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Scaling behavior of second-order phase transitions

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Using the Landau Hamiltonian for the description of second-order phase transitions, we give a proof of scaling for any continuous number of dimensions below four. The proof is based on a summation of diagrams having a power-law divergence and standard renormalization-group methods. The proof is constructive in that it leads to an unambiguous calculation for the critical exponents η and γ. We present in this paper a detailed discussion of the proof; we also compare our method with the ϵ expansion leading to an interesting aspect of that theory: We find that the contribution to the critical exponents of order ϵ can be gotten without any calculation of diagrams. In this paper we have only made a lowest-order calculation in three dimensions. To this order we are of course unable to locate the relevant fixed point, but it leads to a relation between η and γ—also to lowest order—which is such that if γ is fixed to be 1.25, then η turns out to be 0.12.

I. INTRODUCTION

The methods based on the renormalization-group approach initiated by Wilson have had a tremendous success in our understanding of second-order phase transitions.[1,2] Especially the ϵ expansion of Wilson and collaborators using the formalism of field theory led to a deep understanding of the essential properties of phase transitions. However, one of the drawbacks of this method lies in the expansion parameter ϵ itself: in principle, one has to restrict oneself to small ϵ. Nevertheless, one strange feature of the ϵ-expansion is the reasonable success one obtains when one puts in the calculations $\epsilon = 1$ (and even $\epsilon = 2$), which naturally makes this expansion very interesting.

In this paper we present a field-theory-like formulation of second-order phase transitions which is, in principle, valid for all continuous dimensions below four: the formulation itself constitutes a proof of scaling—accepting renormalization-group arguments and some other qualitative assumptions—and allows an explicit and analytic evaluation of the critical exponents.

This approach starts from the Landau Hamiltonian[3]

$$\mathcal{H} = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\Phi^2 + \tfrac{1}{4}\Lambda\Phi^4 + \cdots, \qquad (1)$$

where $\phi(x)$ is the microscope order parameter. In this section we do not discuss the effect of higher powers of $\phi(x)$ in (1); we consider here only one degree of freedom and restrict ourselves to the three-dimensional case. The problem of second-order phase transitions consists then in finding the full infrared singular behavior of the correlation functions generated by (1). We will set $m = 0$ (which corresponds to $T = T_c$) from the beginning and determine the behavior of Green's functions when the external momenta go to zero.

Inspection of the Feynman diagrams one obtains from (1), shows that the one loop diagram of Fig. 1(a) (bubble diagram) behaves like k^{-1}; clearly then, a chain of m bubbles [see Fig. 1(b)] will then behave like k^{-m}. The first step in our procedure consists in "summing up" these strong infrared divergences; in Sec. II, we show that this sum goes like k for small k. The sum of all original diagrams can then be replaced by a new sum, in which each diagram contains as elementary "exchanges" the summed up chain diagrams. Power counting then shows that all the infrared divergences are only of logarithmic type.

We then establish that these logarithmic singularities can be "summed" with a Callan-Symanzik type of equation, familiar from the renormalization-group methods of field theory. It then follows automatically that we have scaling behavior for all the correlation functions in the limit $k \to 0$. At the same time, this equation then also allows us to calculate explicitly the critical exponents η and γ without any ambiguity. In this paper we have only performed a lowest-order calculation, which does not yet give the critical exponents, but it leads to a not unreasonable consistency check of the formulation.

The paper is organized as follows: Sec. II contains an extensive discussion and justification of the bubble summation for Φ^4; in Sec. III we establish scaling of this theory via a Callan-Symanzik

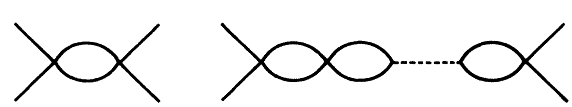

FIG. 1. One-bubble diagram (a) and the diagram with a sequence of bubbles (b), also called the chain diagram.

equation; Sec. IV contains a detailed discussion of the meaning of the bubble summation by comparing it with the ϵ expansion; Sec. V contains a discussion of the effect of higher-order interaction terms.

II. FORMULATION OF THEORY

In this paper, the formulation of second-order phase transitions is based on the Landau Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 + (\lambda_4/4!)\phi^4 + (\lambda_6/6!)\phi^6 + \cdots, \tag{2}$$

where it is understood that the real order parameter ϕ has N degrees of freedom, i.e., each ϕ^2 stands for $\sum_{a=1}^{N}(\phi^a\phi^a)$. All indicated parameters are renormalized ones; counterterms have not been written explicitly. The coupling constants λ_4, $\lambda_6, \ldots$ have a dimension depending on the number of dimensions, called n, in which we consider the problem: in general, the coupling constant in front of $(\phi^2)^p$, called λ_{2p}, has the dimension

$$\lambda_{2p} \sim M^{n-[(n-2)/2]2p}. \tag{3}$$

The problem of critical behavior is to study the infrared behavior of the Green's functions, or correlation functions of (1) when the temperature T approaches the critical temperature T_c, the limit $T = T_c$ corresponding to $m^2 = 0$. For our purposes, it is enough to consider only the case $T = T_c$ and we will from now on always put $m^2 = 0$.

Only these interaction terms in (1) will influence the infrared behavior for which the coupling constant gets a dimension with a non-negative power of M, i.e., for integers $p \geq 2$, which satisfy

$$p(2-n) + n \geq 0. \tag{4}$$

Clearly, for $n < 2$, all powers contribute. If $n > 2$, one gets $p < n/(n-2)$. These results are easily understandable by observing that the field ϕ is dimensionless for $n = 2$.

For $n = 3$, only the terms ϕ^4 and ϕ^6 contribute. Naturally, we will be interested mostly in this case. Below $n = 3$ the closer one comes to $n = 2$, more and more terms contribute.

From general principles, we should in (1) also include terms of the form $(\nabla\phi)^2\phi^2, \ldots$. However, these terms do not in general modify the infrared behavior; a term of the form $(\nabla\phi)^2\phi^2$ would only start to be relevant for $n = 2$.

It is important for what follows to give a better specification of what one means with the limit $k \to 0$. From (3), one sees that there are in (2) different "coupling" constants with a dimension; from the meaning of $\phi(x)$ as microscopic order parameter, these constants must somehow have a dimension specified by the microscopic dimensional parameters, which one can parametrize by the range of the microscopic forces, called a (this could also be the lattice distance; we assume as usual that there is no essential difference between the two). The limit of small k is then assumed to mean that $k \lll 1/a$. Later on, we will neglect terms which are on order (ka) smaller than the leading ones.

For $T \neq T_c$, one has another dimensional parameter ξ, the correlation length: by considering directly $T = T_c$ (and therefore $\xi = \infty$) one has one parameter less and the formulation of what one means with small k becomes much simpler (especially in the calculations). Moreover, for the determination of the critical exponents η and γ, the only exponents that one obtains directly in a field theory approach, one does not need to consider $T \neq T_c$.

These points can be understood better by looking at the Ising model. As is well known,[4] the singular behavior around $T = T_c$ of the Ising model is equivalent to (2), as it should, since (2) is the most general form that one can write down (assuming of course "locality," i.e., short-range microscopic interactions). The results one obtains in this case about the role of a and the coupling constants with a dimension are equivalent with the discussion above.

In the first part of this paper, we will not consider interaction terms in $\mathcal{H}$ except for ϕ^4, which would be correct for $n > 3$. However, in general, one cannot do that for reasons of consistency, e.g., for $n = 3$, the interaction ϕ^6 should be included in order to arrive at a consistent formulation of the field theory. One can in this case easily draw diagrams involving only the ϕ^4 interaction which contribute to the six-point Green's function $\Gamma_{(6)}$, which diverges logarithmically; clearly one then needs a counter term in ϕ^6. What we really have in mind when we neglect the ϕ^6 term is that the "fixed-point" value of the coupling constant of Φ^6 is assumed to vanish or such that its effect can be neglected (for more details, see below).

