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Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region

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We compute exactly the spin-spin correlation functions $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for the two-dimensional Ising model on a square lattice in zero magnetic field for $T> T_c$ and $T< T_c$. We then analyze the correlation functions in the scaling limit $T\rightarrow T_c$, $M^2+N^2\rightarrow\infty$ such that $(M^2+N^2)^{1/2}(T-T_c)$ is fixed. In this scaling limit $\langle\sigma_{0,0}\sigma_{M,N}\rangle = R^{-1/4}F_\pm(t) + R^{-5/4}F_{1\pm}(t)+ o(R^{-5/4})$, where t is the scaling variable R/ξ and $F_\pm(t)$ and $F_{1\pm}(t)$ are the scaling functions (ξ is the correlation length). We derive exact expressions for these scaling functions, in terms of a Painlevé function of the third kind and analyze both the small- and large-t behavior. A table of values for $F_\pm(t)$ (good to ten significant digits) is also given. As an application we compute the coefficients $C_{0\pm}$ and $C_{1\pm}$ in the expansion $k_BT\chi(T) = C_{0\pm}|1-T_c/T|^{-7/4} + C_{1\pm}|1-T_c/T|^{-3/4}+O(1)$ of the zero-field susceptibility $\chi(T)$ as $T\rightarrow T_c^\pm$.

I. INTRODUCTION

In this paper we calculate,[1] *in the scaling limit*, the spin-spin correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for the two-dimensional Ising model[2-5] in the absence of a magnetic field. By scaling limit we mean the limit

$$T\rightarrow T_c \qquad (1.1)$$

and

$$M^2+N^2\rightarrow\infty , \qquad (1.2)$$

such that

$$(M^2+N^2)^{1/2}(T-T_c) \qquad (1.3)$$

is fixed. Here T_c is the critical temperature and the quantity in (1.3) may be either positive or negative (or zero).

In this limit the correlation function is expressible in terms of a function of one variable, which is a simple combination M, N, T, and the interaction energies. This function is known as the scaling function. Scaling functions have been found useful under many circumstances, and general properties of such scaling functions have been discussed by many authors[6-13] in the last ten years. However, the calculation reported here represents the first time that an *exact* calculation of any nontrivial scaling two-spin correlation function has ever been achieved.

As an application of our result we will obtain, for the zero-field magnetic susceptibility $\chi(T)$, all terms which are unbounded at T_c. More precisely, we will compute exactly the four numbers $C_{0\pm}$ and $C_{1\pm}$ in the expansion, valid as $T\rightarrow T_c$,

$$k_BT\chi(T)=C_{0\pm}|1-T_c/T|^{-7/4} + C_{1\pm}|1-T_c/T|^{-3/4}+O(1) . \qquad (1.4)$$

Here k_B is Boltzmann's constant and the subscript + (−) means that $T\rightarrow T_c$ from above (below). Our results supercede the extensive numerical studies of these coefficients made during the last fifteen years by means of series expansions.[14]

II. SUMMARY OF RESULTS AND NOTATION

In view of the length and complexity of this paper we collect together our principal results and notation in this section so that they are readily accessible for easy reference.

A. Definitions and notations

The two-dimensional Ising model on a square lattice is specified by the interaction energy

$$\mathcal{E}=-E_1\sum_{j,k}\sigma_{j,k}\sigma_{j,k+1}-E_2\sum_{j,k}\sigma_{j,k}\sigma_{j+1,k} , \qquad (2.1)$$

where the first (second) index of $\sigma_{j,k}$ specifies the row (column) of the lattice and $\sigma_{j,k}=\pm 1$.

In this section we will use the following notation with regard to equation numbers. First, if an equation number is followed by an S, such as (2.2S), that equation applies only to the symmetrical case $E_1=E_2=E$. Second, in order to facilitate the location of the place in this paper where a particular result is derived, the equation number for a result is sometimes followed by a number in square brackets. Thus, for example, (2.9)-[3.66] means the result (2.9) is derived in the text shortly after (3.66).

We define

$$z_1=\tanh(\beta E_1)\ ,\quad z_2=\tanh(\beta E_2)\ , \tag{2.2}$$

and for the symmetrical lattice,

$$z=z_1=z_2=\tanh(\beta E)\ , \tag{2.2S}$$

where $\beta=(k_BT)^{-1}$. At $T=T_c$

$$\sinh 2\beta_c E_1 \sinh 2\beta_c E_2=1 \tag{2.3}$$

or, equivalently,

$$z_{1c}z_{2c}+z_{1c}+z_{2c}-1=0\ . \tag{2.4}$$

In particular, for the symmetrical lattice,

$$z_c=\sqrt{2}-1\ . \tag{2.4S}$$

The scaled variable t is defined by

$$\begin{aligned} t&=|z_1z_2+z_1+z_2-1|\left(\frac{M^2}{z_2(1-z_1^2)}+\frac{N^2}{z_1(1-z_2^2)}\right)^{1/2}\\ &=|z_1z_2+z_1+z_2-1|\,[z_1z_2(1-z_1^2)(1-z_2^2)]^{-1/4}R\ , \end{aligned} \tag{2.5}$$

where the appropriate radial measure of spatial distance is

$$\begin{aligned} R&=\left[\left(\frac{z_1(1-z_2^2)}{z_2(1-z_1^2)}\right)^{1/2}M^2+\left(\frac{z_2(1-z_1^2)}{z_1(1-z_2^2)}\right)^{1/2}N^2\right]^{1/2}\\ &=\left[\left(\frac{\sinh 2\beta E_1}{\sinh 2\beta E_2}\right)^{1/2}M^2+\left(\frac{\sinh 2\beta E_2}{\sinh 2\beta E_1}\right)^{1/2}N^2\right]^{1/2}\ . \end{aligned} \tag{2.6}$$

For the symmetrical case one has simply

$$t=|z^2+2z-1|\,[z(1-z^2)]^{-1/2}R \tag{2.5S}$$

and

$$R=(M^2+N^2)^{1/2}\ . \tag{2.6S}$$

As $T\to T_c$ we find

$$z_1z_2+z_1+z_2-1\sim(\beta-\beta_c)4z_{1c}z_{2c}[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}] \tag{2.7}$$

and

$$\begin{aligned} t/R&=|\beta-\beta_c|\,2\sqrt{2}\,(z_{1c}z_{2c})^{1/2}[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}]+O((\beta-\beta_c)^2)\\ &=|\beta-\beta_c|\,2(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{-1/2}\\ &\quad\times(E_1\coth 2\beta_cE_1+E_2\coth 2\beta_cE_2)+O((\beta-\beta_c)^2)\ . \end{aligned} \tag{2.8}$$

For the symmetric lattice the behavior is

$$t/R=|\beta-\beta_c|\,4E[1-2^{-1/2}E(\beta-\beta_c)+O((\beta-\beta_c)^2)]\ . \tag{2.8S}$$

B. Results for correlation functions

When $T<T_c$, one has

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle=S_\infty^<\exp(-F_{<MN})\ , \tag{2.9)-[3.66}$$

where

$$S_\infty^<=[1-(\sinh 2\beta E_1\sinh 2\beta E_2)^{-2}]^{1/4} \tag{2.10}$$

and

$$F_{<MN}=\sum_{n=1}^{\infty}F_{<MN}^{(2n)}\ , \tag{2.11}$$

$$\begin{aligned} F_{<MN}^{(2n)}&=(-1)^n[2z_2(1-z_1^2)]^{2n}(2n)^{-1}(2\pi)^{-4n}\\ &\quad\times\int_{-\pi}^{\pi}d\phi_1\int_{-\pi}^{\pi}d\phi_2\cdots\int_{-\pi}^{\pi}d\phi_{4n}\\ &\quad\times\prod_{j=1}^{2n}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\,\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)\ . \end{aligned} \tag{2.12)-[3.84}$$

Here

$$\phi_{4n+1}\equiv\phi_1\ ,\quad \phi_{4n+2}\equiv\phi_2\ ;$$

$$\mathrm{Im}\phi_j<0\ ,\quad j=1,2,\ldots,4n\ ;$$

and

$$\begin{aligned} \Delta(\phi_{2j-1},\phi_{2j})&=(1+z_1^2)(1+z_2^2)\\ &\quad-2z_2(1-z_1^2)\cos\phi_{2j-1}\\ &\quad-2z_1(1-z_2^2)\cos\phi_{2j}\ . \end{aligned} \tag{2.13}$$

When $T>T_c$, the result is

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle=S_\infty^>\left(\sum_{k=1}^{\infty}x_{>MN}^{(2k-1)}\right)\exp\left(-\sum_{n=1}^{\infty}F_{>MN}^{(2n)}\right)\ , \tag{2.14)-[4.87}$$

where

$$\begin{aligned} x_{>MN}^{(k)}&=[-i2z_2(1-z_1^2)]^{k-1}(2\pi)^{-2k}\int_{-\pi}^{\pi}d\phi_1\int_{-\pi}^{\pi}d\phi_2\cdots\int_{-\pi}^{\pi}d\phi_{2k}\\ &\quad\times\prod_{j=1}^{k}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\prod_{j=1}^{k-1}\left(\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\cos\tfrac{1}{2}(\phi_{2j-1}+\phi_{2j+1})\right) \end{aligned} \tag{2.15}$$

[if $k=1$, the second product is replaced by unity],

$$F_{>MN}^{(2n)}=(-1)^n[2z_2(1-z_1^2)]^{2n}(2n)^{-1}(2\pi)^{-4n}\int_{-\pi}^{\pi}d\phi_1\int_{-\pi}^{\pi}d\phi_2\cdots\int_{-\pi}^{\pi}d\phi_{4n}$$

$$\times\prod_{j=1}^{2n}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\cos\tfrac{1}{2}(\phi_{2j-1}+\phi_{2j+1}) \tag{2.16}$$

[with $\phi_{4n+1}=\phi_1$ and $\phi_{4n+2}=\phi_2$], $\mathrm{Im}\phi_j<0$, and

$$S_\infty^{>}=[4z_1z_2(1-z_1^2)(1-z_2^2)]^{1/2}[(\sinh 2\beta E_1\sinh 2\beta E_2)^{-2}-1]^{1/4}\,. \tag{2.17}$$

The above results hold for all T. From here on we concentrate on the scaling limit (1.1)–(1.3). Equivalently, this limit is

$$T\to T_c\,, \tag{2.18}$$

$$R\to\infty\,, \tag{2.19}$$

with

$$t \text{ fixed}\,. \tag{2.20}$$

In this limit we find that

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle=|1-(\sinh 2\beta E_1\sinh 2\beta E_2)^{-2}|^{1/4}\times[\hat{F}_\pm(t)+o(R^{-1})]\,, \tag{2.21}$$

where $\hat{F}_\pm(t)$ depends upon E_1, E_2, M, and N only through the variable t, and the subscript + (−) means $T>T_c$ ($T<T_c$). The first factor may be expanded as a function of t/R, and we find

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle=R^{-1/4}F_\pm(t)+R^{-5/4}F_{1\pm}(t)+o(R^{-5/4})\,, \tag{2.22}$$

where

$$F_\pm(t)=(2t)^{1/4}(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}\hat{F}_\pm(t) \tag{2.23}$$

and

$$F_{1+}(t)/F_+(t)=-F_{1-}(t)/F_-(t)=-tR_1\,, \tag{2.24}$$

where

$$R_1=(16\sqrt{2})^{-1}(z_{1c}z_{2c})^{-1/2}(z_{1c}+z_{2c})^{-1}[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}]^{-1}[E_1(1-z_{2c})^{-1}(-5z_{2c}^2-3z_{1c}^2+6z_{2c}+10z_{1c}-8)+E_2(1-z_{1c})^{-1}(-5z_{1c}^2-3z_{2c}^2+6z_{1c}+10z_{2c}-8)]\,. \tag{2.25}$$

For the symmetric lattice R_1 reduces to

$$R_1=-2^{-3/2}\,. \tag{2.25S}$$

The ratio R_1 is plotted in Fig. 1 for fixed T_c as a function of the ratio E_1/E_2.

Furthermore, we find explicitly

$$\hat{F}_-(t)=\exp\Big(-\sum_{n=1}^{\infty}f^{(2n)}(t)\Big)\,, \tag{2.26}$$

with

$$f^{(2n)}(t)=(-1)^n\pi^{-2n}n^{-1}\int_1^\infty dy_1\cdots\int_1^\infty dy_{2n}\times\prod_{j=1}^{2n}\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}(y_j+y_{j+1})}\prod_{j=1}^{n}(y_{2j}^2-1) \tag{2.27)-[3.128]$$

$[y_{2n+1}\equiv y_1]$

and also

$$\hat{F}_+(t)=G(t)\hat{F}_-(t)\,, \tag{2.28}$$

where

$$G(t)=\sum_{k=0}^{\infty}g^{(2k+1)}(t)\,, \tag{2.29}$$

with

$$g^{(2k+1)}(t)=(-1)^k\pi^{-2k-1}\int_1^\infty dy_1\cdots\int_1^\infty dy_{2k+1}\times\prod_{j=1}^{2k+1}\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}}\prod_{j=1}^{2k}(y_j+y_{j+1})^{-1}\prod_{j=1}^{k}(y_{2j}^2-1) \tag{2.30)-[4.150]$$

[for $k=0$ the last two products are replaced by unity].

As $t\to\infty$ we have

$$F_+(t)=(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}\times(2t)^{1/4}\pi^{-1}K_0(t)+O(e^{-3t}) \tag{2.31a)-[4.155]$$

and

$$F_-(t)=(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}(2t)^{1/4}\times(1+\pi^{-2}\{t^2[K_1^2(t)-K_0^2(t)]-tK_0(t)K_1(t)+\tfrac{1}{2}K_0^2(t)\}+O(e^{-4t}))\,. \tag{2.31b)-[3.151]$$

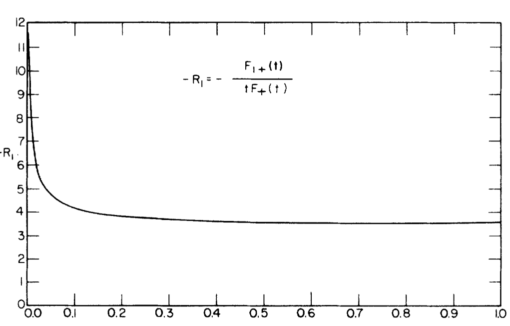

FIG. 1. Ratio $R_1=F_{1+}(t)/[tF_+(t)]$ as a function of $x=E_1/E_2$ for fixed T_c. For $x=1$, $R_1=-2^{-3/2}$.

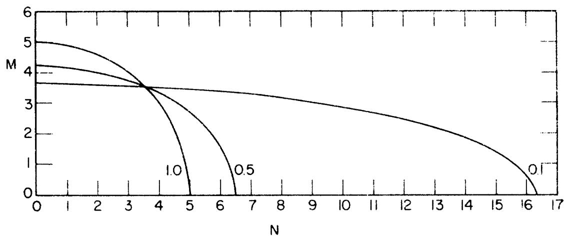

FIG. 2. Ellipses in the M, N plane along which $R^{-1/4}F_{\pm}(t)$ (for fixed t) is a constant. Three different ellipses corresponding to the ratios $E_2/E_1 = 1.0$, 0.5 and 0.1 are shown. Note that for $M=N$ the value of $R^{-1/4}F_{\pm}(t)$ is independent of the ratio E_1/E_2.

Conversely, as $t \to 0$ we obtain

$$F_{\pm}(t) = (\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8} e^{1/4} \times 2^{1/12} A^{-3} [1 \pm \tfrac{1}{2} t\Omega + \tfrac{1}{16} t^2 \pm \tfrac{1}{32} t^3 \Omega + \tfrac{1}{256} t^4(-\Omega^2 + \Omega + \tfrac{1}{8}) + O(t^5\Omega^4)] , \quad \text{(2.32)-[5.53], [6.123]}$$

where

$$\Omega = \ln(t/8) + \gamma_E , \qquad (2.33)$$

$$\gamma_E = 0.577\,215\,664\,901\,53\cdots \qquad (2.34)$$

is Euler's constant, and

$$A = 1.282\,427\,129\,100\,62\cdots \qquad (2.35)$$

is Glaisher's constant.

In Fig. 2, we plot, for various values of the ratio E_2/E_1, the ellipses in the M, N plane along which $R^{-1/4}F_{\pm}(t)$ is constant for fixed t as $T \to T_c$. Note in particular that for $M=N$ the value of $R^{-1/4}F_{\pm}(t)$ is independent of the ratio E_1/E_2.

The functions $F_{\pm}(t)$ may also be expressed in terms of a Painlevé function (of the third kind)[33] which we denote by $\eta(0)$. This function satisfies the second-order nonlinear differential equation

$$\frac{d^2\eta}{d\theta^2} = \frac{1}{\eta}\left(\frac{d\eta}{d\theta}\right)^2 - \eta^{-1} + \eta^3 - \theta^{-1}\frac{d\eta}{d\theta} , \qquad (2.36)$$

with the boundary conditions

$$\eta(\theta) = -\theta[\ln(\tfrac{1}{4}\theta) + \gamma_E] + O(\theta^5 \ln^3\theta) \qquad \text{(2.37)-[B8]}$$

as $\theta \to 0$, and

$$\eta(\theta) = 1 - 2\pi^{-1} K_0(2\theta) + O(e^{-4\theta}) \qquad \text{(2.38)-[6.95]}$$

as $\theta \to \infty$ [see Sec. VI for further discussion]. We then have

$$F_{\pm}(t) = 2^{-1/2}(\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8}\theta^{1/4}[1 \mp \eta(\theta)]\eta^{-1/2}(\theta) \exp\left(\int_\theta^\infty dx\, \tfrac{1}{4} x\eta^{-2}(x)\{[1-\eta^2(x)]^2 - [\eta'(x)]^2\}\right)$$

$$= 2^{-1/2}(\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8}\theta^{1/4}[1 \mp \eta(\theta)] \exp\left(\int_\theta^\infty dx\, x \ln x [1-\eta^2(x)] - h(\theta)\right) , \qquad \text{(2.39)-[6.67]}$$

where

$$\theta = \tfrac{1}{2} t \qquad (2.40)$$

and, in the last line, we have

$$h(\theta) = \left(\frac{\theta\eta'}{2\eta} + \frac{\theta^2}{4\eta^2}[(1-\eta^2)^2 - \eta'^2]\right)\ln\theta . \qquad (2.41)$$

The functions $F_{\pm}(t)$ for $E_1 = E_2$ are plotted in Fig. 3. The function $[F_-(t) - (\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8} \times (2t)^{1/4}]$ for $E_1 = E_2$ is plotted in Fig. 4.

If one believes in some form of universality, then in the scaling limit (2.39) gives (aside from some lattice-dependent factors which are independent of t) the spin-spin correlation not only for the two-dimensional Ising model but also for a large class of two-dimensional models with short-range interactions.

C. Results for magnetic susceptibility

The magnetic susceptibility at $H=0$,

$$\chi(T) = \left.\frac{\partial \mathfrak{M}(T,H)}{\partial H}\right|_{H=0} , \qquad (2.42)$$

is related to $\langle \sigma_{0,0}\sigma_{M,N}\rangle$ by

$$\beta^{-1}\chi(T) = \sum_{M=-\infty}^{+\infty}\sum_{N=-\infty}^{+\infty} [\langle \sigma_{0,0}\sigma_{M,N}\rangle - \mathfrak{M}^2(T,0)] , \qquad (2.43)$$

where

$$\mathfrak{M}(T,0) = (S_\infty^<)^{1/2} \quad \text{if } T < T_c$$
$$= 0 \quad \text{if } T \geq T_c . \qquad (2.44)$$

As $T \to T_c$ we find that

$$\beta^{-1}\chi(T) = C_{0\pm}|1 - T_c/T|^{-7/4} + C_{1\pm}|1 - T_c/T|^{-3/4} + O(1) , \qquad (2.45)$$

where

$$C_{0+} = D\int_0^\infty d\theta\, \theta[1-\eta(\theta)] \times \exp\left(\int_\theta^\infty dx[1-\eta^2(x)]x\ln x - h(\theta)\right) , \qquad \text{(2.46a)-[7.6]}$$

$$C_{0-} = D\int_0^\infty d\theta\, \theta\left[[1+\eta(\theta)] \times \exp\left(\int_\theta^\infty dx[1-\eta^2(x)]x\ln x - h(\theta)\right) - 2\right] , \qquad \text{(2.46b)-[7.17]}$$

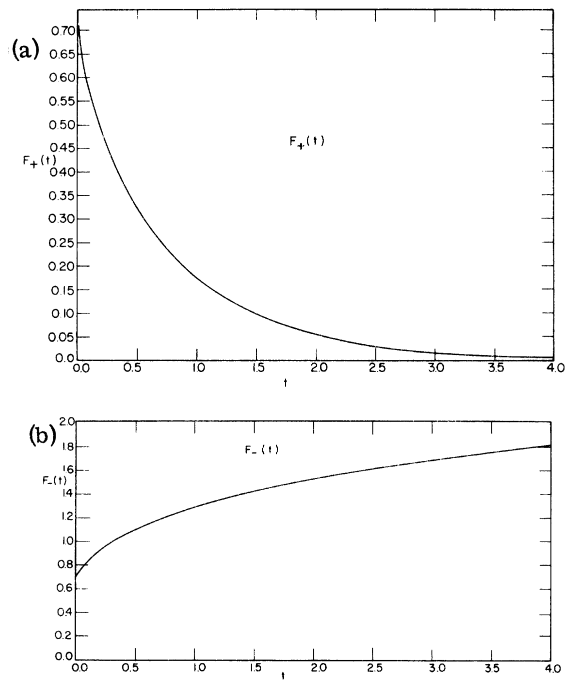

FIG. 3. (a) Scaling function $F_+(t)$ as a function of t for $E_1=E_2$. Note that $F_+(0)=0.703\,380\,157\,7\cdots$. (b) Scaling function $F_-(t)$ as a function of t for $E_1=E_2$. At $t=0$ $F_-(0)=F_+(0)$.

and

$$C_{1+}/C_{0+}=-C_{1-}/C_{0-}=-R_0\beta_c\,, \tag{2.47}$$

with

$$\begin{aligned}D&=2^{-1/2}\pi(z_{1c}z_{2c})^{-1}(z_{1c}+z_{2c})^{1/4}\\&\quad\times\{\beta_c[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}]\}^{-7/4}\\&=2^{1/2}\pi(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)\\&\quad\times\{\beta_c[E_1\coth 2\beta_cE_1+E_2\coth 2\beta_cE_2]\}^{-7/4}\end{aligned} \tag{2.48}$$

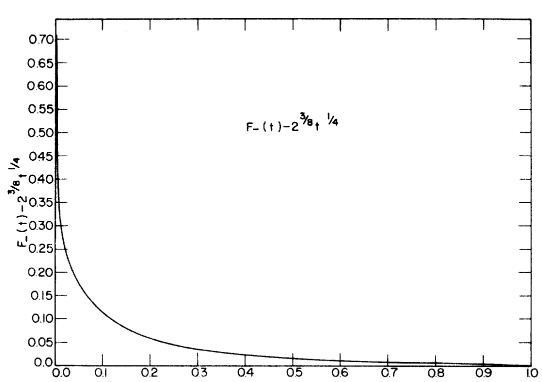

FIG. 4. Function $F_-(t)-2^{3/8}t^{1/4}$ as a function of t for $E_1=E_2$. Note that for $E_1=E_2$ $S_\infty^< \sim 2^{3/8}t^{1/4}$.

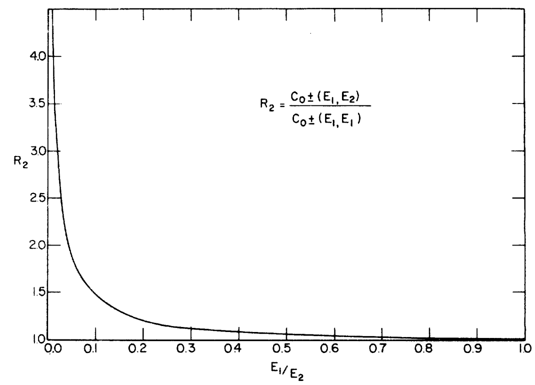

FIG. 5. Amplitude ratio $R_2=C_{0\pm}(E_1,E_2)/C_{0\pm}(E_1,E_1)$ as a function of the ratio E_1/E_2 for fixed T_c.

and

$$\begin{aligned}R_0=&[E_1^2z_{2c}^2(1+6z_{1c}^2+z_{1c}^4)+E_2^2z_{1c}^2(1+6z_{2c}^2+z_{2c}^4)\\&-8E_1E_2z_{1c}z_{2c}(z_{1c}+z_{2c})^2]\{8z_{1c}z_{2c}(z_{1c}+z_{2c})\\&\times[E_1(1-z_{1c})+E_2(1-z_{2c})]\}^{-1}\,.\end{aligned} \tag{2.49}$$

In the symmetric case we have

$$D=2^{5/8}\pi[\ln(1+\sqrt{2}\,)]^{-7/4}\,, \tag{2.48S}$$

$$R_0=-\tfrac{1}{8}\sqrt{2}E\,. \tag{2.50S}$$

In Fig. 5 we plot

$$R_2=C_{0\pm}(E_1,E_2)/C_{0\pm}(E_1,E_1) \tag{2.51}$$

for fixed T_c as a function of E_1/E_2.

In Fig. 6 we plot the dimensionless quantity $R_0(E_1+E_2)^{-1}$ for fixed T_c as a function of E_1/E_2. Note that this quantity is $\frac{1}{2}$ at $E_1/E_2=0$ and $-\frac{1}{16}\sqrt{2}$ at $E_1=E_2$.

To compare these results for $C_{0\pm}$ and $C_{1\pm}$ with the available series-expansion results,[14] we specialize to the symmetric lattice and numerically evaluate $C_{0\pm}$ and $C_{1\pm}$ to find

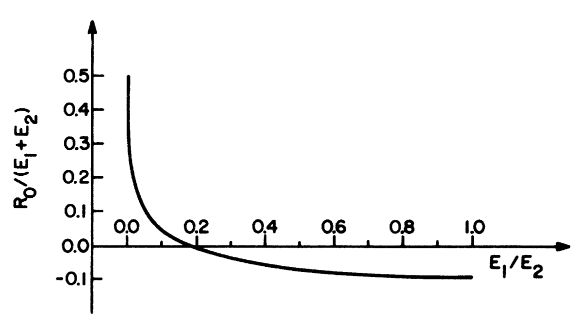

FIG. 6. Dimensionless quantity $R_0/(E_1+E_2)$ as a function of $x=E_1/E_2$. When $x=1$ the value is $-\frac{1}{16}\sqrt{2}$. As $x\to 0$ the function behaves as $\frac{1}{2}+3/(2\ln x)$. Curve crosses zero at ~ 0.1993.

$$C_{0+}=0.962\,581\,732\,2\cdots\,,$$
$$C_{0-}=0.025\,536\,971\,9\cdots\,,$$
$$C_{1+}=0.074\,988\,153\,8\cdots\,, \qquad (2.52S)$$
$$C_{1-}=-0.001\,989\,410\,7\cdots\,.$$

These are to be compared with the series-expansion estimates of Sykes, Gaunt, Roberts, and Wyles[14a] above T_c

$$C_{0+}=0.962\,59\pm0.000\,03\,, \qquad (2.53S)$$

$$C_{1+}\simeq 0.0742\,, \qquad (2.54S)$$

and of Essam and Hunter[14b] below T_c

$$C_{0-}=0.025\,68\pm0.000\,12\,. \qquad (2.55S)$$

Some insight into the close agreement of the exact results (2.52S) and the series results (2.53S)–(2.55S) is obtained by expanding (2.26) as

$$\hat{F}_-(t)=1-f^{(2)}(t)-\{f^{(4)}(t)-\tfrac{1}{2}[f^{(2)}(t)]^2\}+\cdots\,, \qquad (2.56)$$

and expanding (2.28) as

$$\hat{F}_+(t)=g^{(1)}(t)+[g^{(3)}(t)-g^{(1)}(t)f^{(2)}(t)] + (g^{(5)}(t)-g^{(3)}(t)f^{(2)}(t) - g^{(1)}(t)\{f^{(4)}(t)-\tfrac{1}{2}[f^{(2)}(t)]^2\})+\cdots\,. \qquad (2.57)$$

In (2.56), $f^{(2)}(t)$ behaves as e^{-2t} as $t\to\infty$, and the third term behaves as e^{-4t}. In (2.57), $g^{(1)}(t)$ behaves as e^{-t} as $t\to\infty$, the second term behaves as e^{-3t}, and the third term behaves as e^{-5t}. Here the phrase "behaves as $e^{-\alpha t}$" means that as $t\to\infty$ the quantity is asymptotically equal to $e^{-\alpha t}$ times some power of t. Then if we define $C_{0+}^{(2n+1)}$ as the contribution to C_{0+} coming from the term in (2.57) behaving as $e^{-(2n+1)t}$ as $t\to\infty$, and $C_{0-}^{(2n)}$ as the contribution to C_{0-} coming from the terms in (2.56) behaving as e^{-2nt} as $t\to\infty$, we have

$$C_{0+}=\sum_{n=0}^{\infty} C_{0+}^{(2n+1)}\,, \qquad (2.58a)$$

$$C_{0-}=\sum_{n=1}^{\infty} C_{0-}^{(2n)}\,. \qquad (2.58b)$$

Numerically we find for $E_1=E_2$

$$C_{0+}^{(1)}=[2\ln(1+\sqrt{2})]^{-7/4}\,2^{11/8} = 0.961\,797\,616\,76\cdots\,, \qquad (2.59S)\text{-}[7.8]$$

$$C_{0+}^{(3)}=0.000\,783\,348\,15\cdots\,, \qquad (2.60S)$$

$$C_{0+}^{(5)}=0.000\,000\,766\,5\cdots\,, \qquad (2.61S)$$

and

$$C_{0-}^{(2)}=[2\ln(1+\sqrt{2})]^{-7/4}\,2^{11/8}(12\pi)^{-1} = 0.025\,512\,474\,16\cdots\,, \qquad (2.62S)\text{-}[7.20]$$

$$C_{0-}^{(4)}=0.000\,024\,476\cdots\,. \qquad (2.63S)$$

These perturbation-expansion results are compared with the exact results (2.52S) and the series-expansion results (2.53S)–(2.55S) in Table I.

Finally a remark must be made concerning the $O(1)$ term of (1.4). This term arises from short-range contributions to the sum (2.43) and hence cannot be obtained from the scaling functions $F_\pm(t)$.

TABLE I. As $T\to T_c^\pm$ the zero-field susceptibility $\chi(T)$ behaves as $\beta^{-1}\chi(T)=C_{0\pm}|1-T_c/T|^{-7/4}+C_{1\pm}|1-T_c/T|^{-3/4}+O(1)$. The four numbers $C_{0\pm}$ and $C_{1\pm}$ for the symmetric lattice $E_1=E_2$ are given. See the text for the definition of the numbers $C_{0\pm}^{(n)}$.

Quantity	Exact result	Series-expansion result	Perturbation-expansion approximation
C_{0+}	$0.962\,581\,732\,2\cdots$	$0.962\,59\pm0.000\,03$[a]	$C_{0+}^{(1)}=0.961\,797\,616\,76\cdots$
			$C_{0+}^{(1)}+C_{0+}^{(3)}=0.962\,580\,964\,91\cdots$
			$C_{0+}^{(1)}+C_{0+}^{(3)}+C_{0+}^{(5)}=0.962\,581\,731\,4\cdots$
C_{0-}	$0.025\,536\,971\,9\cdots$	$0.025\,68\pm0.000\,12$[b]	$C_{0-}^{(2)}=0.025\,512\,474\,16\cdots$
			$C_{0-}^{(2)}+C_{0-}^{(4)}=0.025\,536\,950\cdots$
C_{0+}/C_{0-}	$37.693\,652\,01\cdots$		$C_{0+}^{(1)}/C_{0-}^{(2)}=12\pi=37.699\,111\,84$
			$(C_{0+}^{(1)}+C_{0+}^{(3)})/(C_{0-}^{(2)}+C_{0-}^{(4)})=37.693\,654\,3$
C_{1+}	$0.074\,988\,153\,8\cdots$	0.0742[a]	
C_{1-}	$-0.001\,989\,410\,7\cdots$		

[a]Reference 14a. [b]Reference 14b.

From the series-expansion results[14] this term is known to be nonzero.

D. Outline of main text

As a guide to the remainder of the paper, we present the following outline.

III. PERTURBATION EXPANSION FOR $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ FOR $T<T_c$ AND LARGE (M^2+N^2)

Using the Pfaffian approach[15,16] to the two-dimensional Ising model, Cheng and Wu[17] (CW) developed a perturbation expansion for $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for $T<T_c$ (and $T>T_c$) and large M^2+N^2 which they then analyzed to leading order. Here we examine in this perturbation expansion scheme the nth-order terms. To facilitate the readability of this paper, we summarize all relevant results of CW that we will need. The interested reader should consult either CW or Chap. 12 of Ref. 5 whenever we say "it has been shown that . . . " for a demonstration of the stated result. In this way the reader is able to follow the derivations of this section (and Sec. IV) without being familiar with the results of CW.

