_i)$.

D. Renormalization group to all orders in ϵ

The justification for our renormalization-group analyses, to arbitrary order in ϵ, in the limit $\epsilon \to 0$, closely follows the lines of reasoning given by Wilson and Kogut[7] and by Wilson[8] for the static case. We shall not retrace all of their steps, but shall indicate the principal modifications necessary for the dynamic case.

The first step is to show that the self-energies and vertices generated by the renormalization group are regular functions of the wave-vectors and frequencies, in the limit $k_i \to 0$ and $\omega_i \to 0$. It is clear that the integrals arising from any diagram in our perturbation theory have integrands which remain finite and regular for any frequency on the real axis as long as all intermediate momenta are restricted to a shell, with $b^{-1}\Lambda < p < \Lambda$. Furthermore the integrals over intermediate frequency variables can be carried out from $-\infty$ to $+\infty$, without any difficulties, because the integrands fall off sufficiently rapidly at infinity and contain no poles or singularities closer to the real axis than $\Gamma_l\Lambda^2 b^{-2}$.

The integrations over intermediate wave vectors can lead to no divergences, because the regions of integration are finite. Unfortunately, however, the use of sharp cutoffs in k space does lead to nonanalyticities of a weaker sort, e.g., a term in the self-energy proportional to $|k|$, because the volumes of integration in various diagrams are nonanalytic functions of the incoming momenta.[9] These spurious singular terms lead to no fundamental change in the renormalization group because the singularities introduced at any stage l are canceled by singularities of the opposite sign in the next stage of the integration. Alternatively, one may employ a renormalization group with "soft" cutoffs, and avoid these difficulties.[7,8] We emphasize that these problems of sharp cutoffs are also a feature of the *static* renormalization group, and are not consequences of the extension to dynamics.

Our next task is to examine the effects of the rescaling operations R_b^s, [Eq. (I4.3)] on the dynamic vertices $U_l^{(m)}$. If the vertex in question approaches a constant when all wave vectors and frequencies approach zero, then

$$\lim_{k_i,\omega_i\to 0} R_b^s(U^{(m)}_{l,\sigma_1\cdots\sigma_m}) = b^{d-ma-(q-1)z} \lim_{k_i,\omega_i\to 0} U^{(m)}_{l,\sigma_1\cdots\sigma_m}, \tag{3.14}$$

where q is the total number of plus signs in the set $\{\sigma_1\cdots\sigma_m\}$. Any diagram for a vertex must have at least one external line propagating forward in time, so that $1 \le q \le m$. Near $d = 4$, we have $z \approx 2$ and $a \approx 1$. Thus the multiplicative factor on the right-hand side of (3.14) is $\le b^{-2}$, if $m \ge 6$, or if $m = 4$ and $q \ge 2$. The multiplicative factor is of order b^0, for the case $m = 4$, $q = 1$. However, if we assume that $U_l^{(4)}$ is a regular function of its wave vectors and frequencies, the difference $U^{(4)}_{l,+---} - u_l$ is at least of order ω or k^2, and hence decreases as b^{-2}, under the operation R_b^s. We may also see that $U^{(2)}_{l,+-}$ is at least of order k^4, ωk^2, or ω^2, and similarly decreases as b^{-2}.

The noise vertex $U^{(2)}_{l,++}$ is rescaled under R_b^s by a factor $\approx b^0$. As a manifestation of the fluctuation dissipation theorem, however, it may be shown that

$$U^{(2)}_{l,++} = \frac{1}{\Gamma_l} + \frac{2}{\omega}\mathrm{Im}U^{(2)}_{l,+-} . \tag{3.15}$$

The second term on the right-hand side of (3.15) will be proportional to k^2 or ω, and will decrease as b^{-2}.

We therefore see that the only parameters of the renormalized perturbation theory which do not decrease as b^{-2} or faster, under the action of R_b^s, are r_l, Γ_l, and u_l. It is just these parameters which must be kept in the recursion relations to lowest order in ϵ. As is discussed by Wilson and Kogut[7] for the static case, all other parameters, specifying the higher-order vertices, represent "fast transients" which, for large l, rapidly approach quasistationary values, determined by the values of the slow parameters r_l, Γ_l, and u_l. Therefore, the remaining variables may be eliminated from the recursion relations for the slow variables, and a closed set of recursion relations may be written for the variables r_l, Γ_l, and u_l. Furthermore, the quasistationary values of the eliminated variables are small (generally of order ϵ^2 or higher) and it may be shown that they do not affect the recursion relations for the slow variables, to lowest order in ϵ.

In our actual calculations[10] of various critical exponents and ratios, beyond the lowest order in ϵ [cf. Sec. VI], we do not directly use the recursion relations. Rather we employ a trick, which is a generalization to dynamics of Wilson's Feynman-graph technique.[11] Critical ratios and exponents can be read directly from the coefficients of the various powers of ϵ and of the logarithms in a Feynman-graph expansion of $\chi_s(k,\omega)^{-1}$, *provided* that the initial equations are chosen in such a way

as to eliminate all slow transients. From the renormalization-group analysis we know precisely how many slow transients must be eliminated. In the case of model A, the constant Γ_l determines the overall frequency scale, while r_l must be adjusted so that one approaches the fixed point. The only slow transient corresponds to u_l, and this may be eliminated by proper choice of the bare static coupling constant u_0. The correct value of u_0 may be found in various ways. To lowest order in ϵ it is determined by the recursion relations. To higher order, it is most conveniently determined from the conditions that the expansions for the four-point vertex and static susceptibilities exponentiate properly.[11]

E. Formalism for case C

A number of modifications in the above formalism are necessary in a two-field model. We shall illustrate this by discussing case C. In addition to the order-parameter propagator G_l, defined by (3.1), we must define a propagator for the energy field

$$D_l^{-1}(k,\omega) = -i\omega/\lambda_l^E k^2 + C_l^{-1} . \tag{3.16}$$

Vertices now have m order-parameter lines and n energy lines, with m even, and carry a weight

$$U_l^{(m,n)} = U_{l;\sigma_1\cdots\sigma_{m+n}}^{(m,n)}(k_i,\omega_i)\delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{m-1},\alpha_m} \times (2\pi)^d\delta^d\left(\sum_{i=1}^{m+n}\sigma_i k_i\right)(2\pi)\delta\left(\sum_{i=1}^{m+n}\sigma_i\omega_i\right) . \tag{3.17}$$

The variables entering the recursion relations may be identified as

$$\tilde{r}_l = \lim_{\omega,k\to 0}\{G_0^{-1}(k,\omega)+\Sigma(k,\omega)\}_l , \tag{3.18}$$

$$\Gamma_l^{-1} = \lim_{\omega,k\to 0}\frac{i\partial}{\partial\omega}\{G_0^{-1}+\Sigma\}_l = \lim_{\omega,k\to 0}\{\overline{U}_{++}^{(2,0)}\}_l , \tag{3.19}$$

$$\tilde{u}_l = \lim_{\omega_i,k_i\to 0}\{\overline{U}_{+---}^{(4,0)}\}_l , \tag{3.20}$$

$$\gamma_l = \lim_{\omega_i,k_i\to 0}\{\overline{U}_{+--}^{(2,1)}\}_l , \tag{3.21}$$

$$\delta\beta_l = \{\overline{U}_{+}^{(0,1)}\}_l , \tag{3.22}$$

$$C_l^{-1} = \lim_{k\to 0}\lim_{\omega\to 0}\{D_0^{-1}+\Pi\}_l , \tag{3.23}$$

$$\lambda_l^{-1} = \lim_{k,\omega\to 0} ik^2\frac{\partial}{\partial\omega}\{D_0^{-1}+\Pi\}_l = \lim_{k,\omega\to 0} k^2\{\overline{U}_{++}^{(0,2)}\}_l . \tag{3.24}$$

[The function Π in the last two equations is the self-energy for D defined in (I3.26).] We shall define the propagator G_l in terms of the parameter

$$r_l = \tilde{r}_l - 2\gamma_l C_l\delta\beta_l \tag{3.25}$$

rather than $\tilde{r}_l$, since it is then not necessary to keep track of $\tilde{r}_l$ and $\delta\beta_l$ separately. In order to do this consistently (i.e., in order to make $U_{l;+-}^{(2,0)}$ small) we shall redefine the operation $\{\ \}_l$ to include integration over the zero-wave-vector energy propagator, as soon as the wave vectors of the propagators in the attached loops have been integrated out (see Fig. 1). With this convention, the vertex $\{U_{+}^{(0,1)}\}_l$ is equal to zero, for all l.