The correlation functions can be calculated, in perturbation theory, by using standard Feynman rules which can be obtained from the path integral for the partition function

$$Z(j) \sim \int \exp\left(-\int [\mathcal{H}(\phi) - \phi j]\, dx\right) \mathfrak{D}\phi. \tag{5}$$

The resulting expressions can also be defined for an arbitrary continuous number of dimensions n.[5] When calculating the one-loop corrections to the four-point correlation function $\Gamma_{(4)}(k)$, one finds

$$\Gamma_{(4)}(k) \sim k^{n-4} . \tag{6}$$

Note also that the functions $\Gamma_{(n)}(k)$ represent one-particle irreducible Green's functions; the momenta of the external lines are always chosen—unless stated otherwise—as follows:

$$k_1 = \cdots = k_{n-1} = k; \quad k_n = -(n-1)k.$$

Clearly, one finds a very singular infrared behavior for $n<4$. For $n>4$ there is no singular behavior; as is well known, this means that one has the "classical" Landau theory for $n>4$, which corresponds to only taking tree-diagram contributions into account. The loop corrections, corresponding to statistical fluctuations have no influence on the small momentum behavior of the theory. The case $n=4$ is the dividing line; one has then only logarithmic corrections, characteristic of a renormalizable theory and which can be "summed up" with the renormalization group equations resulting in the limit $k \to 0$ in a scale-invariant result. However, as it turns out, the fixed point value for the coupling constant vanishes and the critical exponents are therefore not modified from the Landau values. All these results are well known and have been formulated precisely by Wilson and others. For $n<4$, Wilson introduces an expansion in the parameter $\epsilon \equiv 4-n$. Clearly, the expansion can only be valid for $\epsilon > 0$, which means that the point $\epsilon = 0$ is not a regular point; the series in ϵ is therefore expected to be only an asymptotic one. For expression (6), one now gets the expansion

$$k^{n-4} = e^{-\epsilon \ln k} = 1 - \epsilon \ln k + \cdots ,$$

and one again finds only logarithmic corrections that can be summed up with the renormalization group equation leading to a nontrivial scaling in the limit $k \to 0$. One now finds that the fixed-point value for the coupling constant is proportional to ϵ; also the critical exponents differ from the Landau value by terms of order ϵ. Note however, that if one loop gives a behavior like k^{n-4}, then clearly going to higher order will give an even stronger singular behavior, e.g., the chain diagrams [Fig. 1(b)], with p loops will diverge like $(k^{n-4})^p$. For small ϵ, this is no problem however, since a diagram with p loops contains the coupling constant to the $(p+1)$th power and is therefore multiplied by ϵ^{p+1}. But this shows also that one can no longer rely on a straightforward expansion in the coupling constant when ϵ ceases to be small, apart from the expansion in ϵ itself.

The first step in our procedure is to "sum" up the very strong infrared singularities coming from the chain diagrams [Fig. 1(b)]. To avoid double counting in a later stage, this summation can be done most elegantly by performing first a canonical transformation[6]: we introduce in (1), after some straightforward changes of notation, an additional field χ as follows:

$$\mathcal{H} = \tfrac{1}{2}(\nabla\phi)^2 + (\lambda/8N)\phi^4 - (N/2\lambda)[\chi + (\lambda/2N)\phi^2]^2 . \tag{7}$$

It is clear from the path integral (5) that the addition of this last term does not change $Z(j)$, since the integration

$$\int \exp \int \frac{N}{2\lambda}\left(\chi + \frac{\lambda}{2N}\phi^2\right)^2 dx\, \mathfrak{D}\chi$$

is Gaussian and gives only a trivial constant. One might have some difficulty with the existence of this functional integral, since the sign in front of χ^2 is "wrong" for the integral to be meaningful. Observe however that we could have added also the term $(N/2\lambda)[\chi - (i\lambda/2N)\phi^2]^2$. The "$i$" in front of $\chi\phi^2$ gives no troubles with hermiticity, since only Green's functions without external χ lines have physical meaning. This implies that the coupling constant is in fact $(i\lambda/2N)^2$ which is real. Actually, one can easily check that the Green's functions for the two different added terms are the same as it should, since we really only use the functional integral of Eq. (5) to read off the Feynman rules. One has then that (6) is equivalent with

$$\mathcal{H} = \tfrac{1}{2}(\nabla\phi)^2 - (N/2\lambda)\chi^2 - \tfrac{1}{2}\chi\phi^2 . \tag{8}$$

From this expression one sees that the Feynman rules involve two propagators and one vertex and are given in Fig. 2.

The strong infrared singularities in the chain diagrams are now present in the diagrams for the full propagator—represented by a double solid line (Fig. 3)—that we call $S(N,n,k)$, instead of the χ propagator, and which is obtained by summing all the bubbles present in the χ propagator. It is clear that the collection of all Feynman diagrams generated by (2) is the same as the one we obtain by taking all Feynman diagrams wherein we re-

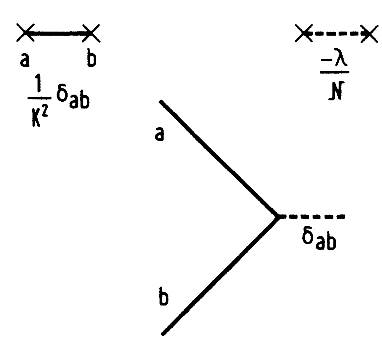

FIG. 2. Feynman rules obtained from (8).

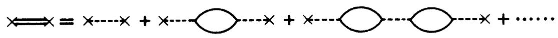

FIG. 3. "Sum" of chain diagrams giving the new propagator $S(N, n, k)$.

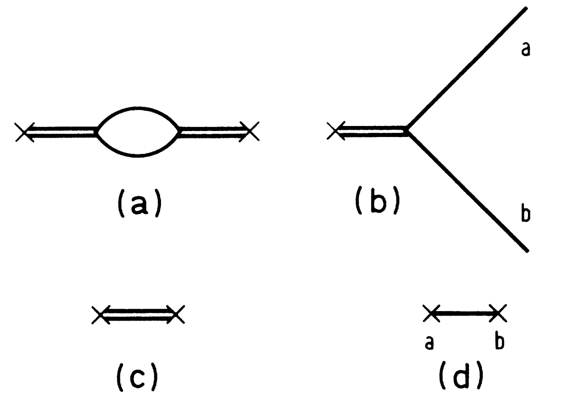

FIG. 4. "New" Feynman rules; contributions containing diagram (a) have to be omitted.

place the χ propagator by the new propagator $S(N, n, k)$, except that we have to leave out diagrams that contain the bubble diagram of Fig. 4(a).

The calculation of $S(N, n, k)$ is straightforward: from the diagrams of Fig. 3, we obtain

$$S(N, n, k) = -\frac{\lambda}{N} + \left(\frac{-\lambda}{N}\right)^2 \frac{N}{2} \int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2(k-p)^2} + \cdots$$
$$= -\frac{\lambda}{N}\left(1 - \frac{\lambda}{2} \int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2(k-p)^2} + \cdots\right). \tag{9}$$

The n-dimensional integral can be evaluated by consulting the work of 't Hooft and Veltman.[5] We find

$$I_n \equiv \frac{1}{2} \int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2(k-p)^2}$$
$$= F(n) K^{n-4} \tag{10}$$
$$= (2^4\pi)^{(1-n)/2} \frac{\Gamma[\frac{1}{2}(4-n)]\Gamma[\frac{1}{2}(n-2)]}{\Gamma[\frac{1}{2}(n-1)]} k^{n-4}. \tag{11}$$

The power k^{n-4} is expected from power counting and dimensional reasons.

Note the poles at $n=4$ and $n=2$: the pole at $n=4$ expresses the logarithmic divergence of the integral for $n=4$ as expected. The pole at $n=2$ is of a different nature: it is an infrared singularity obviously present in the integral. Clearly, it is impossible to formulate in perturbation theory a massless theory in two dimensions. As we will show later, our formulation, which departs from perturbation theory by our summing of diagrams, allows one to include the two-dimensional case.

If we formally sum up series (9), we find

$$S(N, n, k,) = \frac{-1}{N}\left(\frac{\lambda}{1+\lambda F(n) k^{n-4}}\right); \tag{12}$$

and in the limit of small k ($k^{4-n} \ll \lambda$), this becomes

$$S\,(N, n, k,) \Rightarrow \frac{-1}{NF(n)} k^{4-n} \Rightarrow \frac{-1}{N} (2^4\pi)^{(n-1)/2} \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(4-n)]\Gamma[\frac{1}{2}(n-2)]} k^{4-n}. \tag{13}$$

Note the upside-down effect of the factors in (13) compared with the ones in (11): especially, the pole at $n=2$ in (11) has now become a zero; this remark is important for later.

