PHYSICAL REVIEW B VOLUME 13, NUMBER 7 1 APRIL 1976

Critical dynamics of helium below T_c

Eric D. Siggia*

Bell Laboratories, Murray Hill, New Jersey 07974
and Department of Physics, † *University of Pennsylvania, Philadelphia, Pennsylvania 19174*

(Received 13 October 1975)

The frequency- and wave-number-dependent density correlation function has been calculated below T_c for the symmetric planar-spin model of helium to first order in $4-d$. In the hydrodynamic limit, the ratio of the damping of second sound to its velocity is an order of magnitude smaller than was found by Tyson. The theoretical results seem consistent with the light-scattering measurements however.

In a previous paper, the renormalization group was applied to several models of the critical dynamics of helium and the antiferromagnet above T_c.[1] Based on the simplest model for helium, the symmetric easy-plane ferromagnet (model E), we have investigated the critical dynamics below T_c. We have calculated in a Feynman-graph expansion to order $\epsilon = 4-d$ the full frequency- and wave-number-dependent correlation function of the density which participates in second sound. From it, we extract the velocity u_2 and the damping D_2 of second sound and the leading corrections in $k\xi$ to hydrodynamics. Numerical calculations of the shape function in three dimensions for arbitrary $k\xi$ will appear in a separate publication. The shape function is necessary for a detailed comparison with light-scattering data within a millidegree of T_c when $k\xi \gtrsim 1$.[2] At present we can make contact only with the macroscopic second-sound measurements of Greywall, Ahlers, and Tyson[3-5] and light scattering for $k\xi \lesssim 1$.

The model we use below T_c is given by Eq. (8) of Ref. 6, or Eq. (2.1) of Ref. 1 with $\gamma_0 = 0$ and $C_0 = 1$. (The symmetric model of helium neglects the important corrections to scaling caused by the slow approach of the effective-specific-heat exponent to zero. Computations are much simpler than in the more exact asymmetric model and with a few modifications it describes the antiferromagnet as well.) The Langevin equations of motion, when expressed in terms of the free energy, remain unchanged below T_c. In the free energy, however, one must allow for the effects of spontaneous symmetry breaking.[7] Our model then reproduces the thermodynamic relation for the second-sound velocity correctly.[1-8]

Let m be the appropriate combination of energy and density fluctuations which couples to the transverse component of the order parameter to produce second sound:

$$\langle mm\rangle(k,\omega) = \left(\frac{2\chi_0}{\omega_\kappa}\right)\frac{[(\lambda_0/\chi_0\Gamma_0)k^2 + \mathrm{Re}\Sigma_1]\,|-i\omega+\chi_T^{-1}+\Sigma_2|^2 + (\chi_T^{-1}+\mathrm{Re}\Sigma_2)u_2^2\chi_T^{-1}}{|[-i\omega+(\lambda_0/\chi_0\Gamma_0)k^2+\Sigma_1](-i\omega+\chi_T^{-1}+\Sigma_2)+u_2^2\chi_T^{-1}|^2}\,, \tag{1a}$$

$$\chi_T^{-1} = k^2 + 8u_0\int_0^\infty \frac{d^4q}{(2\pi)^4}\frac{(k+q)^2-q^2}{q^2(1+q^2)[1+(k+q)^2]}\,, \tag{1b}$$

$$\Sigma_1 = \frac{g_0^2}{\lambda_0\Gamma_0}\int_0^\infty \frac{d^4q}{(2\pi)^4}\left(\frac{[(k+q)^2-q^2]^2(-i\omega+\Gamma_+)}{q^2[1+(k+q)^2]D} - \frac{k^2}{2q^2(1+q^2)}\right), \tag{1c}$$

$$\Sigma_2 = \frac{g_0^2}{\lambda_0\Gamma_0}k^2\int_0^\infty \frac{d^4q}{(2\pi)^4}\left(\frac{-i\omega+\Gamma_+}{[1+(k+q)^2]D} - \frac{1}{2q^2(1+q^2)}\right) - 8u_0k^2\int_0^\infty \frac{d^4q}{(2\pi)^4}\frac{2u_2^2q^2-i\omega+\Gamma_+}{q^2[1+(k+q)^2]D}$$
$$-16u_0(-i\omega+k^2)\int_0^\infty \frac{d^4q}{(2\pi)^4}\frac{[(k+q)^2-q^2](u_2^2q^2-i\omega+\Gamma_+)}{q^2[1+(k+q)^2]D}, \tag{1d}$$

