Intrinsic response time of a Josephson tunnel junction*

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The microscopic theory of the Josephson effect is reformulated in the time domain. The four terms in the usual theory are seen to be necessary consequences of intrinsic junction delays. The tunneling current flowing in response to a voltage pulse is shown to rise in a time $\hbar/2\Delta$ and undergo damped energy-gap oscillations, before approaching the expected supercurrent. The theory provides a technique for calculating junction behavior when connected to an arbitrary circuit.

In his original paper Josephson[1] described a formal framework which included essentially all of the details of the effect he was predicting. It has remained for others to develop those details. For example, Werthamer[2] generalized Josephson's work to include time-dependent voltages across the junction and showed that there are two complex-valued frequency-dependent terms, I_{qp} and I_J, in the tunneling current. At low temperatures and voltages one of these terms, $\mathrm{Re} I_J$, sometimes called the sine term, describes the tunneling of pairs and gives rise to Josephson's effect; another, $\mathrm{Im} I_{qp}$, describes the tunneling of quasiparticles (elementary excitations in a superconductor) and leads to the nonlinear current-voltage characteristic discovered by Giaever.[3] Approximations to these two terms have been incorporated into a phenomenological theory[4] (called the resistively shunted junction model) which has been used with success to describe a substantial number of experimental results.

The remaining two terms in the time-dependent theory are important only when the voltage or frequencies at the junction are higher than approximately the energy gap. One of these terms is the cosine term, $\mathrm{Im} I_J$, which has been the subject of a great deal of current interest.[5] Recently Harris[6,7] has shown that, in the small-sinusoidal-signal limit, the cosine term and the remaining reactive part of the quasiparticle term, $\mathrm{Re}\, I_{qp}$, can be interpreted as opposite-phase parts of the sine and quasiparticle terms, respectively.

Because of the success of the phenomenological theory, these latter two terms have been omitted from most analyses of the Josephson effect and their importance has not been widely pursued. Unfortunately, however, a model limited to two terms is incapable of describing the detailed behavior of Josephson devices used in new applications where the frequencies are above the energy gap and intrinsic junction delays are significant. Therefore, in this paper we will demonstrate that all four terms are required to discuss the detailed time dependence of the Josephson effect and show that their frequency-dependent structure determines an intrinsic junction response time of order $\hbar/2\Delta$.[8]

We note first that in linear systems it is useful to deal with Fourier coefficients of, for example, the current density $J(\omega)$ flowing in response to an electric field $E(\omega)$ applied to the system. A frequency-dependent complex-valued proportionality coefficient (the conductivity) relates the two: $J(\omega)=\sigma(\omega)E(\omega)$. It is well known that the frequency dependence of $\sigma(\omega)$ is a reflection of time delays inherent in the system.[9] In fact, if the disturbance is a pulse (a δ function), the time-dependent response is just proportional to the Fourier transform of $\sigma(\omega)$ into the time domain:

$$\sigma(t)=\int_{-\infty}^{\infty}\sigma(\omega)e^{-i\omega t}\,d\omega\,.$$

Similarly, the frequency dependence of each of the four terms in the Josephson tunneling current is a reflection of intrinsic delays in that system. Unfortunately, the nonlinearity of the junction makes its description somewhat more involved. However, the formalism resulting from a description of these delays adds new intuitive understanding of the behavior of the device and provides a computational technique for finding the response of a junction to any electrical stimulus.

To examine the time response of the Josephson junction we rewrite in the time domain the nonlinear microscopic high-frequency theory of the effect[2,6,7,10,11]:

$$I(t)=\frac{1}{2\pi}\,\mathrm{Im}\left(U^*(t)\int_{-\infty}^{\infty}I_{qp}(t-t')\,U(t')\,dt' + U(t)\int_{-\infty}^{\infty}I_J(t-t')\,U(t')\,dt'\right), \tag{1a}$$

where

$$U(t)=\exp[i\phi_s(t)] =\exp\left(-i\,\frac{e}{\hbar}\int_{-\infty}^{t}V(t')\,dt'+\tfrac{1}{2}i\phi_0\right). \tag{1b}$$

In the derivation of this equation, it is assumed that at all times the superconductors on each side of the insulating barrier are described as equilibrium superconductors of negligible spatial extent. Equilibrium is also required indirectly because tunneling is introduced using a perturbation approach, producing a result that is strictly valid only in the limit of low coupling between the two superconducting electrodes. In some real junctions in which the coupling is not small, additional effects may occur as equilibrium is disturbed for the large currents which flow in them.