The summation we performed is by no means justified: for small values of k, each term in the series (9) blows up and the series itself has no sense anymore. However, one sees that the series has alternating signs, a phenomenon familiar from renormalizable field theory, called asymptotic freedom, where one also effectively sums the leading singularities which are in this case logarithms. It is one of the basic results (in fact, also an assumption!) of the renormalization group approach in field theory, that the leading logarithms (which also form a geometric series) can be summed in the case of alternating signs and give then a first-order result; corrections come in this case also from nonleading logarithms. The method one uses here consists in replacing the formal sum obtained in perturbation theory by a differential equation, which expresses essentially the effect on the Green's functions of an increase in the momenta. This equation is obtained from the formal series expansion. One then forgets about the formal series and one solves the differential equation, postulating then that the results one so obtains are meaningful statements about the "summed up" theory (see also, Sec. III).

We can "imitate" this method also here in a straightforward and simple way: one observes for this that the series for $Q(k) \equiv -N S(N, n, k)$ (this new definition is only for the convenience of the following argument) satisfies the following nonlinear equation:

$$\frac{dQ(k)}{d(k^{n-4})}=-F(n)Q^2(k)\equiv\beta(Q)\,. \tag{14}$$

We can integrate this equation by starting from a rather large value of k and then go to small k-values. Figure 5(a) shows that the origin is a fixed point for $k\to 0$; this equation can then easily be integrated to give [Fig. 5(b)]:

$$S(N,n,k)=-(1/N)[1/F(n)]k^{4-n}\,, \tag{15}$$

which is the same result as obtained in (13). Note that the formal summation as in (12) would not have been possible in the case that the signs in (9) would all have been the same; the equation corresponding to (11) would have had a positive β and then the point at infinity would have been a fixed point, which would be in conflict with the limit of the formal sum. This result is also expected since in this case the naive sum is of the form $(1-x)^{-1}$ which has a singularity at $x=1$; one can then no longer connect the domains $x<1$ and $x>1$.

If one now calculates Feynman diagrams using (13) as a propagator, one immediately finds that all the Green's functions have only logarithmic singularities (apart from a definite power of the momenta present for dimensional reasons). One can understand this as follows: the Feynman rules one uses now correspond to a "field theory" in which there are no explicit parameters with a dimension. The only singularities then allowed are of logarithmic type. Note however that "no parameters $\cdots$ with a dimension" does not imply immediately scaling: in fact there is always implicitly a mass parameter present, since one has to cut off the logarithmic divergences somewhere. These remarks imply also that the infrared divergences can also be gotten from the calculation of the ultraviolet divergences: e.g., a diagram will give a contribution $C\ln(k^2/M^2)$ regardless of whether $k^2\ll M^2$ or $k^2\gg M^2$ (M is here an arbitrary scale parameter). This is a well-known result for renormalizable and fully massless theories. But for the determination of the large momentum behavior, one can simply use naive power counting which makes the singularity structure of the

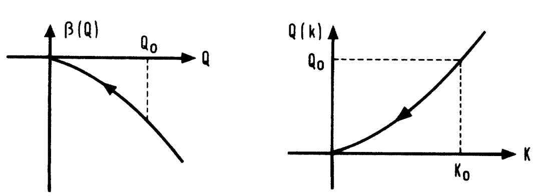

FIG. 5. Diagrams that illustrate the summation of the chain diagrams in the limit $k\to 0$.

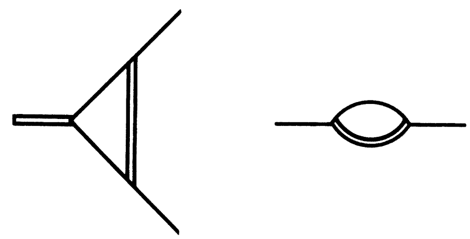

FIG. 6. One-loop diagrams.

Green's functions very simple to discuss and to determine.

The "primitive" Green's functions can now easily be determined: these are the two propagators and the vertex function, i.e., the dressed up diagrams corresponding to Figs. 4(b)–4(d). Note that here we do not consider the Green's function $\Gamma^{(6)}$. One can also easily convince oneself that the logarithmic divergences in $\Gamma^{(4)}$ appear only because of insertions of the dressed up vertex and propagators; this leads to a tremendous simplification of the number of diagrams one has to consider.

The relevant one-loop and two-loop diagrams can be found in Figs. 6 and 7, respectively.

An important question remains: which terms did we neglect in using the expression (15) as a propagator? It is clearly by no means trivial to use this expression, obtained only in the limit for small k, as a propagator since we have now to integrate over the momenta flowing through it. The point is that we integrate only over those momenta which are small compared with the cutoff Λ (the

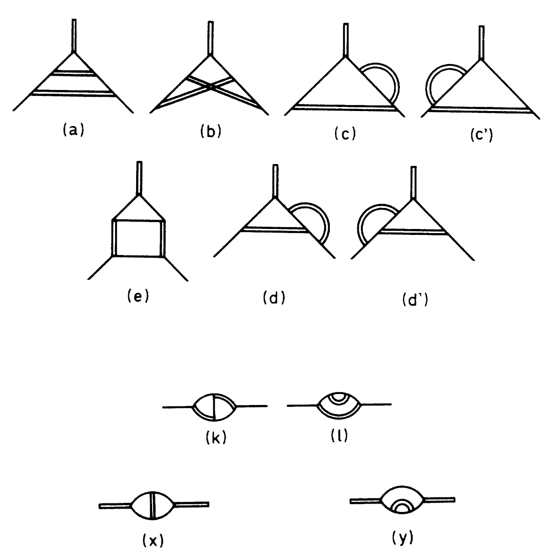

FIG. 7. Set of two-loop diagrams.

inverse lattice distance); this is in fact implicitly assumed in the field-theory formulation of second-order phase transitions. The very use of the order parameter $\phi(x)$ as a "local" field, means that we consider (2) only as a valid description for distances large compared to the "lattice" distance.

More quantitatively, if one rather arbitrarily assumes that the corrections in the "lattice" distance to the theory can be obtained from (2), one will find corrections to the propagator of the form $k^{n-4}(ka)$; i.e., in the small-momentum limit, the singularities in the correlation functions would have to be multiplied by a term $[1+(ka)f(k)]$, where $f(k)$ has only logarithmic singularities. As remarked before, we neglect in this work these corrections.

III. SCALING

Using the results of Sec. II, for the Green's functions below, we will get expressions of the form

$$\Gamma_{(2,1)}(N,n,k)=\alpha+\alpha^3 C_{11}(N,n)\ln\frac{k^2}{M^2}+\alpha^5\left(C_{21}(N,n)\ln\frac{k^2}{M^2}+C_{22}(N,n)\ln^2\frac{k^2}{M^2}\right)+\cdots, \tag{16}$$

$$\Gamma_{(2,0)}(N,n,k)=k^2\left[1-\alpha^2 D_{11}(N,n)\ln\frac{k^2}{M^2}-\alpha^4\left(D_{21}(N,n)\ln\frac{k^2}{M^2}+D_{22}(N,n)\ln^2\frac{k^2}{M^2}\right)-\cdots\right], \tag{17}$$

$$\Gamma_{(0,2)}(N,n,k)=-NF(n)k^{n-4}\left(1-\alpha^4 F_{21}(N,n)\ln\frac{k^2}{M^2}-\cdots\right). \tag{18}$$

Writing these expressions, we have chosen normalization conditions which take a simple form for $k^2=M^2$. In $\Gamma_{(2,1)}$ and $\Gamma_{(2,0)}$, we have left out the factor δ_{ab}, a and b referring to the internal degrees of freedom carried by the fields ϕ. The reader might wonder where the coupling constant α comes from, since the Feynman rules that we obtained in Sec. II imply $\alpha=1$. We do this here only formally: we will introduce the equations of the renormalization group which are valid for the Green's functions [Eqs. (16), (17), and (18)] for any value of α and therefore also for $\alpha=1$. This is, however, only true if α is introduced as we did it.