The next step in the development of the renormalization group for case C is to demonstrate that the vertices and self-energies generated at any finite l are regular functions of wave vector and frequency. The argument may be carried out just as for model A, provided that Γ_{l-1} and λ_{l-1}^E are both finite.

Finally, we must examine the effect of R_b^s on the vertices. It can be shown that

$$\lim_{k_i,\omega_i\to 0} R_b^s(U_{l;\sigma_i}^{(m,n)}) = b^{d-ma-na_E-(q-1)z}\lim_{k_i,\omega_i\to 0}(U_{l;\sigma_i}^{(m,n)}) , \tag{3.26}$$

where q is the number of positive values in the set σ_i. Since $a\approx 1$, $a_E\approx 2$, and $z\approx 2$, for $d\approx 4$, the only parameters for which the factor in (3.26) is not of order b^{-2} or smaller, are the parameters already considered in (3.18)–(3.25). Note that since $\{\Pi(k,\omega)\}_l$ is regular as $\omega,k\to 0$, there will be no renormalization of the transport coefficient λ_l, except for the simple rescaling induced by R_b^s. We may also note that the renormalized vertices and propagators at stage l remain regular functions of k and ω/Γ_l, if λ_0^E is set equal to ∞. In fact the interaction between order-parameter fluctuations mediated by the energy propagator becomes an instantaneous interaction in this limit; it is clear that the dynamic critical exponents and scaling functions for the order parameter will then be the same as in model A.

On the other hand, in the limit $\lambda_0^E\to 0$, the vertices and propagators develop singularities, which invalidate the above derivation of the renormaliza-

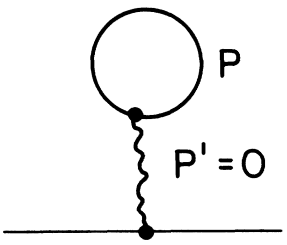

FIG. 1. Example of a term to be included in the partial self-energy $\{\Sigma\}_l$ when the intermediate wave vector p is greater than Λ/b^l even though the wave vector p' of the energy mode is identically zero.

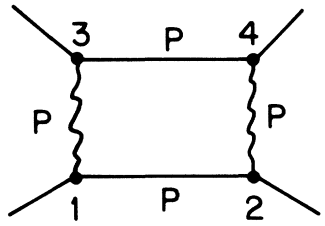

FIG. 2. Contribution to the renormalization of $U_l^{(4,0)}$, which has a singular dependence on the transmitted frequency if $\lambda_l^E = 0$.

tion group. In particular, consider the contribution of the diagram in Fig. 2 to the renormalization of $U_{l;+---}^{(4,0)}$. Let us suppose that all external wave vectors are equal to zero, and that the wave vector p transmitted by the internal propagators is of order Λ/b. If the times associated with the vertices are such that $t_1 \approx t_2$, $t_3 \approx t_4$, but t_1 is much later than t_3, then the value of the diagram will decay as $\exp[-2\lambda_l^E p^2(t_1 - t_3)/C_l]$. This leads to structure in $U_{l;+---}^{4,0}$ when the transmitted frequency is of order $\lambda_l^E \Lambda^2/C_l$, which must be considered a singularity at zero frequency, if $\mu_l \equiv \lambda_l^E/C_l\Gamma_l \approx 0$.

IV. RENORMALIZATION-GROUP ANALYSIS FOR CASE-C REGIONS IN THE *n-d* PLANE

The key equations controlling the dynamic critical behavior of model C in the approximate recursion formulas of Sec. IV of I were the recursion relations for Γ_{l+1} and λ_{l+1}^E [Eqs. (I4.21) and (I4.25)]. The equation for λ_{l+1}^E contained only a trivial rescaling due to the operation R_b^s, because the self-energy corrections to $D^{-1}(k,\omega)$ given by diagrams 2(c) of I do not give a singular contribution behaving like $i\omega/k^2$, as $\omega \to 0$ and $k \to 0$, when the intermediate wave vectors are restricted to a shell $\Lambda/b < p < \Lambda$. In fact, as mentioned above, this is true of any diagram contributing to the self-energy, no matter how complicated. Equation (I4.25) should therefore be correct to all orders of ϵ.

Equation (I4.21) for Γ_{l+1}^{-1} will be modified when terms of order ϵ^2 and higher are taken into account. According to our renormalization-group analysis, the equation for Γ_{l+1}^{-1} may be written for large l, in the form

$$\frac{1}{\Gamma_{l+1}} = b^{2+c\eta-z}\frac{1}{\Gamma_l} f(\mu_l, \gamma_l^2 C_l), \tag{4.1}$$

$$f(\mu_l, \gamma_l^2 C_l) = 1 + \gamma_l^2 C_l f_1(\mu_l) + \gamma_l^4 C_l^2 f_2(\mu_l) + \cdots . \tag{4.2}$$

Here we have assumed that μ_l and γ_l have taken on their fixed-point values, and that all "fast transients" have died out. The quantity $2 + c\eta$ is *by definition* the value of z for model A, at the given n and d.

As we have argued above, the renormalization-group equations for Γ_l must reduce to those of model A as $\mu_l \to \infty$. Furthermore, all diagrams for the renormalization-group equations are regular functions of μ^{-1}, in this limit. Thus we may write

$$f_i(\mu) \sim f_i^{(\infty)}\mu^{-1}, \quad \text{as } \mu \to \infty, \quad \text{for all } i. \tag{4.3}$$

In contrast to this, we have seen in Sec. III E, and shall verify explicitly in Sec. VI, that $f(\mu)$ is singular as $\mu \to 0^+$. Nevertheless, the lowest-order recursion relations are regular as $\mu \to 0^+$, so that

$$f_1(\mu) \sim f_1^{(0)} + f_1^{(1)}\mu + \cdots, \quad \mu \to 0. \tag{4.4}$$

Now, f_1 is a monotonic decreasing function of μ at least for small ϵ, with $f_1^{(0)} > 0$, $f_1^{(1)} < 0$, $f_1^{(\infty)} > 0$. If we assume that γ_l and C_l have reached their fixed-point values, the equation for μ_{l+1} becomes

$$\mu_{l+1} = b^{c\eta - \tilde{\alpha}/\nu}\mu_l[1 + \gamma^{*2}C^* f_1(\mu_l) + \gamma^{*4}C^{*2} f_2(\mu_l) + \cdots]. \tag{4.5}$$

Note that $\gamma^{*2}C^*$ is of order $\tilde{\alpha}/\nu$, which we shall assume to be positive.