We comment briefly on each of the quantities in Eq. (1). First $U(t)$ is a solution of the time-dependent Schrödinger equation with Hamiltonian $eV=e(V_L-V_R)$, where L and R refer to the left and right sides of the junction, respectively. $U(t)$ is analogous to the product of the macroscopic wave functions discussed by Feynman[12]: $\psi_L^*\psi_R$. The phase $\phi_s(t)$ of $U(t)$ is, however, that for a single electron, and is thus half of the usual pair phase. The quantities $I_{qp}(t)$ and $I_J(t)$ are the Fourier transforms of the corresponding quantities in the frequency domain.[13] Since $I_{qp}(\hbar\omega)$ and $I_J(\hbar\omega)$ obey $I(-\hbar\omega)=I^*(\hbar\omega)$ and the Kramers-Kronig relations,[6,7] the corresponding quantities in the time domain are real and vanish for times prior to zero.[9] The two complex-valued terms (four altogether) in the frequency domain thus collapse into two real ones in the time domain.

We have obtained analytic forms for $I_{qp}(t)$ and $I_J(t)$ using Werthamer's formulas for $I_{qp}(\hbar\omega)$ and $I_J(\hbar\omega)$ when the superconductors on each side of the barrier are identical and at absolute zero.[2,14] To evaluate the functions in the time domain it is first helpful to think of the quasiparticle term as divided into a part which is a pure resistance R_N plus whatever correction is necessary. The pure resistance gives rise to an instantaneous response; the remaining part is delayed by the internal dynamics of the junction. Thus, we find that

$$I_{qp}(t)=-2\pi\,(\hbar/eR_N)\,\delta'(t)+I_{qp\,0}(t)\,, \tag{2a}$$

where

$$I_{qp\,0}(t)=\frac{2\pi^2\Delta^2}{\hbar eR_N}\,J_1\!\left(\frac{t\Delta}{\hbar}\right)\,Y_1\!\left(\frac{t\Delta}{\hbar}\right)\,.$$

We also find

$$I_J(t)=\frac{2\pi^2\Delta^2}{\hbar eR_N}\,J_0\!\left(\frac{t\Delta}{\hbar}\right)\,Y_0\!\left(\frac{t\Delta}{\hbar}\right)\,. \tag{2b}$$

The J_n and Y_n are the nth-order Bessel functions of the first and second kinds.

Given the formal result of Eq. (1) we can now consider the case of a voltage pulse applied to the junction at time $t=0$: $V(t)=V_0\tau\delta(t)$.[15] Note that the voltage is zero at all other times. The phase difference $\phi_s(t)$ across the junction decreases at $t=0$ by $\Delta\phi_s=-eV_0\tau/\hbar$. For simplicity we choose $\phi_s(t)=0$ for $t<0$. Thus $\phi_s(t)=\Delta\phi_s=-eV_0\tau/\hbar$ for $t>0$. Now one simply evaluates Eq. (1) and finds the following:

$$I(t)=(V_0\tau/R_N)\,\delta(t)+[S_{qp\,0}(t)-S_J(t)]\,\sin(\Delta\phi_s)-[-\pi\Delta/2eR_N-S_J(t)]\,\sin(2\Delta\phi_s)\,. \tag{3}$$

Here

$$S_{qp\,0}(t)=-\frac{\pi\Delta}{2eR_N}-\frac{1}{2\pi}\int_0^t I_{qp\,0}(t')\,dt'\,,$$

and

$$S_J(t)=-\frac{\pi\Delta}{2eR_N}-\frac{1}{2\pi}\int_0^t I_J(t')\,dt'\,.$$

The integrals $S(t)$ are plotted in Fig. 1. Both the delayed part of the quasiparticle current $S_{qp\,0}(t)$ and the Josephson current $S_J(t)$ rise from $-\pi\Delta/2eR_N$, oscillate at the energy gap frequency, and approach zero. They differ in that $S_{qp\,0}(t)$ initially rises linearly while $S_J(t)$ rises with infinite initial slope because of a logarithmic singularity in $I_J(t)$ at $t=0$. One can now see clearly that while the Josephson part of the current does not turn on as fast as the instantaneous part of the quasiparticle current, the Josephson current does rise in a time comparable to $\hbar/2\Delta$ (0.23×10^{-12} sec for niobium). The delayed part of the quasiparticle current $S_{qp\,0}(t)$ rises on the same time scale but somewhat more slowly.