A. Preliminary notation

We first define the following infinite matrices:

$$S_{mn}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\phi\, e^{-i(n-m)\phi}S(e^{i\phi})\,, \tag{3.1}$$

$$V_{mn}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\phi\, e^{-i(m+n)\phi}\,V(e^{i\phi})\,, \tag{3.2}$$

$$U_{mn}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\phi\, e^{-i(m+n)\phi}\,U(e^{i\phi})\,, \tag{3.3}$$

$$T_{mn}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\phi\, e^{-i(m+n)\phi}\,T(e^{i\phi})\,, \tag{3.4}$$

for m, $n=0, 1, 2, \ldots$, where the generating functions $S(e^{i\phi})$, $V(e^{i\phi})$, $U(e^{i\phi})$, and $T(e^{i\phi})$ are given[18]

$$S(e^{i\phi})=[(1-\alpha_1e^{i\phi})(1-\alpha_2e^{-i\phi})\times(1-\alpha_1e^{-i\phi})^{-1}(1-\alpha_2e^{i\phi})^{-1}]^{1/2}\,, \tag{3.5}$$

$$V(e^{i\phi_2})=(1-z_1^2)(2\pi)^{-1}\times\int_{-\pi}^{\pi}d\phi_1\frac{e^{-iM\phi_1-iN\phi_2}}{\Delta(\phi_1,\phi_2)}e^{-2i\phi_2}\lambda(\phi_1,-\phi_2)\,, \tag{3.6}$$

$$U(e^{i\phi_2})=-e^{-i\phi_2}(1-z_1^2)(2\pi)^{-1}\times\int_{-\pi}^{\pi}d\phi_1\frac{e^{-iM\phi_1-iN\phi_2}}{\Delta(\phi_1,\phi_2)}2iz_2\sin\phi_1\,, \tag{3.7}$$

and

$$T(e^{i\phi_2})=-(1-z_1^2)(2\pi)^{-1}\times\int_{-\pi}^{\pi}d\phi_1\frac{e^{-iM\phi_1-iN\phi_2}}{\Delta(\phi_1,\phi_2)}\lambda(\phi_1,\phi_2)\,, \tag{3.8}$$

and with z_1 and z_2 given by (2.2), $\Delta(\phi_1, \phi_2)$ by (2.13),

$$\lambda(\phi_1, \phi_2) = 1 - z_2^2 - z_1(1 + z_2^2 + 2z_2\cos\phi_1)e^{-i\phi_2} \tag{3.9}$$

and

$$\alpha_1 = z_1(1-z_2)(1+z_2)^{-1},$$
$$\alpha_2 = z_1^{-1}(1-z_2)(1+z_2)^{-1}. \tag{3.10}$$

The square root in (3.5) is defined so that $S(e^{i\psi}) > 0$. It is convenient to define (following Refs. 15 and 17)

$$a = (1+z_1^2)(1+z_2^2),$$
$$\gamma_1 = 2z_2(1-z_1^2), \tag{3.11}$$

and

$$\gamma_2 = 2z_1(1-z_2^2).$$

From (2.4) it follows that the $T = T_c$ condition can be expressed either as

$$\alpha_2 = 1 \tag{3.12}$$

or as

$$a - \gamma_1 - \gamma_2 = (z_1 z_2 + z_1 + z_2 - 1)^2 = 0. \tag{3.13}$$

We also note that for $T < T_c$ $(0 < \alpha_1 < \alpha_2 < 1)$ the index[19] of the generating function (3.5) is zero.

We define the matrices A, B, and C by

$$A = \begin{bmatrix} 0 & S & 0 & 0 \\ -S^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -S \\ 0 & 0 & S^T & 0 \end{bmatrix}, \tag{3.14}$$

$$B = \begin{bmatrix} 0 & 0 & T & U \\ 0 & 0 & -U & V \\ -T & U & 0 & 0 \\ -U & -V & 0 & 0 \end{bmatrix}, \tag{3.15}$$

and

$$C = A + B, \tag{3.16}$$

with the entries in A and B being the operators S, V, U, and T defined by (3.1), (3.2), (3.3), and (3.4), respectively. The superscript "T" denotes the transpose operation.

B. Perturbation expansion, $T < T_c$

If we define for $M, N \geq 0$ the quantity

$$f_{MN} = \langle \sigma_{0,0}\sigma_{M,N+1}\rangle / \langle \sigma_{0,0}\sigma_{M,N}\rangle, \tag{3.17}$$

then CW have shown that

$$(f_{MN})^2 = x_{30}x'_{40} - x'_{30}x_{40}, \tag{3.18}$$

where x_{30}, x_{40}, x'_{30}, and x'_{40} are the zeroth component of the vectors x_3, x_4, x'_3, and x'_4, respectively, which are defined by

$$Cx = y_3 \quad \text{and} \quad Cx' = y_4, \tag{3.19}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad x' = \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 0 \\ 0 \\ \delta \\ 0 \end{bmatrix}, \quad y_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{bmatrix}, \tag{3.20}$$

and δ is the unit vector in l_2 given by

$$\delta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}. \tag{3.21}$$

Since C is antisymmetric it follows that $x_{30} = x'_{40} = 0$ and $x'_{30} = -x_{40}$. Thus from (3.18)

$$f_{MN} = |x'_{30}|. \tag{3.22}$$

That x and x' exist is guaranteed by the fact that C^{-1} exists.[19]

The perturbation expansion of CW consists in expanding C^{-1} as

$$C^{-1} = (1 + A^{-1}B)^{-1}A^{-1}$$
$$= \sum_{n=0}^{\infty} (-1)^n (A^{-1}B)^n A^{-1}. \tag{3.23}$$

This will be a valid expansion for $T < T_c$ and large $M^2 + N^2$ since the elements of S are order 1 while those of T, U, and V are exponentially small.

We can easily see that

$$A^{-1} = \begin{bmatrix} 0 & -(S^T)^{-1} & 0 & 0 \\ S^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & (S^T)^{-1} \\ 0 & 0 & -S^{-1} & 0 \end{bmatrix} \tag{3.24}$$

and

$$A^{-1}B = \begin{bmatrix} 0 & 0 & (S^T)^{-1}U & -(S^T)^{-1}V \\ 0 & 0 & S^{-1}T & S^{-1}U \\ -(S^T)^{-1}U & -(S^T)^{-1}V & 0 & 0 \\ S^{-1}T & -S^{-1}U & 0 & 0 \end{bmatrix}. \tag{3.25}$$

It thus follows from (3.22)–(3.25) that

$$f_{MN} = 1 + \delta^T \sum_{n=1}^{\infty} [(A^{-1}B)^{2n}]_{33}(S^T)^{-1}\delta. \tag{3.26}$$

The subscripts refer to the matrix elements, i.e., $(A^{-1}B)_{32} = -(S^T)^{-1}V$.

The Wiener-Hopf[19] method can be applied to obtain the matrix elements of S^{-1} by solving the equation

$$\sum_{l=0}^{\infty} S_{ml}(S^{-1})_{ln} = \delta_{m,n} . \tag{3.27}$$

The result is

$$(S^{-1})_{mn} = (2\pi i)^{-2} \oint d\xi\, \xi^{-(n+1)} P(\xi) \times \oint d\xi' \frac{\xi'^m}{\xi' - \xi} Q(\xi'^{-1}) , \tag{3.28}$$

where

$$P(\xi) = [(1-\alpha_2\xi)(1-\alpha_1\xi)^{-1}]^{1/2} \tag{3.29}$$

and

$$Q(\xi) = [(1-\alpha_1\xi)(1-\alpha_2\xi)^{-1}]^{1/2} . \tag{3.30}$$

The contours of integration in (3.28) are the unit circles, except that the one for ξ' is to be indented outward near $\xi' = \xi$. We note the P and Q functions have the property that

$$P(\xi)Q(\xi) = 1 . \tag{3.31}$$

C. Recursion relations

It is both useful and convenient to define the operators

$$P_1 = U(S^T)^{-1}U + VS^{-1}T , \tag{3.32}$$

$$P_2 = -VS^{-1}U + U(S^T)^{-1}V , \tag{3.33}$$

and

$$P_3 = T(S^T)^{-1}U - US^{-1}T . \tag{3.34}$$

Then using (3.25) and (3.32)–(3.34) we see that

$$[(A^{-1}B)^{2k}]_{33} = -(S^T)^{-1}\{P_1[(A^{-1}B)^{2k-2}]_{33} - P_2[(A^{-1}B)^{2k-2}]_{43}\} \tag{3.35}$$

and

$$[(A^{-1}B)^{2k}]_{43} = S^{-1}\{P_3[(A^{-1}B)^{2k-2}]_{33} - P_1^T[(A^{-1}B)^{2k-2}]_{43}\} \tag{3.36}$$

for $k \geq 2$, and for $k=1$

$$[(A^{-1}B)^2]_{33} = -(S^T)^{-1}P_1 \tag{3.37}$$

and

$$[(A^{-1}B)^2]_{43} = S^{-1}P_3 . \tag{3.38}$$

Equations (3.35)–(3.38) are the basic recursion relations. They will prove most useful if we restate these relations in generating function terms. Accordingly we define

$$S^{-1}(\xi, \xi') = \sum_{m,n=0}^{\infty} \xi^m (S^{-1})_{mn} (\xi')^n , \tag{3.39}$$

$$P_j(\xi, \xi') = \sum_{m,n=0}^{\infty} \xi^m (P_j)_{mn} (\xi')^n , \quad j = 1, 2, \text{ and } 3, \tag{3.40}$$

and

$$R_{\alpha\beta}^{(2k)}(\xi, \xi') = \sum_{m,n=0}^{\infty} \xi^m \{[(A^{-1}B)^{2k}]_{\alpha\beta}(S^T)^{-1}\}_{mn}(\xi')^n ,$$

$$\alpha = 3, 4 \text{ and } \beta = 3, 4 . \tag{3.41}$$

In (3.39), $S^{-1}(\xi, \xi')$ should not be confused with $[S(\xi)]^{-1}$. Using (3.28)–(3.30) we see that (3.39) becomes

$$S^{-1}(\xi, \xi') = Q(\xi)P(\xi')(1 - \xi\xi')^{-1} . \tag{3.42}$$

From (3.32)–(3.34) and (3.42) the generating functions $P_j(\xi, \xi')$ of (3.40) are

$$P_1(1,2) = U(1)S^{-1}(\bar{2}, \bar{1})U(2) + V(1)S^{-1}(\bar{1}, \bar{2})T(2) , \tag{3.43}$$

$$P_2(1,2) = -V(1)S^{-1}(\bar{1}, \bar{2})U(2) + U(1)S^{-1}(\bar{2}, \bar{1})V(2) , \tag{3.44}$$

and

$$P_3(1,2) = T(1)S^{-1}(\bar{2}, \bar{1})U(2) - U(1)S^{-1}(\bar{1}, \bar{2})T(2) . \tag{3.45}$$

The notation $S^{-1}(1, 2)$ means $S^{-1}(e^{i\phi_1}, e^{i\phi_2})$, and $S^{-1}(\bar{1}, \bar{2})$ means $S^{-1}(e^{-i\phi_1}, e^{-i\phi_2})$. Likewise for the other functions $P_j(1, 2)$, $j = 1, 2,$ and 3, and the functions $U(1)$, $V(1)$, etc. The steps leading to (3.42) are valid only if $|\xi|$ and $|\xi'|$ are less than 1. Hence when we write $S^{-1}(\bar{1}, \bar{2})$ we must have $\operatorname{Im}\phi_1 < 0$, and $\operatorname{Im}\phi_2 < 0$, so that (3.43)–(3.45) have the restriction $\operatorname{Im}\phi_j < 0$, $j = 1, 2$.

Then in the notation (3.39)–(3.45), the recursion relations (3.35)–(3.38) become

$$R_{33}^{(2k)}(\xi, \xi') = (2\pi)^{-2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d(2)d(4) S^{-1}(\bar{2}, \xi) \times [-P_1(2,4)R_{33}^{(2k-2)}(\bar{4}, \xi') + P_2(2,4)R_{43}^{(2k-2)}(\bar{4}, \xi')] \tag{3.46}$$

and

$$R_{43}^{(2k)}(\xi, \xi') = (2\pi)^{-2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d(2)d(4) S^{-1}(\xi, \bar{2}) \times [P_3(2,4)R_{33}^{(2k-2)}(\bar{4}, \xi') - P_1^T(2,4)R_{43}^{(2k-2)}(\bar{4}, \xi')] \tag{3.47}$$

for $k \geq 2$, where $d(2)$ denotes $d\phi_2$, etc., and $P_1^T(2, 4)$ is the generating function corresponding to the transpose operator P_1^T, i.e.,

$$P_1^T(1,2) = U(1)S^{-1}(\bar{1}, \bar{2})U(2) + T(1)S^{-1}(\bar{2}, \bar{1})V(2) . \tag{3.48}$$

For $k = 1$ we have

$$R_{33}^{(2)}(\xi, \xi') = -(2\pi)^{-2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d(2)d(4) \times S^{-1}(\bar{2}, \xi)P_1(2,4)S^{-1}(\xi', \bar{4}) \tag{3.49}$$

and

$$R_{43}^{(2)}(\xi, \xi') = (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d(2)\, d(4) \times S^{-1}(\xi, \bar{2}) P_3(2,4) S^{-1}(\xi', \bar{4}) . \quad (3.50)$$

To summarize we see from (3.26) and (3.41) with $\alpha = \beta = 3$ that

$$f_{MN} = 1 + \sum_{n=1}^{\infty} R_{33}^{(2n)}(0,0) . \quad (3.51)$$

D. Quantity f_{MN}

In this section we will show that all positive integers n and all $M, N \geq 0$ that

$$R_{33}^{(2n)}(0,0) = (2\gamma_1)^{2n} (2\pi)^{-4n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} d\phi_1 \cdots d\phi_{4n} \prod_{j=1}^{2n} \frac{e^{-iM\phi_{2j-1} - i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1}, \phi_{2j})} \times \cos\tfrac{1}{2}\phi_1 \prod_{j=1}^{2n-1} \left(\frac{\sin\frac{1}{2}(\phi_{2j-1} - \phi_{2j+1})}{1 - e^{-i(\phi_{2j} + \phi_{2j+2})}} \right) \sin\tfrac{1}{2}\phi_{4n-1} , \quad (3.52)$$

where $\Delta(\phi_1, \phi_2)$ is defined by (2.13) and $\mathrm{Im}(\phi_j) < 0$, $j = 1, 2, \ldots, 4n$.

To establish (3.52) we compute $R_{33}^{(2n)}(\xi, \xi')$ and $R_{43}^{(2n)}(\xi, \xi')$ by induction. Once we have computed $R_{33}^{(2)}(\xi, \xi')$ and $R_{43}^{(2)}(\xi, \xi')$ we use the recursion relations (3.46) and (3.47) to establish the general case.

For $n = 1$ we use (3.42)–(3.45) in (3.49) and (3.50) to find

$$R_{33}^{(2)}(\xi, \xi') = -(2\pi)^{-2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d(2)\, d(4) P(\xi) Q(\xi') [(1 - \xi e^{-i\phi_2})(1 - \xi' e^{-i\phi_4})(1 - e^{-i(\phi_2 + \phi_4)})]^{-1} \times Q(\bar{2}) [U(4) Q(\bar{4}) P(\bar{2}) U(2) + V(2) Q(\bar{2}) P(\bar{4}) T(4)] P(\bar{4}) \quad (3.53)$$

and

$$R_{43}^{(2)}(\xi, \xi') = (2\pi)^{-2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d(2)\, d(4) Q(\xi) Q(\xi') [(1 - \xi e^{-i\phi_2})(1 - \xi' e^{-i\phi_4})(1 - e^{-i(\phi_2 + \phi_4)})]^{-1} \times P(\bar{2}) [T(2) Q(\bar{4}) P(\bar{2}) U(4) - U(2) P(\bar{4}) Q(\bar{2}) T(4)] P(\bar{4}) . \quad (3.54)$$

Since the values of the functions $T(e^{i\phi_2})$, $U(e^{i\phi_2})$, and $V(e^{i\phi_2})$ are determined by the residue at the pole coming from $\Delta(\phi_1, \phi_2) = 0$, we may make use of the identity $\cos\phi_2 = \gamma_2^{-1}(a - \gamma_1 \cos\phi_1)$ when working with an integral over ϕ_1. This leads to the following identities:

$$T(2)[P(\bar{2})]^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} d\phi_1 \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1, \phi_2)} [-2z_2 z_1^{-1}(1 - z_1^2) e^{-i\phi_2}(1 - \cos\phi_1)] \quad (3.55)$$

and

$$V(2)[Q(\bar{2})]^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} d\phi_1 \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1, \phi_2)} [-2z_1 z_2 (1 - z_1^2) e^{-i\phi_2}(1 + \cos\phi_1)] . \quad (3.56)$$

Making use of (3.31), (3.55), and (3.56) in (3.53) and (3.54) we obtain

$$R_{33}^{(2)}(\xi, \xi') = -(2\pi)^{-4} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} d(1) \cdots d(4) \frac{P(\xi) Q(\xi')}{(1 - \xi e^{-i\phi_2})(1 - \xi' e^{-i\phi_4})} \times \prod_{j=1}^{2} \frac{e^{-iM\phi_{2j-1} - i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1}, \phi_{2j})} (1 - e^{-i(\phi_2 + \phi_4)})^{-1} \times [-(1 - z_1^2)^2 4z_2^2 \sin\phi_1 \sin\phi_3 + 4z_2^2 (1 - z_1^2)^2 (1 + \cos\phi_1)(1 - \cos\phi_3)] \quad (3.57a)$$

and

$$R_{43}^{(2)}(\xi, \xi') = (2\pi)^{-4} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} d(1) \cdots d(4) \frac{Q(\xi) Q(\xi')}{(1 - \xi e^{-i\phi_2})(1 - \xi' e^{-i\phi_4})} \times \prod_{j=1}^{2} \frac{e^{-iM\phi_{2j-1} - i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1}, \phi_{2j})} 4iz_1^{-1} z_2^2 (1 - z_1^2)^2 [(1 - \cos\phi_1)\sin\phi_3 - \sin\phi_1 (1 - \cos\phi_3)] (1 - e^{-i(\phi_2 + \phi_4)})^{-1} . \quad (3.57b)$$

Using the trigonometric identities

$$(1-\cos\phi_1)\sin\phi_3-\sin\phi_1(1-\cos\phi_3)=4\sin\tfrac{1}{2}\phi_1\sin\tfrac{1}{2}(\phi_1-\phi_3)\sin\tfrac{1}{2}\phi_3\,,$$
$$(1+\cos\phi_1)(1-\cos\phi_3)-\sin\phi_1\sin\phi_3=-4\cos\tfrac{1}{2}\phi_1\sin\tfrac{1}{2}(\phi_1-\phi_3)\sin\tfrac{1}{2}\phi_3 \tag{3.58}$$

in (3.57), we obtain the final form for $R_{33}^{(2)}(\xi,\xi')$ and $R_{43}^{(2)}(\xi,\xi')$:

$$R_{33}^{(2)}(\xi,\xi')=(2\gamma_1)^2(2\pi)^{-4}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}d(1)\cdots d(4)\frac{P(\xi)Q(\xi')}{(1-\xi e^{-i\phi_2})(1-\xi' e^{-i\phi_4})}$$
$$\times\prod_{j=1}^{2}\frac{e^{-iM\phi_{2j-1}-i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}(1-e^{-i(\phi_2+\phi_4)})^{-1}\cos\tfrac{1}{2}\phi_1\sin\tfrac{1}{2}(\phi_1-\phi_3)\sin\tfrac{1}{2}\phi_3 \tag{3.59a}$$

and

$$R_{43}^{(2)}(\xi,\xi')=iz_1^{-1}(2\gamma_1)^2(2\pi)^{-4}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}d(1)\cdots d(4)\frac{Q(\xi)Q(\xi')}{(1-\xi e^{-i\phi_2})(1-\xi' e^{-i\phi_4})}$$
$$\times\prod_{j=1}^{2}\frac{e^{-iM\phi_{2j-1}-i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}(1-e^{-i(\phi_2+\phi_4)})^{-1}\sin\tfrac{1}{2}\phi_1\sin\tfrac{1}{2}(\phi_1-\phi_3)\sin\tfrac{1}{2}\phi_3\,. \tag{3.59b}$$

We have used the definition (3.11) of γ_1 in obtaining (3.59). If we set $\xi=\xi'=0$ in (3.59a) we obtain the result originally derived by CW.

For k any even integer, we now assume that

$$R_{33}^{(k)}(\xi,\xi')=(2\gamma_1)^k(2\pi)^{-2k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}d(1)\cdots d(2k)\frac{P(\xi)Q(\xi')}{(1-\xi e^{-i\phi_2})(1-\xi' e^{-i\phi_{2k}})}$$
$$\times\prod_{j=1}^{k}\frac{e^{-iM\phi_{2j-1}-i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\cos\tfrac{1}{2}\phi_1\prod_{j=1}^{k-1}\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{1-e^{-i(\phi_{2j}+\phi_{2j+2})}}\sin\tfrac{1}{2}\phi_{2k-1} \tag{3.60a}$$

and

$$R_{43}^{(k)}(\xi,\xi')=iz_1^{-1}(2\gamma_1)^k(2\pi)^{-2k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}d(1)\cdots d(2k)\frac{Q(\xi)Q(\xi')}{(1-\xi e^{-i\phi_2})(1-\xi' e^{-i\phi_{2k}})}$$
$$\times\prod_{j=1}^{k}\frac{e^{-iM\phi_{2j-1}-i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\sin\tfrac{1}{2}\phi_1\prod_{j=1}^{k-1}\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{1-e^{-i(\phi_{2j}+\phi_{2j+2})}}\sin\tfrac{1}{2}\phi_{2k-1}\,. \tag{3.60b}$$

We now proceed to demonstrate that these relations are true for k replaced by $k+2$. Using the recursion relations (3.46) and (3.47) and the induction hypothesis we have

$$R_{33}^{(k+2)}(\xi,\xi')=(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(2)d(4)S^{-1}(\bar{2},\xi)[-P_1(2,4)R_{33}^{(k)}(\bar{4},\xi')+P_2(2,4)R_{43}^{(k)}(\bar{4},\xi')]$$
$$=(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(2)d(4)\frac{P(\xi)Q(\bar{2})}{1-\xi e^{-i\phi_2}}\{-[U(4)S^{-1}(\bar{4},\bar{2})U(2)+V(2)S^{-1}(\bar{2},\bar{4})T(4)]$$
$$\times R_{33}^{(k)}(\bar{4},\xi')+[U(2)S^{-1}(\bar{4},\bar{2})V(4)-V(2)S^{-1}(\bar{2},\bar{4})U(4)]R_{43}^{(k)}(\bar{4},\xi')\}$$
$$=(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(2)d(4)P(\xi)(1-\xi e^{-i\phi_2})^{-1}(1-e^{-i(\phi_2+\phi_4)})^{-1}\{-[U(2)U(4)Q(\bar{4})$$
$$+V(2)[Q(\bar{2})]^2P(\bar{4})T(4)]R_{33}^{(k)}(\bar{4},\xi')+[U(2)Q(\bar{4})V(4)-V(2)[Q(\bar{2})]^2P(\bar{4})U(4)]R_{43}^{(k)}(\bar{4},\xi')\}$$
$$=(2\pi)^{-2-2k}(2\gamma_1)^k\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}d(2)d(4)d(5)d(6)\cdots d(2k+4)\frac{P(\xi)Q(\xi')}{(1-\xi e^{-i\phi_2})(1-\xi' e^{-i\phi_{2k+4}})}$$
$$\times\prod_{j=3}^{k+2}\frac{e^{-iM\phi_{2j-1}-i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\prod_{j=1}^{k+1}(1-e^{-i(\phi_{2j}+\phi_{2j+2})})^{-1}D(2,4,5)\prod_{j=3}^{k+1}\sin\tfrac{1}{2}(\phi_{2j-1}-\phi_{2j+1})\sin\tfrac{1}{2}\phi_{2k+3}\,, \tag{3.61}$$

where in the last line

$$D(2,4,5)=-[U(2)U(4)+V(2)[Q(\bar{2})]^2T(4)[P(\bar{4})]^2]\cos\tfrac{1}{2}\phi_5$$
$$+[U(2)[Q(\bar{4})]^2V(4)-V(2)[Q(\bar{2})]^2U(4)](iz_1^{-1})\sin\tfrac{1}{2}\phi_5\,. \tag{3.62}$$

Using (3.7), (3.55), and (3.56) in (3.62) we have

$$D(2,4,5)=(2\pi)^{-4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(1)d(3)\frac{e^{-iM(\phi_1+\phi_3)-i(N+1)(\phi_2+\phi_4)}}{\Delta(\phi_1,\phi_2)\Delta(\phi_3,\phi_4)}$$
$$\times\{-[-4z_2^2(1-z_1^2)^2\sin\phi_1\sin\phi_3+4z_2^2(1-z_1^2)^2(1+\cos\phi_1)(1-\cos\phi_3)]\cos\tfrac{1}{2}\phi_5$$
$$+[2iz_2\sin\phi_1 2z_1z_2(1+\cos\phi_3)-2z_1z_2(1+\cos\phi_1)2iz_2\sin\phi_3](iz_1^{-1})\sin\tfrac{1}{2}\phi_5\}, \tag{3.63a}$$

$$D(2,4,5)=(2\gamma_1)^2(2\pi)^{-4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(1)d(3)\frac{e^{-iM(\phi_1+\phi_3)-i(N+1)(\phi_2+\phi_4)}}{\Delta(\phi_1,\phi_2)\Delta(\phi_3,\phi_4)}$$
$$\times[\cos\tfrac{1}{2}\phi_1\sin\tfrac{1}{2}(\phi_1-\phi_3)\sin\tfrac{1}{2}\phi_3\cos\tfrac{1}{2}\phi_5-\cos\tfrac{1}{2}\phi_1\sin\tfrac{1}{2}(\phi_1-\phi_3)\cos\tfrac{1}{2}\phi_3\sin\tfrac{1}{2}\phi_5], \tag{3.63b}$$

$$D(2,4,5)=(2\gamma_1)^2(2\pi)^{-4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(1)d(3)\frac{e^{-iM(\phi_1+\phi_3)-i(N+1)(\phi_2+\phi_4)}}{\Delta(\phi_1,\phi_2)\Delta(\phi_3,\phi_4)}\cos\tfrac{1}{2}\phi_1\sin\tfrac{1}{2}(\phi_1-\phi_3)\sin\tfrac{1}{2}(\phi_3-\phi_5), \tag{3.64}$$

where we used the trigonometric identities (3.58) in going from (3.63a) to (3.63b). Substitution of the expression (3.64) for $D(2,4,5)$ into (3.61) gives (3.60a) with k replaced by $k+2$.

The induction proof for $R_{33}^{(k)}(\xi,\xi')$, k any even integer, is similar. Substituting $\xi=\xi'=0$ into (3.60a) and writing the even integer k as $2n$ we obtain (3.52). Equations (3.51) and (3.52) give us the perturbation expansion of the quantity f_{MN} defined by (3.17).

It will be convenient to define

$$f_{MN}^{(2k)}=R_{33}^{(2k)}(0,0), \tag{3.65}$$

where $R_{33}^{(2k)}(0,0)$ is given by (3.52).

E. $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ in terms of f_{MN}

1. Basic notation

The correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ can in view of (3.17) be expressed as

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle=S_\infty^<\left(\prod_{k=N}^{\infty}f_{Mk}\right)^{-1}$$
$$=S_\infty^<\exp\left(-\sum_{k=N}^{\infty}\ln f_{Mk}\right), \tag{3.66}$$

where $S_\infty^<$ is given by (2.10). Recalling (3.51) and definition (3.65) we define $F_{<MN}$ by

$$F_{<MN}=\sum_{k=N}^{\infty}\ln\left(1+\sum_{n=1}^{\infty}f_{Mk}^{(2n)}\right). \tag{3.67}$$

Expanding the logarithm in (3.67) we obtain the expansion

$$F_{<MN}=\sum_{n=1}^{\infty}F_{<MN}^{(2n)}, \tag{3.68}$$

with

$$F_{<MN}^{(2)}=\sum_{k=N}^{\infty}f_{Mk}^{(2)}, \tag{3.69a}$$

$$F_{<MN}^{(4)}=\sum_{k=N}^{\infty}[f_{Mk}^{(4)}-\tfrac{1}{2}(f_{Mk}^{(2)})^2], \tag{3.69b}$$

$$F_{<MN}^{(6)}=\sum_{k=N}^{\infty}[f_{Mk}^{(6)}-f_{Mk}^{(2)}f_{Mk}^{(4)}+\tfrac{1}{3}(f_{Mk}^{(2)})^3], \tag{3.69c}$$

etc.

The remainder of this section and all of Sec. III F will be devoted to proving that $F_{<MN}^{(2n)}$ is given by (2.12). We first show that (2.12) is true for $n=1$ and $n=2$. This will then motivate the general case which is proved in Sec. III F.

2. $F_{<MN}^{(2)}$

Using (3.52) for $n=1$ [though first let $\phi_1\to\phi_3$ and $\phi_2\to\phi_4$ and add this to (3.52)], (3.65), and (3.69a) we have

$$F_{<MN}^{(2)}=\tfrac{1}{2}(2\gamma_1)^2(2\pi)^{-4}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_4$$
$$\times\frac{e^{-iM(\phi_1+\phi_3)-i(N+1)(\phi_2+\phi_4)}}{\Delta(\phi_1,\phi_2)\Delta(\phi_3,\phi_4)}(1-e^{-i(\phi_2+\phi_4)})^{-2}$$
$$\times\sin\tfrac{1}{2}(\phi_1-\phi_3)\sin\tfrac{1}{2}(\phi_3-\phi_1)$$

(recall $\mathrm{Im}\,\phi_j<0$, $j=1,2,3,4$).

Multiplying numerator and denominator by $e^{i(\phi_2+\phi_4)}$ we have

$$F_{<MN}^{(2)}=(-1)\gamma_1^2\tfrac{1}{2}(2\pi)^{-4}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_4$$
$$\times\prod_{j=1}^{2}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}, \tag{3.70}$$

where $\phi_5\equiv\phi_1$ and $\phi_6\equiv\phi_2$. This is (2.12) for $n=1$.

3. $F_{<MN}^{(4)}$

Using (3.52) for $n=2$ and (3.65) we have

$$\sum_{k=N}^{\infty}f_{Mk}^{(4)}=(2\gamma_1)^4(2\pi)^{-8}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_8\prod_{j=1}^{4}\frac{e^{-iM\phi_{2j-1}-i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}$$

$$\times(1-e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)})^{-1}\cos\tfrac{1}{2}\phi_1\prod_{j=1}^{3}\left(\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{1-e^{-i(\phi_{2j}+\phi_{2j+2})}}\right)\sin\tfrac{1}{2}\phi_7\,. \tag{3.71}$$

Making the change of variables $\phi_1\to\phi_7$, $\phi_3\to\phi_5$, $\phi_5\to\phi_3$, $\phi_7\to\phi_1$ and similarly for ϕ_{2j} in (3.71) we obtain the equivalent expression

$$\sum_{k=N}^{\infty} f_{Mk}^{(4)}=-(2\gamma_1)^4(2\pi)^{-8}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_8\prod_{j=1}^{4}\left(\frac{e^{-iM\phi_{2j-1}-i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)$$

$$\times(1-e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)})^{-1}\cos\tfrac{1}{2}\phi_7\prod_{j=1}^{3}\left(\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{1-e^{-i(\phi_{2j}+\phi_{2j+2})}}\right)\sin\tfrac{1}{2}\phi_1\,. \tag{3.72}$$

Adding (3.71) and (3.72) and dividing by 2 we obtain

$$\sum_{k=N}^{\infty} f_{Mk}^{(4)}=\tfrac{1}{2}(2\gamma_1)^4(2\pi)^{-8}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_8\prod_{j=1}^{4}\left(\frac{e^{-iM\phi_{2j-1}-i(N+1)\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\sin\tfrac{1}{2}(\phi_{2j-1}-\phi_{2j+1})\right)$$

$$\times\left(\prod_{j=1}^{3}(1-e^{-i(\phi_{2j}+\phi_{2j+2})})^{-1}\right)(1-e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)})^{-1}\,, \tag{3.73}$$

with $\phi_{10}\equiv\phi_2$, $\phi_9\equiv\phi_1$.