Next we note that if Eq. (4.5) has a stable fixed point at a finite positive value of μ^*, for $\epsilon \to 0$, and if f_2, f_3, etc., are regular functions of μ near this value, then the higher-order corrections will at most perturb the fixed-point value of μ_l by an amount of order ϵ, since $\gamma^{*2}C^*$ is of order ϵ or smaller. Comparing (4.5) and (4.2), we see that if μ_l approaches a finite nonzero value as $l \to \infty$, *we must have* $z = 2 + \tilde{\alpha}/\nu$ *to all orders in* ϵ. This, of course, is the situation in region II.

From (4.2), (4.5) we see that there always exists a fixed point with $\mu^* = \infty$, finite Γ^*, and $z = 2 + c\eta$. It is also clear that if $\gamma^{*2}C^* f_1^{(\infty)} + \gamma^{*4}C^{*2} f_2^{(\infty)} + \cdots$ is positive, as will certainly be the case for $\gamma^{*2}C^*$ small and ϵ not too large, then the fixed point at $\mu^* = \infty$ will be stable if and only if $c\eta > \alpha/\nu$. This then is the condition for the existence of region I. The above analysis is valid to all orders in ϵ, and establishes the behavior in regions I and II, as well as the criterion for the boundary between the two regions.

The situation in region III ($\mu \to 0$) is more problematical. Consider the case when

$$c\eta - \frac{\alpha}{\nu} + \frac{\gamma^{*2}C^* f_1^{(0)}}{\ln b} < 0, \tag{4.6}$$

which occurs when $2 < n < 4$ and $\epsilon \to 0$. It is then clear that if $\gamma^{*2}C^*$ were sufficiently small, and if $f_2(\mu) + \gamma^{*2}C^* f_3(\mu) + \cdots$ were *finite and bounded for all* μ, then the only stable fixed point of (4.5) would be at $\mu^* = 0$, and we would have

$$2 + c\eta < z < 2 + \alpha/\nu, \tag{4.7a}$$

$$z = 2 + c\eta + \ln f(0)/\ln b\,. \tag{4.7b}$$

The inequalities in (4.7a) are just the defining relation for region III, and would imply that the recursion relations are indeed correct to order ϵ, for $2<n<4$. However, our previous discussion has shown that the recursion relations (4.2) and (4.5) are not expected to be correct, even to lowest order in ϵ, when $\mu=0$. Furthermore, our explicit calculations of $\Sigma(k,\omega)$ in Sec. VI indicate that $f_2(\mu)$ diverges as $|\ln\mu|$, when $\mu\to 0^+$, for $\epsilon\to 0$. If we neglect the contributions to (4.5) of f_3 and higher terms we see that the "fixed point" $\mu^*=0^+$ is never stable, and that there is always a fixed point at finite positive μ^*, with $z=2+\alpha/\nu$, provided that $\alpha/\nu>c\eta$ [see Eq. (4.5)]. In the region $2<n<4$, however, the fixed-point value μ^* would be very small, with $|\ln\mu|$ of order ϵ^{-1}. This reasoning would predict that region III does not exist, and that the phase diagram is that given by Fig. 4(b) of I.

In fact, the situation is not entirely clear. It seems likely that f_3 contains terms that behave like $|\ln\mu|^2$, as $\mu\to 0$, and it is not known whether the sum of all the terms in (4.2) will be convergent or divergent as $\mu\to 0$. To put the problem in a slightly different language, it is not clear that a well-behaved recursion relation can be written for the *single* slow variable μ_l, when μ_l is extremely small. It may be necessary to introduce one or more additional parameters, in order to develop a renormalization-group analysis which will correctly describe the situation $\epsilon\to 0$, $2<n<4$. Thus, for the moment at least, we must leave open the possibility of the different phase diagrams indicated in Figs. 4 and 5 of I.

V. RELAXATION OF THE ENERGY

A. Case C

For sufficiently small values of ω, the energy response function $\chi_E(k,\omega)$ may be written in the approximate form

$$\chi_E^{-1}(k,\omega)\approx\left(\frac{-i\omega}{\lambda_l^E k^2}\,b^{(z-2)l}-\frac{i\omega}{\Gamma_l^E}\,b^{zl}+C_l^{-1}\right)b^{-\tilde{\alpha}l/\nu}\,, \tag{5.1}$$

where b^l is the minimum of Λ/k or Λ/κ. The coefficient $1/\Gamma_l^E$ is of order ϵ, and is defined as

$$\frac{1}{\Gamma_l^E}=\lim_{\omega,k\to 0}\frac{i\partial}{\partial\omega}\{\Pi(k,\omega)\}_l\,. \tag{5.2}$$

Thus

$$\frac{i\partial}{\partial\omega}\chi_E^{-1}\bigg|_{\omega=0}\approx\left(\frac{1}{\lambda_l^E k^2}+\frac{1}{\Gamma_l^E}\,b^{2l}\right)b^{l(z-2-\tilde{\alpha}/\nu)}\,. \tag{5.3}$$

For model C, in region II, we have $z=2+\tilde{\alpha}/\nu$, while both λ_l^E and Γ_l^E remain finite for large l. The second term on the right-hand side of (5.3) is negligible compared to the first if $k\ll\kappa$, and is small but finite (order ϵ) relative to the first term, if $k\gtrsim\kappa$. It follows that

$$\begin{aligned}\omega_E(k)&\propto\kappa^{\alpha/\nu}k^2 \quad \text{for } k\le\kappa\,,\\ &\propto k^{2+\alpha/\nu} \quad \text{for } k\gtrsim\kappa\,.\end{aligned} \tag{5.4}$$

In region I, where $z=2+c\eta>2+\alpha/\nu$, the situation is somewhat more complicated. We must distinguish between the cases where $\alpha>0$ and $\alpha<0$. In the former case, $1/\Gamma_l^E$ approaches a finite value, of order ϵ relative to $1/\Gamma^*$, while $1/\lambda_l^E\to 0$ as $b^{-l(c\eta-\alpha/\nu)}$. In the limit $k\to 0$, with fixed κ, we see that the first term on the right-hand side of (5.3) is dominant, and $\omega_E(k)\propto\kappa^{\alpha/\nu}k^2$. On the other hand, if $k\gtrsim\kappa$, the second term in (5.3) is dominant, for sufficiently large l, and we have $\omega_E(k)\propto k^{2+c\eta}$.

When $\alpha<0$, the coupling constant $\gamma_l^2 C_l$ approaches zero for large l, as $b^{\alpha l/\nu}$. Consequently $1/\Gamma_l^E$ approaches zero as $b^{\alpha l/\nu}$, while $1/\lambda_l^E\to 0$ as $b^{-c\eta l}$. If $|\alpha|/\nu\gtrsim c\eta$, then the first term in (5.3) is dominant for all values of k/κ. If $|\alpha|/\nu<c\eta$, then the first term dominates for $k\to 0$, but the second term dominates for $k\gtrsim\kappa$. Using the definition (I3.13) we find that for $c\eta>\alpha/\nu>-c\eta$,

$$\begin{aligned}\omega_E^{\text{sing}}(k)&\approx\kappa^{|\alpha|/\nu}k^2 \quad \text{for } k\to 0, \text{ fixed } \kappa\\ &\approx k^{2+c\eta} \quad \text{for } k\gtrsim\kappa\,,\end{aligned} \tag{5.5}$$

whereas for $\alpha/\nu<-c\eta$, we find

$$\begin{aligned}\omega_E^{\text{sing}}(k)&\approx\kappa^{|\alpha|/\nu}k^2 \quad \text{for } k\to 0, \text{ fixed } \kappa\\ &\approx k^{2+|\alpha|/\nu} \quad \text{for } k\gtrsim\kappa\,.\end{aligned} \tag{5.6}$$

The existence of this last region was overlooked in I, and hence the description of energy relaxation in region II was incomplete.

