Specific heat of superconducting films: Effect of inhomogeneities*

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(Received 13 October 1975)

Contributions to the specific heat per component of an n-component two-dimensional Ginzburg-Landau field are computed to order g for $n = \infty$, where $g \propto \lambda^2 \langle [\delta T_c(\vec{x})]^2 \rangle_c$ is a measure of the strength of the system inhomogeneity and $\lambda \ll \xi$ is the range of correlations for variations $\delta T_c(\vec{x})$ in the local transition temperature. For a two-component alloy we find $g \propto c(1-c)(d \ln T_c/dc)^2$, where c is the concentration of one component. The order-g contribution $C^{(g)}$ is compared to the order-$1/n$ contribution $C^{(1/n)}$ to the specific heat of the homogeneous system. We find that $C^{(g)}$ is negative in the temperature range for which $C^{(1/n)}$ is positive, and tends to cancel the peak in the specific heat associated with the $O(1/n)$ contribution. This result may explain why no peak was observed in the experiment of Zally and Mochel.

I. INTRODUCTION

It has long been an interesting question whether superconducting films exhibit any specific-heat anomaly at the transition point. The theoretical interest centers on the fact that a superconducting film, with thickness much less than the coherence length, is a physical realization of a two-dimensional system for which the number of components of the order parameter is greater than one (namely two: the real and imaginary parts of the complex order parameter). It has long been known that the traditional type of phase transition, with broken symmetry and long-range order in the low-temperature phase, is impossible for such systems.[1,2] This does not rule out, however, a phase transition of the Stanley-Kaplan type[3] in which the response to an external field, coupled to the order parameter, becomes infinite at and below a critical temperature, but with no long-range order. At present it is not known whether such a transition is accompanied by an anomaly in the specific heat.

On the experimental side, the results of Zally and Mochel[4] indicate no anomalous behavior, with the specific heat a monotonically decreasing function of temperature in the critical region. Such behavior was observed to be in good agreement with existing theory.[5] The theory, however, is equivalent to the exact solution of the n-component Ginzburg-Landau model in the limit $n \to \infty$, whereas a superconductor corresponds to $n = 2$. The use of this approximation (Hartree approximation) was motivated by the lack of an exact solution for finite n, and by the reasonableness of the results. Furthermore, the $n = \infty$ limit serves as a basis for expansion in powers of $1/n$.[6-10] When order-$1/n$ contributions to the specific heat are included, however, a peaking of the specific heat is predicted,[11] and the good agreement with experiment disappears.

All of the above considerations apply to homogeneous superconductors, for which the transition temperature is uniform throughout the system. In the present paper we wish to broaden the discussion to include inhomogeneous systems for which the transition temperature T_c can have local variations $\delta T_c(\vec{x})$. We have in mind specifically the case of two (or more) component alloys, for which the transition temperature usually depends on the relative concentrations of the components. In the experiment of Zally and Mochel, for example, a Bi-Sb alloy was used, for which the transition temperature varies from 6.1 to 0 K as the Sb concentration varies from 0 to 70 at.%. A given alloy will not be perfectly uniform, and this will lead to local variations in T_c. We will show that such variations can explain why no specific-heat peak was observed in the Zally-Mochel experiment. Our approach will be to take the $n = \infty$ limit of the homogeneous system as starting point and to use perturbation theory to compute corrections to order $1/n$ and to order g, where

$$g \propto [\lambda/\xi(0)]^2 \langle [\delta T_c(\vec{x})]^2 \rangle_c / T_c(\Delta T_c)$$

is a measure of the amount of inhomogeneity in the system. Here λ is the range of the correlations in $\delta T_c(\vec{x})$, $\xi(0)$ is the bare ($T = 0$) correlation length for fluctuations of the order parameter, ΔT_c is the width of the critical region for the homogeneous system, and $\langle \rangle_c$ means a configuration average over an ensemble of distributions $\delta T_c(\vec{x})$. We assume that $\lambda \ll \xi$, the temperature-dependent correlation length. As mentioned above, the $O(1/n)$ contribution $C^{(1/n)}$ is responsible for a peak in the specific heat. We will find, however, that the $O(g)$ contribution $C^{(g)}$ is negative in the temperature range for which $C^{(1/n)}$ is positive, and tends to cancel the peak.

The plan of the paper is as follows. Section II is concerned with a presentation of the model in terms of the usual Ginzburg-Landau parameters, a reduction to dimensionless variables, and the

solution of the model for $n=\infty$. In Sec. III we compute the corrections of $O(1/n)$ and $O(g)$ to the specific heat. In Sec. IV the perturbation theory of Sec. III is used to motivate a self-consistent type of calculation which is expected to be more accurate than simple perturbation theory for intermediate values of the parameters, $1/n, g \sim 1$. Finally Sec. V contains a discussion of the results. Here we show that for a binary alloy the parameter $g \propto c(1-c)(d\ln T_c/dc)^2$, where c is the concentration of one component. An order-of-magnitude estimate for the Zally and Mochel experiment gives $g \sim 1$. This is of the correct order to account for the lack of a specific-heat peak in their results.

II. n-COMPONENT GINZBURG-LANDAU MODEL AND LIMIT $n\to\infty$

In the Ginzburg-Landau (GL) model of an inhomogeneous superconductor the partition function is given by

$$Z_{\mathrm{GL}} = \int D^2\psi(\vec{\mathbf{x}})\, e^{-F[\psi]/T}, \tag{2.1}$$

where $\int D^2\psi(\vec{\mathbf{x}})\cdots$ means a functional integration over the real and imaginary parts of the order parameter $\psi(\vec{\mathbf{x}})$ and the free-energy functional $F[\psi]$ is given by the GL form

$$\mathfrak{F}[\psi] = \frac{F[\psi]}{T} = \int d^3x \left(\frac{\alpha}{T} |\psi(\vec{\mathbf{x}})|^2 + \frac{\delta}{T} |\vec{\nabla}\psi(\vec{\mathbf{x}})|^2 + \frac{1}{2}\frac{\beta}{T}|\psi(\vec{\mathbf{x}})|^4 + \frac{\alpha'(\vec{\mathbf{x}})}{T}|\psi(\vec{\mathbf{x}})|^2 \right). \tag{2.2}$$

Here α, δ, β are the usual GL parameters,

$$\alpha = \delta\epsilon/\xi^2(0), \quad \delta = 1/2m, \quad \beta = \xi_0^2/nm\xi^4(0), \tag{2.3}$$

where

$$\begin{aligned} &\xi(0) = \xi_0 \simeq 0.133 v_F/T_c, \quad \xi_0 \ll l \\ &\xi(0) \simeq 0.99(\xi_0 l)^{1/2}, \quad \xi_0 \gg l \end{aligned} \tag{2.4}$$

is the bare (temperature-independent) coherence length.[12] In Eqs. (2.3) and (2.4), $\epsilon = (T - T_c)/T_c$, where T_c is the mean transition temperature, m is the electronic mass, n is the conduction-electron density, v_F is the Fermi velocity, and l is the electronic mean free path. We use units such that the velocity of light, Boltzmann's constant, and Planck's constant divided by 2π are all unity. The coefficient $\alpha'(\vec{\mathbf{x}})$ in Eq. (2.2) represents the fluctuations in transition temperature produced by the inhomogeneities and is given by $\alpha'(\vec{\mathbf{x}}) = [\delta/\xi^2(0)]\epsilon'(\vec{\mathbf{x}})$, where $\epsilon'(\vec{\mathbf{x}}) = \delta T_c(\vec{\mathbf{x}})/T_c$.