Having defined the quantities in Eq. (3), we can turn to that result itself. The total current it describes is plotted in Fig. 2 for the case of a pair phase difference $2\Delta\phi_s$ of $-\frac{1}{2}\pi$. The figure shows the instantaneous part of the quasiparticle current

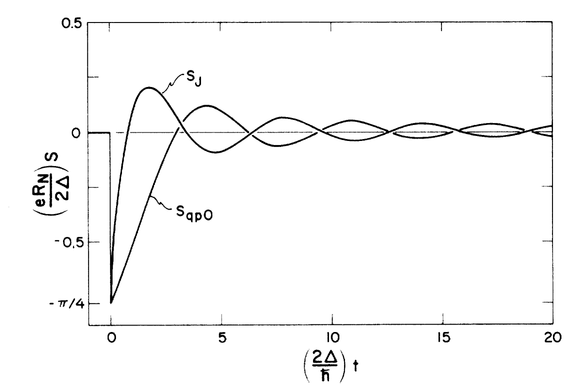

FIG. 1. Josephson (S_J) and delayed quasiparticle (S_{qp0}) parts of the current through a tunnel junction as a function of time.

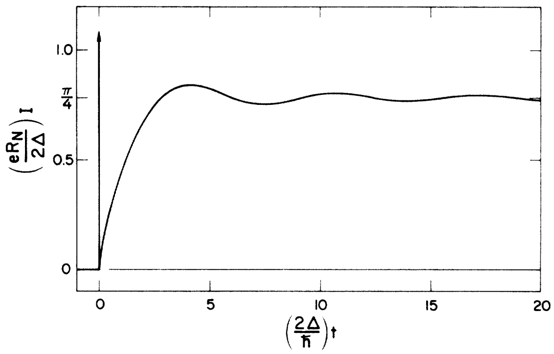

FIG. 2. Total current through a Josephson tunnel junction as a function of time. Junction is subjected to a voltage pulse at time zero sufficient to change the pair phase by $-\frac{1}{2}\pi$.

at time zero described by the δ-function term of Eq. (3). At later times the total current rises rapidly to the expected supercurrent, undergoes damped energy-gap oscillations, and stabilizes at the supercurrent. The second term in Eq. (3) (in square brackets) contains both Josephson and delayed quasiparticle contributions. It is transient in the sense that both contributions vanish for large times. The third term is the one which approaches the expected Josephson supercurrent for large times. The argument of the sine function is $2\Delta\phi_s$, the usual pair phase difference since $\Delta\phi_s$ is the phase difference per single electron.

It is interesting to consider the energy flow in this problem. The energy delivered electrically to the junction from time t to $t+\Delta t$ is of course

$$\int_t^{t+\Delta t} I(t')V(t')\,dt' .$$

Since the voltage is zero at all times except $t=0$, all of the electrical power which flows into the junction does so at time zero. It may at first glance seem somewhat surprising that the current continues to change with time after the energy has been delivered to the junction. Clearly energy storage and dissipative processes must be occuring without being reflected in the electrical portion of the system. One can see from Fig. 2 that most of the stored energy is released within a time $\Delta t \sim \hbar/2\Delta$. If the energy uncertainty ΔE is of the order of the gap 2Δ, then $\Delta E\ \Delta t \sim \hbar$, and the uncertainty principle applies properly. Dissipation is closely connected with the remark in Ref. 7 that the theory is derived assuming that the wave functions on each side of the insulating barrier retain their equilibrium values even after the current is allowed to flow through the junction. If this is to be the case, then an implicit mechanism is assumed which permits the instant thermalization of excess quasiparticles injected by the current into the electrodes. This implicit transfer of energy is what allows the oscillations in the junction current to damp out after all of the electrical energy has been delivered at $t=0$.