$$D = (-i\omega+\Gamma_+)^2 + u_2^2q^2,\quad \Gamma_+ = 1+q^2+(q+k)^2.$$

Equation (1) has been written in a scaled form which is universal except for the first factor provided that the bare parameters (denoted by a subscript "0"), are given their ϵ-expanded values. They have the following meaning: χ_0, the static susceptibility of m or specific heat; u_0, the four-spin coupling constant; λ_0, the thermal conductivity; Γ_0, the kinetic coefficient of the order parameter; and g_0, the dynamic coupling constant. The reader should consult Ref.1 for details. We have scaled all lengths with

$$\kappa_< = (8u_0\psi_0^2\Lambda^{-\epsilon})^{1/(2-\epsilon)},$$

where ψ_0 is the spontaneous value of the order parameter and Λ the ultraviolet cutoff. In four dimensions $\kappa_<$ is just $(-2t)^{1/2}$, the inverse longitudinal correlation length (t is the reduced temperature), and in any dimension is related by a universal factor to κ_- [see Eq. (5)], which is determined experimentally from ρ_s.[1,9] The nonuniversal frequency scale is defined by

$$\omega_\kappa = \Gamma_0\kappa_<^{2-\epsilon/2}\Lambda^{\epsilon/2}, \qquad (2)$$

and again is related by a universal factor to measurable properties such as the damping of second sound. The scaled second-sound velocity is

$$u_2^2 = \frac{g_0^2\psi_0^2}{\chi_0\Gamma_0^2\kappa_<^{2-\epsilon}\Lambda^\epsilon} = \frac{g_0^2}{\chi_0\Gamma_0^2 8u_0}$$

and χ_T is the static transverse correlation function. At the fixed point[1,7]

$$u_0 = \tfrac{3}{5}\epsilon(1+\tfrac{3}{5}\epsilon)\Lambda^\epsilon/(4!K_d), \qquad (3a)$$

$$g_0^2/\lambda_0\Gamma_0 = (\epsilon - 0.284\epsilon^2)\Lambda^\epsilon/K_d, \qquad (3b)$$
$$\lambda_0/\chi_0\Gamma_0 = 1+1.659\epsilon, \quad u_2^2 = 5+3.875\epsilon,$$

where K_d is the phase-space volume element; equal to $1/2\pi^2$ in three dimensions and $1/8\pi^2$ in four dimensions.

Our calculation was done with the formalism of Ref. 10 which we found rather cumbersome below T_c. Its application to helium was discussed in Ref. 1. Several remarks are in order about Eq. (1) to assist those who wish to reproduce it. Since we work to lowest order in ϵ, any universal quantity occurring within the integrals has been evaluated in four dimensions. We were able to write Eq. (1) in a form suggestive of hydrodynamics although it is valid for arbitrary k and ω. Nonhydrodynamic effects appear only through Σ_1 and Σ_2. Our formalism, however, generated six distinct self-energies. We have freely rearranged the order-ϵ contributions, dropping or adding ϵ^2 terms where necessary, to make the result take its present form. While we must obtain hydrodynamics to all orders in ϵ, there is no guarantee that all the corrections to hydrodynamics occur only through the functions (Σ_1 and Σ_2) which renormalize the hydrodynamic parameters.

Although we will not derive Eq. (1), several remarks might make it at least plausible. In the hydrodynamic limit, both Σ_1 and Σ_2 are real, of order k^2, and contribute to the damping of second sound. The form of the equation insures that all sum rules are satisfied. The velocity of second sound agrees with its thermodynamic definition in terms of χ_T [see Eqs. (4.3)–(4.5) of Ref. 1]. At T_c, u_2^2 in physical units vanishes, and we recover the same expression one would calculate from above T_c.

Subtractions were required to make Σ_1 and the first term of Σ_2 independent of the cutoff Λ. The counter terms correspond to the relevant variables λ_I and Γ_I we found in the recursion formulas above T_c.[1] As functions of $\kappa_<$, they have been exponentiated to give the frequency scale in Eq. (2). The exponent $\epsilon/2$ is the same as we found above T_c, in accord with dynamic scaling.[6,8,11] We have found no divergences in the density correlation function other than those either associated with static quantities or with dynamic parameters already encountered above T_c. When $k\xi$ tends to infinity, the terms which required a subtraction diverge logarithmically. This simply means that the frequency scale at T_c is $\propto k^{2-\epsilon/2}$ as required by dynamic scaling.[8] The accuracy of Eq. (1) is not uniform in $k\xi$ for finite ϵ. For $k\xi \gtrsim 1$, we encounter the same difficulties as did Aharony and Fisher when they calculated the static order-parameter correlation function.[12] The singularities predicted by an operator product expansion appear as logarithms in a Feynman-graph expansion.[13] Unless they are properly exponentiated, the ϵ expansion is numerically unreliable.[14] For small $k\xi$, the corrections to Ornstein-Zernike behavior or to hydrodynamics appear to be regular.[12] Since the leading singularity at T_c is fixed by scaling, its coefficient can be found by a Feynman-graph expansion. [See Eq. (9a).]