We now make the assumption of "two dimensionality," namely that both $\psi(\vec{\mathbf{x}})$ and $\alpha'(\vec{\mathbf{x}})$ vary on a scale large compared to the film thickness d, i.e., that $\xi \gg d$ and $\lambda \gg d$. The latter constraint is not essential but is a convenient assumption for the time being. Then Eq. (2.2) becomes

$$\mathfrak{F}[\psi] = d\int d^2x \left(\frac{\alpha}{T}|\psi|^2 + \frac{\delta}{T}|\vec{\nabla}\psi|^2 + \frac{1}{2}\frac{\beta}{T}|\psi|^4 + \frac{\alpha'(\vec{\mathbf{x}})}{T}|\psi|^2 \right). \tag{2.5}$$

In the usual way it is convenient to rescale the order parameter and length variables so as to reduce Eq. (2.5) to a simple dimensionless form. To this end we set

$$\psi = a\phi, \quad \vec{\mathbf{x}} = b\vec{\mathbf{y}}, \tag{2.6}$$

and choose the scale factors a, b so that the coefficients of the terms in $|\vec{\nabla}\phi|^2$ and $|\phi|^4$ are $\frac{1}{2}$ and $\frac{1}{8}$, respectively. This yields

$$a = (T/2\delta d)^{1/2}, \quad b = (\delta^2 d/\beta T)^{1/2}, \tag{2.7}$$

and gives

$$\mathfrak{F}[\phi] = \int d^2y \left[\tfrac{1}{2}\tau_0 |\phi^2(\vec{\mathbf{y}})| + \tfrac{1}{2}|\vec{\nabla}\phi(\vec{\mathbf{y}})|^2 + \tfrac{1}{8}|\phi(\vec{\mathbf{y}})|^4 + \tfrac{1}{2}\tau'(\vec{\mathbf{y}})|\phi(\vec{\mathbf{y}})|^2 \right], \tag{2.8}$$

where

$$\tau_0 = \frac{d\delta}{\beta T}\alpha = \frac{k_F d}{6\pi^2}\left(\frac{\mu}{T_c}\right)\left(\frac{\xi(0)}{\xi_0}\right)^2 \epsilon. \tag{2.9}$$

Here k_F is the Fermi momentum and μ is the Fermi energy, and we have made the usual simplification of replacing T by T_c except in the factor ϵ; τ' is given by Eq. (2.9) with α, ϵ replaced by α', ϵ', respectively.

The model is now conveniently generalized to an n-component GL theory by writing

$$\mathfrak{F}[\phi_i] = \int d^2y \frac{1}{2}\sum_{i=1}^{n} \left[(\vec{\nabla}\phi_i)^2 + \phi_i^2 \left(\tau_0 + \tau'(\vec{\mathbf{y}}) + \frac{1}{2n}\sum_{j=1}^{n}\phi_j^2 \right) \right], \tag{2.10}$$

with partition function

$$Z = \int \prod_{i=1}^{n} [D\phi_i(\vec{\mathbf{y}})] e^{-\mathfrak{F}[\phi_i]}. \tag{2.11}$$

The superconductor is the case $n=2$.

Consider first the homogeneous system with $\tau' = 0$. In the limit $n\to\infty$ (Hartree limit) the problem is readily solved.[11] For then the component ϕ_i is acted upon by an interaction averaged over all the other components. The coefficient of ϕ_i^2 defines an inverse correlation length κ_1 by

$$\kappa_1^2 = \tau_0 + \frac{1}{n}\sum_{j=1}^{n}\langle\phi_j^2\rangle$$

$$= \tau_0 + G(0). \tag{2.12}$$

Angular brackets denote thermal averages. The order-parameter–order-parameter correlation function, or propagator, $G(\vec{y})$ (which is independent of j because of the isotropy of the model) is given by

$$G(\vec{y}) = \langle\phi_j(\vec{y})\phi_j(0)\rangle$$

$$= \int \frac{d^2p}{(2\pi)^2} g(p) e^{i\vec{p}\cdot\vec{y}}, \tag{2.13}$$

with the Fourier transform having the Ornstein-Zenike form

$$g(p) = (p^2 + \kappa_1^2)^{-1}. \tag{2.14}$$

The integral in Eq. (2.13) diverges for $\vec{y} = 0$ and requires a Debye cutoff at the maximum wave number p_D. With $p_D \gg \kappa_1$ we find

$$G(0) = (1/2\pi)\ln(p_D/\kappa_1). \tag{2.15}$$

The cutoff can be absorbed into the renormalized temperature variable

$$\tau_1 = \tau_0 + (1/2\pi)\ln p_D, \tag{2.16}$$

in terms of which Eq. (2.12) becomes

$$\kappa_1^2 = \tau_1 - (1/2\pi)\ln\kappa_1, \tag{2.17}$$

independent of the cutoff. Differentiating Eq. (2.17) gives

$$\frac{d\kappa_1}{d\tau_0} = \frac{2\pi\kappa_1}{1 + 4\pi\kappa_1^2}. \tag{2.18}$$

Introducing the thermodynamic free energy $F = -\ln Z$, one sees on differentiating inside the functional integral of Eq. (2.11), that the entropy per component is related to the propagator by

$$S = -\frac{1}{n}\frac{dF}{d\tau_0} = -\tfrac{1}{2}G(0) = -\frac{1}{4\pi}\ln\left(\frac{p_D}{\kappa_1}\right). \tag{2.19}$$

Differentiating Eq. (2.19) and using Eq. (2.18) gives the specific heat per component in the Hartree limit as

$$C = 2\frac{dS}{d\tau_0} = \frac{1}{1 + 4\pi\kappa_1^2}, \tag{2.20}$$

where the factor 2 serves to normalize the specific heat to unity in the low-temperature range $\tau_1 \ll -1$. C is a monotonic function of τ_1, decreasing smoothly from its asymptotic value of unity for $\tau_1 \ll -1$ to $(4\pi\tau_1)^{-1}$ for $\tau_1 \gg 1$. The full dependence of C on τ_1 is shown by the curve labeled "C" in Fig. 3, where the abscissa is labeled τ, rather than τ_1, for a reason which will become clear in Sec. III.

III. ORDER-$1/n$ AND ORDER-g CONTRIBUTIONS TO THE SPECIFIC HEAT

The calculation of the order-$1/n$ contribution $C^{(1/n)}$ has been given in Ref. 11. We repeat the derivation here in order to emphasize the similarity between the $O(1/n)$ calculation and the $O(g)$ calculation which follows. The expansion in powers of $1/n$ is most easily performed by graphical means (see, for example, Ref. 13). The interaction vertex associated with the quartic term in Eq. (2.10) is of order $1/n$, while each closed loop appearing in the graphical expansion contributes a factor n. The Hartree free energy F is of order n, so the correction of order $1/n$ relative to F which we seek is of order unity and is contributed by the ring graphs depicted in Fig. 1. The graph with r loops has rotational symmetry r. Hence the sum over ring graphs gives

$$F^{(1/n)} = \frac{1}{2}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}\sum_{q}[\pi_0(q)]^r, \tag{3.1}$$

$$= \frac{1}{2}\sum_{q}\ln[1 + \pi_0(q)], \tag{3.2}$$

where

$$\pi_0(q) = \sum_{p} g(\vec{p})g(\vec{p} + \vec{q}) = \frac{1}{4\pi\kappa_1^2} l\left(\frac{q}{2\kappa_1}\right) \tag{3.3}$$

is the "polarization propagator." Performing the integration in Eq. (3.3) gives

$$l(x) = x^{-1}(1 + x^2)^{-1/2}\ln[x + (1 + x^2)^{1/2}]. \tag{3.4}$$

Differentiating Eq. (3.2) gives the $O(1/n)$ contribution to the entropy per component as

$$S^{(1/n)} = -\frac{1}{n}\frac{dF^{(1/n)}}{d\tau_0} = \frac{1}{n}v(0)\sum_{q}v(q)T(q), \tag{3.5}$$

where

$$v(q) = [1 + \pi_0(q)]^{-1} \tag{3.6}$$

and

$$T(q) = -\frac{1}{2}\frac{d\pi_0(q)}{d(\kappa_1^2)} = \sum_{p} g^2(\vec{p})g(\vec{p} + \vec{q}) \tag{3.7}$$

$$= \frac{1}{16\pi\kappa_1^4}\frac{1 + l(q/2\kappa_1)}{1 + (q/2\kappa_1)^2}. \tag{3.8}$$