We want to obtain the factor

$$\prod_{j=1}^{4}(1-e^{-i(\phi_{2j}+\phi_{2j+2})})^{-1}\,,$$

from the factor

$$\prod_{j=1}^{3}(1-e^{-i(\phi_{2j}+\phi_{2j+2})})^{-1}\,,$$

that occurs in (3.73). To accomplish this we cyclically permute the integration variables in (3.73). Under these cyclic permutations the only noninvariant factor in (3.73) is

$$\prod_{j=1}^{3}(1-e^{-i(\phi_{2j}+\phi_{2j+2})})^{-1}\,.$$

Doing this (3.73) becomes

$$\sum_{k=N}^{\infty} f_{Mk}^{(4)}=\tfrac{1}{2}(2\gamma_1)^4(2\pi)^{-8}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_8\prod_{j=1}^{4}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\,\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{1-e^{-i(\phi_{2j}+\phi_{2j+2})}}\right)A_4(\phi_2,\phi_4,\phi_6,\phi_8)\,, \tag{3.74}$$

where

$$A_4(\phi_2,\phi_4,\phi_6,\phi_8)=e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)}(1-e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)})^{-1}\frac{1}{4}\left(4-\sum_{j=1}^{4}e^{-i(\phi_{2j}+\phi_{2j+2})}\right). \tag{3.75}$$

We can rewrite A_4 as

$$A_4(\phi_2,\phi_4,\phi_6,\phi_8)=\tfrac{1}{2}e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)}+\tfrac{1}{4}e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)}(1-e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)})^{-1}$$

$$\times\left(2+2e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)}-\sum_{j=1}^{4}e^{-i(\phi_{2j}+\phi_{2j+2})}\right). \tag{3.76}$$

Using (3.76) in (3.74)

$$\sum_{k=N}^{\infty} f_{Mk}^{(4)}=F_{<MN}^{(4)}+R_{MN}^{(4)}\,,$$

where $F_{<MN}^{(4)}$ is the right-hand side of (2.12) for $n=2$ and

$$R_{MN}^{(4)}=\tfrac{1}{2}(2\gamma_1)^4(2\pi)^{-8}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_8\prod_{j=1}^{4}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\,\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{1-e^{-i(\phi_{2j}+\phi_{2j+2})}}\right)$$

$$\times\left[\tfrac{1}{4}e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)}(1-e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)})^{-1}\left(2+2e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)}-\sum_{j=1}^{4}e^{-i(\phi_{2j}+\phi_{2j+2})}\right)\right]. \tag{3.77}$$

Thus $F^{(4)}_{<MN}$ as defined by (3.69b) will be given by (2.12) for $n=2$ if we can show that

$$R^{(4)}_{MN}-\frac{1}{2}\sum_{k=N}^{\infty}(f^{(2)}_{Mk})^2=0\ . \tag{3.78}$$

Letting $\phi_2 \leftrightarrow \phi_4$ and $\phi_1\leftrightarrow\phi_3$ in (3.52) for $n=1$, adding this to (3.52) for $n=1$ and summing we obtain

$$-\frac{1}{2}\sum_{k=N}^{\infty}(f^{(2)}_{Mk})^2=-\tfrac{1}{8}(2\gamma_1)^4(2\pi)^{-8}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_8\left(\prod_{j=1}^{4}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)$$
$$\times e^{-i(\phi_2+\cdots+\phi_8)}(1-e^{-i(\phi_2+\cdots+\phi_8)})^{-1}\sin^2\tfrac{1}{2}(\phi_1-\phi_3)\sin^2\tfrac{1}{2}(\phi_5-\phi_7)\left[(1-e^{-i(\phi_2+\phi_4)})(1-e^{-i(\phi_6+\phi_8)})\right]^{-1}. \tag{3.79}$$

In (3.79) the sine factors loop as $1\to3\to1$ and $5\to7\to5$ as compared with $1\to3\to5\to7\to1$ of (3.77). To compare $R^{(4)}_{MN}$ of (3.77) with (3.79) it is useful to make one loop in (3.79) rather than two separate loops. This is done by use of the identity

$$\sin\tfrac{1}{2}(\phi_a-\phi_b)\sin\tfrac{1}{2}(\phi_c-\phi_d)+\sin\tfrac{1}{2}(\phi_a-\phi_d)\sin\tfrac{1}{2}(\phi_b-\phi_c)=\sin\tfrac{1}{2}(\phi_a-\phi_c)\sin\tfrac{1}{2}(\phi_b-\phi_d)\ . \tag{3.80}$$

Letting $a\to1$, $b\to5$, $c\to3$, and $d\to7$ in (3.80) and using this in (3.79) we obtain

$$-\frac{1}{2}\sum_{k=N}^{\infty}(f^{(2)}_{Mk})^2=-\tfrac{1}{8}(2\gamma_1)^4(2\pi)^{-8}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_8\prod_{j=1}^{4}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}$$
$$\times e^{-i(\phi_2+\cdots+\phi_8)}(1-e^{-i(\phi_2+\cdots+\phi_8)})^{-1}\left[(1-e^{-i(\phi_2+\phi_4)})(1-e^{-i(\phi_6+\phi_8)})\right]^{-1}$$
$$\times\left[\sin\tfrac{1}{2}(\phi_1-\phi_5)\sin\tfrac{1}{2}(\phi_3-\phi_7)+\sin\tfrac{1}{2}(\phi_3-\phi_5)\sin\tfrac{1}{2}(\phi_7-\phi_1)\right]\sin\tfrac{1}{2}(\phi_1-\phi_3)\sin\tfrac{1}{2}(\phi_5-\phi_7)\ . \tag{3.81}$$

One part of (3.81) has the correct $1\to3\to\cdots7\to1$ structure. The other part can be brought into this form by relabeling the integration variables ($5\leftrightarrow7$, $6\leftrightarrow8$). Doing this (3.81) becomes

$$-\frac{1}{2}\sum_{k=N}^{\infty}(f^{(2)}_{Mk})^2=-\tfrac{1}{4}(2\gamma_1)^4(2\pi)^{-8}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_8\prod_{j=1}^{4}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}$$
$$\times\sin\tfrac{1}{2}(\phi_{2j-1}-\phi_{2j+1})e^{-i(\phi_2+\cdots+\phi_8)}\left[1-e^{-i(\phi_2+\cdots+\phi_8)}\right]^{-1}\left[(1-e^{-i(\phi_2+\phi_4)})(1-e^{-i(\phi_6+\phi_8)})\right]^{-1}. \tag{3.82}$$

Finally, if we add to (3.82) the relabeled version $2\to4\to6\to8\to2$ and $1\to3\to5\to7\to1$, and divide by 2, then

$$-\frac{1}{2}\sum_{k=N}^{\infty}(f^{(2)}_{Mk})^2=-\tfrac{1}{8}(2\gamma_1)^4(2\pi)^{-8}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_8\prod_{j=1}^{4}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right.$$
$$\left.\times\sin\tfrac{1}{2}(\phi_{2j-1}-\phi_{2j+1})(1-e^{-i(\phi_{2j}+\phi_{2j+2})})^{-1}\right)\left(2+2e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)}-\sum_{j=1}^{4}e^{-i(\phi_{2j}+\phi_{2j+2})}\right)$$
$$\times e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)}\left[1-e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)}\right]^{-1}. \tag{3.83}$$

Comparing (3.83) and (3.77) we see that (3.78) holds; and hence $F^{(4)}_{<MN}$ is given by (2.12) for $n=2$.

F. $F^{(2n)}_{<MN}$ for arbitrary n

We now prove (2.12) for arbitrary n. We begin with the definition

$$\tilde F^{(2n)}_{<MN}=(-1)^n\gamma_1^{2n}(2n)^{-1}(2\pi)^{-4n}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4n}$$
$$\times\prod_{j=1}^{2n}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\ \frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)$$
$$\times(1-e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n})})\ . \tag{3.84}$$

Suppose we can prove that

$$\tilde F_{MN}\equiv\sum_{n=1}^{\infty}\tilde F^{(2n)}_{<MN}=\ln\left(1+\sum_{n=1}^{\infty}f^{(2n)}_{MN}\right), \tag{3.85}$$

where $\tilde F^{(2n)}_{<MN}$ is given by (3.84) and $f^{(2n)}_{MN}$ by (3.52). Then from (3.85) it follows that $F^{(2n)}_{<MN}$ is given by (2.12). To see this we note that by definition [see (3.66)–(3.68)] of $F^{(2n)}_{<MN}$

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle=S^{<}_{\infty}\exp\left(-\sum_{n=1}^{\infty}F^{(2n)}_{<MN}\right), \tag{3.86}$$

and from (3.66) and the hypothesis (3.85)

$$\langle \sigma_{0,0}\sigma_{M,N}\rangle = S_\infty^< \exp\left(-\sum_{k=N}^{\infty} \ln\left(1+\sum_{n=1}^{\infty} f_{Mk}^{(2n)}\right)\right) = S_\infty^< \exp\left(-\sum_{k=N}^{\infty}\sum_{n=1}^{\infty} \tilde{F}_{<Mk}^{(2n)}\right) = S_\infty^< \exp\left(-\sum_{n=1}^{\infty}\sum_{k=N}^{\infty} \tilde{F}_{<Mk}^{(2n)}\right). \quad (3.87)$$

Then from (3.86) and (3.87)

$$F_{<MN}^{(2n)} = \sum_{k=N}^{\infty} \tilde{F}_{<Mk}^{(2n)}. \quad (3.88)$$

But doing the summation in (3.88) is trivial [see (3.84)]; and hence, $F_{<MN}^{(2n)}$ is given by (2.12). Hence if we establish (3.85) with $\tilde{F}_{<MN}^{(2n)}$ defined by (3.84) we have proved (2.12).

We define the generating functions

$$f(x) = \sum_{n=0}^{\infty} f_{MN}^{(2n)} x^n \quad (3.89)$$

and

$$F(x) = \sum_{n=0}^{\infty} \tilde{F}_{<MN}^{(2n)} x^n, \quad (3.90)$$

where $f_{MN}^{(0)} = 1$, $\tilde{F}_{<MN}^{(0)} = 0$, then (3.85) is a special case ($x=1$) of

$$f(x) = e^{F(x)}. \quad (3.91)$$

Differentiation of (3.91) gives

$$f'(x) = F'(x) f(x) \quad (3.92)$$

or, equivalently,

$$n f_{MN}^{(2n)} = \sum_{l=1}^{n} l \tilde{F}_{<MN}^{(2l)} f_{MN}^{2(n-l)}. \quad (3.93)$$

Therefore, (3.93) and the boundary conditions $f(0) = 1$, $F(0) = 0$ will imply (2.12). We now prove (3.93).

The variables in the identity

$$e^{-i(\phi_2+\phi_4+\phi_6+\cdots+\phi_{4n})} - 1 = \left(e^{-i(\phi_2+\phi_4+\phi_6+\cdots+\phi_{4n})} - e^{-i(\phi_6+\phi_8+\cdots+\phi_{4n})}\right) + \left(e^{-i(\phi_6+\phi_8+\cdots+\phi_{4n})} - e^{-i(\phi_{10}+\phi_{12}+\cdots+\phi_{4n})}\right) + \left(e^{-i(\phi_{10}+\phi_{12}+\cdots+\phi_{4n})} - e^{-i(\phi_{14}+\phi_{16}+\cdots+\phi_{4n})}\right) + \cdots + \left(e^{-i(\phi_{4n-2}+\phi_{4n})} - 1\right) \quad (3.94)$$

may be cyclically permuted and the identity (3.94) will continue to be valid *when* used in the integrand of (3.84) since the integrand in (3.84) is invariant under cyclic permutations of the integration variable labels.

Thus the identity

$$e^{-i(\phi_2+\phi_4+\phi_6+\cdots+\phi_{4n})} - 1 = -\left(1 - e^{-i(\phi_{4n}+\phi_2)}\right)\left(e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-2})} + e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-6})} + e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-10})} + \cdots + 1\right) \quad (3.95)$$

when substituted into (3.84) gives, with use of the identity

$$1 - e^{-i(\phi_{4n}+\phi_2)} = 2i \sin\tfrac{1}{2}(\phi_{4n}+\phi_2) e^{-i(\phi_{4n}+\phi_2)/2}, \quad (3.96)$$

$$\tilde{F}_{<MN}^{(2n)} = (-1)^n \gamma_1^{2n} (2n)^{-1} (2\pi)^{-4n} \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{4n} \prod_{j=1}^{2n} \left\{ \frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})} \sin\tfrac{1}{2}(\phi_{2j-1}-\phi_{2j+1}) \right\} \times \prod_{j=1}^{2n-1} \frac{1}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})} 2i e^{-i(\phi_{4n}+\phi_2)/2} \sum_{j=0}^{n-1} e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-2-4j})}, \quad (3.97)$$

where the last term in the sum in (3.97) is to be interpreted as 1 if $n=1$. From (3.52) and (3.65) we have

$$f_{MN}^{(2n)} = (2\gamma_1)^{2n} (2\pi)^{-4n} \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{4n} \prod_{j=1}^{2n} \left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})} \right) \times \prod_{j=1}^{2n-1} \left(\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{2i\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})} \right) e^{-i(\phi_2+\phi_{4n})/2} \cos\tfrac{1}{2}\phi_1 \sin\tfrac{1}{2}\phi_{4n-1}. \quad (3.98)$$

Letting $\phi_1 \leftrightarrow \phi_{4n-1}$, $\phi_2 \leftrightarrow \phi_{4n-2}$, ... in (3.98), adding the result to (3.98), and dividing by two we obtain,

$$f_{MN}^{(2n)} = \gamma_1^{2n} (2\pi)^{-4n} (-1)^n \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{4n} \prod_{j=1}^{2n} \left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})} \sin\tfrac{1}{2}(\phi_{2j-1}-\phi_{2j+1}) \right) \times \prod_{j=1}^{2n-1} \left(\frac{1}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})} \right) i e^{-i(\phi_2+\phi_{4n})/2}. \quad (3.99)$$

Using (3.97) and (3.99) we find

$$\sum_{l=1}^{n-1} l\tilde{F}^{(2l)}_{<MN} f^{2(n-l)}_{MN} = (-1)^n \gamma_1^{2n} (2\pi)^{-4n} \sum_{l=1}^{n-1} \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{4l} \prod_{j=1}^{2l} \left(\frac{e^{-iM\phi_{2j-1} - iN\phi_{2j}}}{\Delta(\phi_{2j-1}, \phi_{2j})} \sin\tfrac{1}{2}(\phi_{2j-1} - \phi_{2j+1}) \right)$$
$$\times \prod_{j=1}^{2l-1} \left(\frac{1}{\sin\frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \right) ie^{-i(\phi_2 + \phi_{4l})/2} \sum_{j=0}^{l-1} e^{-i(\phi_4 + \phi_6 + \cdots + \phi_{4l-2-4j})}$$
$$\times \int_{-\pi}^{\pi} d\phi_{4l+1} \cdots \int_{-\pi}^{\pi} d\phi_{4n} \prod_{j=2l+1}^{2n} \frac{e^{-iM\phi_{2j-1} - iN\phi_{2j}}}{\Delta(\phi_{2j-1}, \phi_{2j})} \sin\tfrac{1}{2}(\phi_{2j-1} - \phi_{2j+1})$$
$$\times \prod_{j=2l+1}^{2n-1} \left(\frac{1}{\sin\frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \right) ie^{-i(\phi_{4l+2} + \phi_{4n})} . \tag{3.100}$$

Using (3.80) ($a \to 4l-1$, $b \to 1$, $c \to 4n-1$, and $d \to 4l+1$) in (3.100), we find that one term has the correct $1 \to 3 \to 5 \to \cdots \to 4n-1 \to 1$ structure, and the other term can be brought to this form by relabeling variables [this is the same argument as that following (3.81)]. Using this new form of (3.100) and adding $n\tilde{F}^{(2n)}_{<MN}$ to this [see (3.97)] we obtain

$$n\tilde{F}^{(2n)}_{<MN} + \sum_{l=1}^{n-1} l\tilde{F}^{(2l)}_{<MN} f^{2(n-l)}_{MN} = i(-1)^n \gamma_1^{2n} (2\pi)^{-4n} \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{4n} \prod_{j=1}^{2n} \frac{e^{-iM\phi_{2j-1} - iN\phi_{2j}}}{\Delta(\phi_{2j-1}, \phi_{2j})} \sin\tfrac{1}{2}(\phi_{2j-1} - \phi_{2j+1})$$
$$\times \prod_{j=1}^{2n-1} \frac{1}{\sin\frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \tilde{H}(\phi_2, \phi_4, \ldots, \phi_{4n}) , \tag{3.101}$$

where

$$\tilde{H}(\phi_2, \phi_4, \ldots, \phi_{4n}) = e^{-i(\phi_2 + \phi_{4n})/2} \sum_{j=0}^{n-1} e^{-i(\phi_4 + \phi_6 + \cdots + \phi_{4n-2-4j})} + \sum_{l=1}^{n-1} e^{-i(\phi_2 + \phi_{4l})/2}$$
$$\times \left(\sum_{k=0}^{l-1} e^{-i(\phi_4 + \phi_6 + \cdots + \phi_{4l-2-4k})} e^{-i(\phi_{4l+2} + \phi_{4n})/2} [e^{i(\phi_{4l} + \phi_{4l+2})/2} - e^{-i(\phi_{4l} + \phi_{4l+2})/2}] \right) . \tag{3.102}$$

If we can show that $\tilde{H}(\phi_2, \phi_4, \ldots, \phi_{4n})$ can be replaced by $ne^{-i(\phi_2 + \phi_{4n})/2}$ in (3.101), we will have established (3.93) [see (3.99)].

We can rewrite (3.102) as

$$\tilde{H}(\phi_2, \phi_4, \ldots, \phi_{4n}) = e^{-i(\phi_2 + \phi_{4n})/2} \left(\sum_{l=0}^{n-1} e^{-i(\phi_4 + \phi_6 + \cdots + \phi_{4n-2-4l})} + \sum_{l=1}^{n-1} (1 - e^{-i(\phi_{4l} + \phi_{4l+2})}) \right.$$
$$\left. \times \sum_{k=0}^{l-1} e^{-i(\phi_4 + \phi_6 + \cdots + \phi_{4l-2-4k})} \right) , \tag{3.103a}$$

$$\tilde{H}(\phi_2, \phi_4, \ldots, \phi_{4n}) = e^{-i(\phi_2 + \phi_{4n})/2} H(\phi_2, \ldots, \phi_{4n}) , \tag{3.103b}$$

where Eq. (3.103b) defines the function $H(\phi_2, \ldots, \phi_{4n})$. Now

$$\sum_{l=1}^{n-1} \sum_{k=0}^{l-1} e^{-i(\phi_4 + \phi_6 + \cdots + \phi_{4l-2-4k})} = (n-1) + (n-2)e^{-i(\phi_4 + \phi_6)} + (n-3)e^{-i(\phi_4 + \phi_6 + \phi_8 + \phi_{10})} + \cdots + e^{-i(\phi_4 + \phi_6 + \cdots + \phi_{4n-6})},$$

so that

$$H(\phi_2, \ldots, \phi_{4n}) = R(\phi_2, \ldots, \phi_{4n}) + n , \tag{3.104}$$

where

$$R(\phi_2, \ldots, \phi_{4n}) = \sum_{l=0}^{n-2} (l+1) e^{-i(\phi_4 + \phi_6 + \cdots + \phi_{4n-2-4l})} - \sum_{l=0}^{n-1} e^{-i(\phi_{4l} + \phi_{4l+2})} \sum_{k=0}^{l-1} e^{-i(\phi_4 + \phi_6 + \cdots + \phi_{4l-2-4k})} . \tag{3.105}$$

We now show that the integral (3.101) with $\tilde{H}$ replaced by $e^{-i(\phi_2 + \phi_{4n})/2} R$ is zero. From (3.104) it will then follow that (3.93) is true. If we multiply the numerator and denominator in (3.101) by $\sin\frac{1}{2}(\phi_2 + \phi_{4n})$, then the factor

$$\prod_{j=1}^{2n} \left(\frac{e^{-iM\phi_{2j-1} - iN\phi_{2j}}}{\Delta(\phi_{2j-1}, \phi_{2j})} \frac{\sin\frac{1}{2}(\phi_{2j-1} - \phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \right)$$

occurring in (3.101) is invariant under cyclic permutations of the variables $(\phi_2, \phi_4, \ldots, \phi_{4n})$. Hence the value of the integral (3.101) (with $\tilde{H} \to e^{-i(\phi_2+\phi_{4n})/2}R$) is not affected if we cyclically permute the variables of the factor

$$R' \equiv e^{i(\phi_2+\phi_{4n})/2} - e^{-i(\phi_2+\phi_{4n})/2}e^{-i(\phi_2+\phi_{4n})/2}R. \tag{3.106}$$

From (3.105) we obtain

$$R' = A_1 - A_2 - A_3 + A_4, \tag{3.107}$$

where

$$A_1 = \sum_{l=0}^{n-2}(l+1)e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-2-4l})}, \tag{3.108}$$

$$A_2 = \sum_{l=1}^{n-1} e^{-i(\phi_{4l}+\phi_{4l+2})} \times \sum_{k=0}^{l-1} e^{-i(\phi_4+\phi_6+\cdots+\phi_{4l-2-4k})}, \tag{3.109}$$

$$A_3 = \sum_{l=0}^{n-2}(l+1)e^{-i(\phi_2+\phi_{4n})}e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-2-4l})}, \tag{3.110}$$

$$A_4 = \sum_{l=1}^{n-1} e^{-i(\phi_{4l}+\phi_{4l+2})}e^{-i(\phi_2+\phi_{4n})} \times \sum_{k=0}^{l-1} e^{-i(\phi_4+\phi_6+\cdots+\phi_{4l-2-4k})}. \tag{3.111}$$

In (3.108) we cyclically permute the variables so that each term in the sum A_1 starts at ϕ_2. That is,

$$A_1 = \sum_{l=0}^{n-2}(l+1)e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-4-4l})}. \tag{3.112}$$

Note that we use an equality sign in (3.112) only in the sense that A_1 [as defined by (3.108)] and the right-hand side of (3.112) yield the same value when integrated over in (3.101). We will continue to use this sense of equality for the remainder of the proof. We rewrite A_2 as follows:

$$A_2 = \sum_{j=2}^{n-1} e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-4j})} \times \sum_{l=0}^{j-2} e^{-i(\phi_{4n-4j+2+4l}+\phi_{4n-4j+4l+4})} + (n-1)e^{-i(\phi_2+\phi_4)}. \tag{3.113}$$

To see how (3.113) follows from (3.109) we note the following transformations: from the $l=n-1$ and $k=0$ term

$$e^{-i(\phi_{4n-4}+\phi_{4n-2})}e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-6})} \to e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-8})}e^{-i(\phi_{4n-6}+\phi_{4n-4})};$$

from the $l=n-2$, $k=0$ and $l=n-1$, $k=1$ terms

$$e^{-i(\phi_{4n-8}+\phi_{4n-6})}e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-10})} + e^{-i(\phi_{4n-4}+\phi_{4n-2})}e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-10})}$$
$$\to e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-12})}(e^{-i(\phi_{4n-10}+\phi_{4n-8})} + e^{-i(\phi_{4n-6}+\phi_{4n-4})})$$
$$= e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-12})}\sum_{l=0}^{1} e^{-i(\phi_{4n-12+2+4l}+\phi_{4n-12+4+4l})};$$

from the $l=n-3$, $k=0$, $l=n-2$, $k=1$, and $l=n-1$, $k=2$ terms

$$e^{-i(\phi_{4n-12}+\phi_{4n-10})}e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-14})} + e^{-i(\phi_{4n-8}+\phi_{4n-6})}e^{-i(\phi_4+\phi_6+\cdots+\phi_{4n-14})}$$
$$+ e^{-i(\phi_{4n-4}+\phi_{4n-2})}e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-14})} \to e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-16})}\sum_{l=0}^{2} e^{-i(\phi_{4n-16+2+4l}+\phi_{4n-16+4+4l})}, \text{ etc.}$$

The $(n-1)e^{-i(\phi_2+\phi_4)}$ term comes from the $l=1$, $k=0$; $l=2$, $k=1$; ... ; $l=n-1$, $k=n-2$ terms.

We now make the following transformations on the terms in (3.110):

the $l=0$ term

$$\phi_2+\phi_{4n}+\phi_4+\phi_6+\cdots+\phi_{4n-2} \to \phi_2+\phi_4+\cdots+\phi_{4n},$$

the $l=1$ term

$$\phi_2+\phi_{4n}+\phi_4+\phi_6+\cdots+\phi_{4n-6} \to \phi_2+\phi_4+\cdots+\phi_{4n-4},$$

the $l=2$ term

$$\phi_2+\phi_{4n}+\phi_4+\phi_6+\cdots+\phi_{4n-10} \to \phi_2+\phi_4+\cdots+\phi_{4n-8},$$

etc. Thus A_3 becomes

$$A_3 = \sum_{l=0}^{n-2}(l+1)e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-4l})}, \tag{3.114a}$$

and hence

$$A_1 - A_3 = -\sum_{l=0}^{n-2} e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-l})} + (n-1)e^{-i(\phi_2+\phi_4)}. \tag{3.114b}$$

And finally for A_4 we can write [this is analogous to the transformations leading to (3.113)]

$$A_4 = \sum_{j=1}^{n-1} e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-4j})} \times \sum_{l=0}^{j-1} e^{-i(\phi_{4n-4j+2+4l}+\phi_{4n-4j+4+4l})}. \tag{3.115}$$

From (3.115) we can see that

$$A_4=e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n})}+2e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-4})}+e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-8})}+\cdots+e^{-i(\phi_2+\phi_4+\phi_6+\phi_8)}$$
$$+e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-12})}\sum_{l=1}^{2}e^{-i(\phi_{4n-12+2+4l}+\phi_{4n-12+4+4l})}$$
$$+e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-16})}\sum_{l=1}^{3}e^{-i(\phi_{4n-16+2+4l}+\phi_{4n-16+4+4l})}+\cdots+e^{-i(\phi_2+\phi_4)}\sum_{l=1}^{n-2}e^{-i(\phi_{4l+8}+\phi_{4l+6})}, \quad (3.116)$$

where we singled out the $j=1,\ l=0;\ j=2,\ l=0,\ 1;\ j=3,\ l=0;\ \ldots;\ j=n-1,\ l=0$ terms to give the first terms in (3.116). Note that the $j=2,\ l=1$ term must be cyclically permuted to give form in (3.116).

From (3.113) and (3.116) it follows that (one must permute indices in certain terms to obtain cancellations)

$$-A_2+A_4=\sum_{l=0}^{n=2}e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n-4l})}-(n-1)e^{-i(\phi_2+\phi_4)}. \quad (3.117)$$

Using (3.117) and (3.114b) in (3.107) it follows that

$$R'=0. \quad (3.118)$$

G. $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ in scaling limit ($T<T_c$)

1. Analysis of $F^{(2)}_{<MN}$ in scaling limit

In this section we examine (2.12) in the scaling limit (1.1)–(1.3). By Cauchy's theorem we are allowed to expand the contours of integration in (2.12) out to the unit circle [recall the restriction $\mathrm{Im}\phi_j<0$ in (2.12)] provided that we do not integrate through any singularities. The only possible singularities are those coming from the factors $\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})$ occurring in the denominator of (2.12) [$\Delta(\phi_{2j-1},\phi_{2j})\neq 0$ on the unit circle for $T<T_c$]. We first expand the ϕ_2 (and ϕ_1) integration to the unit circle. Now expand ϕ_4 (and ϕ_3) to the unit circle, except now at $\phi_4=-\phi_2$ we indent the ϕ_4 integration contour inward. Thus (2.12) can be interpreted as integration on the unit circle except at $\phi_{2j+2}=-\phi_{2j}$ the contour of integration for ϕ_{2j+2} is indented inward.

If the integrand in (2.12) is bounded, then the integral goes exponentially to zero in the limit $(M^2+N^2)\to\infty$. However at $T=T_c$ the denominator contains singular factors due to the vanishing of $\Delta(\phi_{2j-1},\phi_{2j})$ at $\phi_{2j-1}=\phi_{2j}=0$. Thus the leading contribution to the integral (2.12) for $T\to T_c^-$ comes from the behavior of the integrand around $\phi_j=0$, $j=1,2,\ldots,4n$. Thus to examine (2.12) in the scaling limit (1.1)–(1.3) we expand the integrand about $\phi_j=0$, $j=1,2,\ldots,4n$.

Since

$$a-\gamma_1\cos\phi_{2j-1}-\gamma_2\cos\phi_{2j}=(a-\gamma_1-\gamma_2)$$
$$\times[1+\tfrac{1}{2}\gamma_1(a-\gamma_1-\gamma_1)^{-1}\phi_{2j-1}^2+\tfrac{1}{2}(a-\gamma_1-\gamma_2)^{-1}\phi_{2j}^2$$
$$+O(\phi_{2j-1}^4(a-\gamma_1-\gamma_2)^{-1})+O(\phi_{2j}^4(a-\gamma_1-\gamma_2)^{-1})], \quad (3.119)$$

we define the scaled variables (or stretching variables)

$$x_j=[\tfrac{1}{2}\gamma_1(a-\gamma_1-\gamma_2)^{-1}]^{1/2}\phi_{2j-1}, \quad (3.120a)$$
$$x_j=[z_2(1-z_1^2)]^{1/2}\,|z_1z_2+z_1+z_2-1|^{-1}\phi_{2j-1},$$
$$j=1,2,\ldots,2n \quad (3.120b)$$

and

$$y_j=[\tfrac{1}{2}\gamma_2(a-\gamma_1-\gamma_2)^{-1}]^{1/2}\phi_{2j} \quad (3.121a)$$
$$y_j=[z_1(1-z_2^2)]^{1/2}\,|z_1z_2+z_1+z_2-1|^{-1}\phi_{2j},$$
$$j=1,2,\ldots,2n. \quad (3.121b)$$

The quantity $|z_1z_2+z_1+z_2-1|$ near T_c is given by (2.7). Then we have

$$\Delta(\phi_{2j-1},\phi_{2j})=|z_1z_2+z_1+z_2-1|^2$$
$$\times[1+x_j^2+y_j^2+O(\phi_{2j-1}^2x_j^2)+O(\phi_{2j}^2y_j^2)]. \quad (3.122)$$

Note that $O(\phi_{2j-1}^2x_j^2)=O(\phi_{2j-1}^2)=O[(T-T_c)^2]$.

Using (3.120)–(3.122) we see that in the scaling limit (1.1)–(1.3)

$$\prod_{j=1}^{2n}\frac{d\phi_{2j-1}d\phi_{2j}}{\Delta(\phi_{2j-1},\phi_{2j})}\sim\left(\frac{4}{\gamma_1\gamma_2}\right)^n\prod_{j=1}^{2n}\frac{dx_jdy_j}{1+x_j^2+y_j^2}. \quad (3.123)$$

$$e^{-iM(\phi_1+\cdots+\phi_{4n-1})-iN(\phi_2+\cdots+\phi_{4n})}$$
$$\sim e^{-i\bar{M}(x_1+\cdots+x_{2n})-i\bar{N}(y_1+\cdots+y_{2n})}, \quad (3.124)$$

and

$$\prod_{j=1}^{2n}\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\sim\left(\frac{\gamma_2}{\gamma_1}\right)^n\prod_{j=1}^{2n}\frac{x_j-x_{j+1}}{y_j+y_{j+1}}$$

$$(x_{2n+1}\equiv x_1\text{ and }y_{2n+1}\equiv y_1), \quad (3.125)$$

where

$$\bar{M}=M(2/\gamma_1)^{1/2}\,|z_1z_2+z_1+z_2-1|, \quad (3.126a)$$
$$\bar{M}=M[z_2(1-z_1^2)]^{-1/2}\,|z_1z_2+z_1+z_2-1| \quad (3.126b)$$

and

$$\bar{N}=N(2/\gamma_2)^{1/2}\,|z_1z_2+z_1+z_2-1|, \quad (3.127a)$$
$$\bar{N}=N[z_1(1-z_2^2)]^{-1/2}\,|z_1z_2+z_1+z_2-1|. \quad (3.127b)$$

Also note that the correction terms in (3.123) and (3.125) are $O((T-T_c)^2)$.

Applying (3.123)–(3.127) to (2.12) we have

$$F^{(2n)}_{<MN} \sim (-1)^n 2^{-2n} \pi^{-4n} (2n)^{-1} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{2n} \times \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_{2n} \times \prod_{j=1}^{2n} \frac{e^{-i\bar{M}x_j - i\bar{N}y_j}}{1+x_j^2+y_j^2} \, \frac{x_j - x_{j+1}}{y_j + y_{j+1}} , \tag{3.128}$$

where the contours of the y_{j+1} integration are to be indented downward at $-y_j$. The symbol "~" in (3.128) means that $F^{(2n)}_{<MN}$ is asymptotically equal to the right-hand side of (3.128) in the scaling limit (1.1)–(1.3). We denote the right-hand side of (3.128) by $f^{(2n)}(t)$; so that (3.128) becomes

$$F^{(2n)}_{<MN} \sim f^{(2n)}(t) \quad \text{(scaling limit)} , \tag{3.129}$$

and the quantity t is defined by

$$t^2 = \bar{M}^2 + \bar{N}^2 , \tag{3.130}$$

where $\bar{M}$ and $\bar{N}$ are given by (3.126) and (3.127), respectively. From (3.126) and (3.127) we see that the definition (3.130) of t is equivalent to (2.5). In terms of the interaction energies E_1 and E_2, the expression (2.5) for t can be written equivalently as

$$t^2 = \cosh\beta E_1 \cosh\beta E_2 (z_1 z_2)^{-1/2} (z_1 z_2 + z_1 + z_2 - 1)^2 [M^2 (\sinh 2\beta E_1 / \sinh 2\beta E_2)^{1/2} + N^2 (\sinh 2\beta E_2 / \sinh 2\beta E_1)^{1/2}] . \tag{3.131}$$

Furthermore, as $T \to T_c$ we find

$$t \sim 4(\beta - \beta_c)(z_{1c} z_{2c})^{1/4} (\sinh\beta_c E_1 \sinh\beta_c E_2)^{1/2} [M^2 (\sinh 2\beta_c E_1 / \sinh 2\beta_c E_2)^{1/2} + N^2 (\sinh 2\beta_c E_2 / \sinh 2\beta_c E_1)^{1/2}]^{1/2} \tag{3.132a}$$

$$t = (\beta - \beta_c) 4 (z_{1c} z_{2c})^{1/4} (\sinh\beta_c E_1 \sinh\beta_c E_2)^{1/2} [E_1 (1 - z_{2c})^{-1} + E_2 (1 - z_{1c})^{-1}] (M^2 \sinh 2\beta_c E_1 + N^2 \sinh 2\beta_c E_2)^{1/2} . \tag{3.132b}$$

We also define the quantity R by (2.6). Note that for the symmetric lattice $E_1 = E_2$, R is just $(M^2 + N^2)^{1/2}$ [see (2.65)].