In region III, if it exists, we expect that relations (5.4) should apply.

B. Cases A and B

The energy relaxation in cases A and B may be most easily discussed by using the equivalent two-field models A′ and B′. Equations (5.1)–(5.3) are still applicable in these cases, if we set $1/\lambda_l^E=0$. The exponent z is chosen so that Γ_l remains finite in case A′, or so that λ_l remains finite in case B′. As in case C, we have

$$\begin{aligned}&1/\Gamma_l^E\to\text{const} \quad \text{if } \alpha>0\,,\\ &1/\Gamma^E\propto b^{l\alpha/\nu} \quad \text{if } \alpha<0\,.\end{aligned} \tag{5.7}$$

We thus find, for cases A and B,

$$\begin{aligned}\omega_E^{\text{sing}}(k)&\sim\text{const}\times\kappa^{-z} \quad \text{for } k\le\kappa\\ &\sim\text{const}\times k^{-z} \quad \text{for } k\gtrsim\kappa\,.\end{aligned} \tag{5.8}$$

The constants on the right-hand side of (5.8) are of

order ϵ, relative to the corresponding constants in ω_ψ.

C. Case D

As was discussed in Sec. II, the energy relaxation in case D is the same as for case B, given by (5.8), provided that $k/\kappa \gtrsim \kappa$. In the limit $k/\kappa \ll \kappa$, however, $\omega_E(k)$ is equivalent to the first lines of Eqs. (5.4), (5.5), or (5.6). These results follow from Eqs. (5.3) and (5.8), if we note that for case D,

$$1/\lambda_l^E = (1/\lambda_0^E) b^{-l(z-2+\tilde{\alpha}/\nu)}, \tag{5.9}$$

with $z = 4 - \eta$.

VI. FEYNMAN-GRAPH EXPANSION FOR CASE C TO ORDER ϵ^2

A. General formalism

In this section we shall carry out the renormalization-group calculations for case C, using a generalization of Wilson's Feynman-graph expansion.[11] This method consists in matching the bare perturbation theory for the response functions $\chi_\psi(k,\omega)$ and $\chi_E(k,\omega)$ to the expected critical behavior. As is well known, the series will only exponentiate if certain parameters in the starting equations are fixed at particular values[11] which depend on ϵ. These parameters correspond to the slow transients of the recursion-relation analysis, namely, u_0, $\gamma_0^2 C_0$, and μ_0 [Eq. (I4.28)].

The general diagrammatic formalism for carrying out the renormalization-group analysis was described in Sec. III. For the present purposes we need only the bare perturbation theory mentioned there. In order to carry out calculations most conveniently, however, we shall redefine the bare propagators as

$$G_0(k,\omega) = \frac{1}{r_0 + k^2 - i\omega/\Gamma_0} - \frac{1}{1 + k^2 - i\omega/\Gamma_0}, \tag{6.1}$$

$$C_0^{-1} D_0'(k,\omega) = \frac{\mu_0 k^2}{\mu_0 k^2 - i\omega} - 1 \equiv C_0^{-1} D_0 - 1, \tag{6.2}$$

and use the bare three- and four-point vertices γ_0 and u_0. The coefficient μ_0 is equal to

$$\mu_0 \equiv \lambda_0^E / \Gamma_0 C_0. \tag{6.3}$$

The subtraction in Eq. (6.1) provides a smooth ultraviolet cutoff, so that wave vectors may be integrated from 0 to ∞. (This contrasts with the sharp cutoff Λ employed earlier.) The use of the propagator D_0' which vanishes at $\omega = 0$, in place of D_0, means that the four-point vertex should include the static value of the energy-mediated interaction. The four-point vertex is therefore given the value $u_0 = \tilde{u}_0 - \frac{1}{2}\gamma_0^2 C_0$, instead of $\tilde{u}_0$. [See Eq. (I2.20).]

When computing a contribution to the self-energy Π such as Π_b in Fig. 3, however, we must add back a term $\frac{1}{2}\gamma_0^2 C_0$ to the vertex, so as not to include the reducible part of the energy propagator. We shall eventually choose u_0 to have the usual Ginzburg-Landau-Wilson value[7,11]

$$u_0 = 8\pi^2 \epsilon / 4(n+8) + O(\epsilon^2). \tag{6.4}$$

Note also that when $\lambda_0^E \to \infty$ (or $\mu_0 \to \infty$), D_0' goes to zero, and the present model reduces to case A.

B. Vertex $\gamma_0(\epsilon)$

The vertex $\gamma_0(\epsilon)$ is determined by calculating the static energy correlation function at $k = 0$, and identifying its ϵ expansion with that of the specific heat

$$\chi_E(k=0) = C \sim r^{-\alpha/\gamma}, \tag{6.5}$$

where $r \equiv \chi_\psi^{-1}(k=0, \omega=0)$. The calculation is quite analogous to Wilson's determination[11] of $u_0(\epsilon)$, Eq. (6.4). The specific heat is obtained by calculating the static self-energy $\Pi(k=0, \omega=0)$ to second order in ϵ. The relevant diagrams are shown in Fig. 3, and lead to the following contributions at $k = \omega = 0$:

$$\Pi_a = -2n\gamma_0^2 \Pi_0(0,r), \tag{6.6}$$

$$\Pi_b = 2n\gamma_0^2 [8u_0 + \tfrac{1}{2}n(8u_0 + 4\gamma_0^2 C_0)][\Pi_0(0,r)]^2, \tag{6.7a}$$

$$\Pi_c = \Pi_d = 0, \tag{6.7b}$$

where

$$\Pi_0(k,r) \equiv \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{r + p^2} - \frac{1}{1 + p^2}\right) \times \left(\frac{1}{r + (p-k)^2} - \frac{1}{1 + (p-k)^2}\right), \tag{6.8}$$

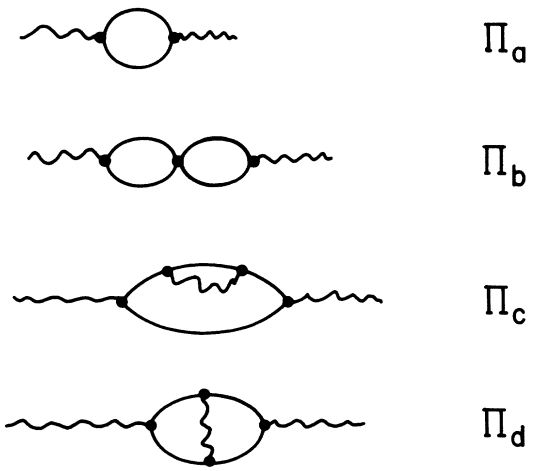

FIG. 3. First- and second-order diagrams for the self-energy Π of the energy field.