In the hydrodynamic limit, Eq (1) has a well defined pole which may be found by expanding Σ_1 and Σ_2 in both k and ω and inserting the fixed point values of Eq. (3). The coefficients of the expansion are rather complicated integrals which were evaluated numerically. We find

$$\omega_2 = 5^{1/2}k(1.0+0.412\epsilon+0.0032\epsilon k^2) - ik^2(1.0+0.440\epsilon-0.0051\epsilon k^2), \qquad (4)$$

where the frequency and wave number are in units of ω_κ and $\kappa_<$, respectively. To compare with experiment, it is convenient to work in terms of the inverse transverse correlation length κ_- defined by by[1]

$$\chi_T \equiv \psi_0^2/\kappa_-^{2-\epsilon}k^2 \equiv \psi_0^2/\rho_s k^2 = (1-\tfrac{1}{20}\epsilon)/k^2 + O(\epsilon^2) \qquad (5)$$

for $k\xi \ll 1$. In Ref. 9 κ_- in three dimensions is calculated from ρ_s,[15]

$$\kappa_- = 0.3\,|t|^{0.67}\,\text{Å}^{-1} \quad \text{at all pressures.}$$

From their respective definitions, it follows that

$$\kappa_-/\kappa_< = [5K_d(1-\tfrac{11}{20}\epsilon)/\epsilon]^{1/(2-\epsilon)}. \qquad (6)$$

The ratio of the damping of second sound times κ_- to its velocity is a universal quantity R_2. It follows from Eq. (4) as $k \to 0$ and Eq. (6) that

$$R_2 = D_2\kappa_-/u_2 = (1+0.028\epsilon)\kappa_-/5^{1/2}\kappa_< . \quad (7)$$

We extrapolate to three dimensions by setting $\epsilon = 1$ and $K_d = 1/2\pi^2$:

$$R_2 = 0.06 - 0.13. \quad (8)$$

The range of values results from extrapolating $1+a\epsilon$ instead of $1/(1-a\epsilon)$ and retaining or omitting ϵ-dependent numbers in the surd when raising to an ϵ-dependent power. The k^2 corrections in Eq. (4) should be decreased by ~ 0.05 when using units of κ_-.

In Ref. 1, we defined a characteristic frequency in terms of the height at zero frequency of the correlation function. [See Eq. (2.16) and Appendix D.] Although this definition facilitates the theoretical calculations, a more suitable method would be to fit the entire shape function. If we adopt the former definition for the characteristic frequencies of the order parameter ω_ψ and density ω_m at T_c, two additional universal ratios may be defined

$$R_m^{\text{crit}} = [\omega_m(k, T=T_c)/u_2\kappa_-](\kappa_-/k)^{2-\epsilon/2},$$
$$R_\psi^{\text{crit}} = [\omega_\psi(k, T=T_c)/u_2\kappa_-](\kappa_-/k)^{2-\epsilon/2}. \quad (9a)$$

In an ϵ expansion we find

$$R_m^{\text{crit}} = (K_d/\epsilon)^{1/2}(1+1.4\epsilon),$$
$$R_\psi^{\text{crit}} = (K_d/\epsilon)^{1/2}(1-0.3\epsilon). \quad (9b)$$

The light-scattering measurements of Winterling *et al.* and Vinen *et al.* measure the density correlation function as it crosses over from the hydrodynamic to the critical regime.[2] Figure 4 of Winterling *et al.* plots the line width and position of the second-sound peak along with an extrapolation in k and ω of the macroscopic measurements of damping and velocity. The frequency shift coincides exactly with the extrapolated velocity, while the linewidth is approximately temperature independent and deviates from the extrapolated damping. The extrapolations cross when $T_c - T = 0.8 \times 10^{-3}$ mK or $k/\kappa_- = 1.0$ in our units (k is held fixed in the experiments). Equations (4) and (8) imply that the linewidth should be an order of magnitude smaller than the frequency shift. We feel that the experiments provide only an upper limit on R_2 of around 0.3. It should be noted that the hydrodynamic form does not fit the measurements within a millidegree of T_c and the instrumental width is somewhat larger than the reported damping. Corrections to hydrodynamics are on the order of 10^{-3}–10^{-4} when $k/\kappa_- = 1$ and presumably not measurable as was found. Light scattering also determines R_m^{crit} in Eq. (9). The ϵ expansion for R_m is poorly convergent.