In the limit $q \to \infty$, $v(q) \to 1$, and $T(q) \to (4\pi\kappa_1^2 q^2)^{-1}$ so

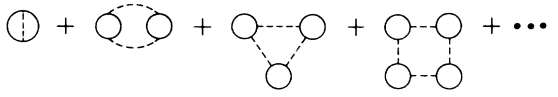

FIG. 1. Graphs for the free-energy contribution $F^{(1/n)}$. Solid lines represent the Hartree propagator, short-dash lines the quartic interaction in the GL functional.

that the integration in Eq. (3.5) fails to converge and has to be cut off at p_D in the usual way. The evaluation of the integral is simplified, however, by writing

$$S^{(1/n)} = \frac{1}{n} v(0) \sum_q T(q) - \frac{1}{n} v(0) \sum_q [1 - v(q)] T(q) . \quad (3.9)$$

The second integration now converges, so the upper limit may be extended to infinity. The first integration is readily performed by using the right-hand member of Eq. (3.7) and performing the sum over q first. The result is

$$S^{(1/n)} = (1/2\pi n)(1 + 4\pi\kappa_1^2)^{-1} [\ln(p_D/\kappa_1) - I(\kappa_1)] , \quad (3.10)$$

where

$$I(\kappa_1) = \int_0^\infty \frac{u\,du}{1+u^2} \frac{l(u)[1+l(u)]}{4\pi\kappa_1^2 + l(u)} . \quad (3.11)$$

We now show that it is possible to remove the dependence of $S^{(1/n)}$ on the cutoff by making a further order $1/n$ renormalization of the temperature variable. We set

$$\tau_1 = \tau - \delta\tau , \quad \kappa_1 = \kappa - \delta\kappa , \quad (3.12)$$

where $\delta\tau, \delta\kappa$ will be $O(1/n)$. Substituting into Eq. (2.17) and demanding that the Hartree relation

$$\kappa^2 = \tau - (1/2\pi) \ln\kappa \quad (3.13)$$

hold also between the new variables yields

$$\delta\kappa = 2\pi\kappa(1 + 4\pi\kappa^2)^{-1}\, \delta\tau . \quad (3.14)$$

In terms of the new variable κ, the Hartree entropy, Eq. (2.19), becomes

$$S = -(1/4\pi) \ln(p_D/\kappa) - (1/4\pi)\delta\kappa/\kappa = -(1/4\pi) \ln(p_D/\kappa) - \tfrac{1}{2}(1 + 4\pi\kappa^2)^{-1}\, \delta\tau . \quad (3.15)$$

Now in Eq. (3.10) we may replace κ_1 by κ, since the resulting error is $O(1/n^2)$. Then the final term in Eq. (3.15) exactly cancels the cutoff-dependent part of $S^{(1/n)}$ provided we choose

$$\delta\tau = (2/n)(1/2\pi) \ln p_D ,$$

giving

$$\tau = \tau_0 + (1 + 2/n)(1/2\pi) \ln p_D . \quad (3.16)$$

The remainder of Eq. (3.10) is cutoff independent and gives

$$S^{(1/n)} = (1/2\pi n)(1 + 4\pi\kappa^2)^{-1} [\ln(1/\kappa) - I(\kappa)] . \quad (3.17)$$

The specific-heat contribution is obtained by differentiating Eq. (3.17), $C^{(1/n)} = 2\, dS^{(1/n)}/d\tau_0$. The result of a numerical computation of $nC^{(1/n)}$ is shown in Fig. 2. The effect of adding $C^{(1/n)}$ to the Hartree specific heat C is shown in Fig. 3, for the special case $n = 2$, by the curve labeled "$C + C^{(1/n)}$." The total specific heat exhibits a peak, in contrast to the Hartree specific heat C, also shown in Fig. 3.

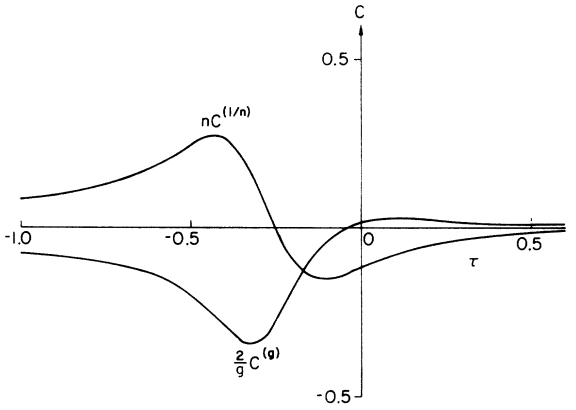

FIG. 2. Contributions of $O(1/n)$ and $O(g)$ to the specific heat of the two-dimensional GL model, where g is the "inhomogeneity parameter" given by Eqs. (3.31), (3.31a), and (3.31b).

Of particular interest is the low-temperature limit $\tau \ll -1$, when we can use the asymptotic form

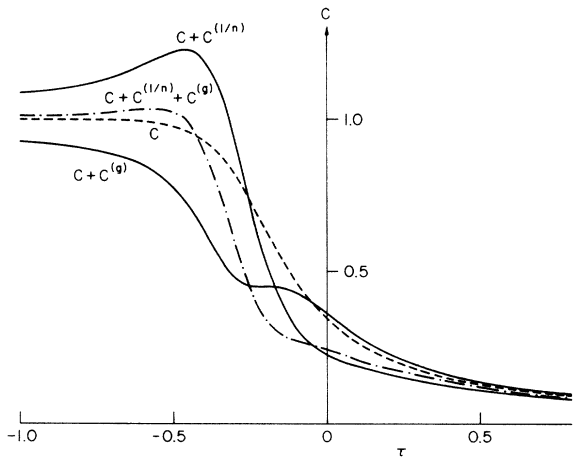

FIG. 3. Specific heat of the two-dimensional GL model. The curve labeled C is the Hartree theory. That labeled $C + C^{(1/n)}$ includes $O(1/n)$ corrections with $n = 2$ taken, as appropriate for superconductors. The curve labeled $C + C^{(g)}$ includes $O(g)$ corrections to the Hartree theory with $g = 1$ taken. The curve labeled $C + C^{(1/n)} + C^{(g)}$ includes both $O(1/n)$ and $O(g)$ corrections (with $n = 2, g = 1$) and qualitatively resembles the Hartree result C. The shoulder in the total specific heat is presumably a result of using perturbation theory beyond its strict region of validity. It is much less pronounced in Figs. 7 and 8, obtained from the self-consistent calculation of Sec. IV.