$$\Pi_0(0,r) = -K_d(1+\tfrac{1}{2}\ln r)(1-\tfrac{1}{4}\epsilon\ln r)+O(\epsilon^2)\,, \quad (6.9)$$

and

$$K_d = 2^{1-d}\pi^{-d/2}/\Gamma(d/2) \quad (6.10)$$

is the volume element in d-dimensional space (K_4 was denoted B in I). Adding up the contributions from Eqs. (6.6) and (6.7) we find

$$\begin{aligned} C_0C^{-1} &= C_0D^{-1}(k=0,\omega=0) \\ &= 1+2v_0n(1+\tfrac{1}{2}\ln r) \\ &\quad\times\{1-\tfrac{1}{4}\epsilon\ln r+[2v_0n+4K_4u_0(n+2)](1+\tfrac{1}{2}\ln r)\} \\ &\quad+O(\epsilon^3)\,, \end{aligned} \quad (6.11)$$

with

$$v_0 \equiv K_d\gamma_0^2C_0\,. \quad (6.12)$$

Comparing Eq. (6.11) with the expansions[7]

$$C_0C^{-1} = 1+(\tilde{\alpha}/\gamma)\ln r+\tfrac{1}{2}(\tilde{\alpha}/\gamma)^2\ln^2 r+O(\epsilon^3)\,, \quad (6.13)$$

where $\tilde{\alpha}=\max(\alpha,0)$, and

$$\begin{aligned} \frac{\alpha}{\gamma} &= \frac{\alpha}{2\nu}+O(\epsilon^3) \\ &= \frac{\epsilon}{2}\left(\frac{4-n}{n+8}\right)-\epsilon^2\frac{(n+2)(13n+44)}{2(n+8)^3}+O(\epsilon^3)\,, \end{aligned} \quad (6.14)$$

we find from the exponential condition

$$v_0(\epsilon)\equiv K_d\gamma_0^2(\epsilon)C_0=(\alpha/2n\nu)(1-\tfrac{1}{2}\epsilon)+O(\epsilon^3)\,, \quad (6.15)$$

for $\alpha>0$, and

$$v_0(\epsilon)=0\,, \quad (6.16)$$

for $\alpha<0$.

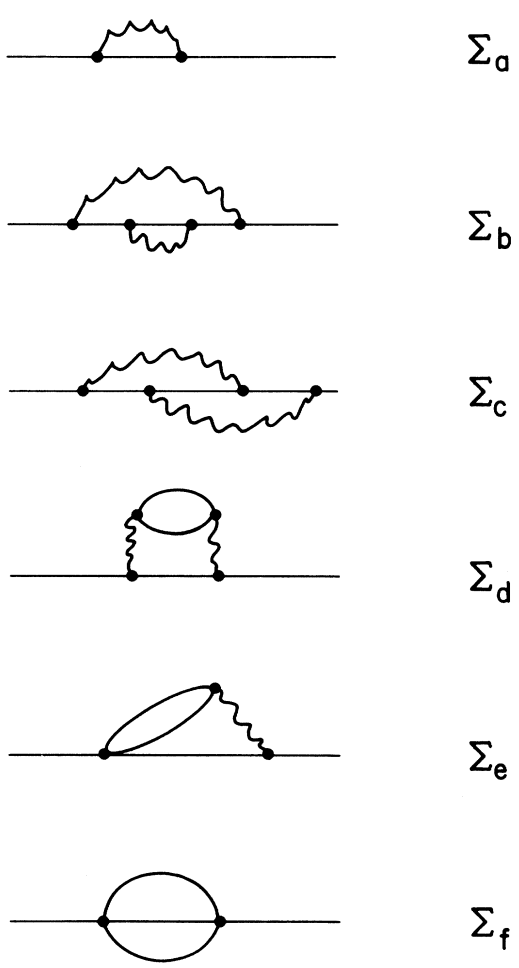

FIG. 4. First- and second-order diagrams for the self-energy Σ of the order parameter.

C. Determination of the dynamic exponent z for $\mu_0 > 0$

In order to determine the dynamic exponents from the Feynman-graph expansion, it is necessary to calculate the self-energies $\Sigma(k,\omega)$ and $\Pi(k,\omega)$ in perturbation theory. This may be done either by using the general rules of Sec. III, or, in the present case of a "bare" perturbation theory, by a number of more compact techniques (see, e.g., the rules given in Refs. 5 or 6).

We shall determine the exponent $z\equiv z_\psi$ and the parameter μ_0, by requiring the exponentiation of $G(k,\omega)$ at $k=0$ and $T=T_c$. The condition $T=T_c$ is equivalent to $r=0$, and is imposed by making the usual "mass renormalization."[11] The self-energy $\Sigma(k,\omega)$ [Eq. (I3.25)] may be obtained to second order in ϵ by calculating the contributions from the diagrams in Fig. 4. For $k=0$, $r=0$, with ω small but finite, we find[12]

$$\Sigma_a = 2i\tilde{\omega}v_0\left(\frac{\ln(-i\tilde{\omega})}{1+\mu_0}-\frac{1}{\mu_0}\ln(1+\mu_0)\right)-\frac{i\tilde{\omega}v_0\epsilon}{1+\mu_0}\left[\tfrac{1}{2}\ln^2(-i\tilde{\omega})-\ln(-i\tilde{\omega})\ln(1+\mu_0)\right], \quad (6.17)$$

$$\Sigma_b = -4i\tilde{\omega}v_0^2\left[\frac{\ln^2(-i\tilde{\omega})}{2(1+\mu_0)^3}+\frac{\ln(-i\tilde{\omega})}{(1+\mu_0)^2}\left(\frac{\ln 4}{\mu_0}+\frac{\ln\mu_0}{1+\mu_0}+\frac{(1-\mu_0)\ln(1+\mu_0)}{\mu_0(1+\mu_0)}-\frac{(2+\mu_0)\ln(2+\mu_0)}{\mu_0(1+\mu_0)}\right)\right], \quad (6.18)$$

$$\Sigma_c = -4i\tilde{\omega}v_0^2\frac{\ln(-i\tilde{\omega})}{(1+\mu_0)^2}\left(\ln\mu_0-\frac{\ln 4}{\mu_0}+3\ln\tfrac{3}{4}-\frac{2}{\mu_0}(1+\mu_0)\ln(1+\mu_0)+\frac{1}{\mu_0}(2+\mu_0)\ln(2+\mu_0)\right), \quad (6.19)$$

$$\Sigma_d = \frac{2i\tilde{\omega}v_0^2n}{(1+\mu_0)^2}\left\{\tfrac{1}{2}\ln^2(-i\tilde{\omega})-\ln(-i\tilde{\omega})[3\ln\tfrac{3}{4}+\mu_0+\ln(1+\mu_0)]\right\}, \quad (6.20)$$

$$\Sigma_e = 8i\tilde{\omega}K_4 u_0 v_0 \left(\frac{n+2}{1+\mu_0}\right)\{\tfrac{1}{2}\ln^2(-i\tilde{\omega}) - \ln(-i\tilde{\omega})[3\ln\tfrac{4}{3} + \ln(1+\mu_0)]\}, \tag{6.21}$$

$$\Sigma_f = -8i\tilde{\omega}K_4^2 u_0^2 (n+2) 3\ln\tfrac{4}{3}\ln(-i\tilde{\omega}), \tag{6.22}$$

with

$$\tilde{\omega} \equiv \omega/\Gamma_0 . \tag{6.23}$$

Putting together Eqs. (6.17)–(6.22), we may express $G(k=0,\omega)$ via Eq. (I3.25) in the form

$$G^{-1}(k=0,\omega,r=0) = \text{const} \times (-i\tilde{\omega})[1 + a_1 \ln(-i\tilde{\omega}) + a_2 \ln^2(-i\tilde{\omega}) + O(\epsilon^3)], \tag{6.24}$$

with

$$\begin{aligned} a_1 = {} & \left(\frac{-2v_0}{1+\mu_0}\right)[1 + \tfrac{1}{2}\epsilon \ln(1+\mu_0)] \\ & + \left(\frac{4v_0^2}{(1+\mu_0)^2}\right)\left[\frac{1+\mu_0}{\mu_0}\ln(1+\mu_0) + \left(\frac{2+\mu_0}{1+\mu_0}\right)[\ln\mu_0 + \ln(2+\mu_0)] + \left(\frac{n}{2} - \frac{2\mu_0^2+5\mu_0+1}{\mu_0(1+\mu_0)}\right)\ln(1+\mu_0) + (1+\tfrac{1}{2}n)3\ln\tfrac{4}{3} + \tfrac{1}{2}n\mu_0\right] \\ & + 8K_4 u_0 v_0 \left(\frac{n+2}{1+\mu_0}\right)[3\ln\tfrac{4}{3} + \ln(1+\mu_0)] + 8K_4^2 u_0^2 (n+2) 3\ln\tfrac{4}{3}, \end{aligned} \tag{6.25}$$

$$a_2 = \left(\frac{v_0}{1+\mu_0}\right)\left[\tfrac{1}{2}\epsilon + \frac{2v_0}{(1+\mu_0)^2}[1 - \tfrac{1}{2}n(1+\mu_0)] - 4K_4 u_0 (2+n)\right]. \tag{6.26}$$