Moreover, in solving the renormalization-group equations, the perturbation theory value of α is not relevant: only the fixed-point value is relevant (see below).

It is also clear that the coefficients in these expressions will depend only on N, the number of degrees of freedom and on n, the number of dimensions; this will have as a consequence that the critical exponents will depend only on dimensionality and degrees of freedom, an expected and welcome result.[7]

Actually, the N dependence can be made explicit, using the fact that each time the "new" propagator occurs, we will get a factor $1/N$ and that every closed loop of ϕ lines will contribute a factor N. From the structure of the diagrams on Figs. 6 and 7, we will be able to write

$$\begin{aligned} &C_{11}(N,n)=(1/N)c_{11}(n),\\ &D_{11}(N,n)=(1/N)d_{11}(n),\\ &C_{2i}(N,n)=(1/N^2)c_{2i}^{(\alpha)}(n)+(1/N)c_{2i}^{(\beta)}(n),\\ &D_{2i}(N,n)=(1/N^2)d_{2i}(n),\\ &F_{21}(N,n)=(1/N)f_{21}(n),\quad i=1,2. \end{aligned} \tag{19}$$

Clearly, $c_{2i}^{(\beta)}(n)$ is completely determined by Fig. 7(e).

One can now also see that there are no troubles with the dimension $n=2$ and that the coefficients D_{ij}, C_{ij}, and F_{ij} have a regular behavior at $n=2$. The point is that although each loop integration introduces a pole at $n=2$, it is exactly cancelled by the zero at $n=2$ in the propagator (13), since each new loop contains this propagator once.

We now observe that the Green's functions $\Gamma_{(r,s)}$ satisfy the Callan-Symanzik equation[8]

$$\left(M\frac{\partial}{\partial M}+\beta(\alpha)\frac{\partial}{\partial\alpha}-r\gamma_\phi^{(\alpha)}-s\gamma_\chi^{(\alpha)}\right)\Gamma_{(r,s)}(M,\alpha,k)=0. \tag{20}$$

This is a well-known result for a fully massless and renormalizable theory. The essential criterion for the validity of these equations is not in the first place field theory, but a regularity structure in the coefficients C_{ij}, D_{ij}, and F_{ij} which is satisfied in particular for a renormalizable field theory. This structure, which is equivalent with the validity of (20) for $\Gamma_{(2,1)}$, $\Gamma_{(2,0)}$, and $\Gamma_{(0,2)}$, is such that of all the coefficients of the logarithms, only the ones in front of the terms $\ln(k^2/M^2)$ are independent; once all the coefficients C_{p1}, D_{p1}, F_{p1} ($p=1,\ldots$) are known; all the others are determined. In field theory, these relations follow from the topological structure of Feynman diagrams and power counting. The theory we consider here is certainly no longer a field theory, but the logarithms are generated by the same mechanism as in field theory: the "Green's functions" in this theory will therefore also satisfy the Callan-Symanzik equation.

One can also see the renormalization-group equation in this context as a quantitative expression of the observation we made at the end of Sec. II: we arrived there at a "field theory" with no explicit parameters anymore with a dimension. The parameter M is then nothing but an arbitrary scale that we introduced; clearly, any other value of the scale would have been equally good and (20) expresses nothing but the irrelevance of the particular scale value M that we have chosen.[9]

The functions $\beta(\alpha)$, $\gamma_\phi(\alpha)$, and $\gamma_\chi(\alpha)$ are determined from the perturbation expansions (16), (17), and (18) by demanding the validity of Eq. (20) in each order of the perturbation theory; by doing so one also automatically recovers the constraints of the renormalization group on the coefficients of the logarithms, which are of no direct interest to us.

Introducing $t=\frac{1}{2}\ln(k^2/M^2)$, one obtains $dM/M = -dt$ (for fixed k^2); and Eq. (20) then gives

$$\gamma_\phi(\alpha)=\sum_{p=1}^{\infty}\alpha^{2p}D_{p1}, \qquad (21)$$

$$\gamma_\chi(\alpha)=\sum_{p=2}^{\infty}\alpha^{2p}F_{p1}, \qquad (22)$$

$$\beta(\alpha)=2\sum_{p=1}^{\infty}\alpha^{2p+1}C_{p1}+\alpha[2\gamma_\phi(\alpha)+\gamma_\chi(\alpha)]. \qquad (23)$$

The N dependence, including only diagrams up to two loops, can be introduced explicitly and one gets

$$\gamma_\phi(\alpha)=\alpha^2(1/N)d_{11}+\alpha^4(1/N^2)d_{21}, \qquad (24)$$

$$\gamma_\chi(\alpha)=\alpha^4(1/N)f_{21}, \qquad (25)$$

$$\beta(\alpha)=\alpha^3\left(2\frac{1}{N}c_{11}+2\frac{1}{N}d_{11}\right) + \alpha^5\left(2\frac{1}{N^2}c_{21}^{(\alpha)}+2\frac{1}{N}c_{21}^{(\beta)}+\frac{2}{N^2}d_{21}+\frac{1}{N}f_{21}\right). \qquad (26)$$

One now forgets about the perturbation expansion and one tries to directly solve Eq. (20) using expressions (21)–(23) for $\beta(\alpha)$, $\gamma_\phi(\alpha)$, $\gamma_\chi(\alpha)$ or in an approximate way (24)–(26). At the same time, one hopes that those aspects of the solution of this equation which are relevant for the problem of phase transitions will involve only small values of α^2 so that one can use the perturbation-expansion results. Observe, however, that even small values of α^2 do not really justify the use of the perturbation results; this is an assumption.

The solution of the Callan-Symanzik equation is determined as follows[10]: (i) Using the variable $t=\frac{1}{2}\ln(k^2/M^2)$ which gives $-\partial/\partial t=M\partial/\partial M$ and introducing

$$\Gamma_{(r,s)}(\alpha,k,M)=k^{n-r(n-2)/2-2s}F_{(r,s)}(\alpha,t) \qquad (27)$$

(the power of k in front of $F_{(r,s)}$ is for dimensional reasons), one equivalently obtains

$$\left(-\frac{\partial}{\partial t}+\beta(\alpha)\frac{\partial}{\partial\alpha}-r\gamma_\phi-s\gamma_\chi\right)F_{(r,s)}(\alpha,t)=0. \qquad (28)$$

(ii) Then solve the nonlinear equation

$$\frac{d\alpha'(\alpha,t)}{dt}=\beta(\alpha'),$$

with (29)

$$\alpha'(0,\alpha)=\alpha$$

(one can of course take here $\alpha=1$!).

Below, we will be interested in the fixed points of this equation; these are values of α' that do not change anymore when one varies t. Clearly, Eq. (29) shows that these points occur when $\beta(\alpha')=0$. Moreover, when one reaches such a fixed point one often has $|t|\to\infty$. This last point is always true when $\beta(\alpha)$ is calculated in perturbation theory and truncated at a given order, which one always does in practice. In this case we generally have

$$\beta(\alpha')\sim c(\alpha'-\alpha^*)^m R(\alpha')$$

where α^* is such a fixed point. In this expression m is a positive integer and $R(\alpha^*)\neq 0$. The solution of (29) is then

$$t-t_0=\int_{\alpha(t_0)}^{\alpha(t)}\frac{d\alpha'}{c(\alpha'-\alpha^*)^m R(\alpha')}. \qquad (30)$$

When $\alpha(t)\to\alpha^*$, one clearly sees that $|t|\to\infty$. (iii) The solution of (20) can then be written (as one can easily verify)

$$\Gamma_{(r,s)}(\alpha,t,k)=k^{n-r(n-2)/2-2s}F_{(r,s)}(\alpha'(t,\alpha),0)\times\exp\left(-\int_0^t[r\gamma_\phi(\alpha'(t',\alpha))+s\gamma_\chi(\alpha'(t',\alpha))]\,dt'\right). \qquad (31)$$

When $k\to 0$ (i.e., $t\to -\infty$), and $\alpha'(t,\alpha)$ goes to the corresponding fixed point α^*, one obtains

$$\Gamma_{(r,s)} \underset{k\to 0}{\sim} k^{n-r(n-2)/2-2s}(k^2/M^2)^{-r\gamma_\phi(\alpha^*)/2-s\gamma_\chi(\alpha^*)/2}\,. \tag{32}$$

This is only the leading term in the asymptotic expansion: we are not interested here in getting the next-order terms, which are suppressed by a solid power of (k^2/M^2). It would even not make much sense to include these "correction" terms, since we already neglected earlier such powers.