$l(x) \sim x^{-2} \ln 2x$ in Eqs. (3.3) and (3.8) to give (with $\kappa_1 \to \kappa$)

$$\pi_0(q) \simeq (1/\pi q^2) \ln(q/\kappa) \tag{3.18}$$

and

$$T(q) \simeq 1/4\pi\kappa^2 q^2 . \tag{3.19}$$

Using also $v(0) \simeq 4\pi\kappa^2$, and approximating

$$v(q) \simeq [1 + (1/\pi q^2) \ln(q/\kappa)]^{-1}$$

by

$$v(q) \simeq q^2/(q^2 + q_\kappa^2) , \tag{3.20}$$

with q_κ determined self-consistently from

$$q_\kappa^2 = (1/\pi) \ln(q_\kappa/\kappa) , \tag{3.21}$$

Eq. (3.5) becomes

$$S^{(1/n)} \simeq \frac{1}{n} \sum_q \frac{1}{q^2 + q_\kappa^2} = \frac{1}{2\pi n} \ln\left(\frac{p_D}{q_\kappa}\right) . \tag{3.22}$$

The term in $\ln p_D$ may be absorbed into a renormalization of the temperature variable, as described above. Then the cutoff-independent part of Eq. (3.22) is

$$S^{(1/n)} = -(1/2\pi n) \ln q_\kappa \simeq -(1/4\pi n) \ln|\tau| , \tag{3.23}$$

giving

$$C^{(1/n)} \simeq (2\pi n|\tau|)^{-1} , \quad \tau \ll -1 . \tag{3.24}$$

This behavior is to be contrasted with that of the Hartree specific heat [Eq. (2.20) with $\kappa_1 \to \kappa$] in the limit $\tau \ll -1$:

$$C \simeq 1 - 4\pi\kappa^2$$

$$\simeq 1 - 4\pi e^{-4\pi|\tau|} , \quad \tau \ll -1 . \tag{3.25}$$

The sum $C + C^{(1/n)}$ is therefore greater than unity for sufficiently large negative τ, and the specific heat therefore exhibits a peak, no matter how large we choose n.

We return now to Eq. (2.10) and consider the case $\tau'(\vec{y}) \neq 0$, in the limit $n \to \infty$. The extra term in $\mathcal{F}[\phi_i]$ becomes, in Fourier-transformed variables,

$$\frac{1}{2} \sum_q \tau'(\vec{q}) \sum_{i=1}^{n} \sum_p \phi_i(\vec{p} - \vec{q}) \phi_i(-\vec{p}) . \tag{3.26}$$

This acts as a one-body "scattering potential" which we represent diagrammatically by a cross and treat in perturbation theory. Since $\langle\tau'\rangle_c$ can be taken zero without loss of generality (a nonzero $\langle\tau'\rangle_c$ can be absorbed by a redefinition of τ_0), the leading terms are the second-order contributions to the free energy depicted in Fig. 4. As before, the short-dash lines represent the quartic interaction in $\mathcal{F}[\phi_i]$. The long-dash lines joining the

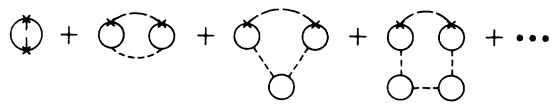

FIG. 4. Graphs for the free-energy contribution $F^{(g)}$. The crosses represent the inhomogeneity term in the GL functional. The long-dash lines represent "conservation of momentum" after averaging over all configurations of the inhomogeneity distribution.

crosses indicate that momentum is conserved "on the average," i.e., that only the diagonal terms $\langle\tau'(\vec{q})\tau'(-\vec{q})\rangle_c$ survive when we take the configuration average. Note that the graphs of Fig. 4 are strikingly similar to those of Fig. 1, the principal difference being that the graphs of Fig. 4 have no rotational symmetry. Their contribution to the free energy is

$$F^{(g)} = -\tfrac{1}{4} n \sum_{r=1}^{\infty} (-1)^{r+1} \sum_q [\pi_0(q)]^r \langle\tau'(\vec{q})\tau'(-\vec{q})\rangle_c$$

$$= -\tfrac{1}{4} n \sum_q \frac{\pi_0(q)}{1 + \pi_0(q)} \langle\tau'(\vec{q})\tau'(-\vec{q})\rangle_c . \tag{3.27}$$

Differentiating Eq. (3.27) gives the contribution to the entropy per component

$$S^{(g)} = -\frac{1}{n} \frac{dF^{(g)}}{d\tau_0}$$

$$= -\tfrac{1}{2} v(0) \sum_q v^2(q) T(q) \langle\tau'(\vec{q})\tau'(-\vec{q})\rangle_c . \tag{3.28}$$

Now the correlation function $\langle\tau'(\vec{y})\tau'(0)\rangle_c$ is peaked about $\vec{y} = 0$ with width $\sim \lambda/b$, where b is the length scale parameter of Eq. (2.7). Therefore its Fourier transform $\langle\tau'(\vec{q})\tau'(-\vec{q})\rangle_c$ is peaked about $\vec{q} = 0$ with width $k_0 \sim b/\lambda$. On the other hand, the variation of the product $v^2(q)T(q)$ with q is on a scale set by $\kappa \simeq \tau^{1/2}$ for $\tau \gg 1$ and on a scale set by $q_\kappa \simeq (2|\tau|)^{1/2}$ for $\tau \ll -1$. Therefore, provided $k_0 \gg |\tau|^{1/2}$, which is equivalent to the condition $\lambda \ll \xi_{GL} = \xi(0)/|\epsilon|^{1/2}$, we may replace $\langle\tau'(\vec{q})\tau'(-\vec{q})\rangle_c$ in Eq. (3.28) by its value g at $q = 0$, and take it outside the integration, k_0 playing the role of a large-momentum cutoff on the logarithmically divergent integration. Thus

$$S^{(g)} = -\tfrac{1}{2} g v(0) \sum_{q<k_0} v^2(q) T(q) , \tag{3.29}$$

where

$$g = \langle\tau'(\vec{q})\tau'(-\vec{q})\rangle_c\big|_{\vec{q}=0}$$

$$= \int d^2y \, \langle\tau'(\vec{y})\tau'(0)\rangle_c \tag{3.30}$$

$$= A(\lambda^2/b^2) \langle\tau'^2\rangle_c , \tag{3.31}$$

the numerical coefficient A depending on the precise form of the correlation function. For example, if $\langle\tau'(\vec{y})\tau'(0)\rangle_c \propto \exp[-(b/\lambda)|\vec{y}|]$, then $A = 2\pi$. Substituting for b and τ' in terms of the GL parameters from Eqs. (2.7) and (2.9), we find

$$g = A(\lambda^2 d/\beta T)\langle\alpha'^2\rangle_c = A\frac{nd}{4mT}\left(\frac{\lambda}{\xi_0}\right)^2\langle\epsilon'^2\rangle_c,$$

where we have converted to microscopic parameters via Eq. (2.3). Using $n = k_F^3/3\pi^2$ yields

$$g = \frac{A}{6\pi^2}(k_F d)\left(\frac{\mu}{T_c}\right)\left(\frac{\lambda}{\xi_0}\right)^2\frac{\langle[\delta T_c(\vec{x})]^2\rangle_c}{T_c^2}. \tag{3.31a}$$

Alternatively, g can be expressed in terms of the "width of the critical region" for the homogeneous system, ΔT_c, obtained by setting $\tau_0 = 1$. Using Eq. (2.9) we find

$$g = A\left(\frac{\lambda}{\xi(0)}\right)^2\frac{\langle[\delta T_c(\vec{x})]^2\rangle_c}{T_c(\Delta T_c)}. \tag{3.31b}$$

The dimensionless parameter g depends on two parameters—the range of correlation λ and the mean-square fluctuation $\langle[\delta T_c(\vec{x})]^2\rangle_c$—which may not be known at all accurately experimentally. In most applications, however, the randomness is on an atomic scale. In Sec. V we give an argument to show that for binary alloys the dependence of g on these two variables reduces to a dependence on the single variable $c(1-c)(d\ln T_c/dc)^2$, where c is the concentration of one component of the alloy. The dependence of T_c on c is easily determined experimentally and in fact is known for the Bi-Sb alloy used by Zally and Mochel.