2. "Rotational invariance" of $F^{(2n)}_{<MN}$ in scaling limit

In writing (3.129) we have implicitly assumed that (3.128) is only a function of $(\bar{M}^2 + \bar{N}^2)$ and not $\bar{M}$ and $\bar{N}$ separately. We now show this is the case. Defining the new integration variables

$$\begin{aligned} u_j &= x_j \sin\theta + y_j \cos\theta , \\ v_j &= -x_j \cos\theta + y_j \sin\theta , \end{aligned} \tag{3.133}$$

for $j = 1, 2, \ldots, 2n$; and writing (which defines the angle θ)

$$\bar{M} = t \sin\theta , \quad \bar{N} = t \cos\theta , \tag{3.134}$$

t given by (3.130), we see that (3.128) becomes

$$f^{(2n)}(t) = (-1)^n 2^{-2n} \pi^{-4n} (2n)^{-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{2n} \left(\frac{du_j \, dv_j}{1 + u_j^2 + v_j^2} \, e^{-itu_j} \, \frac{(u_j - u_{j+1}) \sin\theta - (v_j - v_{j+1}) \cos\theta}{(u_j + u_{j+1}) \cos\theta + (v_j + v_{j+1}) \sin\theta} \right) , \tag{3.135}$$

$$u_{2n+1} \equiv u_1 \quad \text{and} \quad v_{2n+1} \equiv v_1 .$$

We now perform the v_j integrations $j = 1, 2, \ldots, 2n$ in (3.135) by closing the contour in the upper half-plane and evaluating the residue at $v_j = i(\sqrt{1 + u_j^2})$. Hence (3.135) reduces to

$$f^{(2n)}(t) = (-1)^n 2^{-2n} \pi^{-2n} (2n)^{-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{2n} \left(\frac{du_j}{(1 + u_j^2)^{1/2}} \, e^{-itu_j} \, \frac{(u_j - u_{j+1}) \sin\theta - i[(\sqrt{1 + u_j^2}) - (\sqrt{1 + u_{j+1}^2})] \cos\theta}{(u_j + u_{j+1}) \cos\theta + i[(\sqrt{1 + u_j^2}) + (\sqrt{1 + u_{j+1}^2})] \sin\theta} \right) . \tag{3.136}$$

We now show that the quantity

$$\frac{(u - v) \sin\theta - i[(\sqrt{1 + u^2}) - (\sqrt{1 + v^2})] \cos\theta}{(u + v) \cos\theta + i[(\sqrt{1 + u^2}) + (\sqrt{1 + v^2})] \sin\theta} \tag{3.137}$$

is independent of the angle θ. To do this we introduce $u = \sinh\psi$ and $v = \sinh\chi$ into (3.137) obtaining

$$\frac{(\sinh\psi - \sinh\chi) \sin\theta - i(\cosh\psi - \cosh\chi) \cos\theta}{(\sinh\psi + \sinh\chi) \cos\theta + i(\cosh\psi + \cosh\chi) \sin\theta} . \tag{3.138}$$

Using the hyperbolic addition formulas for $\cosh x \pm \cosh y$ and $\sinh x \pm \sinh y$ in (3.138) and rewriting the resulting expression in terms of the hyperbolic tangent function, (3.138) becomes simply

$$-i\tanh\tfrac{1}{2}(\psi-\chi)\,, \qquad (3.139)$$

a quantity independent of the angle θ. Thus we may set $\theta=0$ in (3.137) to obtain the identity

$$\frac{(u-v)\sin\theta - i[(\sqrt{1+u^2})-(\sqrt{1+v^2})]\cos\theta}{(u+v)\cos\theta + i[(\sqrt{1+u^2})+(\sqrt{1+v^2})]\sin\theta} = -i\,\frac{(\sqrt{1+u^2})-(\sqrt{1+v^2})}{u+v} \qquad (3.140)$$

Using (3.140) in (3.136), we obtain the explicitly "rotational invariant" form for $f^{(2n)}(t)$

$$f^{(2n)}(t)=2^{-2n}\pi^{-2n}(2n)^{-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{2n}\left(\frac{du_j}{(1+u_j^2)^{1/2}} \times e^{-itu_j}\,\frac{(\sqrt{1+u_j^2})-(\sqrt{1+u_{j+1}^2})}{u_j+u_{j+1}}\right). \qquad (3.141)$$

3. Final integral form for $f^{(2n)}(t)$

We can simplify (3.141) if we deform each u_j-integration contour into the lower half-plane so that we are now integrating on the loop about the branch point at $u_j=-i$. Letting $u_j=-iy_j$, we see that (3.141) will be of the form

$$f^{(2n)}(t)=2^{-2n}\pi^{-2n}(2n)^{-1}\int_1^{\infty}\cdots\int_1^{\infty}\prod_{j=1}^{2n}\left(\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}}\,dy_j\right) \times P_n(y_1,y_2,\ldots,y_{2n})\,, \qquad (3.142)$$

where $P_n(y_1,y_2,\ldots,y_{2n})$ is a sum of 2^{2n} terms coming from all possible combinations of plus and minus signs of the product term

$$\prod_{j=1}^{2n}\frac{(\sqrt{1+u_j^2})-(\sqrt{1+u_{j+1}^2})}{u_j+u_{j+1}}$$

in (3.141). This is so because of the square-root nature of the product term. Clearly the denominator (u_j+u_{j+1}) presents no difficulty; and so, we concentrate on the numerator where the signs of the square roots take on all possible plus and minus signs. We can represent a term in this sum as $(x_1-x_2)(x_2-x_3)\cdots(x_n-x_1)$. Summing over all possible sign changes means that we must evaluate the sum

$$S_{2n}=\sum_{(\epsilon_j=\pm1)}(\epsilon_1x_1-\epsilon_2x_2) \times(\epsilon_2x_2-\epsilon_3x_3)\cdots(\epsilon_{2n}x_{2n}-\epsilon_1x_1)\,. \qquad (3.143)$$

The only terms in (3.143) which contribute are those quadratic in each of the ϵ_j. Thus we have

$$S_{2n}=(-1)^n2^{2n}(x_1^2x_3^2x_5^2\cdots x_{2n-1}^2+x_2^2x_4^2x_6^2\cdots x_{2n}^2)\,. \qquad (3.144)$$

If we apply (3.144) to (3.142) we obtain[20] (2.27) [when one uses (3.144) in (3.142) the two terms resulting from (3.144) can be written as one term as we have done in (2.27)].

Since (2.27) is of the standard Laplace type, it is straightforward to analyze the large-t behavior. We find that for fixed n and $t\to\infty$

$$f^{(2n)}(t)=(-1)^n\pi^{-n}(2n)^{-1}2^{-3n+1}(e^{-2nt}/t^{2n}) \times[1-\tfrac{7}{4}nt^{-1}+\tfrac{1}{128}n(196n+272)t^{-2}+O(t^{-3})]\,. \qquad (3.145)$$

For small t we have

$$f^{(2n)}(t)=C_{2n}(\ln t)^{2n}+O(\ln^{2n-1}t)\quad(t\to0)\,, \qquad (3.146)$$

where C_{2n} is a constant.

4. $f^{(2)}(t)$ in terms of Bessel functions

The function $f^{(2)}(t)$ can be explicitly computed. From (2.27) we see that

$$\frac{d^2f^{(2)}(t)}{dt^2}=-\pi^{-2}\int_1^{\infty}dy_1 \times\int_1^{\infty}dy_2\,\frac{e^{-t(y_1+y_2)}}{[(y_1^2-1)(y_2^2-1)]^{1/2}}\,(y_2^2-1)\,. \qquad (3.147)$$

Using the integral representation

$$K_0(t)=\int_1^{\infty}\frac{dx}{(x^2-1)^{1/2}}\,e^{-tx} \qquad (3.148)$$

for the modified Bessel function $K_0(t)$, we see that (3.147) becomes

$$\frac{d^2f^{(2)}(t)}{dt^2}=\pi^{-2}[K_1'(t)K_0(t)+K_0^2(t)] =\pi^{-2}t^{-1}K_0(t)K_1(t)\,. \qquad (3.149)$$

Differentiation verifies that

$$-f^{(2)}(t)=\pi^{-2}\{t^2[K_1^2(t)-K_0^2(t)] -tK_0(t)K_1(t)+\tfrac{1}{2}K_0^2(t)\}\,. \qquad (3.150)$$

From (2.23), (2.26), (3.145), and (3.150) the result (2.32b) follows.

5. $\hat{F}^{(2)}(t)$

From (2.26) we see that we can write $\hat{F}_-(t)$ as

$$\hat{F}_-(t)=1+\sum_{n=1}^{\infty}\hat{F}_-^{(2n)}(t)\,, \qquad (3.151)$$

where

$$\hat{F}_-^{(2)}(t)=-f^{(2)}(t)\quad[\text{recall } (3.150)]\,, \qquad (3.152a)$$

$$\hat{F}_-^{(4)}(t)=-f^{(4)}(t)+(1/2!)[f^{(2)}(t)]^2\,, \qquad (3.152b)$$

$$\hat{F}_-^{(6)}(t)=-f^{(6)}(t)+f^{(2)}(t)f^{(4)}(t)-(1/3!)[f^{(2)}(t)]^3\,, \qquad (3.152c)$$

etc. From (3.145) we see that $\hat{F}_-^{(2n)}(t)$ is asymptotically equal to e^{-2nt} times some power of t as $t\to\infty$.

If one uses (3.145) in (3.152b) or (3.152c) in order to compute (for $t\to\infty$) the asymptotic behavior of $\hat{F}_-^{(4)}(t)$ or $\hat{F}_-^{(6)}(t)$, respectively, then one finds that not only does the leading term in (3.145) cancel but so do the two correction terms. This suggests we should examine (3.152) in more detail.

We discuss here only the case for $n=2$, i.e., $\hat{F}_-^{(4)}(t)$. Using (3.152b) and (2.27) we see that

$$\hat{F}_-^{(4)}(t)=\pi^{-4}\int_1^\infty dy_1\cdots\int_1^\infty dy_4\prod_{j=1}^4\left(\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}}\right)(y_2^2-1)(y_4^2-1)$$
$$\times\{-\tfrac{1}{2}[(y_1+y_2)(y_2+y_3)(y_3+y_4)(y_4+y_1)]^{-1}+\tfrac{1}{4}[(y_1+y_2)^2(y_3+y_4)^2]^{-1}+\tfrac{1}{4}[(y_1+y_4)^2(y_2+y_3)^2]^{-1}\}$$
$$=4^{-1}\pi^{-4}\int_1^\infty dy_1\cdots\int_1^\infty dy_4\prod_{j=1}^4\left(\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}}\right)(y_2^2-1)(y_4^2-1)\left(\frac{(y_2-y_4)(y_1-y_3)}{(y_1+y_2)(y_2+y_3)(y_3+y_4)(y_4+y_1)}\right)^2. \quad (3.153)$$

The cancellation that occurs in (3.152b) has been explicitly carried out in (3.153) in the sense that the leading term in the large-t expansion of (3.153) is nonvanishing. An elementary computation shows that as $t\to\infty$

$$\hat{F}_-^{(4)}(t)=3\,2^{-12}\pi^{-2}(e^{-4t}/t^8)[1-\tfrac{31}{2}t^{-1}+O(t^{-2})], \quad (3.154)$$

6. *Scaling functions $F(t)$ and $F_1(t)$*

From the error estimates of Sec. IIIG 1 it is clear that

$$\exp(-F_{<MN})=\hat{F}_-(t)+o(R^{-1}) \text{ (scaling limit)}. \quad (3.155)$$

The essential point is that the integrand in (2.12) has correction terms of order $(T-T_c)^2$ in the scaling limit since the variables ϕ_j appear only as sine and cosine factors. See, for instance, (3.119) and (3.122). Thus from (2.9) follows (2.21). To obtain $F_-(t)$ and $F_{1-}(t)$ we must expand $|1-(\sinh 2\beta E_1 \times\sinh 2\beta E_2)^{-2}|^{1/4}$ to second order in the scaling limit.[21] The result of this expansion is given by (2.23), (2.24), and (2.25). Note that at $t=0$ it follows from (2.24) $F_{1-}(0)=0$, as it must to agree with the known[5,31] $T=T_c$ expansion of $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ and $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ for $N\to\infty$.

7. *Convergence proof*

It is straightforward to study the convergence of (2.26). Using the inequality

$$(y_j+y_{j+1})^{-1}\le(1+y_{j+1})^{-1} \text{ for } y_j, y_{j+1}\ge 1, \quad (3.156)$$

and the fact that the integrand in (2.27) is nonnegative, we have

$$|f^{(2n)}(t)|\le\pi^{-2n}(2n)^{-1}\int_1^\infty dy_1\cdots\int_1^\infty dy_{2n}\prod_{j=1}^{2n}\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}}\prod_{j=1}^{n}\frac{y_{2j}^2-1}{(1+y_{2j})^2}$$
$$=\pi^{-2n}(2n)^{-1}\left(\int_1^\infty dy\,\frac{e^{-ty}}{(y^2-1)^{1/2}}\right)^n\left(\int_1^\infty dy'\,\frac{e^{-ty'}}{(y'^2-1)^{1/2}}\,\frac{y'-1}{y'+1}\right)^n. \quad (3.157)$$

We define t^* as the solution to the equation

$$\frac{1}{\pi^2}\int_1^\infty dy\,\frac{e^{-ty}}{(y^2-1)^{1/2}}\int_1^\infty dy'\,\frac{e^{-ty'}}{(y'^2-1)^{1/2}}\,\frac{y'-1}{y'+1}=1, \quad (3.158)$$

and we find numerically that

$$t^*=0.03302. \quad (3.159)$$

Then for all $t>t^*$ the sum

$$\sum_{n=1}^\infty f^{(2n)}(t)$$

converges. Clearly this is only a sufficient condition.

IV. PERTURBATION EXPANSION FOR $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ FOR $T>T_c$ AND LARGE M^2+N^2

A. Perturbation-expansion formalism

The index of the function $S(e^{i\phi})$ defined by (3.5) is not zero for $T>T_c$. Consequently we modify the Toeplitz determinant of Sec. III so that we work with zero index generating functions. So following CW we define the following "barred" quantities[22]:

$$\bar{S}_{mn}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\phi\,e^{-i(n-m)\phi}\bar{S}(e^{i\phi}), \quad (4.1)$$

$$\bar{V}_{mn}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\phi\,e^{-i(m+n)\phi}\bar{V}(e^{i\phi}), \quad (4.2)$$

$$\bar{U}_{mn}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\phi\,e^{-i(m+n)\phi}\bar{U}(e^{i\phi}), \quad (4.3)$$

for $m, n=0, 1, 2, \ldots$, where the generating functions $\bar{S}(e^{i\phi})$, $\bar{V}(e^{i\phi})$, and $\bar{U}(e^{i\phi})$ are given by

$$\bar{S}(e^{i\phi})=e^{-i\phi}S(e^{i\phi})=[(1-\alpha_1 e^{i\phi})(1-\alpha_2^{-1}e^{i\phi})\times(1-\alpha_1 e^{-i\phi})^{-1}(1-\alpha_2^{-1}e^{-i\phi})^{-1}]^{1/2}, \quad (4.4)$$

$$\bar{V}(e^{i\phi_2})=e^{2i\phi_2}V(e^{i\phi_2})=(1-z_1^2)(2\pi)^{-1}\int_{-\pi}^{\pi}d\phi_1\frac{e^{-iM\phi_1-iN\phi_2}}{\Delta(\phi_1,\phi_2)}\times\lambda(\phi_1,-\phi_2), \quad (4.5)$$

$$\bar{U}(e^{i\phi_2})=e^{i\phi_2}U(e^{i\phi_2})=-(1-z_1^2)(2\pi)^{-1}\int_{-\pi}^{\pi}d\phi_1\frac{e^{-iM\phi_1-iN\phi_2}}{\Delta(\phi_1,\phi_2)}\times 2iz_2\sin\phi_1, \quad (4.6)$$

with $S(e^{i\phi})$, $V(e^{i\phi})$, and $U(e^{i\phi})$ given by (3.5)–(3.7), respectively, $\Delta(\phi_1,\phi_2)$ and $\lambda(\phi_1,\phi_2)$ given by (2.13) and (3.9), respectively.

We now introduce the barred version of the matrices A, B, and C of Sec. III:

$$\bar{A}=\begin{bmatrix}0&\bar{S}&0&0\\-\bar{S}^T&0&0&0\\0&0&0&-\bar{S}\\0&0&\bar{S}^T&0\end{bmatrix}, \quad (4.7)$$

$$\bar{B}=\begin{bmatrix}0&0&T&\bar{U}\\0&0&-\bar{U}&\bar{V}\\-T&\bar{U}&0&0\\-\bar{U}&-\bar{V}&0&0\end{bmatrix}, \quad (4.8)$$

and

$$\bar{C}=\bar{A}+\bar{B} \quad (4.9)$$

The operators $\bar{S}$, $\bar{V}$, and $\bar{U}$ appearing in (4.7) and (4.8) are defined by the matrices (4.1)–(4.3), respectively. The operator T is the same as in Sec. III and is defined by (3.4).

Then it is known[5,17] that for arbitrary $T>T_c$ and arbitrary $M, N\geq 0$

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle=[r(M,N)]^{1/2}\det(\bar{C}), \quad (4.10)$$

with

$$r(M,N)=x_{20}''x_{40}'-x_{20}'x_{40}'', \quad (4.11)$$

and $\bar{C}$ defined by (4.9). Furthermore x_{20}'', x_{40}'', x_{20}', and x_{40}' are the zeroth components of the vectors x_2'', x_4'', x_2', and x_4', respectively, which are solutions to the equations

$$\bar{C}x''=y_2 \quad\text{and}\quad \bar{C}x'=y_4, \quad (4.12a)$$

with

$$x''=\begin{pmatrix}x_1''\\x_2''\\x_3''\\x_4''\end{pmatrix},\quad y_2=\begin{pmatrix}0\\\delta\\0\\0\end{pmatrix}, \quad (4.12b)$$

and x' and y_4 of the form (3.20). Since $\bar{S}(e^{i\phi})$ is of index zero for $T>T_c$, the methods of Sec. III can be applied to compute $\det(\bar{C})$ in (4.10). That is to say, if we define in analogy with (3.17)

$$\bar{f}_{MN}=\frac{\det(\bar{C}_{M,N+1})}{\det(\bar{C}_{M,N})}, \quad (4.13)$$

where we made the dependence of $\bar{C}$ on M and N explicit by writing $\bar{C}_{M,N}$, then

$$(\bar{f}_{MN})^2=x_{30}x_{40}'-x_{30}'x_{40}, \quad (4.14a)$$

where

$$\bar{C}x=y_3, \quad (4.14b)$$

x and y_3 are of the form (3.20), and x_{30} and x_{40} (x_{30}' and x_{40}') are the zeroth components of the vectors x_3 and x_4 (x_3' and x_4') that are solutions to (4.14b) [(4.12)].

Thus in view of the results of Sec. III we may write

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle=\bar{S}_\infty[r(M,N)]^{1/2}\exp(-\bar{F}_{MN}), \quad (4.15)$$

with

$$\bar{F}_{MN}=\sum_{k=N}^{\infty}\ln\bar{f}_{Mk}, \quad (4.16)$$

$$\bar{f}_{MN}=1+\delta^T\sum_{n=1}^{\infty}[(\bar{A}^{-1}\bar{B})^{2n}]_{33}(\bar{S}^T)^{-1}\delta, \quad (4.17)$$

$$\bar{S}_\infty=\lim_{N\to\infty}\det(\bar{C}_{M,N}), \quad (4.18)$$

and the quantity δ is defined by (3.21). As shown by CW[23]

$$\bar{S}_\infty=(1-z_2^2)^{-1}(\gamma_1\gamma_2)^{1/2}\times[(\sinh 2\beta E_1\sinh 2\beta E_2)^{-2}-1]^{1/4}, \quad (4.19)$$

$$\bar{S}_\infty=(1-z_2^2)^{-1}S_\infty^{>}, \quad (4.20)$$

where we used the definition (2.17) of $S_\infty^{>}$ in going from (4.19) to (4.20).

Using the fact that $\bar{C}$ is antisymmetric we obtain from (4.11) and (4.12) the result

$$r(M,N)=(x_{20}')^2. \quad (4.21)$$

We can compute x_{20}' from (4.12) by writing $\bar{C}^{-1}$ as

$$\bar{C}^{-1}=\sum_{n=0}^{\infty}(-1)^n(\bar{A}^{-1}\bar{B})^n\bar{A}^{-1}, \quad (4.22)$$

so that x_{20}' can be written perturbatively as

$$-x_{20}'=\delta^T\sum_{n=0}^{\infty}[(\bar{A}^{-1}\bar{B})^{2n+1}]_{23}(\bar{S}^T)^{-1}\delta. \quad (4.23)$$

Note that in (4.23) only odd powers of $(\bar{A}^{-1}\bar{B})$ contribute to x'_{20}. This should be compared with (3.26), the perturbation expansion for f_{MN}, where only even powers of $(A^{-1}B)$ contribute.

The method of Wiener-Hopf[19] can be applied to obtain the matrix elements of $\bar{S}^{-1}$ which are needed in (4.23). The result is

$$(\bar{S}^{-1})_{mn} = (2\pi i)^{-2}\oint d\xi^{-(n+1)}\bar{P}(\xi)\oint d\xi'\frac{\xi'^m}{\xi'-\xi}\bar{Q}(\xi'^{-1}), \tag{4.24}$$

where now $\bar{P}(\xi)$ and $\bar{Q}(\xi)$ are given by[24]

$$\bar{P}(\xi) = [(1-\alpha_1\xi)(1-\alpha_2^{-1}\xi)]^{-1/2} \tag{4.25}$$

and

$$\bar{Q}(\xi) = [(1-\alpha_1\xi)(1-\alpha_2^{-1}\xi)]^{1/2}, \tag{4.26}$$

with the property that

$$\bar{P}(\xi)\bar{Q}(\xi) = 1. \tag{4.27}$$

The contours of integration in (4.24) are the unit circles, except that the one for ξ' is to be indented outward near $\xi'=\xi$.

We now turn to the problem of computing the $(2n+1)$th term, $n=0, 1, 2, \ldots$ in (4.23). As in Sec. III we introduce certain recursion relations which when solved will give us $\delta^T[(\bar{A}^{-1}\bar{B})^{2n+1}]_{23}(\bar{S}^T)^{-1}\delta$ of (4.23) for $n=0, 1, 2, \ldots$.

B. Recursion relations

We define the operators $\bar{P}_1$, $\bar{P}_2$, and $\bar{P}_3$ as the "barred" version of (3.32)–(3.34). That is,

$$\bar{P}_1 = \bar{U}(\bar{S}^T)^{-1}\bar{U} + \bar{V}\bar{S}^{-1}T, \tag{4.28}$$

$$\bar{P}_2 = -\bar{V}\bar{S}^{-1}\bar{U} + \bar{U}(\bar{S}^T)^{-1}\bar{V}, \tag{4.29}$$

and

$$\bar{P}_3 = \bar{T}(\bar{S}^T)^{-1}\bar{U} - \bar{U}\bar{S}^{-1}\bar{T}. \tag{4.30}$$

Then our basic recursion relations for $T > T_c$ are (for $k \geq 1$)

$$[(\bar{A}^{-1}\bar{B})^{2k+1}]_{23} = -\bar{S}^{-1}\{\bar{P}_3[(\bar{A}^{-1}\bar{B})^{2k-1}]_{13} + \bar{P}_1^T[(\bar{A}^{-1}\bar{B})^{2k-1}]_{23}\} \tag{4.31}$$

and

$$[(\bar{A}^{-1}\bar{B})^{2k+1}]_{13} = -(\bar{S}^T)^{-1}\{\bar{P}_1[(\bar{A}^{-1}\bar{B})^{2k-1}]_{13} + \bar{P}_2[(\bar{A}^{-1}\bar{B})^{2k-1}]_{23}\}. \tag{4.32}$$

For $k=0$ $(\bar{A}^{-1}\bar{B})_{13}$ and $(\bar{A}^{-1}\bar{B})_{23}$ are given below [(4.92) and (4.93)].

We introduce the generating functions

$$\bar{S}^{-1}(\xi, \xi') = \sum_{m,n=0}^{\infty}\xi^m(\bar{S}^{-1})_{mn}(\xi')^n, \tag{4.33}$$

$$\bar{S}^{-1}(\xi, \xi') = \bar{Q}(\xi)\bar{P}(\xi')(1-\xi\xi')^{-1}, \tag{4.34}$$

and

$$\bar{R}_{\alpha\beta}^{(k)}(\xi, \xi') = \sum_{m,n=0}^{\infty}\xi^m\{[(\bar{A}^{-1}\bar{B})^k]_{\alpha\beta}(\bar{S}^T)^{-1}\}_{mn}\xi'^n. \tag{4.35}$$

In going from (4.33) to (4.34) we made use of (4.24)–(4.26). Using (4.35) we see that (4.31) and (4.32) can be written equivalently as

$$\bar{R}_{23}^{(2k+1)}(\xi, \xi') = -(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(2)d(4)\bar{S}^{-1}(\xi, \bar{2})\times\{\bar{P}_3(2, 4)\bar{R}_{13}^{(2k-1)}(\bar{4}, \xi') + \bar{P}_1^T(2, 4)\bar{R}_{23}^{(2k-1)}(\bar{4}, \xi')\} \tag{4.36}$$

and

$$\bar{R}_{13}^{(2k+1)}(\xi, \xi') = -(2\pi)^{-2}\int\int_{\pi}^{\pi}d(2)d(4)\bar{S}^{-1}(2, \xi)\times\{\bar{P}_1(2, 4)\bar{R}_{13}^{(2k-1)}(\bar{4}, \xi') + \bar{P}_2(2, 4)\bar{R}_{23}^{(2k-1)}(\bar{4}, \xi')\}, \tag{4.37}$$

where $k=1, 2, \ldots$, and

$$\bar{P}_1(1, 2) = \bar{U}(2)\bar{S}^{-1}(\bar{2}, \bar{1})\bar{U}(1) + \bar{V}(1)\bar{S}^{-1}(\bar{1}, \bar{2})T(2), \tag{4.38}$$

$$\bar{P}_2(1, 2) = -\bar{V}(1)\bar{S}^{-1}(\bar{1}, \bar{2})\bar{U}(2) - \bar{U}(1)\bar{S}^{-1}(\bar{2}, \bar{1})\bar{V}(2), \tag{4.39}$$

$$\bar{P}_3(1, 2) = T(1)\bar{S}^{-1}(\bar{2}, \bar{1})\bar{U}(2) - \bar{U}(1)\bar{S}^{-1}(\bar{1}, \bar{2})T(2). \tag{4.40}$$

As in Sec. III the notation $\bar{S}^{-1}(1, 2)$ means $\bar{S}^{-1}(e^{i\phi_1}, e^{i\phi_2})$ and $\bar{S}^{-1}(\bar{1}, \bar{2})$ means $\bar{S}^{-1}(e^{-i\phi_1}, e^{-i\phi_2})$. Likewise for the functions $\bar{P}_j(1, 2)$ and $j=1$, 2, and 3, and the functions $\bar{U}(1)$, $\bar{V}(1)$, etc.

In view of (4.23) and (4.36) we have

$$-x'_{20} = \sum_{k=0}^{\infty}\bar{R}_{23}^{(2k+1)}(0, 0), \tag{4.41a}$$

and we define

$$\bar{x}_{MN}^{(2k+1)} = -(1-z_2^2)^{-1}\bar{R}_{23}^{(2k+1)}(0, 0), \quad k=0, 1, 2, \ldots. \tag{4.41b}$$

For $k=0$ we have

$$(\bar{A}^{-1}\bar{B})_{13} = (\bar{S}^T)^{-1}\bar{U} \tag{4.42}$$

and

$$(\bar{A}^{-1}\bar{B})_{23} = \bar{S}^{-1}T, \tag{4.43}$$

so that $\bar{R}_{13}^{(1)}(\xi, \xi')$ and $\bar{R}_{23}^{(1)}(\xi, \xi')$ are given by

$$\bar{R}_{13}^{(1)}(\xi, \xi') = (2\pi)^{-1}\int_{-\pi}^{\pi}d(2)\bar{S}^{-1}(\bar{2}, \xi)\bar{U}(2)\bar{S}^{-1}(\xi', 2), \tag{4.44}$$

$$\bar{R}_{13}^{(1)}(\xi, \xi') = -i\gamma_1\bar{P}(\xi)\bar{Q}(\xi')(2\pi)^{-2}\times\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\phi_1\,d\phi_2\frac{e^{-iM\phi_1-iN\phi_2}}{\Delta(\phi_1, \phi_2)}\times[(1-\xi e^{-i\phi_2})(1-\xi' e^{-i\phi_2})]^{-1}\sin\phi_1, \tag{4.45}$$

and

$$\overline{R}_{23}^{(1)}(\xi, \xi') = (2\pi)^{-1} \int_{-\pi}^{\pi} d(2)\overline{S}^{-1}(\xi, \overline{2})T(2)\overline{S}^{-1}(\xi', \overline{2}), \tag{4.46}$$

$$\overline{R}_{23}^{(1)}(\xi, \xi') = -(1 - z_2^2)\overline{Q}(\xi)\overline{Q}(\xi')(2\pi)^{-2} \times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1\, d\phi_2\, \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1, \phi_2)} \times [(1 - \xi e^{-i\phi_2})(1 - \xi' e^{-i\phi_2})]^{-1}. \tag{4.47}$$

We used (4.25)–(4.27) and (4.34) in going from (4.44) to (4.45). To obtain (4.47) from (4.46) one uses the identity

$$T(e^{i\phi_2})[\overline{P}(e^{-i\phi_2})]^2 = -(1 - z_2^2)(2\pi)^{-1} \times \int_{-\pi}^{\pi} d\phi_1\, \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1, \phi_2)}\ . \tag{4.48}$$

This identity is derived in a manner analogous to (3.55). The barred version of (3.56) is the identity

$$V(e^{i\phi_2})[\overline{Q}(e^{-i\phi_2})]^2 = -\gamma_1^2(1 - z_2^2)^{-1}(2\pi)^{-1} \times \int_{-\pi}^{\pi} d\phi_1\, \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1, \phi_2)} \sin^2\phi_1\ . \tag{4.49}$$

Hence the first term in (4.41), $\overline{x}_{MN}^{(1)}$, is given by the $\xi = \xi' = 0$ value of (4.47) [recall definition (4.41b)],

$$\overline{x}_{MN}^{(1)} = (2\pi)^{-2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1\, d\phi_2\, \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1, \phi_2)}\ . \tag{4.50}$$

Equation (4.50) was first derived by CW.

Since the general structure of $\overline{R}_{13}^{(2k+1)}(\xi, \xi')$ and $\overline{R}_{23}^{(2k+1)}(\xi, \xi')$ is not revealed by the $k=0$ case, we proceed to the $k=1$ case. Making use of (4.34), (4.38)–(4.40), (4.45), and (4.47) in (4.36) for $k=1$ we obtain

$$\overline{R}_{23}^{(3)}(\xi, \xi') = -\overline{Q}(\xi)\overline{Q}(\xi')(2\pi)^{-4} \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} d(2)d(4)d(5)d(6)\, \frac{e^{-M\phi_5 - iN\phi_6}}{\Delta(\phi_5, \phi_6)} [(1 - \xi e^{-i\phi_2})(1 - \xi' e^{-i\phi_6})]^{-1} \prod_{j=1}^{2} (1 - e^{-i(\phi_{2j} + \phi_{2j+2})})^{-1}$$
$$\times (\{[\overline{P}(\overline{2})]^2 T(2)\overline{U}(4) - \overline{U}(2)[\overline{P}(4)]^2 T(4)\}(-i\gamma_1 \sin\phi_5) + \{\overline{U}(2)\overline{U}(4) + [\overline{P}(\overline{2})]^2 T(2)[\overline{Q}(\overline{4})]^2$$
$$\times \overline{V}(4)\}[-(1 - z_2^2)])\ . \tag{4.51}$$

Now use the definition (4.6) of $\overline{U}(2)$ and identities (4.48) – (4.49) in (4.51) to obtain

$$\overline{R}_{23}^{(3)}(\xi, \xi') = -(1 - z_2^2)\gamma_1^2\, \overline{Q}(\xi)\overline{Q}(\xi')(2\pi)^{-6} \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} d(1)\cdots d(6) \prod_{j=1}^{3} \frac{e^{-iM\phi_{2j-1} - iN\phi_{2j}}}{\Delta(\phi_{2j-1}, \phi_{2j})}$$
$$\times [(1 - \xi e^{-i\phi_2})(1 - \xi' e^{-i\phi_6})]^{-1} \left(\prod_{j=1}^{2} (1 - e^{-i(\phi_{2j} + \phi_{2j+2})})^{-1} \right) (\sin\phi_1 - \sin\phi_3)(\sin\phi_3 - \sin\phi_5). \tag{4.52}$$

Similarly,

$$\overline{R}_{13}^{(3)}(\xi, \xi') = -\overline{P}(\xi)\overline{Q}(\xi')(2\pi)^{-4} \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} d(2)d(4)d(5)d(6)\, \frac{e^{-iM\phi_5 - iN\phi_6}}{\Delta(\phi_5, \phi_6)} [(1 - \xi e^{-i\phi_2})(1 - \xi' e^{-i\phi_6})]^{-1} \prod_{j=1}^{2} (1 - e^{-i(\phi_{2j} + \phi_{2j+2})})^{-1}$$
$$(\{\overline{U}(4)\,\overline{U}(2) + \overline{V}(2)[\overline{Q}(\overline{2})]^2 T(4)[\overline{P}(\overline{4})]^2\}(-i\gamma_1 \sin\phi_5) + \{-[\overline{Q}(\overline{2})]^2\, \overline{V}(2)\, \overline{U}(4) + \overline{U}(2)[\overline{Q}(\overline{4})]^2\, \overline{V}(4)\}[-(1 - z_2^2)])$$
$$= -i\gamma_1^3\, \overline{P}(\xi)\, \overline{Q}(\xi')(2\pi)^{-6} \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} d(1)\cdots d(6) \prod_{j=1}^{3} \frac{e^{-iM\phi_{2j-1} - iN\phi_{2j}}}{\Delta(\phi_{2j+1}, \phi_{2j})} [(1 - \xi e^{-i\phi_2})(1 - \xi' e^{-i\phi_6})]^{-1}$$
$$\times \prod_{j=1}^{2} (1 - e^{-i(\phi_{2j} + \phi_{2j+2})})^{-1} \sin\phi_1 (\sin\phi_1 - \sin\phi_3)(\sin\phi_3 - \sin\phi_5)\ . \tag{4.53}$$

The general structure of $\overline{R}_{23}^{(2k+1)}(\xi, \xi')$ and $\overline{R}_{13}^{(2k+1)}(\xi, \xi')$ is now clear from (4.52) and (4.53), respectively. That is we have for $k \geq 1$

$$\overline{R}_{13}^{(2k+1)}(\xi, \xi') = -i\gamma_1^{2k+1}\, \overline{P}(\xi)\, \overline{Q}(\xi')(2\pi)^{-4k-2} \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} d\phi_1 \cdots d\phi_{4k+2} \prod_{j=1}^{2k+1} \frac{e^{-iM\phi_{2j-1} - iN\phi_{2j}}}{\Delta(\phi_{2j-1}, \phi_{2j})} [(1 - \xi e^{-i\phi_2})(1 - \xi' e^{-i\phi_{4k+2}})]^{-1}$$
$$\times \prod_{j=1}^{2k} (1 - e^{-i(\phi_{2j} + \phi_{2j+2})})^{-1} \sin\phi_1 \prod_{j=1}^{2k} (\sin\phi_{2j-1} - \sin\phi_{2j+1}) \tag{4.54}$$

and

$$\overline{R}_{23}^{(2k+1)}(\xi, \xi') = -(1 - z_2^2)\, \gamma_1^{2k}\, \overline{Q}(\xi)\, \overline{Q}(\xi')(2\pi)^{-4k-2} \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} d\phi_1 \cdots d\phi_{4k+2}$$

$$\times\prod_{j=1}^{2k+1}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}[(1-\xi e^{-i\phi_2})(1-\xi' e^{-i\phi_{4k+2}})]^{-1}\prod_{j=1}^{2k}(1-e^{-i(\phi_{2j}+\phi_{2j+2})})^{-1}\prod_{j=1}^{2k}(\sin\phi_{2j-1}-\sin\phi_{2j+1}) . \quad (4.55)$$

For $k=1$ (4.54) and (4.55) have been established. The method of proof for the general case is to assume (4.54) and (4.55) hold for $2k-1$ and show that the truth of (4.54) and (4.55) follows for $2k+1$. To do this we use the induction hypothesis in the recursion relations (4.36) and (4.37). Then we make use of (4.38)–(4.40) and identities (4.48) and (4.49) to simplify the resulting expression. The details of this argument is just repeating the $k=1$ case but this time carrying along the extra factor $(\sin\phi_5-\sin\phi_7)\cdots(\sin\phi_{4k-1}-\sin\phi_{4k+1})$. Thus we spare the reader and consider (4.54) and (4.55) established.