The most accurate measurements on second sound in helium are those of Greywall and Ahlers[3,5] and Tyson.[4] Combining their measurements, one finds for R_2

$$R_2^{\text{exp}} \simeq 0.65\,|t|^{-0.06} \simeq 1.0.$$

The theoretical value of R_2 disagrees with experiment by an order of magnitude. Although Eq. (6) indicates that the ϵ expansion is rather poorly convergent, our experience above T_c led us to expect agreement within 100%.[1] The calculation has been carried out to the same order in ϵ in both cases. Above T_c, graphs of order ϵ^2 were needed to determine Eqs. (3a) and (3b); but the same fixed point values apply below T_c. For both helium and the antiferromagnet above T_c, the amplitude ratios analogous to R_2 were a factor of ~ 2.5 too large in four dimensions and decreased to nearly their experimental value to next order in ϵ. Most of the change came from $K_d^{1/2}$ which varies by a factor of 2 between three and four dimensions. It was treated exactly and not expanded in ϵ. (There is a longitudinal and transverse correlation length below T_c and a single correlation length above T_c which are known experimentally in helium. The two universal ratios that they determine agree with theory within a factor of 2.[9]) To lowest order in ϵ Eq. (7) gives

$$R_2 = (8\pi^2\epsilon)^{-1/2},$$

which is the same (for $\epsilon = 1$) as Eq. (8), contrary to what happened above T_c.

Strictly speaking, the asymmetric planar ferromagnet (model ***F***) is required to describe helium.[1] It contains a coupling between the energy and order parameter which is absent from the symmetric model we used below T_c. Because the specific-heat exponent is slightly negative in helium, the asymptotic properties of both models are identical, although there are large correction terms. To second order in ϵ, calculations could only be done in the symmetric model. To first order in ϵ, however, fixed point parameters differ by 25% between the two models, and the correction terms make a significant contribution to the effective exponents. There is no reason to expect that below T_c the asymmetric model would change Eq. (8) by more than 25%.

We have not resolved the conflict between the two types of experiments discussed above. Among conceivable explanations might be (a) an unanticipated extrinsic source of damping in the macroscopic measurements due to the experimental cavity; (b) a breakdown of hydrodynamics leading to a term proportional to $|\ln(k\xi)|$ in the damping.[8,11]

ACKNOWLEDGMENTS

The author has benefited from conversations with A. Aharony and G. Ahlers and collaboration with B. I. Halperin and P. C. Hohenberg at Bell Laboratories. B. I. H. and P. C. H. also made valuable comments on the first draft of this paper. Mrs. A. Kane carried out the necessary programming.

---

*Supported in part by the NSF.

†Permanent address.

[1]B. I. Halperin, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. B 13, 1299 (1976).

[2]G. Winterling, F. S. Holmes, and T. J. Greytak, Phys. Rev. Lett. 30, 427 (1973); W. F. Vinen *et al.*, in *Proceedings of LT14*, edited by M. Krusius and M. Vuorio (Elsevier, New York, 1975), Vol. I, p. 191.

[3]D. S. Greywall and G. Ahlers, Phys. Rev. Lett. 28, 1251 (1972).

[4]J. A. Tyson, Phys. Rev. Lett. 21, 1235 (1968).

[5]G. Ahlers, in *The Physics of Liquid and Solid Helium*, edited by J. B. Ketterson and K. H. Bennemann (Wiley, New York, to be published), Vol. I, Chap. II.

[6]B. I. Halperin, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. Lett. 32, 1289 (1974).

[7]E. Brézin, D. J. Wallace, and K. G. Wilson, Phys. Rev. B 7, 232 (1973). Note that in Ref. 1 the vertex u_0 is defined without a factor 4!

[8]B. I. Halperin and P. C. Hohenberg, Phys. Rev. 177, 952 (1969).

[9]P. C. Hohenberg, A. Aharony, B. I. Halperin, and E. D. Siggia, Phys. Rev. B (to be published).

[10]P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973).

[11]B. I. Halperin, and P. C. Hohenberg, Phys. Rev. 188, 898 (1969).

[12]A. Aharony and M. E. Fisher, Phys. Rev. B 10, 2818 (1974).

[13]E. Brézin, J. C. le Guillou, and J. Zinn-Justin, Phys. Rev. Lett. 32, 473 (1974).

[14]M. Combescot, M. Droz, and J. M. Kosterlitz, Phys. Rev. Lett. 33, 705 (1974).

[15]There is a factor of 4π difference between the definition of κ_- in Ref. 1, and that of Ref. 8.