Returning to the evaluation of $S^{(g)}$, the sum over q in Eq. (3.29) may be carried out in entirely analogous fashion to the evaluation of $S^{(1/n)}$. That is, we write

$$S^{(g)} = -\tfrac{1}{2}gv(0)\sum_{q<k_0}T(q) + \tfrac{1}{2}gv(0)\sum_q[1-v^2(q)]T(q)$$

$$= -(g/4\pi)(1+4\pi\kappa_1^2)^{-1}[\ln(k_0/\kappa_1) - J(\kappa_1)], \tag{3.32}$$

where

$$J(\kappa_1) = \int_0^\infty \frac{u\,du}{1+u^2}\,\frac{l(u)[1+l(u)][8\pi\kappa_1^2 + l(u)]}{[l(u)+4\pi\kappa_1^2]^2}. \tag{3.33}$$

The term in $\ln k_0$ in Eq. (3.32) can be removed by renormalizing the temperature variable [cf. Eq. (3.16)] to

$$\tau = \tau_0 + (1+2/n)(1/2\pi)\ln p_D - (g/2\pi)\ln k_0, \tag{3.34}$$

leaving the cutoff-independent form

$$S^{(g)} = -(g/4\pi)(1+4\pi\kappa^2)^{-1}[\ln(1/\kappa) - J(\kappa)], \tag{3.35}$$

with κ related to τ by Eq. (3.13). Differentiating gives the specific-heat contribution $C^{(g)} = 2dS^{(g)}/d\tau_0$. The result of a numerical computation of $(2/g)C^{(g)}$ is shown in Fig. 2. Note that in the low-temperature range, where $C^{(1/n)}$ is positive and leads to a peak in the total heat capacity, $C^{(g)}$ is negative and, if $\frac{1}{2}g$ is of the same order as or greater than $1/n$, tends to cancel out the peak. This is demonstrated in Fig. 3, where $g = 1$ has been used. The curve labeled C is the uncorrected Hartree result. That labeled $C + C^{(1/n)}$ includes the $O(1/n)$ correction for $n = 2$. The curve labeled $C + C^{(1/n)} + C^{(g)}$ includes both $O(1/n)$ and inhomogeneity corrections and resembles qualitatively the uncorrected Hartree curve. Finally the curve labeled $C + C^{(g)}$ includes the inhomogeneity correction only, and demonstrates the effect of inhomogeneities in broadening the transition. Note that the term $C^{(g)}$ leads to a shoulder in the total specific heat which is presumably an artifact of using perturbation theory beyond its strict region of validity. This shoulder was not observed experimentally[4] and is much less pronounced in the self-consistent version of the theory which we introduce in Sec. IV.

Of particular interest is the low-temperature form of $C^{(g)}$. Using Eqs. (3.19) and (3.20) together with $v(0) \simeq 4\pi\kappa^2$, Eq. (3.29) becomes, for $\tau \ll -1$,

$$S^{(g)} \simeq -\tfrac{1}{2}g\sum_{q<k_0}\frac{q^2}{(q^2+q_\kappa^2)^2} = -\frac{g}{4\pi}\ln\left(\frac{k_0}{q_\kappa}\right). \tag{3.36}$$

The cutoff-independent part gives, on differentiation,

$$C^{(g)} \simeq -g/4\pi|\tau|, \quad \tau \ll -1. \tag{3.37}$$

Combining Eqs. (3.24), (3.25), and (3.37) yields

$$C + C^{(1/n)} + C^{(g)} \simeq 1 + (2/n - g)(1/4\pi|\tau|), \quad \tau \ll -1. \tag{3.38}$$

Hence if $g < 2/n$, the total specific heat approaches unity from above for large negative τ and therefore exhibits a peak. If $g > 2/n$, however, the specific heat approaches unity from below, indicating that the broad peak associated with $C^{(1/n)}$ is wiped out by the inhomogeneity term. For finite n, there may, however, be a sharper structure around $\tau \sim 0$ which is only seen when all orders in $1/n$ are included in the calculation. It is quite possible that such structure, if it exists, will not be removed by inhomogeneities. We will discuss this point further in Sec. V.

IV. A SELF-CONSISTENT APPROACH

So far we have used perturbation theory to correct the Hartree limit for the effects of finite n

and inhomogeneities. However, we wish to apply the results to cases in which the "small parameters" of the perturbation theory, $1/n$ and g, are not strictly small, i.e., $n=2$, $g\sim 1$. In this section, therefore, we present a self-consistent calculation which is expected to be more reliable than simple perturbation theory for these intermediate values of $1/n$ and g.

The exact propagator $g_E(p)$ has the form

$$g_E(p)=[\tau_0+p^2+\sigma(p)]^{-1}, \tag{4.1}$$

where $\sigma(p)$ is the self-energy. Hartree, $O(1/n)$ and $O(g)$ contributions to $\sigma(p)$ are shown in Figs. 5(a)–5(c). Here a single bold line represents the Hartree propagator, Eq. (2.14), while a wavy line represents the "screened potential," defined in Fig. 5(d), and the shaded circle is the vertex function defined in Fig. 5(e). If q is the momentum carried by the wavy line, then the graphical equation of Fig. 5(d) is easily solved to yield for the screened potential $(-1/n)v(q)$, where $v(q)=[1+\pi_0(q)]^{-1}$ was introduced in Eq. (3.6). Similarly, the graphical equation of Fig. 5(e) yields for the shaded circle $(g)^{1/2}v(q)$, where q is the momentum carried by the long dash line and we have associated a factor $(g)^{1/2}$ with each cross after carrying out the configuration average.

The calculation is made self-consistent by using, in the computation of $\sigma(p)$, self-consistently determined propagators instead of Hartree propagators. Then we get the self-energy graphs shown in Fig. 6, where double bold lines represent self-consistent propagators $g_{sc}(p)$. Thus

$$g_{sc}(p)=[\tau_0+p^2+\sigma_{sc}(p)]^{-1}, \tag{4.2}$$

where

$$\sigma_{sc}(p)=\sum_q g_{sc}(q)+\frac{2}{n}\sum_q v(q)g_{sc}(\vec{q}+\vec{p}) -g\sum_q v^2(q)g_{sc}(\vec{q}+\vec{p}), \tag{4.3}$$

$$v(q)=[1+\pi_0(q)]^{-1}, \tag{4.4}$$

and

$$\pi_0(q)=\sum_k g_{sc}(k)g_{sc}(\vec{k}+\vec{q}) \tag{4.5}$$

is also determined using self-consistent propagators. This approach has been rather successful in describing the specific heat of the one-dimensional homogeneous GL model.[14] It has also been used as an approximate scheme for determining the critical exponent η for dimensionalities D in the range $2\le D\le 4$.[15] The set of coupled integral equations (4.2)–(4.5) is rather intractable as it stands. We therefore make the further simplification of