The above expressions may be related to the dynamic exponent z by observing that according to the general scaling forms (I3.14)–(I3.17), applied to $G(k,\omega) = \chi_\psi(k,\omega)$, we have

$$G(k \to 0,\ \omega,\ r=0) \sim k^{-2+\eta}(-i\tilde{\omega}/k^z)^{-(1+P)}. \tag{6.27}$$

In Eq. (6.27) we have used the value $x_\psi = -2+\eta$ [see Eq. (I3.16)], and we define $1+P$ as the leading power of the scaling function [Eq. (I3.14)] $f_\psi(u) \sim u^{-(1+P)}$ as $u \to \infty$. In order that the right-hand side of (6.27) remain finite as $k \to 0$ for finite ω we must have

$$z = z_\psi = (2-\eta)/(1+P) \tag{6.28}$$

and

$$\begin{aligned} G^{-1}(k=0,\omega,r=0) &\propto (-i\tilde{\omega})^{1+P} \\ &\sim (-i\tilde{\omega})[1 + P\ln(-i\tilde{\omega}) \\ &\quad + \tfrac{1}{2}P^2\ln^2(-i\tilde{\omega}) + O(P^3)]. \end{aligned} \tag{6.29}$$

Equation (6.29) agrees with the perturbation expansion (6.24)–(6.26) to order ϵ^2 if and only if

$$P = a_1, \tag{6.30}$$

$$a_2 = \tfrac{1}{2}a_1^2. \tag{6.31}$$

The exponent z is then obtained from (6.28) as

$$\begin{aligned} z = {} & 2 + \left(\frac{4v_0}{1+\mu_0}\right)[1 + \tfrac{1}{2}\epsilon\ln(1+\mu_0)] + 8K_4^2 u_0^2 (n+2)(6\ln\tfrac{4}{3} - 1) + \left(\frac{16K_4 u_0 v_0 (n+2)}{1+\mu_0}\right)[3\ln\tfrac{4}{3} - \ln(1+\mu_0)] \\ & + \frac{4v_0^2 n}{(1+\mu_0)^2}[3\ln\tfrac{4}{3} - \mu_0 - \ln(1+\mu_0)] + \frac{8v_0^2}{(1+\mu_0)^3}[1 + \mu_0 - (2+\mu_0)\ln\mu_0 + 3(1+\mu_0)\ln\tfrac{4}{3} \\ & + (3+\mu_0)\ln(1+\mu_0) - (2+\mu_0)\ln(2+\mu_0)]. \end{aligned} \tag{6.32}$$

D. Determination of regions in the *n*-*d* plane

Inserting the expressions (6.25) and (6.26) for the constants a_1 and a_2, into the exponentiation condition (6.31) yields to lowest order in ϵ

$$\frac{(4-n)^2}{n^2(n+8)^2(1+\mu_0)^3}\left[\frac{n}{2}(1+\mu_0)^2 - \left(\frac{n+2}{2}\right)(1+\mu_0) + 1\right] = 0. \tag{6.33}$$

This equation has three possible solutions for μ_0,

$$\mu_0 = \mu_0^{(1)} = \infty, \tag{6.34a}$$

$$\mu_0 = \mu_0^{(2)} = \frac{2}{n} - 1, \tag{6.34b}$$

$$\mu_0 = \mu_0^{(3)} = 0, \tag{6.34c}$$

corresponding to the three solutions found in I from the recursion-relation analysis. The corresponding exponents are, from Eq. (6.32), to

lowest order,

$$z^{(1)} = 2 + (6\ln\tfrac{4}{3} - 1)\frac{(n+2)\epsilon^2}{2(n+8)^2} = 2 + c\eta\,, \qquad (6.35a)$$

$$z^{(2)} = 2 + \tilde{\alpha}/\nu\,, \qquad (6.35b)$$

$$z^{(3)} = 2 + 2\tilde{\alpha}/n\nu\,, \qquad (6.35c)$$

in accordance with the first-order results of Sec. IV of I. In order to find the higher-order corrections to Eqs. (6.35) from the Feynman-graph expansion, we would have to calculate third-order diagrams for Σ. We may, however, apply the general results of the analysis of Secs. III and IV, and see if these are consistent with the explicit ϵ expansion.

We shall first discuss regions I and II, for which the renormalization group is well behaved. We saw in Sec. IV that region I is characterized by the *exact* conditions

$$\mu_0^{(1)} = \infty\,, \qquad (6.36)$$

$$z^{(1)} = 2 + c\eta\,, \qquad (6.37)$$

where $2 + c\eta$ is defined as the dynamic exponent in the absence of energy conservation (case A). Similarly, region II has

$$0 < \mu_0^{(2)} < \infty\,, \qquad (6.38)$$

$$z^{(2)} = 2 + \tilde{\alpha}/\nu\,, \qquad (6.39)$$

to all orders in ϵ. In fact the value of $\mu_0^{(2)}$ can be found to the next order in region II, by imposing the scaling condition (6.39). Inserting the expression for z in terms of μ_0, Eqs. (6.32), into Eq. (6.39) and using (6.16), we find for $n = 1$

$$\mu_0^{(2)} = 1 + \epsilon(\tfrac{55}{6}\ln 2 - \tfrac{14}{3}\ln 3 - \tfrac{17}{18})\,. \qquad (6.40)$$

Note also that for $\mu_0 \to \infty$, Eq. (6.32) reduces to $z = 2 + c\eta$, as in case A.

The boundary between regions I and II, which goes through the point $\epsilon = 0$, $n = 4$, is seen to correspond to the relation $z^{(1)} = z^{(2)}$, i.e., $\tilde{\alpha}/\nu = c\eta$. Inserting the ϵ expansions (6.14), (6.35a), and $\eta = \frac{1}{2}\epsilon^2(n+2)/(n+8)^2 + O(\epsilon^3)$,[11] we find that this is satisfied along the curve $n = n_{21}(\epsilon)$ given by

$$n_{21} = 4 - 4\epsilon(1 + c/16) + O(\epsilon^2)\,, \qquad (6.41)$$

with[10]

$$c = 6\ln\tfrac{4}{3} - 1 + O(\epsilon)\,. \qquad (6.42)$$

For comparison we note that $\alpha = 0$ along the curve $n = 4 - 4\epsilon + O(\epsilon^2)$.

E. Amplitude ratio in region II

A physical result which may be obtained in region II is the value of the universal amplitude ratio

$$\mu = \lambda_E \kappa^2 \chi_\psi(k=0)/\Gamma C\,, \qquad (6.43)$$

where λ_E and Γ are the *dressed* values of the transport and kinetic coefficients, defined in Eqs. (I3.7) and (I3.8), $C = \chi_E(k=0)$ is the dressed specific heat, $\chi_\psi(k=0) = r^{-1}$ is the order-parameter susceptibility, and κ is the inverse correlation range.