From (4.41) and (4.55) we have for $k\geq 1$

$$\bar{x}_{MN}^{(2k+1)}=\gamma_1^{2k}(2\pi)^{-4k-2}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4k+2}\prod_{j=1}^{2k+1}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\prod_{j=1}^{2k}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{1-e^{-i(\phi_{2j}+\phi_{2j+2})}}\right) , \quad (4.56a)$$

$$\bar{x}_{MN}^{(2k+1)}=(-i\gamma_1)^{2k}(2\pi)^{-4k-2}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4k+2}\prod_{j=1}^{2k+1}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\prod_{j=1}^{2k}\left(\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})\cos\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\times e^{i(\phi_{2j}+\phi_{2j+2})/2}\right) . \quad (4.56b)$$

The quantity $\bar{x}_{MN}^{(1)}$ is given by (4.50). Equation (4.56) provides a perturbation expansion for x'_{20} [see (4.41)] for arbitrary $T>T_c$ and arbitrary $M, N\geq 0$. As in Sec. IV, the condition $\mathrm{Im}\phi_j<0$, $j=1, 2, \ldots, 4k+2$ is understood in (4.56).

C. Quantity $\bar{f}_{MN}$

In this section we compute the quantity $\bar{f}_{MN}^{(2k)}$ [defined by (4.13)] for arbitrary $T>T_c$ and $M, N\geq 0$.

If we define

$$\bar{f}_{MN}^{(2k)}=\delta^T[(\bar{A}^{-1}\bar{B})^{2k}]_{33}(\bar{S}^T)^{-1}\delta , \quad (4.57)$$

then the perturbation expansion (4.17) becomes

$$\bar{f}_{MN}=1+\sum_{k=1}^{\infty}\bar{f}_{MN}^{(2k)} . \quad (4.58)$$

From (4.35) and (4.57) we have

$$\bar{f}_{MN}^{(2k)}=\bar{R}_{33}^{(2k)}(0,0) . \quad (4.59)$$

We can compute $\bar{R}_{33}^{(2k)}(\xi,\xi')$ and $\bar{R}_{43}^{(2k)}(\xi,\xi)$ by using for $k\geq 2$ the "barred" version of (3.46) and (3.47), respectively. For $k=1$ $\bar{R}_{33}^{(2)}(\xi,\xi')$ and $\bar{R}_{43}^{(2)}(\xi,\xi')$ are the "barred" version of (3.49) and (3.50), respectively. Thus, for instance,

$$\bar{R}_{33}^{(2)}(\xi,\xi')=-(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(2)\,d(4)\,\bar{S}^{-1}(\bar{2},\xi)\,\bar{P}_1(2,4)\,\bar{S}^{-1}(\xi',\bar{4}) , \quad (4.60)$$

$$\bar{R}_{33}^{(2)}(\xi,\xi')=-(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(2)\,d(4)\,\bar{P}(\xi)\,\bar{Q}(\xi')[(1-\xi e^{-i\phi_2})(1-e^{-i(\phi_2+\phi_4)})(1-\xi' e^{-i\phi_4})]^{-1}\times\{\bar{U}(2)\,\bar{U}(4)+\bar{V}(2)[\bar{Q}(\bar{2})]^2\,T(4)[\bar{P}(\bar{4})]^2\} , \quad (4.61)$$

where we used (4.27), (4.34), and (4.38) in going from (4.60) to (4.61). Using the definition (4.6) of $\bar{U}(2)$ and the identities (4.48) and (4.49), (4.61) becomes

$$\bar{R}_{33}^{(2)}(\xi,\xi')=-\gamma_1^2(2\pi)^{-4}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_4\prod_{j=1}^{2}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\bar{P}(\xi)\,\bar{Q}(\xi')\times[(1-\xi e^{-i\phi_2})(1-e^{-i(\phi_2+\phi_4)})(1-\xi' e^{-i\phi_4})]^{-1}\sin\phi_1(\sin\phi_1-\sin\phi_3) , \quad (4.62)$$

where $\mathrm{Im}\phi_j<0$, $j=1, 2, 3,$ and 4.

Likewise from the "barred" version of (3.50) we obtain, with the help of (4.27), (4.34), and (4.40),

$$\bar{R}_{43}^{(2)}(\xi,\xi')=(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(2)\,d(4)\,\bar{S}^{-1}(\xi,\bar{2})\,\bar{P}_3(2,4)\,\bar{S}^{-1}(\xi',\bar{4}) , \quad (4.63)$$

$$\bar{R}_{43}^{(2)}(\xi,\xi')=(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d(2)\,d(4)\,\bar{Q}(\xi)\,\bar{Q}(\xi')[(1-\xi e^{-i\phi_2})(1-e^{-i(\phi_2+\phi_4)})(1-\xi' e^{-i\phi_4})]^{-1}\times\{T(2)[\bar{P}(\bar{2})]^2\,\bar{U}(4)-\bar{U}(2)\,T(4)[\bar{P}(\bar{4})]^2\} . \quad (4.64)$$

Using (4.6), (4.48), and (4.49) in (4.64) we obtain

$$\bar{R}_{43}^{(2)}(\xi,\xi')=-i\gamma_1(1-z_2^2)(2\pi)^{-4}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_4\prod_{j=1}^{2}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\bar{Q}(\xi)\bar{Q}(\xi')$$
$$\times[(1-\xi e^{-i\phi_2})(1-e^{-i(\phi_2+\phi_4)})(1-\xi' e^{-i\phi_4})]^{-1}(\sin\phi_1-\sin\phi_3)\,, \tag{4.65}$$

where $\mathrm{Im}\phi_j<0$, $j=1$, 2, 3, and 4.

For arbitrary $n=1, 2, 3, \ldots$ we now establish that

$$\bar{R}_{33}^{(2n)}(\xi,\xi')=-\gamma_1^{2n}(2\pi)^{-4n}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4n}\prod_{j=1}^{2n}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\bar{P}(\xi)\bar{Q}(\xi')$$
$$\times[(1-\xi e^{-i\phi_2})(1-\xi' e^{-i\phi_{4n}})]^{-1}\prod_{j=1}^{2n-1}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{(1-e^{-i(\phi_{2j}+\phi_{2j+2})})}\right)\sin\phi_1 \tag{4.66}$$

and

$$\bar{R}_{43}^{(2n)}(\xi,\xi')=-i\gamma_1^{2n-1}(1-z_2^2)(2\pi)^{-4n}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4n}\prod_{j=1}^{2n}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\bar{Q}(\xi)\bar{Q}(\xi')$$
$$\times[(1-\xi e^{-i\phi_2})(1-\xi' e^{-i\phi_{4n}})]^{-1}\prod_{j=1}^{2n-1}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{(1-e^{-i(\phi_{2j}+\phi_{2j+2})})}\right)\,. \tag{4.67}$$

We will assume that (4.66) and (4.67) hold for $n=k-1$ and then show by use of the "barred" version of (3.46) and (3.47) that (4.66) and (4.67) hold for $n=k$. Since (4.62) and (4.65) establish the case $k=1$, we will then have shown (4.66) and (4.67) for arbitrary n.

Using (4.66) and (4.67) for $n=k-1$ in the barred version of (3.46) we obtain after using (4.38)–(4.40) and (4.34)

$$\bar{R}_{33}^{(2k)}(\xi,\xi')=(2\pi)^{-2}\int_{-\pi}^{\pi}d\phi_2\int_{-\pi}^{\pi}d\phi_4[(1-\xi e^{-i\phi_2})(1-e^{-i(\phi_2+\phi_4)})]^{-1}\bar{P}(\xi)\,(-\{\bar{U}(2)\bar{U}(4)\bar{Q}(\bar{4})$$
$$+\bar{V}(2)[\bar{Q}(\bar{2})]^2T(4)\bar{P}(\bar{4})\}\bar{R}_{33}^{(2k-2)}(\bar{4},\xi')+\{\bar{V}(2)[\bar{Q}(\bar{2})]^2\bar{U}(4)\bar{P}(\bar{4})+\bar{U}(2)\bar{V}(4)\bar{Q}(\bar{4})\}\bar{R}_{43}^{(2k-2)}(\bar{4},\bar{\xi}'))\,, \tag{4.68}$$

$$\bar{R}_{33}^{(2k)}(\xi,\xi')=(2\pi)^{-4k+2}\int_{-\pi}^{\pi}d\phi_2\int_{-\pi}^{\pi}d\phi_4\int_{-\pi}^{\pi}d\phi_5\cdots\int_{-\pi}^{\pi}d\phi_{4k}\prod_{j=3}^{2k}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\bar{P}(\xi)\bar{Q}(\xi')$$
$$\times[(1-\xi e^{-i\phi_2})(1-\xi' e^{-i\phi_{4k}})]^{-1}\prod_{j=1}^{2k-1}(1-e^{-i(\phi_{2j}+\phi_{2j+2})})^{-1}\prod_{j=3}^{2k-1}(\sin\phi_{2j-1}-\sin\phi_{2j+1})\,(-\{\bar{U}(2)\bar{U}(4)$$
$$+\bar{V}(2)[\bar{Q}(\bar{2})]^2T(4)[\bar{P}(\bar{4})]^2\}(-\gamma_1^{2k-2}\sin\phi_5)+\{-\bar{V}(2)[\bar{Q}(\bar{2})]^2\bar{U}(4)+\bar{U}(2)\bar{V}(4)[\bar{Q}(4)]^2\}[-i\gamma_1^{2k-3}(1-z_2^2)])\,. \tag{4.69}$$

Using (4.6), (4.48), and (4.49), (4.69) becomes just (4.66) with n replaced by k. The "barred" version of (3.47) establishes (4.67) in a similar manner.

From (4.59) and (4.66)

$$\bar{f}_{MN}^{(2k)}=-\gamma_1^{2k}(2\pi)^{-4k}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4k}\prod_{j=1}^{2k}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\prod_{j=1}^{2k-1}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{1-e^{-i(\phi_{2j}+\phi_{2j+2})}}\right)\sin\phi_1\,. \tag{4.70}$$

D. Quantity $\bar{F}_{MN}$

From (4.16), (4.17), (4.57), and (4.58) we have

$$\bar{F}_{MN}=\sum_{k=N}^{\infty}\ln\left(1+\sum_{n=1}^{\infty}\bar{f}_{Mk}^{(2n)}\right)\,, \tag{4.71}$$

where $\bar{f}_{Mk}^{(2n)}$ is given by (4.70). Equation (4.71) should be compared with (3.67), the analog below T_c. Defining [in analogy with (3.68)] $\bar{F}_{MN}^{(2n)}$

$$\bar{F}_{MN}=\sum_{n=1}^{\infty}\bar{F}_{MN}^{(2n)}\,, \tag{4.72}$$

where

$$\bar{F}_{MN}^{(2)}=\sum_{k=N}^{\infty}\bar{f}_{Mk}^{(2)}\,, \tag{4.73a}$$

$$\bar{F}_{MN}^{(4)}=\sum_{k=N}^{\infty}[\bar{f}_{Mk}^{(4)}-\tfrac{1}{2}(\bar{f}_{Mk}^{(2)})^2]\,, \tag{4.73b}$$

etc., we will show that

$$\bar{F}_{MN}^{(2n)}=(-1)^n\gamma_1^{2n}(2n)^{-1}(2\pi)^{-4n}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4n}$$
$$\times\prod_{j=1}^{2n}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}$$
$$\times e^{i\phi_{2j}}\cos\tfrac{1}{2}(\phi_{2j-1}+\phi_{2j+1})\,, \tag{4.74}$$

with $\phi_{4n+1}\equiv\phi_1$, $\phi_{4n+2}\equiv\phi_2$.

Equation (4.74) should be compared with (2.12).

1. $\bar{F}_{MN}^{(2)}$

Using (4.70) for $k=1$ in (4.73a) we have

$$\bar{F}_{MN}^{(2)} = -\gamma_1^2(2\pi)^{-4}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_4 \times\prod_{j=1}^{2}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\frac{\sin\phi_1-\sin\phi_3}{(1-e^{-i(\phi_2+\phi_4)})^2}\sin\phi_1 . \tag{4.75}$$

Letting $\phi_1\to\phi_3$ and $\phi_2\to\phi_4$ in (4.75), adding this to (4.75) and dividing by 2, gives

$$\bar{F}_{MN}^{(2)} = \tfrac{1}{2}\gamma_1^2(2\pi)^{-4}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_4 \times\prod_{j=1}^{2}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\,\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{(1-e^{-i(\phi_{2j}+\phi_{2j+2})})}\right), \tag{4.76}$$

with $\phi_5\equiv\phi_1$, $\phi_6\equiv\phi_2$.

We can rewrite (4.76) as

$$\bar{F}_{MN}^{(2)} = -\gamma_1^2\tfrac{1}{2}(2\pi)^{-4}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_4 \times\prod_{j=1}^{2}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\,\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})} \times e^{i\phi_{2j}}\cos\tfrac{1}{2}(\phi_{2j-1}+\phi_{2j+1})\right), \tag{4.77}$$

where we used the identity $\sin x-\sin y=\frac{1}{2}\sin\frac{1}{2}(x-y)\cos\frac{1}{2}(x+y)$.

This is just (4.74) for $n=1$.

2. $\bar{F}_{MN}^{(2n)}$ for arbitrary n

We define

$$\bar{F}_{MN}^{(2n)} = (-1)^n\gamma_1^{2n}(2n)^{-1}(2\pi)^{-4n}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4n} \times\prod_{j=1}^{2n}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\,\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})} \times e^{i\phi_{2j}}\cos\tfrac{1}{2}(\phi_{2j-1}+\phi_{2j+1})\right)(1-e^{-i(\phi_2+\phi_4+\cdots+\phi_{4n})}) \tag{4.78}$$

$$\bar{f}(x)=\sum_{n=0}^{\infty}\bar{f}_{MN}^{(2n)}x^{2n}, \tag{4.79}$$

and

$$\bar{F}(x)=\sum_{n=0}^{\infty}\bar{F}_{MN}^{(2n)}x^{2n}, \tag{4.80}$$

with $\bar{f}_{MN}^{(0)}=1$, $\bar{F}_{MN}^{(0)}=0$. Then as in Sec. III F we see that (4.74) will be true if we can prove that

$$n\bar{f}_{MN}^{(2n)}=\sum_{l=1}^{n}l\bar{F}_{MN}^{(2l)}\bar{f}_{MN}^{2(n-l)} . \tag{4.81}$$

In (4.70), if we let $\phi_1\to\phi_{4n-1}$, $\phi_2\to\phi_{4n}$, $\phi_3\to\phi_{4n-3}$, $\cdots$, add this to (4.70), and divide by two, we obtain

$$\bar{f}_{MN}^{(2k)} = 2^{-1}\gamma_1^{2k}(2\pi)^{-4k}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4k}\prod_{j=1}^{2k}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}(\sin\phi_{2j-1}-\sin\phi_{2j+1})\right)\prod_{j=1}^{2k-1}(1-e^{-i(\phi_{2j}+\phi_{2j+2})})^{-1} . \tag{4.82}$$

Furthermore, (4.82) can be written as

$$\bar{f}_{MN}^{(2k)} = \gamma_1^{2k}(2\pi)^{-4k}(-1)^k\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4k}\prod_{j=1}^{2k}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\sin\tfrac{1}{2}(\phi_{2j-1}-\phi_{2j+1})\cos\tfrac{1}{2}(\phi_{2j-1}+\phi_{2j+1})e^{i\phi_{2j}}\right) \times\prod_{j=1}^{2k-1}\left(\frac{1}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)ie^{-(\phi_2+\phi_{4k})/2} . \tag{4.83}$$

Upon comparing (4.78) with (3.84) and (4.83) with (3.99) we see that the proof of (4.81) is the same as (3.93). Hence (4.74) is proved.

Thus we have shown that for $T>T_c$ and $M,N\geq 0$

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle = S_\infty^{>}\sum_{k=1}^{\infty}\bar{x}_{MN}^{(2k-1)}\exp\left(-\sum_{n=1}^{\infty}\bar{F}_{MN}^{(2n)}\right), \tag{4.84}$$

where $\bar{x}_{MN}^{(2k+1)}$ is given by (4.56) for $k\geq 1$, $\bar{x}_{MN}^{(1)}$ by (4.50), $\bar{F}_{MN}^{(2n)}$ by (4.74), and $S_\infty^{>}$ by (4.19) and (4.20).

E. $x_{MN}^{(k)}$ and $F_{>MN}^{(2n)}$

In (2.12) we may make use of the identity, valid in the integrand of (2.12),

$$\gamma_1\sin\tfrac{1}{2}(\phi_{2j-1}-\phi_{2j+1})\sin\tfrac{1}{2}(\phi_{2j-1}+\phi_{2j+1}) + \gamma_2\sin\tfrac{1}{2}(\phi_{2j}-\phi_{2j+2})\sin\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})=0 \tag{4.85}$$

[see discussion following (3.54)] to rewrite (2.12)

$$F_{<MN}^{(2n)} = [4z_1z_2(1-z_1^2)(1-z_2^2)]^n(2n)^{-1}(2\pi)^{-4n} \times\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4n}\prod_{j=1}^{2n}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right) \times\left(\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})\sin\frac{1}{2}(\phi_{2j}-\phi_{2j+2})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})\sin\frac{1}{2}(\phi_{2j-1}+\phi_{2j+1})}\right)^{1/2} . \tag{4.86}$$

From (4.86) it is clear that $F^{(2n)}_{<MN}$ is invariant under the interchange $E_1 \leftrightarrow E_2$ and $M \leftrightarrow N$. And hence from (2.9)–(2.12), it is clear that $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for $T<T_c$ is invariant under the interchange $E_1 \leftrightarrow E_2$ and $M \leftrightarrow N$, as it must be.

For $T>T_c$, $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ must still be invariant under $E_1 \leftrightarrow E_2$ and $M \leftrightarrow N$. However, an examination of $\overline{x}^{(2k+1)}_{MN}$ [see (4.56b)] and $\overline{F}^{(2n)}_{MN}$ [see (4.74)] that occur in (4.84) shows that $\overline{x}^{(2k+1)}_{MN}$ for $k \geq 1$ and $\overline{F}^{(2n)}_{MN}$, $n \geq 1$, are not *individually* invariant under $E_1 \leftrightarrow E_2$ and $M \leftrightarrow N$. Note that $\overline{x}^{(1)}_{MN}$ is invariant under $E_1 \leftrightarrow E_2$, $M \leftrightarrow N$. This suggests that (4.84) is not the best representation for $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ $(T>T_c)$. In this section we show that

$$\sum_{k=1}^{\infty}\overline{x}^{(2k-1)}_{MN}\exp\left(-\sum_{n=1}^{\infty}\overline{F}^{(2n)}_{MN}\right) = \sum_{k=1}^{\infty}x^{(2k-1)}_{>MN}\exp\left(-\sum_{n=1}^{\infty}F^{(2n)}_{>MN}\right), \tag{4.87}$$

where $x^{(2k-1)}_{>MN}$ is given by (2.15) and $F^{(2n)}_{>MN}$ by (2.16). Using the identity (4.85) it is easy to show that $x^{(2n)}_{>MN}$ and $F^{(2n)}_{>MN}$ are individually invariant under $E_1 \leftrightarrow E_2$ and $M \leftrightarrow N$. Also note that

$$\overline{x}^{(1)}_{MN} = x^{(1)}_{>MN}. \tag{4.88}$$

1. *Special case*

If one examines the expression (4.56b) for $\overline{x}^{(2k+1)}_{MN}$, then one notices that if $e^{i(\phi_{2j}+\phi_{2j+2})/2}$ were replaced $\cos\frac{1}{2}(\phi_{2j}+\phi_{2j+2})$, then the resulting expression would be invariant under $E_1 \leftrightarrow E_2$, $M \leftrightarrow N$. A similar remark holds for $\overline{F}^{(2n)}_{MN}$. Hence we guess that

$$\overline{x}^{(3)}_{MN} - \overline{x}^{(1)}_{MN}\overline{F}^{(2)}_{MN} = x^{(3)}_{>MN} - x^{(1)}_{>MN}F^{(2)}_{>MN}, \tag{4.89}$$

where $x^{(3)}_{>MN}$ is defined by the $k=3$ case of (2.15), and $F^{(2)}_{>MN}$ the $n=1$ case of (2.16).

Thus we have

$$\overline{F}^{(2)}_{MN} - F^{(2)}_{>MN} = +\tfrac{1}{2}\gamma_1^2(2\pi)^{-4}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_4\prod_{j=1}^{2}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\left(\frac{\sin\frac{1}{2}(\phi_1-\phi_3)}{\sin\frac{1}{2}(\phi_2+\phi_4)}\cos\tfrac{1}{2}(\phi_1+\phi_3)\right)^2 \times\left[e^{i(\phi_2+\phi_4)} - \cos^2\tfrac{1}{2}(\phi_2+\phi_4)\right]. \tag{4.90}$$

Now

$$e^{i(\phi_2+\phi_4)} - \cos^2\tfrac{1}{2}(\phi_2+\phi_4) = i\sin\tfrac{1}{2}(\phi_2+\phi_4)\left[e^{i(\phi_2+\phi_4)/2}+\cos\tfrac{1}{2}(\phi_2+\phi_4)\right], \tag{4.91}$$

so that

$$\overline{F}^{(2)}_{MN} - F^{(2)}_{>MN} = \tfrac{1}{2}\gamma_1^2(2\pi)^{-4}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_4\prod_{j=1}^{2}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\, i\sin^2\tfrac{1}{2}(\phi_1-\phi_3)\cos^2\tfrac{1}{2}(\phi_1+\phi_3)\,G(\phi_2,\phi_4), \tag{4.92}$$

where

$$G(\phi_2,\phi_4) = \left[e^{i(\phi_2+\phi_4)/2}+\cos\tfrac{1}{2}(\phi_2+\phi_4)\right]/\sin\tfrac{1}{2}(\phi_2+\phi_4). \tag{4.93}$$

Now from (4.56b) and (2.15)

$$\overline{x}^{(3)}_{MN} - x^{(3)}_{>MN} = -\gamma_1^2(2\pi)^{-6}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_6\prod_{j=1}^{3}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\prod_{j=1}^{2}\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})\cos\frac{1}{2}(\phi_{2j-1}+\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})} \times\left[e^{i(\phi_2/2+\phi_4+\phi_6/2)} - \cos\tfrac{1}{2}(\phi_2+\phi_4)\cos\tfrac{1}{2}(\phi_4+\phi_6)\right]. \tag{4.94}$$

Writing

$$e^{i(\phi_2/2+\phi_4+\phi_6/2)} - \cos\tfrac{1}{2}(\phi_2+\phi_4)\cos\tfrac{1}{2}(\phi_4+\phi_6) = i\left[e^{i(\phi_2+\phi_4)/2}\sin\tfrac{1}{2}(\phi_4+\phi_6)+\sin\tfrac{1}{2}(\phi_2+\phi_4)\cos\tfrac{1}{2}(\phi_4+\phi_6)\right] \tag{4.95}$$

in (4.94), changing the integration variable labels by $1 \leftrightarrow 5$ and $2 \leftrightarrow 6$, adding this to (4.94), and dividing by two gives

$$\overline{x}^{(3)}_{MN} - x^{(3)}_{>MN} = -\tfrac{1}{2}i\gamma_1^2(2\pi)^{-6}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_6\prod_{j=1}^{3}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\prod_{j=1}^{2}\left[\sin\tfrac{1}{2}(\phi_{2j-1}-\phi_{2j+1})\cos\tfrac{1}{2}(\phi_{2j-1}+\phi_{2j+1})\right] \times\left[G(\phi_4,\phi_6)+G(\phi_2,\phi_4)\right], \tag{4.96}$$

where we used the definition (4.93). We can rewrite (4.96)

$$\overline{x}^{(3)}_{MN} - x^{(3)}_{>MN} = -\tfrac{1}{8}i\gamma_1^2(2\pi)^{-6}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_6\prod_{j=1}^{3}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)(\sin\phi_1-\sin\phi_3)(\sin\phi_3-\sin\phi_5) \times\left[G(\phi_4,\phi_6)+G(\phi_2,\phi_4)\right]. \tag{4.97}$$

Now consider the three cyclic permutations of the integration variable labels in (4.97). That part of the integrand of (4.97) that is not invariant becomes

$$(\sin\phi_1-\sin\phi_3)(\sin\phi_3-\sin\phi_5)[G(\phi_2,\phi_4)+G(\phi_4,\phi_6)]+(\sin\phi_3-\sin\phi_5)(\sin\phi_5-\sin\phi_1)[G(\phi_4,\phi_6)+G(\phi_6,\phi_2)]$$
$$+(\sin\phi_5-\sin\phi_1)(\sin\phi_1-\sin\phi_3)[G(\phi_6,\phi_2)+G(\phi_2,\phi_4)]=-G(\phi_2,\phi_4)(\sin\phi_1-\sin\phi_3)^2$$
$$-G(\phi_4,\phi_6)(\sin\phi_3-\sin\phi_5)^2-G(\phi_6,\phi_2)(\sin\phi_5-\sin\phi_1)^2\,. \tag{4.98}$$

Hence (4.97) becomes

$$\bar{x}^{(3)}_{MN}-x^{(3)}_{>MN}=\tfrac{1}{8}i\gamma_1(2\pi)^{-6}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_6\prod_{j=1}^{3}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)(\sin\phi_1-\sin\phi_3)^2G(\phi_2,\phi_4)\,. \tag{4.99}$$

Comparing (4.99) with (4.92), and recalling (4.50) and (4.88), we see that (4.89) follows.

2. The general case

Equations (4.88) and (4.89) have established (4.87) to second order. Here we show that (4.87) holds to every order.

We define

$$x(z)=\sum_{k=0}^{\infty}x^{(k)}z^k,\quad \bar{x}(z)=\sum_{k=0}^{\infty}\bar{x}^{(k)}z^k\,, \tag{4.100a}$$

where

$$x^{(k)}=\begin{cases}x^{(k)}_{>MN} & \text{for } k \text{ odd}\\ 0 & \text{for } k \text{ even}\end{cases} \tag{4.100b}$$

and

$$\bar{x}^{(k)}=\begin{cases}\bar{x}^{(k)}_{MN} & \text{for } k \text{ odd}\\ 0 & \text{for } k \text{ even.}\end{cases} \tag{4.100c}$$

Also we define

$$F(z)=\sum_{n=0}^{\infty}F^{(n)}z^n,\quad \bar{F}(z)=\sum_{n=0}^{\infty}\bar{F}^{(n)}z^n\,, \tag{4.101a}$$

where

$$F^{(n)}=\begin{cases}F^{(n)}_{>MN} & \text{for } n \text{ even}\\ 0 & \text{for } n \text{ odd}\end{cases} \tag{4.101b}$$

and

$$\bar{F}^{(n)}=\begin{cases}\bar{F}^{(n)}_{MN} & \text{for } n \text{ even}\\ 0 & \text{for } n \text{ odd,}\end{cases} \tag{4.101c}$$

and finally,

$$G(z)\equiv\bar{F}(z)-F(z)\,, \tag{4.102a}$$

$$G^{(n)}\equiv\bar{F}^{(n)}-F^{(n)} \tag{4.102b}$$

[where this $G(z)$ is not to be confused with that of (2.29)].

Then (4.87) is the $z=1$ case of

$$\bar{x}(z)\exp[-\bar{F}(z)]=x(z)\exp[-F(z)] \tag{4.103}$$

or, equivalently,

$$x(z)=\bar{x}(z)\exp[-G(z)]\,. \tag{4.104}$$

Differentiating (4.104) and multiplying the resulting equation by $\bar{x}(z)$ yields

$$x'(z)\bar{x}(z)=\bar{x}'(z)x(z)-\bar{x}(z)x(z)G'(z)\,. \tag{4.105}$$

Suppose that we can prove (4.105), then (4.87) must follow since (4.105) can be written

$$\frac{x'(z)}{x(z)}=\frac{\bar{x}'(z)}{\bar{x}(z)}-G'(z)\,, \tag{4.106}$$

which upon integrating gives

$$\ln x(z)=\ln\bar{x}(z)-G(z)+(\text{const}) \tag{4.107}$$

or

$$x(z)=[\bar{x}(z)e^{-G(z)}](\text{const})\,. \tag{4.108}$$

Now let z approach zero in (4.108).

For $z\to 0$ we have

$$x(z)\sim zx^{(1)}\,,$$
$$\bar{x}(z)\sim z\bar{x}^{(1)}\,,$$
$$G(z)\sim z^2G^{(2)}\,, \tag{4.109}$$

so that in the limit $z\to 0$ (4.108) becomes

$$x^{(1)}=\bar{x}^{(1)}(\text{const})\,. \tag{4.110}$$

Recalling (4.88) we see that the constant in (4.110) must be equal to one; and hence, from (4.105) follows (4.104).

From (4.100)

$$x'(z)\bar{x}(z)-\bar{x}'(z)z(z)=\sum_{L=0}^{\infty}z^L\sum_{k=0}^{L}[(k+1)x^{(k+1)}\bar{x}^{(L-k)}-(L-k+1)x^{(k)}\bar{x}^{(L-k+1)}]$$
$$=\sum_{L=0}^{\infty}z^L\left((L+1)x^{(L+1)}\bar{x}^{(0)}-(L+1)x^{(0)}\bar{x}^{(L+1)}+\sum_{k=0}^{L-1}(2k-L-1)x^{(k)}\bar{x}^{(L-k+1)}\right)$$
$$=\sum_{L=0}^{\infty}z^L\sum_{k=0}^{L-1}(2k-L-1)x^{(k)}\bar{x}^{(L-k+1)}\,. \tag{4.111}$$

From (4.100)–(4.102)

$$x(z)\bar{x}(z)G'(z)=\sum_{k=0}^{\infty}x^{(k)}z^{k}\sum_{L=0}^{\infty}\bar{x}^{(L)}z^{L}\sum_{m=0}^{\infty}G^{(m+1)}(m+1)z^{m}$$
$$=\sum_{k=0}^{\infty}x^{(k)}z^{k}\sum_{L=0}^{\infty}z^{L}\sum_{m=0}^{L}\bar{x}^{(m)}G^{(L-m+1)}(L-m+1)$$
$$=\sum_{L=0}^{\infty}z^{L}\sum_{k=0}^{L}x^{(k)}\sum_{m=0}^{L-k}\bar{x}^{(m)}G^{(L-m-k+1)}(L-m-k+1)\ . \tag{4.112}$$

Thus from (4.105), (4.111), and (4.112)

$$\sum_{k=1}^{L}(2k-L-1)x^{(k)}\bar{x}^{(L-k+1)}$$
$$=-\sum_{k=0}^{L}x^{(k)}\sum_{m=0}^{L-k}(L-k-m+1)G^{(L-m-k+1)}\bar{x}^{(m)}$$
$$=-\sum_{k=1}^{L-1}x^{(k)}\sum_{m=1}^{L-k}\bar{x}^{(m)}G^{(L-m-k+1)}(L-m-k+1)\ , \tag{4.113}$$

where we used $\bar{x}^{(0)}=x^{(0)}=0$. If we establish (4.113), the identity (4.87) follows.

We define for all positive integers l

$$x^{(l)}=(-\tfrac{1}{2}i\lambda_1)^{l-1}(2\pi)^{-2l}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{2l}\prod_{j=1}^{l}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\prod_{j=1}^{l-1}\left((\sin\phi_{2j-1}-\sin\phi_{2j+1})\frac{\cos\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right), \tag{4.114}$$

$$\bar{x}^{(l)}=(-\tfrac{1}{2}i\gamma)^{l-1}(2\pi)^{-2l}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{2l}\prod_{j=1}^{l}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\prod_{j=1}^{l-1}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}e^{i(\phi_{2j}+\phi_{2j+2})/2}\right), \tag{4.115}$$

and

$$G^{(l)}=\bar{F}^{(l)}-F^{(l)}=(-\tfrac{1}{2}i\gamma_1)^{l-1}l^{-1}(2\pi)^{-2l}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{2l}\prod_{j=1}^{l}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\frac{(\sin\phi_{2j-1}-\sin\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)$$
$$\times\left(\prod_{j=1}^{l}e^{i(\phi_{2j}+\phi_{2j+2})/2}-\prod_{j=1}^{l}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\right)\ . \tag{4.116}$$

If we establish (4.113) with the definitions (4.114)–(4.116) of $x^{(l)}$, $\bar{x}^{(l)}$, and $G^{(l)}$ [as compared with (4.100)–(4.102)], then (4.113) will also follow for (4.100)–(4.102).

Before we analyze the sum (4.113), it is useful to cast $G^{(l)}$ into a different form. As $G^{(l)}$ is represented in (4.116) the factor $\sin\frac{1}{2}(\phi_{2l}+\phi_2)$ appears in the denominator. This factor does not appear in $x^{(l)}$ or $\bar{x}^{(l)}$; and so, it is advantageous to eliminate this term in $G^{(l)}$. To do this we begin by noting

$$e^{ix/2}-\cos\tfrac{1}{2}x=i\sin\tfrac{1}{2}x \tag{4.117}$$

to show that

$$\prod_{j=1}^{l}e^{i(\phi_{2j}+\phi_{2j+2})/2}-\prod_{j=1}^{l}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})=i\sin\tfrac{1}{2}(\phi_2+\phi_4)\prod_{j=2}^{l}e^{i(\phi_{2j}+\phi_{2j+2})/2}+i\cos\tfrac{1}{2}(\phi_2+\phi_4)\sin\tfrac{1}{2}(\phi_4+\phi_6)$$
$$\times\prod_{j=3}^{l}e^{i(\phi_{2j}+\phi_{2j+2})/2}+i\cos\tfrac{1}{2}(\phi_2+\phi_4)\cos\tfrac{1}{2}(\phi_4+\phi_6)\sin\tfrac{1}{2}(\phi_6+\phi_8)\prod_{j=4}^{l}e^{i(\phi_{2j}+\phi_{2j+2})/2}$$
$$+\cdots+i\cos\tfrac{1}{2}(\phi_2+\phi_4)\cos\tfrac{1}{2}(\phi_4+\phi_6)\cdots\cos\tfrac{1}{2}(\phi_{2l-2}+\phi_{2l})\sin\tfrac{1}{2}(\phi_{2l}+\phi_2)\ . \tag{4.118}$$

The first product term in (4.116) is invariant under cyclic permutations of the even and odd integration variable labels ($\phi_1\to\phi_3\to\cdots\to\phi_{2l-1}\to\phi_1$ and $\phi_2\to\phi_4\to\cdots\to\phi_{2l}\to\phi_2$). Thus we may cyclically permute the labels of the right-hand side of (4.118), and (4.118) will continue to remain an identity if used in the intergrand of (4.116). So we cyclically permute each term in (4.118) so that each term contains a factor of $\sin\frac{1}{2}(\phi_{2l}+\phi_2)$. In this manner we obtain

$$G^{(l)}=i(-\tfrac{1}{2}i\gamma_1)^{l}l^{-1}(2\pi)^{-2l}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{2l}\prod_{j=1}^{l}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}(\sin\phi_{2j-1}-\sin\phi_{2j+1})$$
$$\times\prod_{j=1}^{l-1}\frac{1}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\left(\sum_{k=1}^{l}\prod_{j=k}^{l-1}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\prod_{j=1}^{k-1}e^{i(\phi_{2j}+\phi_{2j+2})/2}\right), \tag{4.119}$$

where the products

$$\prod_{j=k}^{l-1}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\quad\text{and}\quad\prod_{j=1}^{k-1}e^{i(\phi_{2j}+\phi_{2j+2})/2}$$

are to be interpreted as unity for $j=k>l-1$ and $j=1>k-1$, respectively.