The result (32) corresponds exactly to the scaling behavior of the correlation functions (for the physical correlation functions, one puts $s=0$).

For the inverse propagator we obtain

$$\Gamma_{(2,0)} \sim k^{2-2\gamma_\phi(\alpha^*)}\,, \tag{33}$$

which leads immediately to the value of the critical exponent η,

$$\eta = 2\gamma_\phi(\alpha^*)\,. \tag{34}$$

In field theory and also in the context of the renormalization group, one can determine the anomalous dimension of an infinite set of other local operators as, e.g., $:\phi^p(x):$. Here one can also include derivatives. Here we will only calculate the anomalous dimension of $:\phi^2(x):$, which is directly related to the critical exponents γ and η.

This operator is also important for getting the behavior of the theory for $T\neq T_c$; it is of no real interest to treat the term $\frac{1}{2}m^2\phi^2$ in (2) to all orders in m^2, i.e., to calculate the correlation functions with $m^2\neq 0$, since only the case of small $m^2 (T\simeq T_c)$ is needed. Therefore one treats $\frac{1}{2}m^2\phi^2$ as a perturbation: all correlation functions for small m^2 can be obtained by first calculating them in the case $m^2=0$ and then adding a correction which can be expressed in terms of correlation functions of the type $\langle :\phi^2(x):\phi(x_1)\cdots\phi(x_n)\rangle$.[11,12] This also shows that the properties of the theory for $m^2\gtrsim 0$ or $m^2 \lesssim 0$ are equivalent, i.e., the scaling behaviors for $T\gtrsim T_c$ and $T\lesssim T_c$ are the same. The relevant equations and definitions follow.[11]

(i) Introduce the one-particle irreducible Green's functions $\Delta\Gamma_{(2)}(k,M,\alpha)$ connected in the usual way with the Fourier transform of

$$\langle :\phi^2(x):\phi(x_1)\phi(x_2)\rangle\,.$$

(ii) This function satisfies a Callan-Symanzik equation of the form

$$\left(M\frac{\partial}{\partial M}+\beta(\alpha)\frac{\partial}{\partial\alpha}-\gamma_{\phi^2}(\alpha)\right)\Delta\Gamma_{(2)}(k,M,\alpha)=0\,. \tag{35}$$

The diagrams contributing to $\Delta\Gamma_{(2)}$ are the same as the ones for $\Gamma_{(2,1)}$; one has

$$\Delta\Gamma_{(2)} = 1+\alpha^2 C_{11}(N,n)\ln(k^2/M^2)+\cdots\,, \tag{36}$$

and we have introduced the normalization condition $\Delta\Gamma_2(M,M,\alpha)=1$. Combining (35) and (36), one finds

$$\gamma_{\phi^2}(\alpha) = -2\sum_p \alpha^{2p}C_{p1}\,. \tag{37}$$

Up to two loops, one obtains [using (26)]

$$\gamma_{\phi^2}(N,n,\alpha^2) = -2\left(\alpha^2\frac{1}{N}c_{11}(n)+\alpha^4\frac{1}{N^2}c_{21}^{(\alpha)}(n)+\alpha^4\frac{1}{N}c_{21}^{(\beta)}(n)\right). \tag{38}$$

The connection with the exponent γ is

$$1/\gamma = 1+\gamma_{\phi^2}/(2-\eta)\,. \tag{39}$$

The singular behavior of some other physical quantities parametrized by other critical exponents are often expressed in terms of η and γ,[13] leading to the so-called scaling laws. We do not go here into a detailed discussion of the possible validity of these relations in this framework: we give only some remarks.

(a) From the scaling behavior of the correlation functions (32) (which also contain the behavior of the correlation functions involving $:\phi^2(x):$—each s corresponds roughly to the insertion of one operator $:\phi^2(x):$), one can obtain the relation $\gamma=(2-\eta)\nu$. In fact, (39) and $\gamma=(2-\eta)\nu$ are on the same footing: a direct reasoning gives first $1/\nu=(2-\eta)+\gamma_{\phi^2}$ and then (39).

(b) An important set of scaling laws involves the behavior of the energy correlation function, which is directly related to the specific heat. The relation $\alpha=2-[\gamma/(2-\eta)]\,n=2-\nu n$ can be obtained by assuming that the operator $:\phi^2(x):$ gives the most singular behavior in the energy fluctuation, which cannot in general be justified. Clearly, at $T=T_c$ the term $:\phi^2(x):$ no longer appears in the "energy operator." One finds by direct calculation that the relation $\alpha=2-\nu n$ does not exist and that α therefore has to be treated as an independent exponent.

(c) The critical exponents β and δ cannot be calculated directly: as far as I know there is no thermodynamic relation that connects β and δ to one of the operators $:\phi^p(x):$. However, there is really no problem: although it is impossible to derive an explicit equation of state or to calculate the singular behavior of the total free energy, one can establish that these quantities scale. General thermodynamic considerations allow us then to obtain β and δ—via scaling relations—as a function of the other already calculated exponents.

A more complete discussion of these points—also in the context of the ϵ expansion—is published elsewhere.[14]

IV. NATURE OF THE FIXED POINT

Before discussing the possible position and nature of the fixed point, it is of some interest to study in detail the effect of the summation of the bubble diagrams. From Eqs. (32) and (34) we obtain for the four-point function

$$\Gamma_{(4,0)}(k) = k^{4-n}(k^2/M^2)^{-\eta}. \tag{40}$$

If we calculate the same expression by including only the "Born" diagrams (see Fig. 8), one easily obtains

$$\Gamma_{(4,0)}(k) \sim k^{4-n}, \tag{41}$$

while to lowest order in the coupling constant λ_4,

$$\Gamma_{(4,0)} \sim \lambda_4 \sim \Lambda^{4-n}.$$

Remembering that η is very small ($\eta \simeq 0.05$), one sees from (40) and (41) that the contribution of the chain diagrams of Fig. 1 has almost produced the correct k dependence of $\Gamma_{(4,0)}(k)$; the additional small power of k has then to come from the logarithms. Since in general the magnitude of this additional power is a measure for the fixed point value of the coupling constant, one expects that this value should be small.

This is a major difference with Wilson's ϵ expansion; in his approach the full k dependence of $\Gamma_{(4,0)}(k)$ has to come from a summation of logarithms. For small $\epsilon \equiv 4-n$, (40) shows that the power one has to generate is also small and the fixed point value of the coupling constant is then also expected to be small. The ϵ expansion shows exactly this behavior: the fixed point value is $\simeq\epsilon$. However, when ϵ gets larger, the relevant coupling constants get also large and one can expect difficulties.

To get some better understanding of this, we have to introduce more quantitative results.

(i) We define the renormalized and dimensionless coupling constant in $4-n$ dimensions as g_r; the usual coupling constant λ_4 itself is then $\lambda_4 = M^{4-n} g_r$.[15] Sometimes one takes $M=m$; this is a misleading procedure. Note that at $T=T_c$, $m=0$, so that then also $\lambda_4=0$. Of course, as long as $m\neq 0$, one can do the identification but then the small momentum limit (but still $k>m$) is not straightforward.

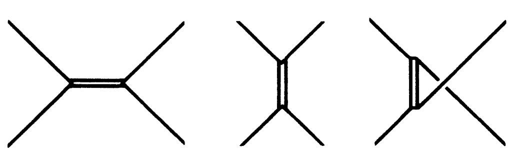

FIG. 8. Set of the "Born" diagrams giving rise to (41).

(ii) If one calculates $\Gamma_{(4,0)}(k)$ for small values of ϵ, one finds (up to the one loop diagrams and expanding in ϵ)

$$\Gamma_{(4,0)}(k) = M^{4-n} g_r + g_r^2 a^2 \ln(k/M) + g_r^2 \times O(\epsilon) + \cdots . \tag{42}$$

To obtain this expression, we have subtracted a constant counter term of the form g_r^2/ϵ; the second term in this expression refers to the four-dimensional value of the bubble diagram. We have here also properly taken into account the crossed diagrams; we also neglect any reference to internal degrees of freedom.