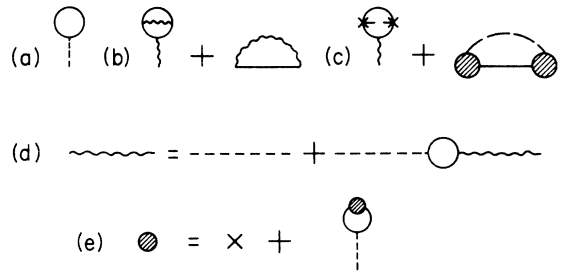

FIG. 5. Graphs for (a) Hartree self-energy; (b) $O(1/n)$ contributions to the self energy; (c) $O(g)$ contributions to the self-energy; (d) the screened potential $(-1/n)$ $v(q)$; (e) the vertex function appearing in (c). In all cases continuous lines represent Hartree propagators.

replacing $\sigma_{sc}(p)$ by $\sigma_{sc}(0)$ in Eq. (4.2). Then $g_{sc}(p)$ has the Ornstein-Zernike form

$$g_{sc}(p)=(\kappa^2+p^2)^{-1}, \tag{4.6}$$

where

$$\kappa^2=\tau_0+\sigma_{sc}(0) =\tau_0+\sum_q g_{sc}(q)+\frac{2}{n}\sum_q v(q)g_{sc}(q) -g\sum_q v^2(q)g_{sc}(q) \tag{4.7}$$

and

$$v(q)=[1+(1/4\pi\kappa^2)l(q/2\kappa)]^{-1}, \tag{4.8}$$

with $l(x)$ given by Eq. (3.4). To extract explicitly the dependence on the cutoffs p_D and k_0 we write

$$\kappa^2=\tau_0+\sum_q g_{sc}(q)+\frac{2}{n}\sum_q g_{sc}(q)-g\sum_q g_{sc}(q) -\frac{2}{n}\sum_q[1-v(q)]g_{sc}(q)+g\sum_q[1-v^2(q)]g_{sc}(q). \tag{4.9}$$

The first two sums in Eq. (4.9) are cutoff at p_D, the third sum at k_0, while the remaining two sums are convergent. Thus

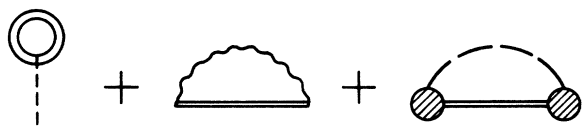

FIG. 6. Self-energy graphs for the self-consistent calculation of Sec. IV. Double lines represent the self-consistently-determined propagator.

$$\kappa^2 = \tau_0 + (1+2/n)(1/2\pi)\ln(p_D/\kappa) - (g/2\pi)\ln(k_0/\kappa) - (1/n\pi)K(\kappa) + (g/2\pi)L(\kappa), \quad (4.10)$$

where

$$K(\kappa) = \int_0^\infty \frac{u\,du}{u^2+\frac{1}{4}} \frac{l(u)}{4\pi\kappa^2 + l(u)}, \quad (4.11)$$

$$L(\kappa) = \int_0^\infty \frac{u\,du}{u^2+\frac{1}{4}} \frac{l(u)[8\pi\kappa^2 + l(u)]}{[4\pi\kappa^2 + l(u)]^2}. \quad (4.12)$$

Hence

$$\kappa^2 = \tau + (1/2\pi)[(1+2/n-g)\ln(1/\kappa) - (2/n)K(\kappa) + gL(\kappa)], \quad (4.13)$$

where $\tau = \tau_0 + (1+2/n)(1/2\pi)\ln p_D - (g/2\pi)\ln k_0$ is the same renormalized temperature variable that appears in Eq. (3.34). The entropy per component is

$$S = -\frac{1}{2}\sum_p g_{sc}(p) = -(1/4\pi)\ln(p_D/\kappa). \quad (4.14)$$

Differentiating, and using Eq. (4.13) to determine $d\kappa/d\tau$, yields the specific heat. The results of a numerical computation, for different values of n and g, are presented[16] in Figs. 7 and 8. The salient features are that, as expected, increasing g has the effect of broadening the transition and that, for $n=2$, the peak associated with the $g=0$ case is virtually wiped out when $g=1$.

In the low-temperature range $\tau \ll -1$ we can obtain analytic results. For then $4\pi\kappa^2 \to 0$ and can be neglected compared to $l(u)$ in Eqs. (4.11) and (4.12), except when u is so large that $l(u) \sim u^{-2}\ln 2u \sim 4\pi\kappa^2$. This defines a maximum value of u,

$$u_m \sim [\kappa(4\pi)^{1/2}]^{-1}[\ln(1/\kappa)]^{1/2}, \quad (4.15)$$

for which $4\pi\kappa^2$ may be neglected compared to $l(u)$. If $4\pi\kappa^2$ is so neglected, then the integrals defining $K(\kappa)$ and $L(\kappa)$ are identical, and diverge logarithmically at their upper limits. To logarithmic accuracy, therefore,

$$K(\kappa) \sim L(\kappa) \sim \ln u_m \sim \ln(1/\kappa) + \tfrac{1}{2}\ln\ln(1/\kappa), \quad \kappa \ll 1. \quad (4.16)$$

Substituting in Eq. (4.13) gives

$$\kappa^2 \sim \tau + \frac{1}{2\pi}\left[\ln\left(\frac{1}{\kappa}\right) - \left(\frac{1}{n}-\frac{g}{2}\right)\ln\ln\left(\frac{1}{\kappa}\right)\right]. \quad (4.17)$$

The specific heat is

$$C = 2\frac{dS}{d\tau} = \left(2\pi\kappa\frac{d\tau}{d\kappa}\right)^{-1}. \quad (4.18)$$

We use Eq. (4.17) to compute $d\tau/d\kappa$:

$$\frac{d\tau}{d\kappa} \sim \frac{1}{2\pi\kappa}\left[1 - \left(\frac{1}{n}-\frac{g}{2}\right)\Big/\ln\left(\frac{1}{\kappa}\right)\right], \quad \kappa \ll 1. \quad (4.19)$$

Therefore, in the low-temperature range $\tau \ll -1$,

$$C \simeq 1 + \left(\frac{1}{n}-\frac{g}{2}\right)\Big/\ln\left(\frac{1}{\kappa}\right) \simeq 1 + (2/n-g)1/4\pi|\tau|, \quad \tau \ll -1. \quad (4.20)$$

Thus the low-temperature behavior agrees with the perturbation-theory result Eq. (3.38). If $g < 2/n$, C approaches unity from above; If $g > 2/n$, C approaches unity from below. In Sec. V we present a mean-field-theory argument which suggests that Eq. (4.20) is exact.

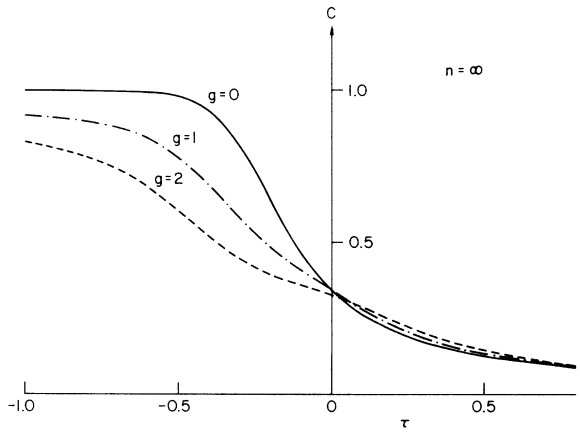

FIG. 7. Specific heat of the two-dimensional GL model as given by the self-consistent calculation of Sec. IV, with $n=\infty$, for three values of the "inhomogeneity parameter" g. The $g=0$ curve is the Hartree result. As expected, the transition broadens with increasing g.