We examine the structure of the product

$$\bar{x}^{(l)}G^{(m)}x^{(n)} , \tag{4.120}$$

where we denote the integration variables associated with $\bar{x}^{(l)}$, $G^{(m)}$, $x^{(n)}$ by ϕ, ϕ', and ϕ'', respectively. The term $(\sin\phi'_{2m-1} - \sin\phi'_1)$ occurring in (4.119) we write as follows:

$$\sin\phi'_{2m-1} - \sin\phi'_1 = (\sin\phi''_{2n-1} - \sin\phi'_1) + (\sin\phi'_{2m-1} - \sin\phi_1) + (\sin\phi_1 - \sin\phi''_{2n-1}) . \tag{4.121}$$

We now define

$$\tilde{G}^{(m)} = i(-\tfrac{1}{2}i\gamma_1)^m m^{-1}(2\pi)^{-2m}\int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{2m}\prod_{j=1}^{m}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)$$
$$\times\prod_{j=1}^{m-1}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)\left(\sum_{k=1}^{m}\prod_{j=k}^{m-1}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\prod_{j=1}^{k-1}e^{i(\phi_{2j}+\phi_{2j+2})/2}\right), \tag{4.122}$$

where we follow the same convention concerning the products [see discussion following (4.119)]. Now if we use (4.121) in (4.119), then the single term (4.120) becomes the sum of three terms; and the factor $(\sin\phi''_{2n-1} - \sin\phi'_1)$ naturally combines $\tilde{G}^{(m)}$ and $x^{(n)}$, the factor $(\sin\phi'_{2m-1} - \sin\phi_1)$ combines $\tilde{G}^{(m)}$ and $\bar{x}^{(l)}$; and the factor $(\sin\phi_1 - \sin\phi''_{2n-1})$ combines $x^{(n)}$ and $\bar{x}^{(l)}$.

Thus we have

$$m\bar{x}^{(l)}G^{(m)}x^{(n)} = S^{(1)}_{lmn} + S^{(2)}_{lmn} + S^{(3)}_{lmn} , \tag{4.123}$$

where

$$S^{(1)}_{lmn} = \bar{x}^{(l)} i(-\tfrac{1}{2}i\gamma_1)^{m+n-1}(2\pi)^{-2m-2n}\int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{2(m+n)}\prod_{j=1}^{m+n}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\prod_{j=1}^{m+n-1}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)$$
$$\times\prod_{j=1}^{n-1}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\sin\tfrac{1}{2}(\phi_{2n}+\phi_{2n+2})\sum_{k=n+1}^{n+m}\prod_{j=k}^{n+m-1}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\prod_{j=n+1}^{k-1}e^{i(\phi_{2j}+\phi_{2j+2})/2} , \tag{4.124}$$

$$S^{(2)}_{lmn} = x^{(n)} i(-\tfrac{1}{2}i\gamma_1)^{l+m-1}(2\pi)^{-2l-2m}\int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{2(l+m)}\prod_{j=1}^{l+n}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\prod_{j=1}^{l+m-1}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)$$
$$\times\left(\sum_{k=1}^{m}\prod_{j=k}^{m-1}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\prod_{j=1}^{k-1}e^{i(\phi_{2j}+\phi_{2j+2})/2}\right)\sin\tfrac{1}{2}(\phi_{2m}+\phi_{2m+2})\prod_{j=m+1}^{m+l-1}e^{i(\phi_{2j}+\phi_{2j+2})/2} , \tag{4.125}$$

and

$$S^{(3)}_{lmn} = -m\tilde{G}^{(m)}(-\tfrac{1}{2}i\gamma_1)^{l+n-2}(2\pi)^{-2(l+n)}\int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{2(l+n)}\prod_{j=1}^{l+n}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\prod_{j=1}^{l+n-1}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)$$
$$\times\left(\prod_{j=1}^{l-1}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\right)\sin\tfrac{1}{2}(\phi_{2n}+\phi_{2n+2})\prod_{j=l+1}^{l+n-1}e^{i(\phi_{2j}+\phi_{2j+2})/2} . \tag{4.126}$$

Letting

$$m = L - l - n + 1 , \tag{4.127}$$

we have from (4.123)–(4.126)

$$\sum_{l=1}^{L-1}\sum_{n=1}^{L-l}\bar{x}^{(l)}(L-l-n+1)G^{(L-l-n+1)}x^{(n)} = S_1 + S_2 + S_3 , \tag{4.128}$$

where

$$S_j = \sum_{l=1}^{L-1}\sum_{n=1}^{L-l}S^{(j)}_{lmn} , \quad j = 1,\ 2,\ \text{and } 3 , \tag{4.129}$$

and m is given by (4.127).

In computing the sum S_1 we need the following result:

$$\sum_{n=1}^{L-l}\left(\prod_{j=1}^{n-1}A_j\right)(B_n - A_n)\sum_{k=n+1}^{L-l+1}\prod_{j=k}^{L-l}A_j\prod_{j=n+1}^{k-1}B_j = \sum_{n=1}^{L-l}\left(\prod_{j=1}^{n}B_j\prod_{j=n+1}^{L-l}A_j\right) - (L-l)\prod_{j=1}^{L-l}A_j , \tag{4.130}$$

where $\{A_j\}$ and $\{B_j\}$ denote sequences of real numbers, and where we follow our product convention [see discussion following (4.119)]. To prove (4.130) we write the left-hand side of (4.130), which we denote by I, as follows:

$$I=B_1\sum_{k=2}^{L-l+1}\prod_{j=k}^{L-l}A_j\prod_{j=2}^{k-1}B_j+\sum_{n=2}^{L-l}\left(\prod_{j=1}^{n-1}A_j\right)B_n\sum_{k=n+1}^{L-l+1}\prod_{j=k}^{L-l}A_j\prod_{j=n+1}^{k-1}B_j-\sum_{n=1}^{L-l}\prod_{j=1}^{n}A_j\sum_{k=n+1}^{L-l+1}\prod_{j=k}^{L-l}A_j\prod_{j=n+1}^{k-1}B_j$$

$$=\sum_{k=2}^{L-l+1}\prod_{j=1}^{k-1}B_j\prod_{j=k}^{L-l}A_j-\prod_{j=1}^{L-l}A_j+\sum_{n=2}^{L-l}\left(\prod_{j=1}^{n-1}A_j\right)B_n\sum_{k=n+1}^{L-l+1}\prod_{j=k}^{L-l}A_j\prod_{j=n+1}^{k-1}B_j-\sum_{n=1}^{L-l-1}\left(\prod_{j=1}^{n}A_j\right)\sum_{k=n+1}^{L-l+1}\left(\prod_{j=k}^{L-l}A_j\right)\prod_{j=n+1}^{k-1}B_j\,. \quad (4.131)$$

The last term in (4.131) can be written as

$$\sum_{n=1}^{L-l-1}\left(\prod_{j=1}^{n}A_j\right)\sum_{k=n+1}^{L-l+1}\left(\prod_{j=k}^{L-l}A_j\right)\prod_{j=n+1}^{k-1}B_j=\sum_{n=1}^{L-l-1}\prod_{j=1}^{L-l}A_j+\sum_{n=1}^{L-l-1}\left(\prod_{j=1}^{n}A_j\right)\sum_{k=n+2}^{L-l+1}\left(\prod_{j=k}^{L-l}A_j\right)\prod_{j=n+1}^{k-1}B_j$$

$$=(L-l-1)\prod_{j=1}^{L-l}A_j+\sum_{n=1}^{L-l+1}\left(\prod_{j=1}^{n}A_j\right)\sum_{k=n+2}^{L-l}\left(\prod_{j=k}^{L-l}A_j\right)\prod_{j=n+1}^{k-1}B_j\,,$$

so that

$$I=\sum_{k=2}^{L-l+1}\prod_{j=1}^{k-1}B_j\prod_{j=k}^{L-l}A_j-(L-l)\prod_{j=1}^{L-l}A_j+\sum_{n=2}^{L-l}\left(\prod_{j=1}^{n-1}A_j\right)B_n\sum_{k=n+1}^{L-l+1}\prod_{j=k}^{L-l}A_j\prod_{j=n+1}^{k-1}B_j-\sum_{n=1}^{L-l-1}\prod_{j=1}^{n}A_j\sum_{k=n+2}^{L-l+1}\prod_{j=k}^{L-l}A_j\prod_{j=n+1}^{k-1}B_j. \quad (4.132)$$

That last two terms in (4.132) cancel [let $n\to n-1$ in the third term of (4.132)], and we thus see that (4.130) follows.

Using (4.117) on the $\sin\frac{1}{2}(\phi_{2n}+\phi_{2n+2})$ term in (4.124) we see that (4.129) for $j=1$ becomes upon using (4.130) [$A_j\to\cos\frac{1}{2}(\phi_{2j}+\phi_{2j+2})$ and $B_j\to e^{i(\phi_{2j}+\phi_{2j+2})/2}$]

$$S_1=\sum_{l=1}^{L-1}\bar{x}^{(l)}(-\tfrac{1}{2}i\gamma_1)^{L-l}(2\pi)^{-2(L-l+1)}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{2(L-l+1)}\prod_{j=1}^{L-l+1}\left(\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\right)\prod_{j=1}^{L-l}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)$$

$$\times\left[\sum_{n=1}^{L-l}\left(\prod_{j=1}^{n}e^{i(\phi_{2j}+\phi_{2j+2})/2}\prod_{j=n+1}^{L-l}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\right)-(L-l)\prod_{j=1}^{L-l}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\right]. \quad (4.133)$$

Using the definitions (4.115) and (4.119) we see that S_1 can be written as

$$S_1=\sum_{l=1}^{L-1}\left[\bar{x}^{(l)}(L-l+1)(\tfrac{1}{2}\gamma_1)^{-1}\tilde{G}^{(L-l+1)}-\bar{x}^{(l)}x^{(L-l+1)}-(L-l)\bar{x}^{(l)}x^{(L-l+1)}\right]$$

$$=\sum_{l=1}^{L-1}\left[-(L-l+1)\bar{x}^{(l)}x^{(L-l+1)}+(L-l+1)(\tfrac{1}{2}\gamma_1)^{-1}\bar{x}^{(l)}\tilde{G}^{(L-l+1)}\right] \quad (4.134)$$

To compute S_2 we need the identity

$$\sum_{l=1}^{L-n}\left(\prod_{j=L-n+2}^{L-n}B_j\right)(B_{L-l-n+1}-A_{L-l-n+1})\sum_{k=1}^{L-l-n+1}\left[\prod_{j=k}^{L-l-n}A_j\prod_{j=1}^{k-1}B_j\right]=\sum_{l=1}^{L-n}\left(\prod_{j=l+1}^{L-n}B_j\right)(B_l-A_l)\sum_{k=1}^{l}\left[\prod_{j=k}^{l-1}A_j\prod_{j=1}^{k-1}B_j\right] \quad (4.135a)$$

$$=(L-n)\prod_{j=1}^{L-n}B_j-\sum_{k=1}^{L-1}\prod_{j=k}^{L-n}A_j\prod_{j=1}^{k-1}B_j\,. \quad (4.135b)$$

The first step in (4.135) is just letting $l'=L-l-n+1$, and the proof from (4.135a) to (4.135b) is similar to (4.130).

To compute S_2 we first interchange sums

$$\sum_{l=1}^{L-1}\sum_{n=1}^{L-l}=\sum_{n=1}^{L-1}\sum_{l=1}^{L-n}\,, \quad (4.136)$$

use the identity (4.117) on the term $\sin\frac{1}{2}(\phi_{2(L-l-n+1)}+\phi_{2(L-l-n+1)+2})$ [recall (4.127)], and thus from (4.135) we obtain

$$S_2=\sum_{n=1}^{L-1}x^{(n)}(-\tfrac{1}{2}i\gamma_1)^{L-n}(2\pi)^{-2(L-n+1)}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{2(L-n+1)}\prod_{j=1}^{L-n+1}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\prod_{j=1}^{L-n}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)$$

$$\times\left((L-n)\prod_{j=1}^{L-n}e^{i(\phi_{2j}+\phi_{2j+2})/2}-\sum_{k=1}^{L-n}\prod_{j=k}^{L-n}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\prod_{j=1}^{k-1}e^{i(\phi_{2j}+\phi_{2j+2})/2}\right)$$

$$= \sum_{n=1}^{L-1} [x^{(n)}(L-n)\bar{x}^{(L-n+1)} - x^{(n)}(\tfrac{1}{2}\gamma_1)^{-1}(L-n+1)\tilde{G}^{(L-n+1)} + x^{(n)}\bar{x}^{(L-n+1)}]$$

$$= \sum_{n=1}^{L-1} [(L-n+1)x^{(n)}\bar{x}^{(L-n+1)} - x^{(n)}(\tfrac{1}{2}\gamma_1)^{-1}(L-n+1)\tilde{G}^{(L-n+1)}]. \qquad (4.137)$$

Letting $s = l+n$ the sum in (4.129) can be written as

$$S_2 = \sum_{s=2}^{L} \sum_{n=1}^{s-1} S^{(3)}_{lmn},$$

so that from (4.126) [and using (4.117)]

$$S_3 = \sum_{s=2}^{L}(L-s+1)i\tilde{G}^{(L-s+1)}(-\tfrac{1}{2}i\gamma_1)^{s-2}(2\pi)^{-2s}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{2s}\prod_{j=1}^{s}\frac{e^{-iM\phi_{2j-1}-iN\phi_{2j}}}{\Delta(\phi_{2j-1},\phi_{2j})}\prod_{j=1}^{s-1}\left(\frac{\sin\phi_{2j-1}-\sin\phi_{2j+1}}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\right)$$
$$\times\sum_{n=1}^{s-1}\left[\left(\prod_{j=1}^{n-1}\cos\tfrac{1}{2}(\phi_{2j}+\phi_{2j+2})\right)\left(e^{i(\phi_{2n}+\phi_{2n+2})/2}-\cos\tfrac{1}{2}(\phi_{2n}+\phi_{2n+2})\right)\prod_{j=n+1}^{s-1}e^{i(\phi_{2j}+\phi_{2j+2})/2}\right]. \qquad (4.139)$$

Now

$$\sum_{n=1}^{s-1}\left(\prod_{j=1}^{n-1}A_j\right)(B_n-A_n)\prod_{j=n+1}^{s-1}B_j = \prod_{j=1}^{s-1}B_j - \prod_{j=1}^{s-1}A_j, \qquad (4.140)$$

so that (4.139) becomes

$$S_3 = \sum_{s=2}^{L}(L-s+1)i\tilde{G}^{(L-s+1)}(\bar{x}^{(s)}-x^{(s)})(-\tfrac{1}{2}i\gamma_1)^{-1} \qquad (4.141a)$$

$$= \sum_{s=1}^{L}(L-s+1)i\tilde{G}^{(L-s+1)}(\bar{x}^{(s)}-x^{(s)})(-\tfrac{1}{2}i\gamma_1)^{-1}, \qquad (4.141b)$$

where we used the fact that $\bar{x}^{(1)} = x^{(1)}$ [recall (4.88)] in going from (4.141a) to (4.141b).

Using (4.134), (4.137), (4.141b), in (4.128) we obtain

$$-\sum_{l=1}^{L-1}\sum_{n=1}^{L-l}\bar{x}^{(l)}(L-l-n+1)G^{(L-l-n+1)}x^{(n)}$$
$$= \sum_{l=1}^{L-1}[(L-l+1)\bar{x}^{(l)}x^{(L-l+1)} - (L-l+1)x^{(l)}\bar{x}^{(L-l+1)}]$$
$$+\tilde{G}^{(1)}(\bar{x}^{(L)}-x^{(L)})(\tfrac{1}{2}\gamma_1)^{-1}. \qquad (4.142)$$

Now [recall (4.122)]

$$(\tfrac{1}{2}\gamma_1)^{-1}\tilde{G}^{(1)} = (2\pi)^{-2}\int_{-\pi}^{\pi}d\phi_1\int_{-\pi}^{\pi}d\phi_2\frac{e^{-iM\phi_1-iN\phi_2}}{\Delta(\phi_1,\phi_2)} = x^{(1)} = \bar{x}^{(1)} \qquad (4.143)$$

so that

$$\tilde{G}^{(1)}(\bar{x}^{(L)}-x^{(L)})(\tfrac{1}{2}\gamma_1)^{-1} = x^{(1)}\bar{x}^{(L)} - \bar{x}^{(1)}x^{(L)}. \qquad (4.144)$$

Using (4.144) in (4.142) we have

$$-\sum_{l=1}^{L-1}\sum_{n=1}^{L-l}\bar{x}^{(l)}(L-l-n+1)G^{(L-l-n+1)}x^{(n)}$$
$$= \sum_{l=1}^{L}[(L-l+1)\bar{x}^{(l)}x^{(L-l+1)} - (L-l+1)x^{(l)}\bar{x}^{(L-l+1)}]$$
$$= \sum_{l=1}^{L}(2l-L-1)\bar{x}^{(L-l+1)}x^{(n)}. \qquad (4.145)$$

Thus (4.113) is proved.

F. $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ in scaling limit ($T > T_c$)

From the analysis of Sec. III G it is clear that in the scaling limit the quantities $x^{(k)}_{>MN}$ [see (2.15)] and $F^{(2n)}_{>MN}$ [see (2.16)] become

$$x^{(2k+1)}_{>MN} \sim (-1)^k(\gamma_1\gamma_2)^{-1/2}(2\pi^2)^{-2k-1}\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{2k+1}\left(\frac{dx_j\,dy_j}{(1+x_j^2+y_j^2)}e^{-i\bar{M}x_j-i\bar{N}y_j}\right)\times\prod_{j=1}^{2k}\left(\frac{x_j-x_{j+1}}{y_j+y_{j+1}}\right) \text{ (scaling limit)} \qquad (4.146)$$

for $k\geq 1$, and $k=0$

$$x^{(1)}_{>MN} \sim (2\pi^2)^{-1}(\gamma_1\gamma_2)^{-1/2}\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_1\,dy_1}{(1+x_1^2+y_1^2)}e^{-i\bar{M}x_1-i\bar{N}y_1} \text{ (scaling limit)} \qquad (4.147)$$

and

$$F^{(2n)}_{>MN} \sim F^{(2n)}_{<MN} \sim f^{(2n)}(t) \text{ (scaling limit)}, \qquad (4.148)$$

where we have used the notation of Sec. III G.

As in Section III G the quantity (4.146) can be simplified by first defining the new integration variables u_j and v_j [see (3.133)], and then performing the v_j integrations to obtain

$$x^{(2k+1)}_{>MN} \sim (\gamma_1\gamma_2)^{-1/2}g^{(2k+1)}(t) \text{ (scaling limit)}, \qquad (4.149)$$

where

$$g^{(2k+1)}(t)=(-1)^k(2\pi)^{-2k-1}\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{2k+1}\frac{du_j e^{-itu_j}}{(1+u_j^2)^{1/2}}\times\prod_{j=1}^{2k}\frac{(1+u_j^2)^{1/2}-(1+u_{j+1}^2)^{1/2}}{u_j+u_{j+1}} \qquad (4.150a)$$

for $k\geq 1$, and for $k=0$

$$g^{(1)}(t)=(2\pi)^{-1}\int_{-\infty}^{+\infty}du_1\frac{e^{-itu_1}}{(1+u_1^2)^{1/2}} \; . \qquad (4.150b)$$

The final simplification occurs when we deform the contours of integration for u_j [in (4.150) $\text{Im}u_j<0$, $j=1,2,\ldots,2k+1$] to a loop enclosing the branch point at $u_j=-i$, $j=1,2,\ldots,2k+1$. The problem then is to determine how the second product in (4.149) sums up. The relevant identity is

$$\sum_{\{\epsilon_j=\pm 1\}}(\epsilon_1x_1-\epsilon_2x_2)(\epsilon_2x_2-\epsilon_3x_3)\cdots(\epsilon_{2n-2}x_{2n-2}-\epsilon_{2n-1}x_{2n-1})=(-1)^{n-1}2^{2n-1}x_2^2x_4^2\cdots x_{2n-2}^2 \; . \qquad (4.151)$$

Thus (4.150) becomes (2.30) for $k\geq 1$, and for $k=0$

$$g^{(1)}(t)=\frac{1}{\pi}\int_1^{\infty}\frac{dy}{(y^2-1)^{1/2}}e^{-ty}=\frac{1}{\pi}K_0(t) \; , \qquad (4.152)$$

where $K_0(t)$ is the modified Bessel function. Hence in the scaling limit (2.14) becomes

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle\sim|1-(\sinh 2\beta E_1\sinh 2\beta E_2)^{-2}|^{1/4}\hat{F}_+(t), \qquad (4.153)$$

where $\hat{F}_+(t)$ is given by (2.28).

Furthermore, the error estimate $o(R^{-1})$ in (2.28) follows in the same manner as in Sec. III G [see discussion following (3.155)]. It should be pointed out that if one examines the representation (4.84) for $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ (for $T>T_c$) in the scaling limit, then the error estimate $o(R^{-1})$ in (2.28) does *not* follow from (4.84). It is only after we proved (4.87) could we obtain the error estimate $o(R^{-1})$.

The large-t behavior of $g^{(2k+1)}(t)$ follows from (2.30) by standard techniques. We find for k fixed and $t\to\infty$

$$g^{(2k+1)}(t)=(-1)^k\pi^{-k-1/2}2^{-3k-1/2}[e^{-(2k+1)t}/t^{2k+1/2}]\times[1-\tfrac{1}{8}(14k+1)t^{-1}+O(t^{-2})] \; . \qquad (4.154)$$

Hence we expand $\hat{F}_+(t)$ as

$$\hat{F}_+(t)=\sum_{k=0}^{\infty}\hat{F}_+^{(2k+1)}(t) \; , \qquad (4.155)$$

where from (2.28), (2.29), and (3.162)

$$\hat{F}_+^{(1)}(t)=g^{(1)}(t)=\pi^{-1}K_0(t) \; , \qquad (4.156a)$$

$$\hat{F}_+^{(3)}(t)=g^{(3)}(t)+g^{(1)}(t)\hat{F}_-^{(2)}(t)=g^{(3)}(t)-g^{(1)}(t)f^{(2)}(t) \; , \qquad (4.156b)$$

$$\hat{F}_+^{(5)}(t)=g^{(5)}(t)+g^{(3)}(t)\hat{F}_-^{(2)}(t)+g^{(1)}(t)\hat{F}_-^{(4)}(t)=g^{(5)}(t)-g^{(3)}(t)f^{(2)}(t)+g^{(1)}(t)\times\{-f^{(4)}(t)+\tfrac{1}{2}[f^{(2)}(t)]^2\} \; , \qquad (4.156c)$$

etc., and the functions $f^{(2n)}(t)$ are given by (2.27). From (3.145) and (4.154) it follows that $\hat{F}_+^{(2k+1)}(t)$ for fixed k and $t\to\infty$ is asymptotically equal to $e^{-(2k+1)t}$ times some power of t. If one uses (4.154) in (4.156b) or (4.156c) [and using (3.145)] one finds not only the leading term in these expansions cancels, but the next correction term.

Of course the large-t expansion of $\hat{F}_+^{(1)}(t)$ is known, and the first few terms are

$$\hat{F}_+^{(1)}(t)=(2\pi)^{-1/2}\frac{e^{-t}}{t^{1/2}}\left(1-\frac{1}{8t}+\frac{9}{128t^2}+O(t^{-3})\right) \quad (t\to\infty). \qquad (4.157)$$

We examine $\hat{F}_+^{(3)}(t)$. From (4.156b) we obtain using (2.27) and (2.30)

$$\hat{F}_+^{(3)}(t)=\pi^{-3}\int_1^{\infty}dy_1\int_1^{\infty}dy_2\int_1^{\infty}dy_3\times\prod_{j=1}^{3}\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}}\left(-\frac{y_2^2-1}{(y_1+y_2)(y_2+y_3)}+\frac{y_2^2-1}{(y_1+y_2)^2}\right)$$
$$=\pi^{-3}\int_1^{\infty}dy_1\int_1^{\infty}dy_2\int_1^{\infty}dy_3\times\prod_{j=1}^{3}\left(\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}}\right)(y_2^2-1)\frac{y_3-y_1}{(y_1+y_2)^2(y_2+y_3)} \; . \qquad (4.158)$$

Letting $y_1\leftrightarrow y_3$ in (4.158), adding the resulting expression to (4.158), and dividing by 2 we obtain

$$\hat{F}_+^{(3)}(t)=2^{-1}\pi^{-3}\int_1^{\infty}dy_1\int_1^{\infty}dy_2\int_1^{\infty}dy_3\times\left(\prod_{j=1}^{3}\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}}\right)(y_2^2-1)\left(\frac{y_1-y_3}{(y_1+y_2)(y_2+y_3)}\right)^2 . \qquad (4.159)$$

From (4.159) it is straightforward to obtain for $t\to\infty$

$$\hat{F}_+^{(3)}(t)=2^{-13/2}\pi^{-3/2}\times(e^{-3t}/t^{9/2})[1-\tfrac{51}{8}t^{-1}+O(t^{-2})] \; . \qquad (4.160)$$

V. PERTURBATION EXPANSION FOR $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ FOR LARGE N AND SMALL $N|T-T_c|$

A. Preliminary notation

For the diagonal case $M=N$ the variables t and R of (2.5) and (2.6), respectively, reduce to

$$t=|z_1z_2+z_1+z_2-1|\left(\frac{1}{z_1(1-z_2^2)}+\frac{1}{z_2(1-z_1^2)}\right)^{1/2}N$$
$$=|z_1z_2+z_1+z_2-1|$$

$$\times[z_1 z_2(1-z_1^2)(1-z_2^2)]^{-1/4}R \tag{5.1}$$

and

$$R=\left[\left(\frac{z_1(1-z_2^2)}{z_2(1-z_1^2)}\right)^{1/2}+\left(\frac{z_2(1-z_1^2)}{z_1(1-z_2^2)}\right)^{1/2}\right]^{1/2}N$$

$$=\left[\left(\frac{\sinh 2\beta E_1}{\sinh 2\beta E_2}\right)^{1/2}+\left(\frac{\sinh 2\beta E_2}{\sinh 2\beta E_1}\right)^{1/2}\right]^{1/2}N. \tag{5.2}$$

Note that for the symmetric case $R=\sqrt{2}N$. We have established that in the scaling limit (2.18)–(2.20)

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle = R^{-1/4}F_\pm(t) + R^{-5/4}F_{1_\pm}(t)+o(R^{-5/4}), \tag{5.3}$$

where for arbitrary M and N, t and R are given by (2.5) and (2.6), respectively, and $F_{1_\pm}(t)$ is related to $F_\pm(t)$ by (2.24). Now if we compute the diagonal correlation function $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ in the scaling limit (2.18)–(2.20) where we identify t by (5.1) and R by (5.2), then any property we establish for $F_\pm(t)$ holds in general for the $M\neq N$ case if we merely use the definitions (2.5) and (2.6) for t and R, respectively. Hence from now on we restrict ourselves to the diagonal correlation function $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ with no loss in generality in the scaling limit (2.18)–(2.20).

It is known[5,25] that the diagonal correlation function can be expressed as a Toeplitz determinant

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle=\det(a_{m-n})_{m,n=0}^{N-1}, \tag{5.4}$$

with

$$a_n=(2\pi)^{-1}\int_{-\pi}^{\pi}e^{-in\theta}W(e^{i\theta})\,d\theta, \tag{5.5a}$$

$$W(\xi)=[(1-k_0\xi^{-1})/(1-k_0\xi)]^{1/2}, \tag{5.5b}$$

and

$$k_0=(\sinh 2\beta E_1\sinh 2\beta E_2)^{-1} \tag{5.6}$$

The square root in (5.5) is defined so that $W(e^{i\theta})$ is positive at $\theta=\pi$.

At $T=T_c$ ($k_0=1$), a_n reduces to

$$a_n=a_n^{(0)}\quad(T=T_c), \tag{5.7}$$

with

$$a_n^{(0)}=\pi^{-1}(n+\tfrac{1}{2})^{-1}. \tag{5.8}$$

The matrix A_0 with elements $(A_0)_{m,n}=a_{m-n}^{(0)}$ is a Cauchy matrix, and so the evaluation of $\det(a_{m-n}^{(0)})_{m,n=0}^{N-1}$ is particularly simple.[26,27] One finds for $N\to\infty$

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle|_{T=T_c}=(e^{1/4}2^{1/12}A^{-3}/N^{1/4})\times[1-\tfrac{1}{64}N^{-2}+O(N^{-4})], \tag{5.9}$$

where A is Glaisher's constant[28] [see (2.35)]. From (5.2), (5.3), and (5.9) it follows

$$F_\pm(0)=(\sinh 2\beta_c E_1+\sinh 2\beta_c E_2)^{1/8}e^{1/4}2^{1/12}A^{-3} \equiv F(0). \tag{5.10}$$

For $E_1=E_2$ we have

$$F(0)=2^{1/8}e^{1/4}2^{1/12}A^{-3} = 0.703\,380\,157\,687\,723\cdots. \tag{5.10S}$$

Denoting by $\tilde{A}$ the matrix (a_{m-n}), a_m given by (5.5), we see from (5.4) that

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle=\det(\tilde{A}) =\det(A_0+\Delta) =\det(A_0)\det(1+\Delta A_0^{-1}), \tag{5.11}$$

with

$$\Delta=\tilde{A}-A_0. \tag{5.12}$$

For T close to T_c we expect Δ to be "small." Thus the expansion

$$\det(1+\Delta A_0^{-1})=\exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\mathrm{Tr}[(\Delta A_0^{-1})^n]\right) \tag{5.13}$$

should be (at least) asymptotic as $T\to T_c$.

This observation that (5.13) should provide a means of computing $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ in the scaling limit (1.1)–(1.3) was first made by Ryazanov[29] and Vaks, Larkin, and Ovchinnikov.[30] They examined (5.13) to leading order and showed (in our notation) that

$$F_+(t)=F(0)[1+\tfrac{1}{2}t\ln t+O(t)]\quad(t\to 0), \tag{5.14}$$

where $F(0)$ is given by (5.10). However their evaluation of the constant "$\frac{1}{2}$" multiplying the $t\ln t$ term is in error (these authors obtain $\frac{1}{4}$).

In this section we will use (5.13) to develop a systematic perturbation expansion for $F_\pm(t)$ in the limit $t\to 0$.

B. Analysis of Δ in scaling limit

The matrix Δ defined by (5.12) has elements

$$\Delta_{m,n}=\Delta_{m-n}=a_{m-n}-a_{m-n}^{(0)}, \tag{5.15}$$

with a_m ($a_m^{(0)}$) given by (5.5) [(5.8)]. We analyze Δ_n in the limit $T\to T_c^\pm$, $n\to\infty$ such that $|T-T_c|n$ is fixed. We consider the $T\to T_c^+$ limit first. We have from (5.5)

$$a_n=(2\pi i)^{-1}\oint d\xi\,\xi^{-n-1}\left(\frac{1-k_0\xi^{-1}}{1-k_0\xi}\right)^{1/2}. \tag{5.16}$$

For $n\geq 0$ we deform the contour in (5.16) outward looping the branch point at $\xi=k_0$ ($k_0>1$ for $T>T_c$) to obtain

$$a_n=\pi^{-1}\int_{k_0}^{\infty}dx\,x^{-n-1}\left(\frac{x-k_0}{x(k_0x-1)}\right)^{1/2}. \tag{5.17}$$

For $n<0$ we contract the contour of integration in (5.16) about the branch points 0 and k_0^{-1} to obtain

$$a_{-n}=-\pi^{-1}\int_0^{k_0^{-1}}dx\,x^{n-1}\left(\frac{k_0-x}{x(1-k_0x)}\right)^{1/2}. \tag{5.18}$$

Letting $x = k_0 y$ in (5.17) we have

$$a_n = \frac{k_0^{-n}}{\pi} \int_1^\infty dy\, y^{-n-1} \left(\frac{y-1}{y(k_0^2 y - 1)}\right)^{1/2}, \qquad (5.19)$$

and with the final change of variables

$$y = 1 + \tfrac{1}{2}(1 - k_0^{-2})(z-1), \qquad (5.20)$$

(5.19) becomes ($T > T_c$, $n \geq 0$)

$$a_n = (k_0^{-n}/\pi)\tfrac{1}{2}(1-k_0^{-2}) \times \int_1^\infty dz \left(\frac{z-1}{z+1}\right)^{1/2} [1 + \tfrac{1}{2}(1-k_0^{-2})(z-1)]^{-n-1/2}. \qquad (5.21)$$

From Appendix C we have

$$\tfrac{1}{2}(1-k_0^{-2}) \sim 2|\beta - \beta_c|(z_{1c} + z_{2c}) \times [E_1(1-z_{2c})^{-1} + E_2(1-z_{1c})^{-1}]. \qquad (5.22)$$

Defining

$$s = |z_1 z_2 + z_1 + z_2 - 1| \times \{[z_1(1-z_2^2)]^{-1} + [z_2(1-z_1^2)]^{-1}\} n, \qquad (5.23)$$

and using (2.4) and (2.7) we see from (5.22) and (5.23) that

$$\tfrac{1}{2}(1-k_0^{-2}) \sim s/n. \qquad (5.24)$$

Using (5.24) in (5.21) we see that

$$a_n \sim \tfrac{1}{2}(1-k_0^{-2})\pi^{-1} \int_1^\infty \frac{dz}{(z^2-1)^{1/2}} (z-1) e^{-sz} \qquad (5.25)$$

in the limit $n \to \infty$, $T \to T_c^+$ such that s as defined by (5.23) is fixed. Evaluating (5.25) we obtain

$$a_n \sim \tfrac{1}{2}(1-k_0^{-2})\pi^{-1}[K_1(s) - K_0(s)]. \qquad (5.26)$$

Similar transformations on (5.18) result in

$$a_{-n} \sim \tfrac{1}{2}(1-k_0^{-2})\pi^{-1}[-K_1(s) - K_0(s)] \qquad (5.27)$$

for $n \to +\infty$, $T \to T_c^+$, s fixed.