(iii) Now we want the logarithmic corrections to build up the correct k dependence of $\Gamma_{(4,0)}$ as expressed by (40). To lowest order in ϵ, this can be achieved as follows:

$$\begin{aligned}\Gamma_{(4,0)}(k) &\simeq g_r + \epsilon g_r \ln M + (1-\delta) g_r^2 a^2 \ln(k/M) \\ &\quad + \delta g_r^2 a^2 \ln(k/M).\end{aligned} \tag{43}$$

Taking $\epsilon = \delta a^2 g_r$, one obtains

$$\begin{aligned}\Gamma_{(4,0)}(k) &\simeq g_r + \epsilon g_r \ln k + (1-\delta) g_r^2 a^2 \ln(k/M) \\ &\simeq g_r k^\epsilon (k/M)^{[(1-\delta)/\delta]\epsilon}.\end{aligned} \tag{44}$$

Comparison with (40) would imply $\eta \sim \epsilon$; below we will see that $\delta = 1$, giving $\eta \sim O(\epsilon^2)$ as it should.

(iv) In our approach, as explained in Sec. II, the chain diagrams led to expression (41) for $\Gamma_{(4,0)}$; to lowest order in ϵ, this leads to the requirement that the total logarithmic contribution of the single bubble should be used up to produce k^ϵ. This implies that one should take $\delta = 1$ in (44).

(v) This result is also obtained directly via the renormalization group, using the ϵ expansion. Within the context of the Callan-Symanzik equation, one can prove quite generally that the fixed-point value of g_r to lowest order in ϵ is given by[15]

$$\beta(g_r, \epsilon) \simeq \beta(g_r) - \epsilon g_r = 0, \tag{45}$$

where $\beta(g_r)$ refers strictly to the four dimensional case.

For Φ^4 theory, one has $\beta(g_r) = a^2 g_r^2 + O(g_r^3)$, leading immediately to the fixed-point value (called g_r^*)

$$g_r^* = \epsilon/a^2 + O(\epsilon^2), \tag{46}$$

which is exactly the same result as obtained under (iii) for $\delta = 1$. Here it is essential that the wave-function renormalization does not contribute to the terms of order g_r^2 in $\beta(g_r)$, a well-known feature of Φ^4 theory.

Of course, the arguments given here cannot be considered as a justification of the summation procedure outlined in Sec. II; they nevertheless give an indication that the assumption allowing a summing up of logarithms via a Callan-Symanzik equation or of powers as we did, are presumably on the same footing.

In three dimensions, we have calculated the coefficients defined by (19) for the one-loop diagrams. Using (24), (26), and (39), we obtain

$$\beta(\alpha)=(2/N)\,\alpha^3\,[c_1(3)+d_1(3)],$$
$$\eta(\alpha)=(2/N)\,\alpha^2\,d_1(3), \tag{47}$$
$$\gamma^{-1}(\alpha)=1-(1/N)\,\alpha^2\{2c_1(3)/[2-\eta(\alpha)]\},$$

with

$$c_1(3)=16/(2\pi)^2 \quad\text{and}\quad d_1(3)=4/3\pi^2.$$

One immediately sees that $\beta(\alpha)>0$ for α close to the origin [i.e., if $\alpha>0$; this is not a restriction for what follows, since one can also write $d\alpha/dt=a^2\alpha^3$ as $d\alpha^2/dt=\frac{1}{2}a^2(\alpha^2)^2$]. This implies that the origin, i.e., $\alpha=0$, is a possible stable fixed point for $k\to 0$; this fixed point is of course not interesting. The next stable fixed point would then be the second zero of $\beta(\alpha)$ away from the origin: to find this we would have to calculate $\beta(\alpha)$ at least up to order α^7, i.e., up to three loops. Logically, it is possible that this happens; the theoretical structure of this paper would then lead to very difficult practical calculations.

In fact, we now present arguments indicating that the first zero of $\beta(\alpha)$ away from the origin is presumably the relevant fixed point. If this actually happens, then it is *a priori* possible to find this zero if $\beta(\alpha)$ is known up to order α^5, i.e., up to two loops (the diagrams are represented in Fig. 7).

The first point consists in the observation that in calculating the logarithmic corrections as explained in Sec. III, we have used the propagator (13) for which we have already taken the small momentum limit; therefore, these corrections are strictly speaking only valid in the small momentum limit. In other words, this expansion should be taken at the fixed point value α^* and the usual arguments[10] that one makes in order to decide whether a fixed point is stable or unstable cannot be used here anymore.

Let us go back to the Wilson expansion to see if we can understand this point better. Equation (46) shows that the fixed-point value of the coupling constant is mainly fixed (here to lowest order in ϵ) by the bubble diagram that builds up the k^ϵ dependence of $\Gamma_{(4)}(k)$. In our method, we loose the explicit connection with the fixed point value of the coupling constant but it seems reasonable to extrapolate that our summation of the bubble diagrams (with $k\to 0$) leads us automatically to the fixed-point value α^*. At the same time, one can understand why the ϵ expansion gives an approximate expression for this fixed point value only by calculating the lowest-order corrections [see (46)], while we have to go to two loop corrections. In the ϵ expansion, the full power present in (40) comes from the summation of logarithms, governed by a Callan-Symanzik equation; in our approach we have split up the limit $k\to 0$ by first doing the bubble summation and then summing the remaining logarithmic corrections by a Callan-Symanzik equation. Then we no longer have an equation similar to (46). It is also no surprise that our $\beta(\alpha)$ is positive around the origin, since a large "negative" piece present in $\beta(g_r,\epsilon)$ has been used up for the bubble summation. However, I have not been able to make these points as quantitative as the ones formalized in (42)–(46).

Clearly, the explicit higher-order calculations can only give us more insight. From (47) and general positivity arguments one finds that the fixed point value α^{*2} has to be in the range 0.1–0.2 in order to have $\eta\simeq 0.05$. One can also treat α^{*2} as an unknown parameter, leading then to a relation between $\eta(\alpha^*)$ and $\gamma(\alpha^*)$; one finds $\eta(\alpha^*)\simeq 0.12$ if $\gamma(\alpha^*)$ is fixed to be 1.25.

It is often argued that the arguments presented in Sec. III are of purely dimensional origin; this is certainly not true, although as usual in physics dimensionality plays a role. To illustrate this point we will show that arguments apparently as "trivial" as those used in (43) and (44) will allow us to get within the ϵ expansion the terms up to order ϵ in the critical exponent γ, without doing any calculation! Accepting then the scaling laws as one usually does, it gives the terms up to order ϵ in all other critical exponents.

To see this, let us "calculate" the contribution to $\Delta\Gamma_{(2)}$ [for the definition, see (35) and (36)]: to lowest order, this contribution will be

$$\Delta\Gamma_{(2)}=1+\tfrac{1}{3}a^2 g_r \ln(k/M), \tag{48}$$

where a^2 is the same number as in (42). Note that we never have to calculate a^2! The factor $\frac{1}{3}$ comes simply from the observation that in (48), only one bubble appears, while in (42) we had to take into account three identical bubbles, coming from crossing.

Summing up (48), using (46), we obtain

$$\Delta\Gamma_{(2)}\sim(k/M)^{\epsilon/3}. \tag{49}$$

This gives immediately $\gamma_{\Phi^2}=-\frac{1}{3}\epsilon$ leading directly [using (39)] to

$$\gamma=1+\tfrac{1}{6}\epsilon+O(\epsilon^2). \tag{50}$$

For $\epsilon=1$, this gives $\gamma=1.17$ to be compared with the correct value $\gamma=1.25$. For fun, if we calculate (39) exactly, we obtain $1/\gamma=1-\frac{1}{6}\epsilon$, which for $\epsilon=1$ gives $\gamma=1.20$.

The same arguments can be given when there are N degrees of freedom. Using then a property of

$I_{abcd} \equiv \delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}$, namely,

$$\sum_{x,y} I_{abxy}I_{xycd} = (N+2)\,\delta_{ab}\delta_{cd}+2I_{abcd},$$

one finds instead of (42),

$$\Gamma_{(4,0)}(k) = I_{abcd}\left(M^{4-n}g_r+(N+8)\,g_r^2 a^2\ln\frac{k}{M}+\cdots\right);$$

instead of (48),

$$\Delta\Gamma_{(2)}(k)=\delta_{ab}\left(1+(N+2)\,g_r^2 a^2\ln\frac{k}{M}+\cdots\right);$$

and instead of (50),

$$\gamma = 1+[(N+2)/2(N+8)]\epsilon+O(\epsilon^2), \tag{51}$$

which is a well-known result.