The simple calculation we have discussed is useful for a number of reasons: First, it shows quantitatively that there is an intrinsic response time in a Josephson tunnel junction, namely, $\hbar/2\Delta$. It also reveals damped energy gap oscillations in the response.

Second, it makes clear that in the frequency domain both the real and imaginary parts of the response, given for example by $I_J(\hbar\omega)$, derive from the delayed response in the time domain $I_J(t)$. Thus in the frequency domain it is necessary to use both real and imaginary parts to completely describe the delays inherent in the system. Therefore in a complete analysis neither the cosine term nor the reactive part of the quasiparticle current may be omitted. On the other hand, although both real and imaginary parts must be retained in the frequency domain, neither term contains any information not revealed in the other because the two are connected by the Kramers-Kronig relations. As an example of this discussion, the Riedel peak[16] in the sine term is simple a reflection of the damped energy-gap oscillations in $I_J(t)$. The step in the cosine term at the gap frequency is a reflection of the same oscillations and thus contains no information about the response which is not manifested also in the Riedel peak.

Finally the time domain formulation of the theory provides a new tool for computations. To develop this tool one projects the junction voltage ahead in time and calculates the current through the junction self-consistently with the voltage and current for the circuit to which the junction is attached. This approach has already been used to calculate points on the time-averaged I-V curve of a current-biased tunnel junction.[17] The results achieved were in agreement with a calculation of the same I-V curve, but done in the frequency domain by McDonald, Johnson, and Harris.[17]

Experimental observation of the result in Fig. 2 has partly been achieved, in the sense that the quasiparticle current[3] and the Riedel peak in the sine term[18] have been observed. These observations were, however, in the frequency domain. One may be able to observe the delays directly using a material having a small energy gap to keep the intrinsic response time as long as possible.[19] One must also keep the junction capacitance small so that the current of interest will not be shunted.[20] Finally, the voltage pulse in the conceptual problem may need to be replaced by a current step, or some more readily achievable shape. One would then examine the voltage response. While extremely difficult, this experiment may be within the scope of present technology.

In conclusion it is clear that, in theory at least, the Josephson tunnel junction is a very fast device by present standards. However, the theory is limited to devices in which the currents are small enough that equilibrium within the electrodes is not substantially disturbed. Recent devices of potentially great technological importance, such as that reported by Broom, Jutzi, and Mohr,[20] have rather high current densities and probably are not accurately described by the present theory. There is therefore a significant need for high-frequency theories of tunnel junctions whose electrodes are not in thermal equilibrium. Furthermore, there are no existing high-frequency theories of such commonly used devices as microbridges, proximity effect bridges, and point contacts. These devices may involve substantial departures from thermal equilibrium and high-frequency theories of them are needed also.

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[8]A similar response time has been discussed phenomenologically by M. J. Stephen [Phys. Lett. A 46, 289 (1973)].

[9]See, for example, J. S. Toll, Phys. Rev. 104, 1760 (1956).

[10]Equation (1) has also been given by L.-E. Hasselberg, M. T. Levinson, and M. R. Samuelson [Phys. Rev. B 9, 3757 (1974)].

[11]Equation (1a) is different from the constant-voltage case, but Eq. (1b) remains the same. A discussion of the latter is given by B. D. Josephson [Adv. Phys. 14, 419 (1965)].

[12]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1965), Vol. III, p. 21-14.

[13]$I_{qp}(t)$ and $I_J(t)$ have units of current per unit time.

[14]One can predict that when $T \neq 0$ the appearance of $I_{qp}(t)$ and $I_J(t)$ will be qualitatively the same as for $T=0$ since $I_{qp}(\hbar\omega)$ and $I_J(\hbar\omega)$ differ only quantitatively in the two cases. When the two superconductors are different and $T \neq 0$, additional structure appears in the frequency domain at $\hbar\omega = \Delta_L - \Delta_R$. This structure will appear in $I_{qp}(t)$ and $I_J(t)$ with repetition time $h/(\Delta_L - \Delta_R)$.

[15]The τ has units of time and compensates for the units of inverse time introduced by the δ function.

[16]See Fig. 1 of Ref. 5.

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[19]Response times of at least 25 psec might be achieved by using materials having T_c less than about 0.1 K or by operating higher T_c junctions close to T_c.

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