The kinetic coefficient Γ may be found to order ϵ from the diagram Σ_a of Fig. 4, whose contribution we rewrite here in terms of the smooth cutoff introduced in Eq. (6.1), as[12]

$$\Sigma_a(k,\omega) = -4\gamma_0^2 C_0 \int \frac{d^d p}{(2\pi)^d}\left(\frac{1}{p^2+r} - \frac{1}{p^2+1}\right) \times\left(\frac{p^2 + r + \mu_0(p-k)^2}{-i\tilde{\omega} + p^2 + r + \mu_0(p-k)^2}\right). \qquad (6.44)$$

Applying the definition (I3.7) and evaluating (6.44) to order ϵ, we find

$$\Gamma = \Gamma_0\left(1 + \frac{2v_0\ln(1+\mu_0)}{\mu_0(1+\mu_0)}\right) \times\left[1 + \left(\frac{2v_0}{1+\mu_0}\right)\ln r\right] + O(\epsilon^2)\,. \qquad (6.45)$$

In region II, μ_0 is given to first order in ϵ by Eq. (6.34b) and v_0 by (6.15), so Eq. (6.45) becomes

$$\Gamma = \Gamma_0\left[1 + \frac{n\tilde{\alpha}}{2(2-n)\nu}\ln\left(\frac{2}{n}\right)\right] r^{\tilde{\alpha}/2\nu} + O(\epsilon^2)\,. \qquad (6.46)$$

The renormalization of C is found from Eqs. (6.6) and (6.8),

$$C = C_0(1 - 2nv_0)(1 - nv_0\ln r) + O(\epsilon^2) = C_0(1 - \tilde{\alpha}/\nu) r^{-\tilde{\alpha}/2\nu}\,. \qquad (6.47)$$

In addition, the inverse correlation range is given by[13] $\kappa^2 = r + O(\epsilon^2)$. Finally, according to the discussion of Sec. III above, the transport coefficient λ_E is unrenormalized, i.e.,

$$\lambda_E = \lambda_0^E\,. \qquad (6.48)$$

Putting together Eqs. (6.43) and (6.46)–(6.48) we find

$$\mu = \mu_0\left[1 + 2v_0\left(n - \frac{\ln(1+\mu_0)}{\mu_0(1+\mu_0)}\right)\right] + O(\epsilon^2)\,. \qquad (6.49)$$

For $n = 1$ we may use (6.40) and the value $v_0 = \epsilon/6$, to find

$$\mu = 1 + \epsilon(9\ln 2 - \tfrac{14}{3}\ln 3 - \tfrac{11}{18}) \approx 1 + 0.5004\epsilon\,. \qquad (6.50)$$

F. Limit $\mu_0 \to 0$ and region III

The third solution of the exponentiation condition (6.33) given in (6.34c) corresponds to the fixed-point value $\mu^* = 0$ discussed in I and in Sec. IV. Although the first-order recursion relations do not

show any singularity in this limit, the second-order expression for a_1 in Eq. (6.25) contains terms of order $\epsilon^2 \ln\mu_0$, which indicate a breakdown of perturbation theory when $\mu_0 \to 0$. Indeed, we may repeat the second-order calculation for the case where μ_0 is identically zero, and find that the contributions from diagrams (b) and (c) of Fig. 4 depend on the ratio ω/μ_0 when $\omega \to 0$, $\mu_0 \to 0$. In the case $\mu_0 = 0$, ω finite, we find

$$\Sigma_b(k=0,\omega,r=0,\mu_0=0) = -4i\tilde{\omega}v_0^2[\ln(-i\tilde{\omega}) + \ln^2(-i\tilde{\omega})], \quad (6.51)$$

$$\Sigma_c(k=0,\omega,r=0,\mu_0=0) = 4i\tilde{\omega}v_0^2[3\ln\tfrac{4}{3}\ln(-i\tilde{\omega}) - \tfrac{1}{2}\ln^2(-i\tilde{\omega})], \quad (6.52)$$

while the contributions from diagrams (a) and (d)–(f) of Fig. 4 are continuous as $\mu_0 \to 0$. We therefore have

$$G^{-1}(k=0,\omega,r=0,\mu_0=0) = \text{const}\times(-i\tilde{\omega})[1 + a_1^0\ln(-i\tilde{\omega}) + a_2^0\ln^2(-i\tilde{\omega})], \quad (6.53)$$

with

$$a_1^0 = -\tfrac{1}{2}v_0 + v_0^2(\tfrac{1}{2} - \tfrac{3}{8}(n+2)\ln\tfrac{4}{3}) - 6K_4u_0v_0(n+2)\ln\tfrac{4}{3} - 24(n+2)K_4^2u_0^2\ln\tfrac{4}{3}, \quad (6.54)$$

$$a_2^0 = \tfrac{1}{8}v_0\epsilon - \tfrac{3}{16}v_0^2(n-2) - (n+2)K_4u_0v_0. \quad (6.55)$$

It is easy to check that the coefficients a_1^0 and a_2^0 do not satisfy the exponentiation condition (6.31) when u_0 and v_0 are given by (6.4) and (6.15). This failure is another indication that there is no well-behaved solution with $\mu_0 = 0$, and the vertices u_0 and v_0 fixed at their static values. The nonuniformity of the $\mu_0 \to 0$ limit was also noted by Brézin and De Dominicis.[14]

On the basis of our second-order calculation, and of the renormalization-group analysis outlined in Sec. IV, we feel that we can have confidence in our calculations of exponents and critical ratios provided that $|\ln\mu| \gg 1/\epsilon$. Using this criterion we can calculate "conservative" boundaries for region II, for $\epsilon \to 0$, and $n \to 2$ or $n \to 4$, namely,

$$n_{23} = 2 + \tfrac{4}{5}\epsilon|\ln\epsilon| + O(\epsilon), \quad (6.56)$$

$$n_{32} = 4 - 4\epsilon(1 + \tfrac{1}{8}c) + O(\epsilon^2). \quad (6.57)$$

Thus we claim with some confidence that scaling applies with $z = 2 + \alpha/\nu$, for $n < n_{23}$ and $n_{32} < n < n_{21}$. It seems likely that $z = 2 + \alpha/\nu$ also for $n_{23} < n < n_{32}$, i.e., that region III does not exist, but further analysis of this case is required, as mentioned in Sec. IV.

G. Energy-response function $D(k,\omega)$

The energy-response function is obtained by calculating the self-energy $\Pi(k,\omega)$, defined in (I3.26). At finite ω, the second-order diagrams for $\Pi(k,\omega)$ are the ones already considered in the static calculation. The generalization of Eq. (6.8) to finite ω is the analog of Eq. (I3.28), with the smooth cutoff, and r_0 replaced by r, i.e.,

$$\Pi_0(k,\omega,r) = \int \frac{d^dp}{(2\pi)^d}\left(\frac{1}{p^2+r} - \frac{1}{p^2-1}\right) \times\left(\frac{1}{(p-k)^2+r} - \frac{1}{(p-k)^2+1}\right) \times\left(\frac{p^2+r+(p-k)^2+r}{-i\tilde{\omega}+p^2+r+(p-k)^2+r}\right). \quad (6.58)$$

In the limit $0 < \tilde{\omega}/\mu_0k^2 \ll 1$, $r = 0$, we have

$$\Pi_0(k,\omega,r=0) = -K_d\ln k - (i\tilde{\omega}/k^2)K_d\ln\tfrac{3}{4}(1 - \epsilon\ln k) + O(\epsilon^2), \quad (6.59)$$

from which Π_a and Π_b can be obtained via Eqs. (6.6) and (6.7a). The contribution from Fig. 3(c) yields, after a lengthy calculation,

$$\Pi_c(k,\omega,r=0) = 8v_0^2n(1+\mu_0)^{-1}\ln\tfrac{3}{4}(-i\tilde{\omega}/k^2)\ln k. \quad (6.60)$$

In addition, we may show that $\Pi_d(k,\omega)$ does not possess any term of the form $(i\tilde{\omega}/k^2)\ln k$ to order ϵ^2. Collecting the terms in Π we find at $r = 0$,

$$\Pi(k,\omega) = \Pi(k,0) + (i\tilde{\omega}/k^2)n\mu_0v_0\ln\tfrac{3}{4}(1 + b_1\ln k) + O(\epsilon^3), \quad (6.61)$$

$$b_1 = (\tilde{\alpha}/\nu)[1 - 2n^{-1}(1+\mu_0)^{-1}] + O(\epsilon^2), \quad (6.62)$$

where we have used Eq. (6.16) for v_0.