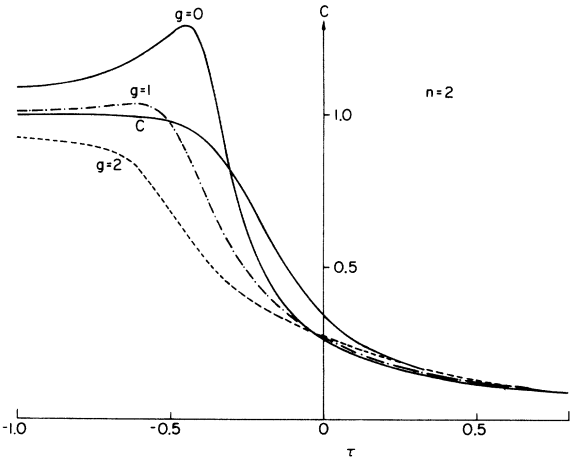

FIG. 8. Specific heat of the two-dimensional GL model according to the self-consistent approach, with $n=2$, for three values of g. For $g \gtrsim 1$ the peak associated with the $g=0$ curve disappears so that the specific heat resembles qualitatively the Hartree result labeled C.

V. DISCUSSION

The effect of inhomogeneities on the specific heat of a superconducting film depends on the dimensionless parameter g, given by Eq. (3.31). It is clearly important to estimate the value of g for a typical system. Consider, for example, a two-component alloy for which T_c depends on the relative concentrations of the components. If the components are well mixed, as in the case for the Bi-Sb alloy used by Zally and Mochel,[4] then the inhomogeneity occurs at the atomic level, that is we may regard the two types of atom as being arranged randomly on the atomic sites. Then the correlation length λ appears to be of the order of the atomic separation. We say "appears" because it is clearly nonsense to associate a local $T_c(\vec{x})$ with a single atom. There is obviously a minimum volume of the system, containing many atoms, required for a meaningful definition of a local T_c. Suppose a characteristic length λ' (assumed $\gg d$ for the moment) defines such a minimum volume ($\sim d\lambda'^2$), and imagine the film to be divided into square slabs of side λ' and thickness d. Then δT_c is completely uncorrelated for neighboring slabs and we have $\lambda \sim \lambda'$. The number of atoms in each slab is

$$N \sim n_A \lambda'^2 d , \tag{5.1}$$

where n_A is the number of atoms per unit volume. If c is the concentration of one component, then the well-known properties of the binomial distribution give the statistical uncertainty in the number of atoms of this type as

$$\delta N \sim [c(1-c)]^{1/2}(n_A d)^{1/2}\lambda' . \tag{5.2}$$

The statistical uncertainty in the concentration is

$$\delta c = \delta N / N \sim [c(1-c)]^{1/2}(n_A d)^{-1/2}(\lambda')^{-1} , \tag{5.3}$$

which leads to a statistical uncertainty in the transition temperature of the slab equal to

$$\delta T_c = \left(\frac{dT_c}{dc}\right)\delta c \sim [c(1-c)]^{1/2}\left(\frac{dT_c}{dc}\right)(n_A d)^{-1/2}(\lambda')^{-1} , \tag{5.4}$$

where dT_c/dc is the rate of change of transition temperature with concentration for the homogeneous system. Using $\lambda \sim \lambda'$ we see, therefore, that the product $\lambda^2\langle(\delta T_c)^2\rangle_c$ appearing in g is independent of λ' and given by

$$\lambda^2\langle(\delta T_c)^2\rangle_c \sim c(1-c)\left(\frac{dT_c}{dc}\right)^2\frac{1}{n_A d} . \tag{5.5}$$

Substituting in Eq. (3.31a) yields

$$g \sim 7Ac(1-c)\left(\frac{n}{n_A}\right)\left(\frac{T_c}{\mu}\right)\left(\frac{d\ln T_c}{dc}\right)^2 . \tag{5.6}$$

For the alloy used by Zally and Mochel, $Bi_{0.4}Sb_{0.6}$, an order of magnitude estimate using $A \sim 2\pi$, $T_c/\mu \sim 10^{-4}$, $d\ln T_c/dc \sim 10$ yields $g \sim 1$. Hence the inhomogeneity effect can explain the absence of a specific heat peak in their experiment.

So far we have assumed that the characteristic lengths λ, λ' are large compared to the film thickness d. What happens in the opposite limit $\lambda, \lambda' \ll d$? Then the two-dimensional integration of Eq. (3.30) has to be replaced by a three-dimensional integration, normalized by a factor b/d. Then g is modified by a factor λ/d and the constant A may also change. But now the characteristic volume necessary to define a local T_c is also modified, by a factor $\lambda'/d \sim \lambda/d$. These two effects cancel to give the same result, Eq. (5.6), for g apart from a factor of order unity.

We have also assumed throughout that $\lambda \ll \xi_{GL} = \xi(0)/|\epsilon|^{1/2}$, which is appropriate to the usual experimental situation. It is of pedagogical interest to consider also the opposite limit, $\lambda \gg \xi_{GL}$, of very-long-range fluctuations in the transition temperature. Recall that the inhomogeneity contribution to the free energy is given by Eq. (3.27),

$$F^{(g)} = -\tfrac{1}{4}n\sum_q \frac{\pi_0(q)}{1+\pi_0(q)}\langle\tau'(\vec{q})\tau'(-\vec{q})\rangle_c . \tag{5.7}$$

The function $\langle\tau'(\vec{q})\tau'(-\vec{q})\rangle_c$ is now sharply peaked about $q=0$ with width $k_0 \sim b/\lambda \ll |\tau|^{1/2}$. The factor $\pi_0(q)/[1+\pi_0(q)]$ may therefore be evaluated at $q=0$ and taken outside the integration. But $\pi_0(0)/[1+\pi_0(0)] = (1+4\pi\kappa^2)^{-1}$ is just the Hartree specific heat C. Therefore

$$F^{(g)} = -\tfrac{1}{4}nC\sum_q \langle\tau'(\vec{q})\tau'(-\vec{q})\rangle_c = -\tfrac{1}{4}nC\langle\tau'^2\rangle_c . \tag{5.8}$$

The inhomogeneity contribution to the heat capacity is then

$$C^{(g)} = -\frac{2}{n}\frac{d^2F^{(g)}}{d\tau_0^2} = \tfrac{1}{2}\langle\tau'^2\rangle_c\frac{d^2C}{d\tau_0^2} . \tag{5.9}$$

This result may also be derived using elementary arguments. If the transition temperature is constant over many correlation lengths then the system may be regarded as composed of subsystems with different, but uniform, transition temperature. If the heat capacity of a single subsystem is $C(\tau_0+\tau')$ then the specific heat of the whole system is

$$\langle C(\tau_0+\tau')\rangle_c = C(\tau_0) + \tfrac{1}{2}\langle\tau'^2\rangle_c\left(\frac{d^2C}{d\tau_0^2}\right)_{\tau'=0} + \cdots . \tag{5.10}$$

The linear term in the Taylor-series expansion vanishes since $\langle\tau'\rangle_c = 0$ by assumption (without loss

of generality). Note that since the second derivative of C vanishes more rapidly than C itself in the low-temperature range [when $O(1/n)$ terms are included in C] this type of long-range inhomogeneity is incapable of canceling the peak in the specific heat of the homogeneous system.