Again if we use the exact result $1/\gamma = 1 - [(N+2)/2(N+8)]\epsilon$ for $N=3$ and $\epsilon=1$, we find $\gamma \simeq 1.30$ as compared with the best-known value $\gamma \simeq 1.38$.

It is amusing to compare these simple arguments with the rather involved calculations in Ref. 1, especially the discussion in Sec. 4 and the result after Eq. (4.50), $\nu = 0.5+\frac{1}{12}\epsilon+O(\epsilon^2)$, which is exactly the same as (50), since $\nu=\frac{1}{2}\gamma$ to this order in ϵ.

It should be emphasized that the arguments presented here are by no means sloppy: it is clear that the transition from (48) to (49) is as rigorous as solving the Callan-Symanzik equation as one can directly verify. In this respect some additional clarification can be gotten by tracing back the origin of the term $-\epsilon g_r$ in (45): this term comes from the dimensional factor M^{4-n} in (42). One has

$$\beta(g_r\epsilon)\equiv\left(M\frac{\partial g_r}{\partial M}\right)_{\lambda_4^u}, \tag{52}$$

where $\lambda_4^u = M^{4-n}g_r+\cdots$ and one gets directly in (52) the term $-\epsilon g_r$. Therefore, $-\epsilon g_r$ is the amount one picks up from $\beta(g_r)$ in order to obtain the jump in $\Gamma_{(4)}$ from M^ϵ to k^ϵ: this is a dynamical phenomenon and a very interesting aspect of the ϵ expansion.

V. CONTRIBUTION OF OTHER INTERACTION TERMS

When n becomes equal to 3 or smaller, one has to include higher order terms in $\mathcal{H}$ as we explained in Sec. II. For $n=3$, only $(\sum_a \Phi^a\Phi^a)^3$ contributes: for many of the later arguments it is instructive to discuss this case first.

The calculation of the corresponding Green's functions is straightforward: we have calculated the diagrams of Fig. 9, also including the many diagrams one obtains by crossing and the already calculated ones of Fig. 6. The details of some of the calculations are collected in the Appendix.

We find

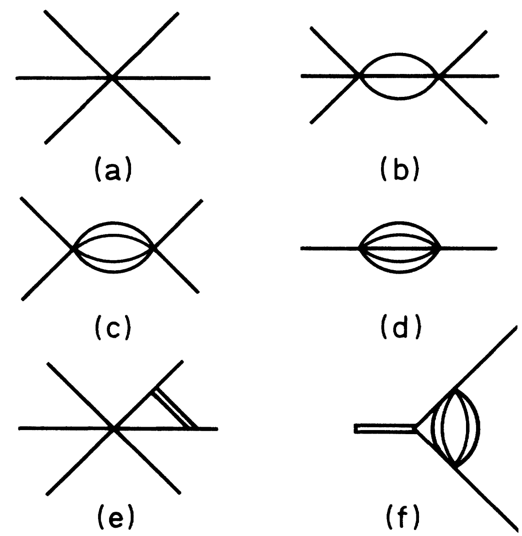

FIG. 9. Typical diagrams involving the vertex Φ^6.

$$\Gamma_{(6,0)} = \lambda+(3N+22)\frac{\pi^3}{(2\pi)^5}\lambda^2\ln\frac{k^2}{M^2} + \frac{16}{N}\frac{15}{(2\pi)^2}\alpha^2\lambda\ln\frac{k^2}{M^2}+\cdots,$$

$$\Gamma_{(2,1)} = \alpha+\frac{16}{N}\frac{1}{(2\pi)^2}\alpha^3\ln\frac{k^2}{M^2} - \frac{5(N+4)\,(N+2)\pi^5}{16(2\pi)^9}\alpha\lambda^2\ln\frac{k^2}{M^2}+\cdots,$$

$$\Gamma_{(2,0)} = k^2\left(1-\frac{4}{3\pi^2N}\alpha^2\ln\frac{k^2}{M^2} - \frac{(N+4)\,(N+2)\pi^5}{48(2\pi)^9}\lambda^2\ln\frac{k^2}{M^2}-\cdots\right), \tag{53}$$

$$\eta(\alpha,\lambda) = \frac{8}{3\pi^2N}\alpha^2+\frac{(N+4)\,(N+2)\pi^5}{24(2\pi)^9}\lambda^2+\cdots. \tag{54}$$

In these expressions λ stands for the coupling constant corresponding to Fig. 9(a); in $\Gamma_{(6,0)}$ we have omitted the over-all factor (completely symmetric in all indices):

$$I_{abcdef}=\delta_{ab}\delta_{cd}\delta_{ef}+\cdots.$$

Similarly, in $\Gamma_{(2,1)}$ and $\Gamma_{(2,0)}$ we left out the factor δ_{ab} and $\Gamma_{(0,2)}$ has no contributions to this order.

The Callan-Symanzik equation will now contain two β functions, $\beta_\lambda(\alpha,\lambda)$ and $\beta_\alpha(\alpha,\lambda)$, which can easily be obtained from (53). We do not go into a detailed discussion of their properties: the results are uninteresting and can be seen directly from (53) and (54); one notes that because of phase space factors—see also the corresponding formulas in the Appendix—the effect on α and the contribution to $\eta(\alpha,\lambda)$ of Φ^6 are totally negligible. Since the expression for $\Gamma_{(6,0)}$ shows that the origin is a stable fixed point for λ when $k\rightarrow 0$, one expects that one simply can put effectively $\lambda=0$. One can also understand easily that the origin is infrared stable for all terms $(\Phi^2)^p$ $(p\geq 2)$, when n and p are related as in (4). These remarks are of course only true if the corresponding interaction constants are positive, which one expects from general positivity arguments of the total energy when one considers fluctuations around the equilibrium value.

We now consider what happens if $n<3$. If n is close to 3 one can use an ϵ expansion for Φ^6, where now $\epsilon=3-n$. When this ϵ is no longer small, one can again do a bubble summation, where the elementary bubble is now as in Fig. 9(b). To do this, one first performs a canonical transformation similar as the one done in (7) by adding a term to the Hamiltonian of the form

$$-\frac{1}{\lambda'}\sum_a (\chi^a+\lambda'\Phi^a\Phi^2)(\chi^a+\lambda'\Phi^a\Phi^2),$$

where λ' is so chosen that the original term Φ^6 is cancelled and instead one then obtains a new interaction of the form $\sum_a \chi^a\Phi^a\Phi^2$, where the propagator for the field χ^a is a constant. One then sums the bubbles present in the propagator for χ^a, leading to a "new" propagator proportional to $k^{2(3-n)}$. After these transformations, the coupling constant with a dimension has disappeared and one will again get only logarithmic divergences for all dimensions.

The procedure sketched above can be repeated each time a new interaction comes in the picture [see (4)]. Obviously, phase space continues to be very unfavorable for these new interactions. It is therefore rather plausible that all these interactions will have no real effect on the problem when n comes down to 2 and the formulation as presented in Sec. III remains valid. But clearly, one will keep scaling for all dimensions even if some of these new interactions would not disappear when $k\rightarrow 0$.

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APPENDIX

To calculate Feynman diagrams coming from the interaction terms $:\Phi^p(x):$, we use the method of dispersion relations. A direct evaluation of the relevant diagrams becomes soon very complicated; via dispersion relation techniques, the evaluation becomes, in general, trivial. We emphasize, however, that the simplification of this method is strongly dependent on the condition $m=0$; but that is all we need.