Combining (5.26) and (5.27) with (5.15) and (5.8) we obtain for Δ_n

$$\Delta_n \sim \tfrac{1}{2}(1-k_0^{-2})\pi^{-1}\Delta_+(s) \quad (n \to \pm\infty,\ T \to T_c^+,\ s \text{ fixed}), \qquad (5.28)$$

with

$$\Delta_+(s) = \theta(s)K_1(|s|) - K_0(|s|) - 1/s, \qquad (5.29)$$

$$\Theta(s) = \begin{cases} +1 & \text{if } s \geq 0 \\ -1 & \text{if } s < 0. \end{cases} \qquad (5.30)$$

The analysis of a_n in the limit $n \to \pm\infty$, $T \to T_c^-$ and s fixed is quite similar, the result is

$$\Delta_n \sim \tfrac{1}{2}(1-k_0^{-2})\pi^{-1}\Delta_-(s) \quad (n \to \pm\infty,\ T \to T_c^-,\ s \text{ fixed}), \qquad (5.31)$$

$$\Delta_-(s) = \Theta(s)K_1(|s|) + K_0(|s|) - 1/s, \qquad (5.32)$$

and $\Theta(s)$ is given by (5.30).

C. Quantity $\mathrm{Tr}[(\Delta A_0^{-1})^n]$ in the limit $n \to \infty$, $T \to T$, s fixed

For simplicity we first analyze the case $n = 1$. To compute $\mathrm{Tr}(\Delta A_0^{-1})$ we must know the matrix elements of A_0^{-1}. Since A_0 is a Cauchy matrix, the inverse matrix elements can be computed. This has been done by Wu[31] who finds

$$(A_0^{-1})_{lk} = \frac{f(k)g(l)}{\pi(k - l + \frac{1}{2})}, \quad k,\ l = 0, 1, 2, \ldots, N-1, \qquad (5.33)$$

with

$$f(k) = \frac{\Gamma(N-k+\frac{1}{2})\Gamma(k+\frac{3}{2})}{\Gamma(N-k+1)\Gamma(k+1)}, \qquad (5.34)$$

$$g(l) = \frac{\Gamma(N-l+\frac{3}{2})\Gamma(l+\frac{1}{2})}{\Gamma(N-l+1)\Gamma(l+1)}, \qquad (5.35)$$

and $\Gamma(x)$ the gamma function. Thus

$$\mathrm{Tr}(\Delta A_0^{-1}) = \sum_{k,\,l=0}^{N-1} \frac{f(k)g(l)}{\pi(k-l+\frac{1}{2})} \Delta_{k-l}, \qquad (5.36)$$

with Δ_k given by (5.15). Equations (5.13) and (5.36) are exact with no approximations having been made. We wish to extract from (5.36) the leading term in the limit $N \to \infty$, $T \to T_c^\pm$ such that t is fixed.

Letting $s = \frac{1}{2}(1-k_0^{-2})k$ and $s' = \frac{1}{2}(1-k_0^{-2})l$ we have, approximating the sum in (5.36) by an integral,

$$\mathrm{Tr}(\Delta A_0^{-1}) = \pi^{-2}\int_0^t ds \int_0^t ds' (s - s' + \tfrac{1}{2}\kappa_d)^{-1} \times f(s/\kappa_d)g(s'/\kappa_d)\Delta_\pm(s-s') + E, \qquad (5.37)$$

where

$$\kappa_d \equiv \tfrac{1}{2}(1-k_0^{-2}), \qquad (5.38)$$

and E is the error in approximating the sum by an integral. Using Stirling's formula we see that as $\kappa_d \to 0$

$$f(s/\kappa_d)g(s'/\kappa_d) = \left(\frac{s(t-s')}{s'(t-s)}\right)^{1/2} + O(\kappa_d). \qquad (5.39)$$

In the limit $N \to \infty$, $T \to T_c^\pm$ such that t is fixed, the error E in (5.37) is negligible to leading order and we have

$$\mathrm{Tr}(\Delta A_0^{-1}) \sim \pi^{-2} P\int_0^t ds\, P\int_0^t ds' \frac{1}{s-s'} \left[\frac{s(t-s')}{s'(t-s)}\right]^{1/2} \times \Delta_\pm(s-s'). \qquad (5.40)$$

Changing variables we have

$$\mathrm{Tr}(\Delta A_0^{-1}) \sim \frac{t}{\pi^2} P\int_0^1 ds\, P\int_0^1 du' \left(\frac{u(1-u')}{u'(1-u)}\right)^{1/2} \times \frac{\Delta \pm [t(u-u')]}{u-u'} \quad (T \to T_c^\pm,\ N \to \infty,\ t \text{ fixed}). \qquad (5.41)$$

The symbol $P\int$ denotes the Cauchy principal value.

Note that although (5.39) is correct, its use in (5.37) is a nonuniform approximation. Clearly s/κ_d is not large when s is in the range $(0, \kappa_d)$. If the integrand were integrable (nonsingular), supplying the estimates that would allow the use of (5.39) in (5.37) would be relatively straightforward. However, the singular factor prevents such a straightforward analysis. Heuristically, the singular part of the denominator goes like $O(\kappa_d^{-1})$ in the range $(0, \kappa_d)$ if we interpret the integrals as Cauchy principal-value integrals. The rest of the integrand is integrable and so the double integral goes like $O(\kappa_d^2)$. Hence the error involved in using (5.39) in (5.37) is $O(\kappa_d)$ as $\kappa_d \to 0$.

The result (5.41) can be easily generalized to $\mathrm{Tr}[(\Delta A_0^{-1})^n]$. We find for $N\to\infty$, $T\to T_c^{\pm}$ such that t is fixed

$$\mathrm{Tr}[(\Delta A_0^{-1})^n] \sim t^n \pi^{-2n} P\int_0^1 du_1 \cdots P\int_0^1 du_{2n} \times \prod_{j=1}^{n} \frac{\Delta_{\pm}[t(u_{2j-1}-u_{2j})]}{u_{2j+1}-u_{2j}} \times \left(\frac{u_{2j+1}(1-u_{2j})}{u_{2j}(1-u_{2j+1})}\right)^{1/2}, \tag{5.42}$$

with $u_{2n+1}=u_1$. We define the right-hand side of (5.42) to be $\Lambda_n^{\pm}(t)$.

D. Small-t expansion of $F_{\pm}(t)$

From (5.9), (5.10), (5.11), and (5.13) we see that to compute the scaling functions $F_{\pm}(t)$ we need (5.13) in the scaling limit. Recalling the definition of $\Lambda_n^{\pm}(t)$ [(5.42)] we have

$$F_{\pm}(t) = F(0)\exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\Lambda_n^{\pm}(t)\right), \tag{5.43}$$

with $F(0)$ given by (5.10). From (5.29), (5.32), and (5.42) we see that

$$\Lambda_n^{\pm}(t) = O(t^n \ln^n t) \quad (t\to 0), \tag{5.44}$$

so (5.43) provides a small-t expansion of $F_{\pm}(t)$. Thus the program is to expand the functions $\Delta_{\pm}[t(u_{2j-1}-u_{2j})]$ appearing in the definition of $\Lambda_n^{\pm}(t)$ [we do this by using the small-t expansion of $K_0(t)$ and $K_1(t)$], evaluate the resulting integrals, and hence obtain $\Lambda_n^{\pm}(t)$. From (5.29) and (5.32) we see that

$$\Delta_-(s) = -\Delta_+(-s). \tag{5.45}$$

Hence from (5.42) and (5.45) it becomes clear that the small-t expansion of $\Lambda_n^-(t)$ can be obtained from the small-t expansion of $\Lambda_n^+(t)$ by replacing all factors of t in the small-t expansion of $\Lambda_n^+(t)$ by $-t$ where we interpret all logarithmic factors $\ln t$ in $\Lambda_n^+(t)$ as $\ln|t|$. This prescription carries over to $F_{\pm}(t)$. Hence we compute the small-t expansion of $F_+(t)$ and apply this prescription to obtain the small-t expansion of $F_-(t)$.

The general structure of $\Lambda_n^+(t)$ is clear from (5.29), (5.42), and the small-x expansions of $K_0(x)$ and $K_1(x)$:

$$\Lambda_n^+(t) = \sum_{k=n}^{\infty}\sum_{l=0}^{n} C_{kl}^{(n)} t^k (\ln t)^l, \tag{5.46}$$

where $C_{kl}^{(n)}$ are constants, and are (in general) expressed as $2n$-dimensional integrals.

Using the expansion for $s\to 0$

$$\Delta_+(s) = \ln(s/2) + \gamma_E + (s/2)(\ln s + \gamma_E - \tfrac{1}{2}) + (s/2)^2(\ln(s/2)+\gamma_E - 1) + \cdots \tag{5.47}$$

in (5.42) for $n=1$ [recall (5.29)] we see that, for example,

$$C_{11}^{(1)} = \pi^{-2}\int_0^1 du\, P\int_0^1 du' \frac{1}{u-u'}\left(\frac{u(1-u')}{u'(1-u)}\right)^{1/2} = \frac{1}{2} \tag{5.48}$$

and

$$C_{10}^{(1)} = (\gamma_E - \ln 2)C_{11}^{(1)} + \pi^{-2}\int_0^1 du\, P\int_0^1 du' \times \frac{1}{u-u'}\left(\frac{u(1-u')}{u'(1-u)}\right)^{1/2}\ln|u-u'| = \tfrac{1}{2}(\gamma_E - 3\ln 2). \tag{5.49}$$

From (5.48) and (5.49) we can already conclude that

$$\Lambda_1^+(t) = \tfrac{1}{2}t\ln t + \tfrac{1}{2}t(\gamma_E - 3\ln 2) + O(t^2\ln^2 t) \tag{5.50a}$$

$$= \tfrac{1}{2}t\Omega + O(t^2\ln^2 t), \tag{5.50b}$$

where Ω is defined by (2.33). From (5.50), (5.44), and (5.43) we have

$$F_{\pm}(t) = F(0)[1 \pm \tfrac{1}{2}t\Omega + O(t^2\ln^2 t)], \tag{5.51a}$$

where we used the prescription of replacing $t\to -t$ to obtain the small-t expansion of $F_-(t)$ from the small-t expansion of $F_+(t)$. Using (5.50a) we can write (5.51) as

$$F_{\pm}(t) = F(0)[1 \pm \tfrac{1}{2}t\ln t \pm \tfrac{1}{2}t(\gamma_E - 3\ln 2) + O(t^2\ln^2 t)]. \tag{5.51b}$$

The coefficient $\frac{1}{2}(\gamma_E - 3\ln 2)$ has the numerical value

$$\tfrac{1}{2}(\gamma_E - 3\ln 2) = -0.751\,112\,938\,389\,14\cdots. \tag{5.52}$$

To obtain the higher-order terms in (5.51) we must keep the higher-order terms in $\Lambda_1^+(t)$ and include the contributions coming from $\Lambda_2^+(t)$, $\Lambda_3^+(t)$, ... In principle, this is straightforward, though by order t^3 it becomes rather messy.

Rather than work through each order in detail we present the results of our computation in Table II. Certain integrals that appear frequently are given in Table III. By using Table II we can conclude that for $t\to 0$

$$\ln[F_{\pm}(t)/F(0)] = \pm\tfrac{1}{2}t\Omega - \tfrac{1}{8}t^2\Omega^2 + \tfrac{1}{16}t^2$$

TABLE II. Coefficients $C_{kl}^{(n)}$ in the small-t expansion of $\Lambda_n^+(t)$. All relevant coefficients to compute $F_\pm(t)$ up to and including the term $t^3\ln t$ are presented. See Eqs. (5.43) and (5.46) in text.

$n=1$	$C_{11}^{(1)}=\frac{1}{2}$
	$C_{10}^{(1)}=\frac{1}{2}(\gamma_E-3\ln 2)$
	$C_{21}^{(1)}=\frac{1}{8}$
	$C_{20}^{(1)}=\frac{1}{8}(\gamma_E-3\ln 2)$
	$C_{31}^{(1)}=\frac{1}{32}$
$n=2$	$C_{22}^{(2)}=\frac{1}{4}$
	$C_{21}^{(2)}=\frac{1}{2}(\gamma_E-3\ln 2)+\frac{1}{4}$
	$C_{20}^{(2)}=\frac{1}{4}(\gamma_E-3\ln 2)^2+\frac{1}{4}(\gamma_E-3\ln 2)-\frac{1}{8}$
	$C_{32}^{(2)}=\frac{1}{8}$
	$C_{31}^{(2)}=\frac{1}{4}(\gamma_E-3\ln 2)+\frac{1}{16}$
$n=3$	$C_{31}^{(3)}=\frac{1}{8}$
	$C_{33}^{(3)}=\frac{3}{8}(\gamma_E-3\ln 2)+\frac{3}{16}$
	$C_{31}^{(3)}=\frac{3}{8}(\gamma_E-3\ln 2)^2+\frac{3}{8}(\gamma_E-3\ln 2)$

$$\pm\tfrac{1}{24}t^3\Omega^3\pm\tfrac{1}{64}t^3\Omega+O(t^4\ln^4 t)\,. \tag{5.53}$$

Actually not enough information is given in Table II to evaluate the coefficient of the t^3 term in (5.53). Strictly from Table II one can compute up to and including the $t^3\ln t$ term in (5.53). The t^3 plus all t^4 terms will be computed by a more efficient method in Sec. VI. We should emphasize it is not necessary to use the techniques of Sec. VI to establish these higher-order terms, it is just easier. For this reason we computed only through $t^3\ln t$ by the techniques of this section.

It is easy to see that

$$C_{nn}^{(n)}=2^{-n} \tag{5.54}$$

since the $2n$-dimensional integral for this coefficient factorizes. Thus,

$$\Lambda_n^+(t)=(\tfrac{1}{2}t\ln t)^n+O(t^n\ln^{n-1}t)\,, \tag{5.55}$$

and hence

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\Lambda_n^+(t)=\ln(1+\tfrac{1}{2}t\ln t)+O\Big(\sum_n t^n\ln^{n-1}t\Big)\,. \tag{5.56}$$

Using (5.56) in (5.43) we see

$$F_+(t)=F(0)[1+\tfrac{1}{2}t\ln t+O(t)]\,. \tag{5.57}$$

The point being that all $t^n(\ln t)^n$ terms in the exponential add up to give the $\frac{1}{2}t\ln t$ term in $F_+(t)$. Hence we cannot have a $t^n(\ln t)^n$ term in the small-t expansion of $F_\pm(t)$. At most we could have is $t^n(\ln t)^{n-1}$.

In Fig. 7 we plot $[1+\frac{1}{2}t\Omega]$ and $F_+(t)/F(0)$ as a function of t. In Fig. 8 we plot $[1-\frac{1}{2}t\Omega]$ and $F_-(t)/F(0)$ as a function of t.

We conclude this section by evaluating one of the nontrivial integrals given in Table III. Let

$$F(x,y)=\pi^{-1}P\int_0^1\frac{du}{u-x}\left(\frac{1-u}{u}\right)^{1/2}\ln|u-y|\,, \tag{5.58}$$

with $0<x<1$ and $0<y<1$. It is not difficult to show that

$$F(x,0)=\lim_{y\to 0^+}F(x,y)\,.$$

Thus we can write

$$F(x,y)-F(x,0)=\pi^{-1}P\int_0^1\frac{du}{u-x}\left(\frac{1-u}{u}\right)^{1/2}\ln\left|\frac{u-y}{u}\right| \tag{5.59}$$

To evaluate (5.59) consider $\int_C f(z)\,dz$ where

$$f(z)=\frac{1}{z-x}\left(\frac{z-1}{z}\right)^{1/2}\ln\left(\frac{z-y}{z}\right)\,, \tag{5.60}$$

and the contour C is shown in Fig. 9. Thus we see that

$$F(x,y)=F(x,0)-\pi\left(\frac{1-x}{x}\right)^{1/2}\quad\text{for } x<y\,,$$

$$=F(x,0)-\frac{\pi}{2}\left(\frac{1-x}{x}\right)^{1/2}\quad\text{for } x=y\,,$$

and

$$=F(x,0)\qquad\text{for } x>y\,. \tag{5.61}$$

It is not difficult to see that

$$\lim_{x\to 1^-}F(x,0)=F(1,0)$$

$$=-\frac{1}{\pi}\int_0^1 du\ln u[u(1-u)]^{-1/2}$$

$$=2\ln 2\,, \tag{5.62}$$

so that

$$F(x,0)-F(1,0)=(1-x)P\int_0^1\frac{du}{u-x}\times\ln u[u(1-u)]^{-1/2}\,. \tag{5.63}$$

Letting

$$\ln u=\int_1^u\frac{dt}{t}=(u-1)\int_0^1\frac{d\xi}{1+\xi(u-1)}\,, \tag{5.64}$$

in (5.63) we see that

$$F(x,0)-F(1,0)=-\int_0^1 d\xi\,P\int_0^1\frac{du}{u-x}\times\left(\frac{1-u}{u}\right)^{1/2}\frac{1}{1-\xi(1-u)}\,, \tag{5.65}$$

TABLE III. Some of the definite integrals needed to compute the coefficients $C_{kl}^{(n)}$. Variables x and y are always restricted to the range $0<x<1$ and $0<y<1$.

1.	$\pi^{-1}\int_0^1 du\,u^n\left(\frac{u}{1-u}\right)^{1/2}=2^{-(2n+2)}\binom{2n+2}{n+1}$
2.	$\pi^{-1}\int_0^1 du\,u^n[u(1-u)]^{1/2}=2^{-(2n+2)}(n+2)^{-1}\binom{2n+1}{n+1}$
3.	$\pi^{-1}\int_0^1 du\left(\frac{1-u}{u}\right)^{1/2}=\frac{1}{2}$
4.	$\pi^{-1}\int_0^1 du\,[u(1-u)]^{-1/2}=1$
5.	$\pi^{-1}\int_0^1 du\,[u(1-u)]^{-1/2}\ln\lvert u-x\rvert=-2\ln 2$
6.	$\pi^{-1}\int_0^1 du\left(\frac{u}{1-u}\right)^{1/2}\ln\lvert u-x\rvert=-x+\frac{1}{2}-\ln 2$
7.	$\pi^{-1}\int_0^1 du\left(\frac{1-u}{u}\right)^{1/2}\ln\lvert u-x\rvert=x-\frac{1}{2}-\ln 2$
8.	$\pi^{-1}\int_0^1 du\,[u(1-u)]^{1/2}\ln\lvert u-x\rvert=\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{16}-\frac{1}{4}\ln 2$
9.	$\pi^{-1}\int_0^1 du\,u[u(1-u)]^{1/2}\ln\lvert u-x\rvert=-\frac{1}{3}x^3+\frac{1}{4}x^2+\frac{1}{8}x+\frac{5}{96}-\frac{1}{8}\ln 2$
10.	$\pi^{-1}P\int_0^1 \frac{du}{u-x}\left(\frac{1-u}{u}\right)^{1/2}=-1$
11.	$\pi^{-1}P\int_0^1 \frac{du}{u-x}\left(\frac{u}{1-u}\right)^{1/2}=+1$
12.	$\pi^{-1}P\int_0^1 \frac{du}{u-x}\,u\left(\frac{1-u}{u}\right)^{1/2}=-x+\frac{1}{2}$
13.	$\pi^{-1}P\int_0^1 \frac{du}{u-x}\,u\left(\frac{u}{1-u}\right)^{1/2}=x+\frac{1}{2}$
14.	$\pi^{-1}P\int_0^1 \frac{du}{u-x}\,u[u(1-u)]^{1/2}=-x^2+\frac{1}{2}x-\frac{1}{8}$
15.	$\pi^{-1}P\int_0^1 \frac{du}{u-x}\,u^2[u(1-u)]^{1/2}=-x^3+\frac{1}{2}x^2+\frac{1}{8}x+\frac{1}{16}$
16.	$\pi^{-1}P\int_0^1 \frac{du}{u-x}\left(\frac{1-u}{u}\right)^{1/2}\ln\lvert u-y\rvert=\begin{cases}2\ln 2+2\left(\frac{1-x}{x}\right)^{1/2}\arccos(x^{1/2}), & x>y\\ 2\ln 2+2\left(\frac{1-x}{x}\right)^{1/2}\arccos(x^{1/2})-\frac{\pi}{2}\left(\frac{1-x}{x}\right)^{1/2}, & x=y\\ 2\ln 2+2\left(\frac{1-x}{x}\right)^{1/2}\arccos(x^{1/2})-\pi\left(\frac{1-x}{x}\right)^{1/2}, & x<y\,.\end{cases}$

where the interchange of the ξ and u integration can be rigorously justified.[32] The inner integral in (5.65) is straightforward to compute, and so after the ξ integration we obtain

$$F(x,0)=2\ln 2+2\left(\frac{1-x}{x}\right)^{1/2}\arccos(x^{1/2})\,. \quad (5.66)$$

Equations (5.61) and (5.66) give $F(x,y)$.

VI. SCALING FUNCTIONS $F_\pm(t)$ IN TERMS OF PAINLEVÉ FUNCTION OF THIRD KIND

In this section we unify the two expansions of the Secs. III-V by demonstrating that $F_\pm(t)$ may be expressed in terms of the Painlevé function of the

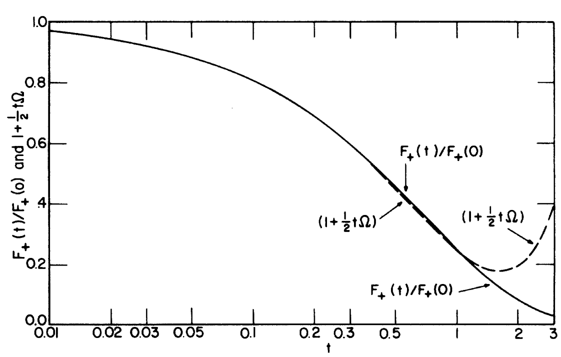

FIG. 7. Functions $(1+\frac{1}{2}t\Omega)$, $\Omega=\ln(\frac{1}{8}t)+\gamma_E$, $\gamma_E=$ Euler's constant, and $F_+(t)/F_+(0)$ as a function of t.

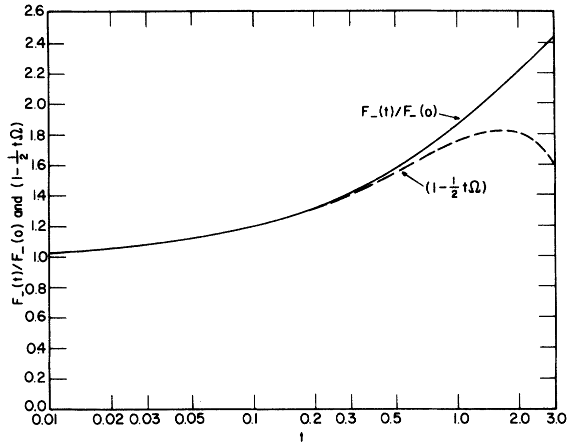

FIG. 8. Functions $(1-\frac{1}{2}t\Omega)$, $\Omega=\ln(\frac{1}{8}t)+\gamma_E$, γ_E = Euler's constant, and $F_-(t)/F_-(0)$ as a function of t.

third kind defined by (2.36)–(2.38).[33]

The Painlevé function arises when we are lead from considering the ratio

$$x_0(N)=\langle\sigma_{0,0}\sigma_{N-1,N-1}\rangle/\langle\sigma_{0,0}\sigma_{N,N}\rangle \tag{6.1}$$

in the scaling limit (1.1)–(1.3) to integral equations of the form

$$\int_0^t K_0(|s-s'|)\,x(s')\,ds'=y(s)\ , \tag{6.2}$$

where $K_0(x)$ is the modified Bessel function and $y(s)$ is taken as a known function. This integral equation has been the subject of much investigation in scattering theory where it arises in the context of electromagnetic scattering from a strip.

Myers[34,35] has expressed the solution to (6.2) [for particular $y(s)$'s] in terms of the Painlevé function $\eta(\theta)$. Therefore when our analysis leads to equations of the form (6.2), we can take over the work of Myers and apply it directly to our problem. Since this work is not well known in the statistical mechanics literature, we have summarized the relevant results of Myers's thesis in Appendix B. A reading of Appendix B is ultimately necessary for an understanding of the computations of this section.

A. Relating $x_0(N)$ to integral equations

We first define

$$S_N=\langle\sigma_{0,0}\sigma_{N,N}\rangle \tag{6.3}$$

and recall that S_N is then given by (5.4). Consider the sum equation

$$\sum_{m=0}^{N-1} a_{n-m}\,x_m=\delta_{n,0}\ . \tag{6.4}$$

Then by Cramer's rule and (5.4) we have $x_0(N)$ being given by (6.1). The quantities a_n in (6.4) are defined by (5.5).

We introduce the auxiliary generating function

$$\overline{W}(\xi)=(1-k_0\xi^{-1})^{-1}\,W(\xi)\ , \tag{6.5}$$

where $W(\xi)$ and k_0 are given by (5.5b) and (5.6), respectively, and define the auxiliary coefficients

$$\overline{a}_n=(2\pi)^{-1}\int_{-\pi}^{\pi} e^{-in\theta}\,\overline{W}(e^{i\theta})\,d\theta\ . \tag{6.6}$$

If we define for $0\le n\le N$

$$y_n=\sum_{m=0}^{N-1}\overline{a}_{n-m}\,x_m\ , \tag{6.7}$$

then from (6.4)–(6.6) it follows that

$$y_n-k_0\,y_{n+1}=\delta_{n,0} \tag{6.8}$$

for $0\le n\le N$. Equation (6.8) has the solution

$$y_n=\delta_{n,0}+\kappa\,k_0^{-n} \tag{6.9}$$

with κ a constant to be determined.

If we define

$$x_N=0\ , \tag{6.10}$$

then (6.7) is equivalent to the system of $(N+1)$ linear equations

$$\sum_{m=0}^{N}\overline{a}_{n-m}\,x_m=\delta_{n,0}+\kappa\,k_0^{-n}\ . \tag{6.11}$$

Let $\overline{x}_m$ and $\overline{\overline{x}}_m$ be the solutions of

$$\sum_{m=0}^{N}\overline{a}_{n-m}\,\overline{x}_m=\delta_{n,0} \tag{6.12}$$

and

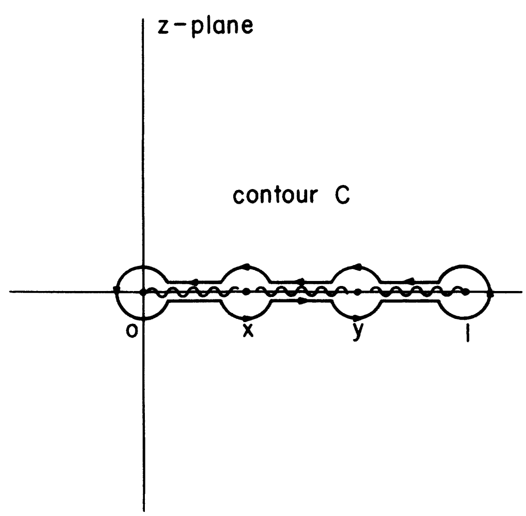

FIG. 9. Contour of integration C.

$$\sum_{m=0}^{N} \bar{a}_{n-m} \tilde{x}_m = k_0^{-n} , \qquad (6.13)$$

respectively, with $0 \le n \le N$. Then

$$\kappa = -\bar{x}_N / \tilde{x}_N , \qquad (6.14)$$

and so

$$x_m = \bar{x}_m - \tilde{x}_m \bar{x}_N / \tilde{x}_N . \qquad (6.15)$$

In the scaling limit we have in a manner similar to that of (5.16)–(5.26)

$$\bar{a}_n \sim \pi^{-1} K_0(s) \quad (n \to \infty,\ T \to T_c^{\pm},\ s \text{ fixed}) , \qquad (6.16)$$

with s being defined by (5.23). Given (6.12), (6.13), and (6.16) it is suggestive to consider the pair of integral equations

$$\int_0^t K_0(|s-s'|)\bar{x}(s';Z)\,ds' = Z^{1/2} e^{-Zs} \qquad (6.17)$$

and

$$\int_0^t K_0(|s-s'|)\tilde{x}_{\pm}(s')\,ds' = e^{\pm s} , \qquad (6.18)$$

with t defined by (5.1) [the $M=N$ case of (2.5)], and Z a large positive parameter so that $Z^{1/2}e^{-Zs}$ mocks the Kronecker δ function $\delta_{0,n}$. Equations (6.17) and (6.18) are, in some sense to be made precise, the continuum analog of (6.12) and (6.13), respectively. For $T > T_c$ ($k_0 > 1$) we associate $\tilde{x}_-(s)$ of (6.18) with (6.13), and for $T < T_c$ ($k_0 < 1$) we associate $\tilde{x}_+(s)$ with (6.13).

We now examine the pair of *infinite* equations

$$\sum_{m=0}^{\infty} \bar{a}_{n-m} \hat{x}_m = \hat{y}_n \qquad (6.19)$$

and

$$\int_0^{\infty} K_0(|s-s'|)\hat{x}(s')\,ds' = \hat{y}(s) , \qquad (6.20)$$

where

$$\hat{y}_n = \hat{y}(s) \qquad (6.21)$$

for s given by (5.23), and we assume that $\hat{y}$ is slowly varying in the sense that $\hat{y}_n - \hat{y}_{nH}$ is small. We can apply the standard methods of Wiener and Hopf[19] to (6.19) and (6.20). Writing

$$[\bar{W}(\xi)]^{-1} = P(\xi)\,Q(\xi^{-1}) , \qquad (6.22)$$

with

$$P(\xi) = \begin{cases} (1-k_0\xi)^{1/2} & \text{for } T < T_c \\ (\xi - k_0)^{1/2} & \text{for } T > T_c \end{cases} \qquad (6.23)$$

and

$$Q(\xi) = \begin{cases} (1-k_0\xi)^{1/2} & \text{for } T < T_c \\ (\xi - k_0)^{1/2} & \text{for } T > T_c , \end{cases} \qquad (6.24)$$

it follows that

$$X(\xi) = P(\xi)[Q(\xi^{-1})Y(\xi)]_+ , \qquad (6.25)$$

where

$$X(\xi) = \sum_{n=0}^{\infty} \hat{x}_n \xi^n , \qquad (6.26a)$$

$$Y(\xi) = \sum_{n=0}^{\infty} \hat{y}_n \xi^n , \qquad (6.26b)$$

and $[\cdots]_+$ denotes the plus part of a function.[36]

Using (6.23)–(6.26) we have

$$\hat{x}_0 = X(0)$$

$$= (2\pi)^{-1}\int_{-\pi}^{\pi} d\theta\,(1-k_0e^{-i\theta})^{1/2}\,Y(e^{i\theta}) . \qquad (6.27)$$

Defining

$$\zeta = \theta(1-k_0)^{-1} , \qquad (6.28)$$

(6.27) becomes

$$\hat{x}_0 = (2\pi)^{-1}\int_{-\pi(1-k_0)^{-1}}^{\pi(1-k_0)^{-1}} d\zeta\,(1-k_0) \times (1-k_0e^{-i\zeta(1-k_0)})^{1/2}\,Y(e^{i\zeta(1-k_0)}) , \qquad (6.29)$$

which to leading order in the scaling limit is

$$\hat{x}_0 \sim (2\pi)^{-1}\int_{-\infty}^{+\infty} d\zeta\,(1-k_0)^{3/2}(1+i\zeta)^{1/2}\,Y(e^{i\zeta(1-k_0)}) . \qquad (6.30)$$

From the Poisson sum formula it follows that

$$\mathcal{Y}(\zeta) \sim (1-k_0)Y(e^{i\zeta(1-k_0)}) , \qquad (6.31)$$

where

$$\mathcal{Y}(\zeta) = \int_0^{\infty} e^{i\zeta t}\,\hat{y}(t)\,dt , \qquad (6.32)$$

and it is assumed that $\hat{y}_n = \hat{y}(s)$ for $s = n(1-k_0)$. Hence (6.30) becomes

$$\hat{x}_0 \sim (1-k_0)^{1/2}\int_{-\infty}^{+\infty} d\zeta\,(1+i\zeta)^{1/2}\mathcal{Y}(\zeta) . \qquad (6.33)$$

Applying the Wiener-Hopf analysis to (6.20) we obtain

$$\chi(\zeta) = \pi^{-1}(1-i\zeta)^{1/2}[(1+i\zeta)^{1/2}\mathcal{Y}(\zeta)]_+ , \qquad (6.34)$$

where

$$\chi(\zeta) = \int_0^{\infty} e^{i\zeta t}\,\hat{x}(t)\,dt . \qquad (6.35)$$

Using the fact that

$$\lim_{s\to 0} s^{1/2}\hat{x}(s) = \lim_{\zeta\to\infty} e^{-i\pi/4}(\zeta/\pi)^{1/2}\chi(\zeta) , \qquad (6.36)$$

and (6.34) we see that (6.33) can be written

$$\hat{x}_0 \sim \pi^{3/2}(1-k_0)^{1/2}\lim_{s\to 0} s^{1/2}\hat{x}(s) , \qquad (6.37)$$

where to summarize $\hat{x}_0$ is the $m=0$ solution to (6.19) and $\hat{x}(s)$ is the solution to (6.20).