Most recent work on the theory of phase transitions in inhomogeneous systems[17-19] is based on the renormalization-group approach,[20] allied to expansions of the critical exponents in powers of $4-D$ where D is the dimensionality of the system. While this approach can say nothing about the $D=2$ case considered here (since the two-dimensional phase transition is, for $n>1$, qualitatively different from that usually encountered), the results do have some bearing on the interpretation of the results presented in this paper. By way of illustration we present a mean-field-theory calculation, for arbitrary dimensionality, of the specific heat in the low-temperature range $\tau \ll -1$.

Recall the free-energy functional for the inhomogeneous system

$$\mathfrak{F} = \int d^D y \left[\tfrac{1}{2}\tau \sum_i \phi_i^2 + \frac{1}{2}\sum_i (\vec{\nabla}\phi_i)^2 + \frac{1}{4n}\left(\sum_i \phi_i^2\right)^2 + \tfrac{1}{2}\tau'(\vec{y}) \sum_i \phi_i^2 \right]. \tag{5.11}$$

If the condensation is (say) in the "1" direction, we set $\phi_1(\vec{y}) = a + \zeta(\vec{y})$, where $a = (n|\tau|)^{1/2}$ is the mean-field-theory value for the order parameter of the homogeneous system, and $\zeta(\vec{y})$ is a small fluctuation. Expanding to second order in the fluctuations $\zeta(\vec{y})$, $\phi_i(\vec{y})$ $(i \geq 2)$, and $\tau'(\vec{y})$ yields

$$\mathfrak{F} = -\tfrac{1}{4}n|\tau|^2 + \frac{1}{2}\int d^D y \left(2|\tau|\zeta^2 + (\vec{\nabla}\zeta)^2 + \sum_{i\geq 2} (\vec{\nabla}\phi_i)^2 + a^2\tau' + 2a\tau'\zeta \right). \tag{5.12}$$

Introducing the Fourier transforms of ζ, ϕ_i, and τ' yields

$$\mathfrak{F} = -\tfrac{1}{4}n|\tau|^2 + \tfrac{1}{2}a^2\tau'(0) + \frac{1}{2}\sum_q \left((2|\tau| + q^2)\zeta(\vec{q})\zeta(-\vec{q}) + q^2 \sum_{i\geq 2} \phi_i(\vec{q})\phi_i(-\vec{q}) + 2a\zeta(\vec{q})\tau'(-\vec{q}) \right). \tag{5.13}$$

The "cross term" (the final term in the large parentheses) is removed by transforming to the variables $\rho(\vec{q}) = \zeta(\vec{q}) + a\tau'(\vec{q})/(2|\tau| + q^2)$ to give

$$\mathfrak{F} = -\tfrac{1}{4}n|\tau|^2 + \tfrac{1}{2}a^2\tau'(0) - \frac{a^2}{2}\sum_q \frac{\tau'(\vec{q})\tau'(-\vec{q})}{2|\tau| + q^2} + \frac{1}{2}\sum_q \left((2|\tau| + q^2)\rho(\vec{q})\rho(-\vec{q}) + q^2 \sum_{i\geq 2} \phi_i(\vec{q})\phi_i(-\vec{q}) \right). \tag{5.14}$$

Integrating over the fluctuations $\rho(\vec{q})$ and $\phi_i(\vec{q})$ $(i \geq 2)$ gives the thermodynamic free energy

$$\begin{aligned} F &= -\ln \int \prod_q \left(d\rho(\vec{q}) \prod_{i\geq 2} d\phi_i(\vec{q}) \right) e^{-\mathfrak{F}[\rho, \phi_i]} \\ &= -\tfrac{1}{4}n|\tau|^2 + \tfrac{1}{2}a\tau'(0) - \frac{a^2}{2}\sum \frac{\tau'(\vec{q})\tau'(-\vec{q})}{2|\tau| + q^2} + \frac{1}{2}\sum_q \ln(2|\tau| + q^2) + \frac{n-1}{2}\sum_q \ln(q^2). \end{aligned} \tag{5.15}$$

On taking the configuration average, the term linear in τ' drops out, and we replace $\langle \tau'(\vec{q})\tau'(-\vec{q}) \rangle_c$ by g. Using $a^2 = n|\tau|$ we compute the specific heat per component

$$\begin{aligned} C &= -\frac{2}{n}\frac{d^2F}{d\tau^2} \\ &= 1 + \frac{4}{n}\sum_q \frac{1}{(2|\tau| + q^2)^2} - 4g\sum_q \frac{q^2}{(2|\tau| + q^2)^3}, \end{aligned} \tag{5.16}$$

$$= 1 + \frac{K_D}{(2|\tau|)^{2-D/2}}\left(\frac{1}{n} - \frac{D}{4}g\right), \quad \tau \ll -1 \tag{5.17}$$

where $K_D = 4\Gamma(2 - \frac{1}{2}D)/(4\pi)^{D/2}$. For $2 < D < 4$, Eq. (5.17) gives the exact low-temperature behavior. Note that it also agrees with our Eq. (4.20) for the two-dimensional problem, suggesting strongly that this result is also exact. Equation (5.17) shows that, for all D in the range $2 < D < 4$, the specific heat approaches its mean-field value from below provided only the g is sufficiently large, $g > 4/nD$. However, Lubensky has shown,[18] using the $(4-D)$-expansion methods, that if the specific-heat exponent of the homogeneous system is negative, $\alpha < 0$, then the inhomogeneous system has the same critical exponents as the homogeneous system, whereas if $\alpha > 0$, there are a new set of "random" exponents. In either case the exponents are independent of g, in accordance with the idea of universality. Therefore, the behavior of C in the low-

temperature range $\tau \ll -1$, which depends on g, is no guide to what happens in the true critical region.

Bearing this fact in mind we will attempt to place our results in perspective. They can be summarized as follows. First, as is well known,[5] the specific heat in the Hartree theory ($n \to \infty$) is a monotonically decreasing function of temperature in the critical region. Second, also known previously,[11] the leading correction, of order $1/n$, to the Hartree theory leads to a peak in the heat capacity, in qualitative disagreement with experiment.[4] Third, the effect of inhomogeneities, treated in lowest order, is to cancel this peak and thus to restore qualitative agreement with experiment, provided that the "inhomogeneity parameter" g is large enough, $g \gtrsim 1$. An order-of-magnitude estimate gives a g of about the right size to account for the absence of a peak in the Zally and Mochel experiment. A self-consistent calculation (Sec. IV) also produces a peak for the homogeneous system, and removes it if $g \gtrsim 1$. The appearance of a peak to order $1/n$ is due in large part to the "slow decay" of $C^{(1/n)}$ in the low-temperature range below T_c, $C^{(1/n)} \propto 1/|\tau|$. The cancellation of the peak by the inhomogeneities is due largely to the cancelling behavior of $C^{(g)} \propto -1/|\tau|$ in this range. It is possible that an exact calculation for finite n would show a sharp structure (e.g., a cusp) in the specific heat as opposed to the broad peak appearing at order $1/n$ (I have no prejudice either way on this point). If the sharp structure has $\alpha < 0$ (e.g., a cusp), then it seems likely[18] that this structure would survive the introduction of inhomogeneities into the system. If $\alpha > 0$, on the other hand, the results of Lubensky[18] suggest that the introduction of inhomogeneities will lead to a renormalized exponent α_R, independent of g. The value of g in this case determines where the "crossover" into the true critical region occurs.

ACKNOWLEDGMENTS

I would like to thank Professor R. A. Ferrell and Professor R. E. Glover for useful discussions and the Fulbright-Hays Committee for a travel grant.

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*Research supported in part by a grant from the NSF.

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