The important formulas that we need to know correspond to the evaluation of the contribution of a Feynman diagram as represented in Fig. 9(d), where k is the external momentum. We consider here a general diagram with r internal lines; again the number of dimensions n is a continous variable. Such a diagram gives a contribution equal to (we consider here only one degree of freedom)

$$I_E^{(r)}(k^2,n)=\left(\frac{1}{(2\pi)^n}\right)^{r-1}\frac{1}{r!}\int\frac{d^np_1}{p_1^2}\cdots\int\frac{d^np_{r-2}}{p_{r-2}^2}\int d^np_{r-1}\Big/ p_{r-1}^2\left(k-\sum_i p_i\right)^2. \tag{A1}$$

The index E refers to euclidean; to get a dispersion relation we make an analytic continuation of the momenta, the so called Wick rotation. The corresponding formula corresponds then to the well-known expressions of relativistic quantum field theory

$$I_M^{(r)}(k^2,n)=-(i)^{r-1}\left(\frac{1}{(2\pi)^n}\right)^{r-1}\frac{1}{r!}\int\frac{d^np_1}{(p_1^2+i\epsilon)}\cdots\int d^np_{r-1}\Big/(p_{r-1}^2+i\epsilon)\left[\left(k-\sum_i p_i\right)^2+i\epsilon\right], \tag{A2}$$

where we have used the correspondence (M stands now for Minkowski space time)

$$\begin{aligned}&(p^2)_M=p^{02}-(\vec{p}\,)^2\rightarrow(p^2)_E=p^{02}+(\vec{p}\,)^2,\\&(p^0)_M=i(p^0)_E,\\&(d^np)_M=i(d^np)_E,\\&I_E^{(r)}(k^2,n)=I_M^{(r)}(-k^2,n).\end{aligned} \tag{A3}$$

Expression (A2) is an analytic function in k^2 with a unitary cut along the positive k^2 axis. From Cauchy's theorem one then gets

$$I_M^{(r)}(k^2,n)=\frac{1}{2\pi}\int_0^\infty\frac{\rho^{(r)}(k'^2,n)}{k'^2-k^2+i\epsilon}\,dk'^2, \tag{A4}$$

where $\rho^{(r)}(k^2,n)$ is up to a factor $1/i$ the jump over the cut of (A2). We can get this jump by making the substitution[16] in (A2),

$$(p_i^2+i\epsilon)^{-1}\to(-2\pi i)\,\delta(p_i^2)\,\theta(p_i^0).$$

We then get

$$\rho^{(r)}(k^2,n)=\left(\frac{1}{(2\pi)^{n-1}}\right)^r\frac{1}{r!}\left(\prod_{i=1}^{r}\int d^np_i\,\delta(p_i^2)\,\theta(p_i^0)\right)\times(2\pi)^n\,\delta^{(n)}\left(k-\sum_i p_i\right), \quad \text{(A5)}$$

which is nothing but the phase space for the decay of a scalar particle of mass k^2 into r identical massless scalar particles.

Below we will derive the following result:

$$\rho^{(r)}(k^2,n)=\frac{(2\pi)^n}{[(2\pi)^{n-1}]^r\,r!}\,R^{(r)}(k^2,n), \quad \text{(A6)}$$

$$R^{(r)}(k^2,n)=\left(\frac{\pi^{(n-1)/2}}{2^{n-2}\,\Gamma[\frac{1}{2}(n-1)]}\right)^{r-1}\times\frac{[\Gamma(n-2)]^{r-1}\,\Gamma[\frac{1}{2}(n-2)]}{\Gamma[\frac{1}{2}r(n-2)]\,\Gamma\{[\frac{1}{2}(r-1)](n-2)\}}\,k^{r(n-2)-n}. \quad \text{(A7)}$$

For the special case of $n=3$, one obtains

$$\rho^{(r)}(k^2,3)=\frac{2(\pi)^{3/2}(8\pi)^{-r+1}}{r!\,\Gamma(\frac{1}{2}r)\,\Gamma[\frac{1}{2}(r-1)]}\,k^{r-3}. \quad \text{(A8)}$$

The derivation of (A7) is obtained as follows:

(i) One first calculates the decay of the particle with mass k^2 into one zero mass particle ($p_r^2=0$) and another particle with mass $(k-p_r)^2$; then the decay of a particle with mass $(k-p_r)^2$ into a zero mass particle ($p_{r-1}^2=0$) and another particle with mass $(k-p_r-p_{r-1})^2$, ...

$$R^{(r)}(k^2,n)=\int d^np_r\,\delta(p_r^2)\,\theta(p_r^0)\times\left[\left(\prod_{i=1}^{r-1}\int d^np_i\,\delta(p_i^2)\,\theta(p_i^0)\right)\times\delta^{(n)}\left(k-p_r-\sum_{i=1}^{r-1}p_i\right)\right].$$

The last factor is clearly $R^{(r-1)}((k-p_r)^2,n)$.

$$\begin{aligned}R^{(r)}(k^2,n)&=\int_0^{k^2}dM_{r-1}^2\int d^nk_{r-1}\int d^np_r\,\delta(k_{r-1}^2-M_{r-1}^2)\,\delta(p_r^2)\,\delta^{(n)}((k-p_r)-k_{r-1})\,R^{(r-1)}(M_{r-1}^2,n)\\&=\int_0^{k^2}dM_{r-1}^2\,S^{(2)}(k^2,M_{r-1}^2,n)\,R^{(r-1)}(M_{r-1}^2,n)\end{aligned} \quad \text{(A9)}$$

where $S^{(2)}(k^2,M^2,n)$ stands for the decay of a particle with mass k^2 into a massless particle and a particle with mass M^2.

(ii) After a trivial calculation, one gets

$$S^{(2)}(k^2,M^2,n)=\Omega_n\,(k^2-M^2)^{n-3}/2(2k)^{n-2}, \quad \text{(A10)}$$

where the factor $\Omega_n=2\pi^{(n-1)/2}/\Gamma[\frac{1}{2}(n-1)]$ comes from the angular integrations.

(iii) Equation (A9) can now be iterated to give ($M_r^2=k^2$)

$$R^{(r)}(k^2,n)=(\Omega_n)^{n-2}\int_0^{M_r^2}dM_{r-1}^2\,\frac{(M_r^2-M_{r-1}^2)^{n-3}}{2(2M_n)^{n-2}}\cdots\times\int_0^{M_3^2}dM_2^2\,\frac{(M_3^2-M_2^2)^{n-3}}{2(2M_3)^{n-2}}\,\frac{(M_2^2)^{n-3}}{2(2M_2)^{n-2}}. \quad \text{(A11)}$$

(iv) In each of these integrals one introduces the scaled variables x_i

$$x_i=M_i^2/M_{i+1}^2,\quad dx_i=dM_i^2/M_{i+1}^2;$$

we then get

$$R^{(r)}(k^2,n)=\left(\frac{\pi^{(n-1)/2}}{2^{n-2}\Gamma[\frac{1}{2}(n-1)]}\right)^{r-1}k^{r(n-2)-n}\times\left(\prod_{j=2}^{r-1}\int_0^1 dx(1-x)^{n-3}x^{[j(n-2)-n]/2}\right). \quad \text{(A12)}$$

The last integrals are well known

$$\int_0^1 dx\,(1-x)^\alpha x^\beta=\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

Writing the factors, one immediately finds (A7).

In integral (A4) one needs to introduce, in general, subtractions and some care with the infrared behavior.

We can now easily calculate, e.g., in three dimensions the contributions of Figs. 9(b)–9(d); using (A8), one gets

$$\rho^{(3)}(k^2,3)=1/3!\,16\pi,$$
$$\rho^{(4)}(k^2,3)=(1/4!\,2^7\pi^2)\,k, \tag{A13}$$
$$\rho^{(5)}(k^2,3)=(1/5!\,3\times 2^9\,\pi^4)k^2.$$

This gives (we give only the renormalized results):

$$I_E^{(3)}(k^2,3)=-(1/3!\,32\pi^2)\ln(k^2/M^2),$$
$$I_E^{(4)}(k^2,3)=(1/4!\,2^8\,\pi^2)\,k, \tag{A14}$$
$$I_E^{(5)}(k^2,3)=(1/5!\,3\times 2^{10}\,\pi^5)\,k^2\ln(k^2/M^2).$$

The diagram of Fig. 9(f) is now also easy to calculate: using $I_E^{(4)}(k^2,3)$, it is proportional to the vertex diagram of Fig. 6.

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[1] A rather complete review of K. Wilson's work and others, together with an extensive list of references may be found in K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974). We apologize for the very subjective list of references given in this paper.

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