If extended scaling[15] holds, we expect that at T_c, for k fixed and $\omega \to 0$, $D(k,\omega)$ will have the form (I3.14),

$$D^{-1}(k,\omega) = k^{\tilde{\alpha}/\nu}[f_E(\omega/k^{z_E})]^{-1} \sim D^{-1}(k,0) + \text{const}\times k^{\tilde{\alpha}/\nu+(2-z_E)(1+q)}(-i\tilde{\omega}/\mu_0k^2)^{1+q} \sim D^{-1}(k,0) + \text{const}\times\{1 + [\tilde{\alpha}/\nu + (2-z_E)(1+q)]\ln k\}(-i\tilde{\omega}/\mu_0k^2)[1 + q\ln(-i\tilde{\omega}/\mu_0k^2)], \quad (6.63)$$

where $[f_E(u)]^{-1} \sim 1 + \text{const}\times u^{1+q}$ for $u \to 0$. We may compare Eq. (6.63) with the perturbation expansion (6.61), and conclude from the absence of a $\ln(-i\tilde{\omega}/\mu_0k^2)$ term in (6.61) that

$$q = 0 + O(\epsilon^2), \quad (6.64)$$

whence

$$\tilde{\alpha}/\nu + 2 - z_E = b_1. \quad (6.65)$$

In region I, $\mu_0 = \mu_0^{(1)} \to \infty$ and we have from Eq. (6.62),

$$b_1^{(1)} = \tilde{\alpha}/\nu + O(\epsilon^2), \tag{6.66}$$

so that

$$z_E^{(1)} = 2 + O(\epsilon^2). \tag{6.67}$$

This means that energy conservation does not influence z_E at T_c, as already seen in Sec. V. In region II, Eq. (6.34b), we have $b_1^{(2)} = 0$, and

$$z_E^{(2)} = 2 + \tilde{\alpha}/\nu + O(\epsilon^2) = z^{(2)} + O(\epsilon^2), \tag{6.68}$$

in accordance with extended scaling (see Sec. V A). In region III, if it exists, we have, at T_c, $\mu_0^{(3)} = 0$, $b_1^{(3)} = (\tilde{\alpha}/\nu)(1 - 2n^{-1})$, and

$$z_E^{(3)} = 2 + 2\tilde{\alpha}/n\nu = z^{(3)}. \tag{6.69}$$

VII. COMPARISON WITH OTHER WORK

The two-field models introduced in I have been considered in two recent papers,[14,16] both of which used the Callan-Symanzik formalism. Murata[16] has carried out calculations for models corresponding A′, B′, C, and D of the present paper. He obtains exponents to order ϵ^2, which are in agreement with ours. Murata also calculates a ratio of relaxation rates at $T = T_c$, for the order parameter and the energy fields, in the various cases. It is not possible to make a direct comparison with our result (6.50), however, because our calculation was performed for the limit $k/\kappa \to 0$ ($T > T_c$). (The ratios are expected to differ by an amount of order ϵ.)

With regard to the ambiguous region $2 < n < 4$, for model C, Murata's calculation suggests that $z = 2 + \alpha/\nu$, with $|\ln \mu| \propto 1/\epsilon$. This is in agreement with the speculations at the end of Sec. IV of the present paper. However, as in the present paper, Murata has not been able to provide support for this conclusion by a detailed renormalization-group analysis or by explicit calculations to order ϵ^3.

Brézin and De Dominicis[14] restrict their attention to model C. They calculate exponents to order ϵ^2 which are in agreement with ours for $n < 2$ and $n > 4$. Similarly their analysis of the boundary between regions I and II, in the vicinity of $n = 4$, agrees with ours. With regard to the ambiguous region where the lowest-order recursion relations give $\mu^* = 0$, however, these authors assume that a breakdown of scaling occurs, and that region III exists as in Figs. 4(a) or 5(a) of I. They propose a form $\omega_\psi(k) \propto k^z \ln^2 k$ in this region, where z is given by $z^{(3)}$ [Eq. (6.35c)] to lowest order in ϵ, but they warn that this may not persist in higher order. Consistent with these assumptions, they have calculated initial slopes of the boundaries between regions II and III. These boundaries coincide with $n_{23}(\epsilon)$ and $n_{32}(\epsilon)$, given in Eqs. (6.56) and (6.57) of the present paper.

Brézin and De Dominicis also consider some of the corrections to scaling, and in a related paper, with Zinn-Justin,[17] they have calculated the exponent z to order ϵ^3, for case A.

It should be remarked that the second-order expressions for z in terms of the ratio $\mu_0(\epsilon)$ in the present paper, R in Ref. 16, or λ in Ref. 14, cannot be directly compared, since the quantities μ_0, λ, and R are defined differently and will presumably differ from each other in order ϵ.

---

*Supported in part by NSF Grant No. MPS 75-08083.

[1]B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. B 10, 139 (1974), hereafter referred to as I. Equations of this paper will be preceded by the symbol I.
[2]J. R. Tucker and B. I. Halperin, Phys. Rev. B 3, 3768 (1971).
[3]P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973).
[4]B. I. Halperin, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. B 13, 1299 (1976).
[5]M. Suzuki, Prog. Theor. Phys. 50, 1769 (1973); Y. Kuramoto, *ibid.* 51, 1712 (1974); C. De Dominicis, Nuovo Cimento Lett. 12, 567 (1975).
[6]S. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, to be published); S. Ma and G. F. Mazenko, Phys. Rev. B 11, 4077 (1975).
[7]K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974).
[8]K. G. Wilson, Rev. Mod. Phys. 47, 773 (1975).
[9]K. G. Wilson, comment in Proceedings of the Conference on the Renormalization Group and Quantum Field Theory, Temple University, 1973, edited by J. D. Gunton and M. S. Green (unpublished), p. 30.
[10]B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. Lett. 29, 1548 (1972).
[11]K. G. Wilson, Phys. Rev. Lett. 28, 548 (1972).
[12]Note the misprint in Eq. (I3.27), corrected in Eq. (2.6).
[13]M. E. Fisher and A. Aharony, Phys. Rev. B 10, 2818 (1974).
[14]E. Brézin and C. De Dominicis, Phys. Rev. B 12, 4954 (1975).
[15]B. I. Halperin and P. C. Hohenberg, Phys. Rev. 177, 952 (1969).
[16]K. K. Murata, Phys. Rev. B 13, 2028 (1976).
[17]C. De Dominicis, E. Brézin, and J. Zinn-Justin, Phys. Rev. B 12, 4945 (1975).