We now apply this result to (6.13) and (6.18). Let us define

$$\hat{y}(s)=\int_0^t ds' K_0(|s-s'|)\tilde{x}_\pm(s') \tag{6.38}$$

for all $s\geq 0$. Then by (6.18) $\hat{y}(s)=e^{\pm s}$ for $0\leq s\leq t$. Similarly, let us put

$$y_n'=\sum_{m=0}^{N}\bar{a}_{n-m}\tilde{x}_m \tag{6.39}$$

for all $n\geq 0$. Then by (6.13) $y_n'=\hat{y}_n$ for $0\leq n\leq N$ with $\hat{y}_n=\hat{y}(s)$ for $s=n(1-k_0)$. Although this does not hold for $n>N$, the difference between y_n and y_n' is small and negligible. We can therefore apply (6.37) to obtain

$$\tilde{x}_0\sim\pi^{3/2}(1-k_0)^{1/2}\lim_{s\to 0}s^{1/2}\tilde{x}_\pm(s). \tag{6.40}$$

Similarly,

$$\tilde{x}_N\sim\pi^{3/2}(1-k_0)^{1/2}\lim_{s\to t}(t-s)^{1/2}\tilde{x}_\pm(s)\ . \tag{6.41}$$

Equation (6.37) cannot be applied to (6.12) and (6.17) as $\delta_{n,0}$ is not a slowly varying function. If we examine the sum equation

$$\sum_{m=0}^{\infty}\bar{a}_{n-m}\bar{x}_m^{(\infty)}=\delta_{n,0}\ , \tag{6.42}$$

it is straightforward to show

$$\bar{x}_0^{(\infty)}=\begin{cases}1 & \text{for } k_0<1\\ k_0 & \text{for } k_0>1\ ,\end{cases} \tag{6.43}$$

and for $n\to\infty$

$$\bar{x}_n^{(\infty)}\sim\begin{cases}-\frac{1}{2}\pi^{-1/2}k_0^n n^{-3/2} & \text{for } k_0<1\\ -\frac{1}{2}\pi^{-1/2}k_0^{-n+1}n^{-3/2} & \text{for } k_0>1.\end{cases} \tag{6.44}$$

Likewise for the integral equation

$$\int_0^\infty K_0(|s-s'|)\bar{x}^{(\infty)}(s';Z)\,ds'=Z^{1/2}e^{-Zs}\ , \tag{6.45}$$

we can show that

$$\bar{x}^{(\infty)}(s;Z)\sim Z^{1/2}(Z+1)^{1/2}\pi^{-3/2}s^{-1/2} \tag{6.46}$$

for $sZ\ll 1$, and

$$\bar{x}^{(\infty)}(s;Z)\sim-\tfrac{1}{2}\pi^{-3/2}e^{-s}s^{-3/2}(1+Z^{-1})^{1/2} \tag{6.47}$$

for $sZ\gg 1$.

Now let

$$y''(s;Z)=\int_t^\infty ds'\,K_0(|s-s'|)\bar{x}^{(\infty)}(s';Z) \tag{6.48}$$

and

$$\hat{y}_n'''=\sum_{m=N+1}^{\infty}\bar{a}_{n-m}\bar{x}_m^{(\infty)}\ . \tag{6.49}$$

Then it follows from (6.46)–(6.49) that

$$\hat{y}_n'''\sim(1-k_0)^{1/2}\lim_{Z\to\infty}\hat{y}''(s;Z) \tag{6.50}$$

for $s=n(1-k_0)$ and $0\leq n\leq N$.

If we consider for $0\leq n\leq N$ the sum equation

$$\sum_{m=0}^{N}\bar{a}_{n-m}(\bar{x}_m-\bar{x}_m^{(\infty)})=\hat{y}_n''', \tag{6.51a}$$

and the corresponding integral equation

$$\int_0^t ds' K_0(|s-s'|)[\bar{x}(s';Z)-\bar{x}^{(\infty)}(s';Z)]=y''(s;Z) \tag{6.51b}$$

then (6.37) can be applied to yield

$$\bar{x}_0(Z)-\bar{x}_0^{(\infty)}(Z)\sim\pi^{3/2}(1-k_0)\lim_{s\to 0}[s^{1/2}\bar{x}(s;Z)-s^{1/2}\bar{x}^{(\infty)}(s;Z)]\ , \tag{6.52}$$

and very similarly

$$\bar{x}_N-\bar{x}_N^{(\infty)}\sim\pi^{3/2}(1-k_0)\lim_{s\to t}(t-s)^{1/2}[\bar{x}(s;Z)-\bar{x}^{(\infty)}(s;Z)]\ . \tag{6.53}$$

Combining all these results we obtain

$$x_0\sim\max(1,k_0)+\pi(1-k_0)\lim_{Z\to\infty}\Big(-Z^{1/2}(Z+1)^{1/2}\pi^{-1}+\lim_{s\to 0}(\pi s)^{1/2}\bar{x}(s;Z)-\pi^{1/2}\lim_{s\to t}\frac{\bar{x}(s;Z)}{\tilde{x}_\pm(s)}\lim_{s\to 0}s^{1/2}\tilde{x}_\pm(s)\Big). \tag{6.54}$$

B. $x_0(N)$ in terms of $\eta(\theta)$

The evaluation of the limits in (6.54) is straightforward if we make use of the results of Myers.[34] Using (B2), (B3), and (B7) of Appendix B we have

$$\lim_{s\to 0}s^{1/2}\tilde{x}_+(s)=\theta^{-1/2}e^{\theta}f_0(1-\eta(\theta))\ , \tag{6.55a}$$

$$\lim_{s\to 0}s^{1/2}\tilde{x}_-(s)=\theta^{-1/2}e^{-\theta}f_0(1+\eta(\theta))\ , \tag{6.55b}$$

$$\lim_{s\to t}(t-s)^{1/2}\tilde{x}_+(s)=\theta^{-1/2}e^{\theta}f_0(1+\eta(\theta))\ , \tag{6.55c}$$

$$\lim_{s\to t}(t-s)^{1/2}\tilde{x}_-(s)=\theta^{-1/2}e^{-\theta}f_0(1-\eta(\theta))\ , \tag{6.55d}$$

$$\lim_{s\to 0}s^{1/2}\bar{x}(s;Z)=-\theta^{-1/2}Z^{1/2}\times e^{-Z\theta}f_0(\lambda_1-\eta(\theta)\lambda_2)\ , \tag{6.55e}$$

$$\lim_{s\to t}(t-s)^{1/2}\bar{x}(s;Z)=-\theta^{-1/2}Z^{1/2}\times e^{-Z\theta}f_0(\lambda_1+\eta(\theta)\lambda_2)\ , \tag{6.55f}$$

with $\theta=\frac{1}{2}t$. The notation follows that of Appendix B. From (B13) we have

$$\lambda_1=\lambda_1(\theta,Z)=M(\theta)[\bar{f}(Z,\theta)-Z\bar{g}(Z,\theta)]\ , \tag{6.56a}$$

$$\lambda_2 = \lambda_2(\theta, Z) = M(\theta)[\bar{g}(Z,\theta) - Z\bar{f}(Z,\theta)] , \quad (6.56b)$$

with the bar operation defined by (B13c).

Equations (6.55) allow one to evaluate the $s \to 0$ and $s \to t$ limits in (6.54). What remains is the $Z \to \infty$ limit. To evaluate this we must know the large-Z behavior of $\lambda_1(\theta, Z)$ and $\lambda_2(\theta, Z)$. From the power-series expansions (B7a) and (B7b) of $f(t)$ and $g(t)$ we can calculate the large-Z behavior of $\bar{f}(Z,\theta)$ and $g(Z,\theta)$.

Thus for $Z \to \infty$ we find

$$\bar{f}(Z,\theta) = \frac{e^{Z\theta}}{Z^{1/2}} \frac{f_0\sqrt{\pi}}{\theta^{1/2}} \left(1 + \frac{f_1}{2Z\theta} + \frac{3f_2}{4Z^2\theta^2} + \cdots\right) \quad (6.57a)$$

and

$$\bar{g}(Z,\theta) = -\frac{e^{Z\theta}}{Z^{1/2}} \frac{f_0\,\eta(\theta)\pi^{1/2}}{\theta^{1/2}} \times\left(1 + \frac{g_1}{2Z\theta} + \frac{3g_2}{4Z^2\theta^2} + \cdots\right) , \quad (6.57b)$$

with f_1, f_2, g_1, and g_2 given by (B7). From (6.57) we can determine the large-Z behavior of (6.56), and hence (6.55e) and (6.55f). We find that

$$\begin{aligned}\bar{x}_0(Z) = \max(1, k_0) + \pi(1-k_0)\{&-[2M(\theta)\\ &\times f_0^2\,\pi\eta(\theta)\theta^{-1} + \pi^{-1}]Z\\ &- M(\theta)f_0^2\pi\theta^{-1}[1 + f_0\eta(\theta)/2\theta\\ &+ g_1\eta(\theta)/2\theta + \eta^2(\theta)] - \tfrac{1}{2}\pi^{-1} + O(1/Z)\}\end{aligned} \quad (6.58)$$

and

$$\begin{aligned}\bar{x}_N = -(1-k_0)f_0^2 M(\theta)\pi^2\theta^{-1}\\ \times[1 - f_1\eta(\theta)/2\theta + g_1\eta(\theta)/2\theta - \eta^2(\theta)] .\end{aligned} \quad (6.59)$$

In Appendix B we show that

$$2\eta(\theta)f_0^2 M(\theta)\pi^2\theta^{-1} = -1 . \quad (6.60)$$

Thus the coefficient of the term proportional to Z in (6.48) vanishes, as it must if the $Z \to \infty$ limit is to exist.

Using (6.60) in (6.58) and (6.59), it follows from (6.15) [or (6.54)] that

$$x_0 = k_0 + |1-k_0| \left(-\frac{3}{2} + \frac{[f_1 - \eta(\theta)g_1]}{2\theta[1-\eta(\theta)]}\right) \quad (T > T_c) , \quad (6.61)$$

$$x_0 = 1 + |1-k_0| \left(\frac{1}{2} + \frac{[f_1 + \eta(\theta)g_1]}{2\theta[1+\eta(\theta)]}\right) \quad (T < T_c) , \quad (6.62)$$

where we used the fact that

$$\frac{\tilde{x}_0}{\tilde{x}_N} = \begin{cases}(1+\eta)/(1-\eta) & \text{for } T > T_c , \quad (6.63a)\\ (1-\eta)/(1+\eta) & \text{for } T < T_c , \quad (6.63b)\end{cases}$$

which follows from (6.40), (6.41), and (6.55).

It is convenient to define the functions $H_\pm(\theta)$ by

$$H_+(\theta) = -\tfrac{3}{2} + (f_1 - \eta g_1)/2\theta(1-\eta) \quad (6.64a)$$

and

$$H_-(\theta) = \tfrac{1}{2} + (f_1 + \eta g_1)/2\theta(1+\eta) , \quad (6.64b)$$

so that (6.61) and (6.62) become

$$x_0 \sim \max(1, k_0) + |1-k_0|\, H_\pm(\theta)$$
$$(T \to T_c^\pm,\ N \to \infty,\ t \text{ fixed}) \quad (6.65)$$

Using (B7c) and (B7d) one can show that

$$H_+(\theta) = 1 + \frac{\eta'}{4\eta}\,\frac{1+\eta}{1-\eta} - \theta\,\frac{(\eta')^2}{8\eta^2} + \frac{\theta(1-\eta^2)^2}{8\eta^2} \quad (6.66a)$$

and

$$H_-(\theta) = \frac{\eta'}{4\eta}\,\frac{1-\eta}{1+\eta} - \theta\,\frac{(\eta')^2}{8\eta^2} + \frac{\theta(1-\eta^2)^2}{8\eta^2} , \quad (6.66b)$$

where the definition (B5a) was used.

C. $F_\pm(t)$ in terms of $\eta(\theta)$

Equation (6.65) gives $x_0(N)$ in the scaling limit. Using this and (6.1) we now construct S_N in the scaling limit. We do this first for $T < T_c$. Choose an N_0, $N_0 \gg N$. Then we can write

$$S_N = S_{N_0} \prod_{n=N+1}^{N_0} x_0(n) = S_{N_0} \exp \sum_{n=N+1}^{N_0} \ln x_0(n) . \quad (6.67)$$

Since we are interested in the limited $T \to T_c$ ($k_0 \to 1$), we can write [in view of (6.65)] to leading order

$$\ln x_0(n) \sim |1-k_0|\, H_-(\theta) . \quad (6.68)$$

Then again to leading order

$$\sum_{n=N+1}^{N_0} \ln x_0(n) \sim 2 \int_{t/2}^{N_0|1-k_0|/2} H_-(\theta)\, d\theta . \quad (6.69)$$

The large-θ behavior of $H_-(\theta)$ follows from that of $\eta(\theta)$. Thus using (2.38) in (6.66b) we find

$$H_-(\theta) = O(e^{-4\theta}) , \quad \theta \to \infty . \quad (6.70)$$

For N_0 large, S_{N_0} becomes independent (again to leading order) of N_0 since it approaches the spontaneous magnetization squared $S_\infty^<$ [see (2.10)]. From (6.69) and (6.70) we see that

$$\exp\left(\sum_{n=N+1}^{\infty} \ln x_0(n)\right) \sim \exp\left(2 \int_{t/2}^{\infty} d\theta'\, H_-(\theta')\right) . \quad (6.71)$$

From (2.10), (5.23), and (5.24) we can conclude that

$$S_\infty^< \sim (2t/N)^{1/4} , \quad (6.72)$$

where t is given by (5.1). Using (6.71) and (6.72) we conclude that

$$S_N^< \sim N^{-1/4}(2t)^{1/4} \exp\left(2 \int_{t/2}^{\infty} d\theta'\, H_-(\theta')\right) . \quad (6.73)$$

In terms of the variable R given (for the diagonal case) by (5.2), (6.73) becomes

$$S_N^< \sim R^{-1/4}(\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8} \times (2t)^{1/4} \exp\left(2\int_{t/2}^{\infty} d\theta' \, H_-(\theta')\right) \quad (T \to T_c^-, \; R\to\infty, \; t \text{ fixed}) . \tag{6.74}$$

We now consider the case $T > T_c$. Again choose $N_0 \gg N$. For $T > T_c$ we have from (6.65)

$$\ln x_0(N) \sim |k_0 - 1| + |k_0 - 1| H_+(\theta) . \tag{6.75}$$

So (6.67) in this case becomes

$$S_N^> \sim S_{N_0} \frac{(k_0)^{N_0}}{(k_0)^N} \exp\left(2\int_{t/2}^{N_0|k_0-1|/2} H_+(\theta')\, d\theta'\right). \tag{6.76}$$

For $\theta\to\infty$ we can show using (2.38)

$$H_+(\theta) = 1/4\theta + O(\theta^{-2}) . \tag{6.77}$$

Using (6.77) in (6.76)

$$S_N^> \sim S_{N_0} k_0^{N_0} k_0^{-N} \exp\left[2\int_{t/2}^{N_0|k_0-1|/2} \left(H_+(\theta') - \frac{1}{4\theta'}\right) d\theta' + \frac{1}{2}\int_{t/2}^{N_0|k_0-1|/2} \frac{d\theta'}{\theta'}\right] . \tag{6.78}$$

Integrating the second term in (6.78) we obtain

$$S_N^> \sim S_{N_0} k_0^{N_0} k_0^{-N} (N_0/N)^{1/2} \times \exp\left[2\int_{t/2}^{N_0|k_0-1|/2} \left(H_+(\theta') - \frac{1}{4\theta'}\right) d\theta'\right] . \tag{6.79}$$

For fixed $T > T_c$ and $N_0\to\infty$ it is known[37] that

$$S_{N_0} = [k_0^{-N}/(\pi N_0)^{1/2}](1 - k_0^{-2})^{-1/4} + O(k_0^{-N}/N_0^{3/2}) . \tag{6.80}$$

Using (6.80) in (6.79) we obtain (note that the N_0 dependence cancels)

$$S_N^> \sim k_0^{-N}(\pi N)^{-1/2}(1 - k_0^{-2})^{-1/4} \times \exp\left[2\int_{t/2}^{\infty} \left(H_+(\theta') - \frac{1}{4\theta'}\right) d\theta'\right], \tag{6.81}$$

where we have replaced the upper limit $\frac{1}{2}N_0|k_0 - 1|$ in (6.79) by ∞ (valid to leading order). Again using (2.10), (5.23), and (5.24) we conclude that

$$S_N^> \sim \frac{\pi^{-1/2} e^{-t} (2t)^{-1/4}}{N^{1/4}} \exp\left[2\int_{t/2}^{\infty} \left(H_+(\theta') - \frac{1}{4\theta'}\right) d\theta'\right], \tag{6.82}$$

or in terms of R

$$S_N^> \sim R^{-1/4}(\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8} \pi^{-1/2} e^{-t} \times (2t)^{-1/4} \exp\left[2\int_{t/2}^{\infty} \left(H_+(\theta') - \frac{1}{4\theta'}\right) d\theta'\right] . \tag{6.83}$$

We can simplify (6.74) and (6.83) by using

$$\int_\theta^\infty \frac{\eta'(s)}{2\eta(s)} \frac{1-\eta(s)}{1+\eta(s)}\, ds = \frac{1}{2}\int_{\eta(\theta)}^{1} \frac{d\eta}{\eta}\frac{1-\eta}{1+\eta} = \ln[1+\eta(\theta)] - \ln 2 - \tfrac{1}{2}\ln\eta(\theta). \tag{6.84}$$

Recalling the definition (6.66b) of $H_-(\theta)$ we see that (6.84) implies that (6.74) can be written as

$$S_N^< = \frac{F_-(t)}{R^{1/4}} + o(R^{-1/4}) , \tag{6.85a}$$

$$F_-(t) = (\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8} \times \frac{1+\eta(\theta)}{2[\sqrt{\eta(\theta)}]} (4\theta)^{1/4} \exp\left(\int_\theta^\infty ds\, \tilde{F}(s)\right) , \tag{6.85b}$$

with

$$\tilde{F}(s) = [s/4\eta^2(s)]\{[1-\eta^2(s)]^2 - [\eta'(s)]^2\} \tag{6.85c}$$

and

$$\theta = \tfrac{1}{2}t . \tag{6.85d}$$

To simplify (6.83) we note from (6.66) the identity

$$H_+(\theta) = -1 + \eta'(\theta)/[1-\eta^2(\theta)] + H_-(\theta) . \tag{6.86}$$

Substituting (6.86) into (6.83) and making use of (6.74) we obtain

$$S_N^> = \pi^{-1/2} e^{-t} (2t)^{-1/2} \times \exp\left[2\int_{t/2}^{\infty} \left(-1 + \frac{\eta'(s)}{1-\eta^2(s)} - \frac{1}{4s}\right) ds\right] S_N^< . \tag{6.87}$$

The integral in (6.87) can be evaluated:

$$\int_{t/2}^{\infty} \left(-1 + \frac{\eta'(s)}{1-\eta^2(s)} - \frac{1}{4s}\right) ds = \lim_{L\to\infty}\left[\int_{t/2}^{L}\left(-1 + \frac{\eta'}{1-\eta^2} - \frac{1}{4s}\right) ds\right]$$
$$= \lim_{L\to\infty}\left\{-L + \tfrac{1}{2}t + \tfrac{1}{2}\ln\frac{1-\eta(t/2)}{1+\eta(t/2)} - \tfrac{1}{2}\ln[1-\eta(L)] + \tfrac{1}{2}\ln[1+\eta(L)] - \tfrac{1}{4}\ln L + \tfrac{1}{4}\ln t/2\right\} . \tag{6.88}$$

Using $\ln[1+\eta(L)] \sim \ln 2$ and $\ln[1-\eta(L)] \sim -2L - \frac{1}{2}\ln\pi - \frac{1}{2}\ln L$ for $L\to\infty$ in (6.88)

$$2\int_{t/2}^{\infty}\left(-1 + \frac{\eta'}{1-\eta^2} - \frac{1}{4s}\right) ds = t + \ln\left(\frac{1-\eta(t/2)}{1+\eta(t/2)}\right) + \tfrac{1}{2}\ln\pi + \tfrac{1}{2}\ln(t/2) + \ln 2. \tag{6.89}$$

Using (6.89) in (6.87) we obtain

$$S_N^> = \frac{1-\eta(t/2)}{1+\eta(t/2)} S_N^< \quad (T\to T_c, \; N\to\infty, \; t \text{ fixed}). \tag{6.90}$$

Thus (6.90) implies that (6.83) can be written

$$S_N^> = F_+(t)/R^{1/4} + o(R^{-1/4}), \tag{6.91a}$$

$$F_+(t) = (\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8}$$

$$\times \frac{1-\eta(\theta)}{2[\eta(\theta)]^{1/2}}(4\theta)^{1/4}\exp\left(\int_\theta^\infty ds\,\tilde{F}(s)\right), \tag{6.91b}$$

and $\tilde{F}(s)$ is given by (6.85c) and $\theta=\frac{1}{2}t$. Another way of stating (6.90) is

$$\frac{F_+(t)}{F_-(t)}=\frac{1-\eta(t/2)}{1+\eta(t/2)}\ . \tag{6.92}$$

Since $\eta(\theta)\to 0$ as $\theta\to 0$ [see (2.37) or Appendix B] we see from (6.92) that

$$\lim_{t\to 0}\ \frac{F_+(t)}{F_-(t)}=1. \tag{6.93}$$

That (6.93) is the case is clear from the small-t perturbation methods of Sec. V. From Secs. III and IV we obtained the representation

$$F_+(t)/F_-(t)=G(t), \tag{6.94}$$

where $G(t)$ is given by (2.29). From (6.94) and (6.92) we obtain

$$\frac{1-\eta(t/2)}{1+\eta(t/2)}=G(t)=\sum_{k=0}^{\infty} g^{(2k+1)}(t), \tag{6.95}$$

with $g^{(2k+1)}(t)$ given by (2.30) and from (6.93) $\lim_{t\to 0} G(t)=1.$

We have numerically solved for the function $\eta(\theta)$ [by solving the nonlinear differential equation for $\eta(\theta)$, see (2.36)–(2.38); and for more details, the reader should consult Appendices A and B], and from this we have obtained $F_\pm(t)$ to ten significant figures. Tables of $F_\pm(t)$ along with all other numerical work can be found in Appendix A. It is from this numerical work that we obtained Fig. 3.

We can write (6.85b) and (6.91b) in an alternative form that is useful in analyzing the small-t limit. For large t, (6.85b) and (6.91b) prove to be more convenient. We first note the identity

$$\int_y^\infty ds\,s[\eta(s)]^{-2}[1-\eta^2(s)]^2$$
$$=-y^2[2\eta^2(y)]^{-1}\{[1-\eta^2(y)]^2-[\eta'(y)]^2\}. \tag{6.96}$$

Then from (6.85) and (6.96) we have

$$\int_\theta^\infty dy\,\tilde{F}(y)=-\frac{1}{2}\int_\theta^\infty dy\int_y^\infty ds\,\frac{s}{\eta^2}(1-\eta^2)^2$$
$$=-\frac{1}{2}\int_\theta^\infty ds\,\frac{s}{\eta^2}(1-\eta^2)^2(\ln s-\ln\theta)$$
$$=-(\theta^2/4\eta)[(1-\eta^2)^2-\eta'^2]\ln\theta$$
$$-\frac{1}{2}\int_\theta^\infty ds\ s\ln s\frac{1}{\eta^2}(1-\eta^2)^2. \tag{6.97}$$

The second term in (6.97) can be written

$$-\frac{1}{2}\int_\theta^\infty ds\ s\ \ln s(1/\eta^2)(1-\eta^2)^2$$
$$=\int_\theta^\infty ds\,s\ln s[1-\eta^2(s)]$$
$$+\frac{1}{2}\int_\theta^\infty ds\,s\ln s(\eta^2-\eta^{-2})\ . \tag{6.98}$$

If we make use of the identity

$$\frac{1}{s}\,\frac{d}{ds}\left(s\,\frac{\eta'}{\eta}\right)=\eta^2-\eta^{-2}, \tag{6.99}$$

the second term in (6.98) becomes

$$\frac{1}{2}\int_\theta^\infty ds\,s\ln s\,(\eta^2-\eta^{-2})$$
$$=\frac{1}{2}\int_\theta^\infty ds\ln s\,\frac{d}{ds}\left(s\,\frac{\eta'}{\eta}\right)$$
$$=-\frac{\theta}{2}\ln\theta\,\frac{\eta'(\theta)}{\eta(\theta)}+\tfrac{1}{2}\ln\eta(\theta). \tag{6.100}$$

Using (6.100) in (6.98) we obtain for (6.97)

$$\int_\theta^\infty dy\,\tilde{F}(y)=\int_\theta^\infty ds\,s\ln s[1-\eta^2(s)]$$
$$+\tfrac{1}{2}\ln\eta(\theta)-h(\theta), \tag{6.101}$$

where $h(\theta)$ is given by (2.41). Note that to derive (6.99) one must use (2.36). Substituting (6.101) into (6.85b) we obtain (2.39) [to obtain $F_+(t)$ we use (6.92)].

Using (2.37) one can show as $\theta\to 0$

$$\exp[-h(\theta)]=\theta^{-1/4}+o(1). \tag{6.102}$$

Combining (6.102) and (2.39) we see

$$\lim_{t\to 0} F_\pm(t)=2^{-1/2}(\sinh 2\beta_c E_1+\sinh 2\beta_c E_2)^{1/8}$$
$$\times\exp\int_0^\infty dx\,x\ln x[1-\eta^2(x)]\,. \tag{6.103}$$

From (5.10) we see that if (6.103) is to agree with the known value of $F_\pm(0)$ we must have the identity

$$\int_0^\infty dx\,x\ln x(1-\eta^2)=\tfrac{1}{4}+\tfrac{7}{12}\ln 2-3\ln A. \tag{6.104}$$

The importance of (6.104) is that the preceding analysis (Sec. VI A in particular) was to a large degree based upon insight gained from the large-t behavior of $F_\pm(t)$. Identity (6.104) shows in some sense that "nothing was left out" in the asymptotic estimates of Sec. VI A.

We have numerically solved (2.36) subject to the boundary conditions (2.37) and (2.38), and then numerically integrated the left-hand side of (6.104). The details of this numerical work can be found in Appendix A. Here we give the results:

$$\int_0^\infty ds\,s\ln s(1-\eta^2)=-0.919\,275\,757\,747\,071\times 10^{-2}$$
$$\text{(numerical work).} \tag{6.105}$$

The right-hand side of (6.104) is known to be

$$\tfrac{1}{4}+\tfrac{7}{12}\ln 2-3\ln A=-0.919\,275\,757\,747\,172\cdots\times 10^{-2}. \tag{6.106}$$

Thus we have verified (6.104) to 13 significant digits.

Dyson[38] has pointed out that Glaisher's constant A can be expressed in terms of the Riemann ζ function. To show this we use the representation[39]

$$\ln A=\tfrac{1}{12}+\lim_{N\to\infty}\left(\sum_{n=1}^{N-1} n\ln n-\tfrac{1}{2}(N^2-N+\tfrac{1}{6})\ln N+\tfrac{1}{4}N^2\right), \tag{6.107}$$

and the relation valid for $\mathrm{Re}\,s>-3$

$$\zeta(s)=\lim_{n\to\infty}\left(\sum_{m=1}^{n} m^{-s}-\frac{n^{1-s}}{1-s}-\tfrac{1}{2}n^{-s}+\tfrac{1}{12}sn^{-s-1}\right), \tag{6.108}$$

where $\zeta(s)$ is the Riemann ζ function.[40] Differentiation of (6.108) with respect to s and comparison with (6.107) shows that

$$\ln A=-\zeta'(-1). \tag{6.109}$$

Thus the identity (6.104) that remains to be *proved* is

$$\int_0^\infty dx\, x\ln x[1-\eta^2(x)]=\tfrac{1}{4}+\tfrac{7}{12}\ln 2+3\zeta'(-1). \tag{6.110}$$

D. $F_\pm(t)$ for $t\to 0$ and $t\to\infty$ as derived from $\eta(\theta)$ representation

From (2.23), (6.85b), and (6.91b) we have

$$F_\pm(t)=\frac{1\mp\eta(\theta)}{2[\eta(\theta)]^{1/2}}\exp\left(\int_\theta^\infty ds\,\tilde{F}(s)\right). \tag{6.111}$$

In Secs. III and IV we computed the large-t expansion of $\hat{F}_\pm(t)$. Here using (6.111) we derive (2.31a) and (2.31b).

For $t\to\infty$ we have

$$\eta(t/2)=1-(2/\pi)K_0(t)+(2/\pi^2)K_0^2(t)+O(e^{-3t}). \tag{6.112}$$

Using this in (6.85) we can show that as $t\to\infty$

$$\int_{t/2}^\infty dx\,\frac{x}{4\eta^2}[(1-\eta^2)^2-\eta'^2]$$
$$=\frac{t^2}{\pi^2}[K_1^2(t)-K_0^2(t)-t^{-1}K_0(t)K_1(t)]+O(e^{-3t}), \tag{6.113}$$

where we used the indefinite integrals[41]

$$\int xK_0^2(x)\,dx=\tfrac{1}{2}x^2[K_0^2(x)-K_1^2(x)], \tag{6.114}$$

$$\int xK_1^2(x)\,dx=\tfrac{1}{2}x^2[K_1^2(x)-K_0^2(x)-(2/x)K_0(x)K_1(x)].$$

Substituting (6.112) and (6.113) into (6.111) we obtain

$$\hat{F}_+(t)=\pi^{-1}K_0(t)+O(e^{-3t}), \tag{6.115}$$

$$\hat{F}_-(t)=\pi^{-2}\{t^2[K_1^2(t)-K_0^2(t)]-tK_0(t)K_1(t)+\tfrac{1}{2}K_0^2(t)\}+O(e^{-4t}). \tag{6.116}$$

To determine the small-t behavior of $F_\pm(t)$ from the Painlevé function representation, it is convenient to start with (2.39). In Appendix B we show that

$$\eta(\theta)=-\theta\Omega-\tfrac{1}{128}\theta^5(8\Omega^3-8\Omega^2+4\Omega-1)+O(\theta^9\Omega^5),\quad \theta\to 0, \tag{6.117}$$

where Ω is given by (2.33). Using (6.117) and (6.104) we have

$$\int_\theta^\infty ds\, s\ln s[1-\eta^2(s)]=\tfrac{1}{4}+\tfrac{7}{12}\ln 2-3\ln A-\int_0^\theta ds\, s\ln[1-\eta^2(s)] \tag{6.118a}$$

$$=\tfrac{1}{4}+\tfrac{7}{12}\ln 2-3\ln A-[I_{11}-I_{33}-2(\gamma_E-2\ln 2)I_{32}-(\gamma_E-2\ln 2)^2I_{31}+O(\theta^8\ln^5\theta)], \tag{6.118b}$$

with

$$I_{k,l}=\int_0^\theta x^k(\ln x)^l\,dx. \tag{6.119}$$

The integrals (6.119) are standard and we obtain for $t\to 0$

$$\int_\theta^\infty ds\, s\ln s[1-\eta^2(s)]=\tfrac{1}{4}+\tfrac{7}{12}\ln 2-3\ln A$$
$$-\{\tfrac{1}{2}\theta^2\ln\theta-\tfrac{1}{4}\theta^2-\tfrac{1}{4}\theta^4[\ln^3\theta-(\tfrac{3}{4}-2\lambda)\ln^2\theta$$
$$+(\tfrac{3}{8}-\lambda+\lambda^2)\ln\theta-(\tfrac{3}{32}-\tfrac{1}{4}\lambda+\tfrac{1}{4}\lambda^2)]\}+O(\theta^8\ln^5\theta), \tag{6.120}$$

with

$$\lambda=\gamma_E-2\ln 2. \tag{6.121}$$

Using (6.117) in (2.41) we obtain for $\theta\to 0$

$$h(\theta)=\ln\theta[\tfrac{1}{4}-\tfrac{1}{2}\theta^2+\tfrac{1}{32}\theta^4(8\Omega^2-4\Omega+1)+\cdots]. \tag{6.122}$$

Using (6.120) and (6.122) in (2.39) we obtain after expanding the exponential

$$F_\pm(t)=F(0)[1\pm\tfrac{1}{2}t\Omega+\tfrac{1}{16}t^2\pm\tfrac{1}{32}t^3\Omega+\tfrac{1}{256}t^4(-\Omega^2+\Omega+\tfrac{1}{8})+O(t^5\Omega^4)], \tag{6.123}$$

where (to summarize)

$$F(0)=(\sinh 2\beta_c E_1+\sinh 2\beta_c E_2)^{1/8}e^{1/4}2^{1/12}A^{-3},$$
$$\Omega=\ln(t/8)+\gamma_E, \tag{6.124}$$
$$\gamma_E=\text{Euler's constant}=0.577\,215\,664\cdots.$$

For the symmetric lattice the value of $F(0)$ is given by (5.10S).

In Sec. V we computed $F_\pm(t)$ for $t\to 0$ through $t^3\ln t$, and as we see from (2.32) the two methods agree (as they must, of course). The error term $O(t^5\Omega^4)$ in (6.123) follows from the remarks following (5.57). It may well happen that the coefficient of this term vanishes, but without explicit computation we know that the correction term cannot exceed $t^5(\ln t)^4$. One should also note that the method using the Painlevé function representation (2.39) is much easier to carry out than is the perturbation scheme given in Sec. V.

VII. SUSCEPTIBILITY $\chi(T)$

A. Leading divergence of $\chi(T)$ as $T\to T_c^\pm$

The zero-field susceptibility $\chi(T)$ defined by (2.42) is related to $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ by (2.43). As $T\to T_c^\pm$, $\chi(T)$ diverges, and it is this leading divergence we compute in this section. Since it is the large-M^2+N^2 behavior that determines the singularity of $\chi(T)$ [any finite sum of terms in (2.43) is finite and continuous at T_c], we may start the sum in (2.43) at any large, but finite, N_0 and M_0. Hence to leading order we write

$$\beta^{-1}\chi(T)\sim 4\sum_{N=N_0}^{\infty}\sum_{M=M_0}^{\infty}[\langle\sigma_{0,0}\sigma_{M,N}\rangle-\mathfrak{M}^2(T,0)]\,. \tag{7.1}$$

We consider the $T\to T_c^+$ case first. Recalling the definitions of $\bar{M}$ and $\bar{N}$ [see (3.126) and (3.127)] we may be leading order approximate (7.1) by an integral,

$$\beta^{-1}\chi(T)\sim 4(\tfrac{1}{4}\gamma_1\gamma_2)^{1/2}(z_1z_2+z_1+z_2-1)^{-2}\times\int_0^\infty d\bar{N}\int_0^\infty d\bar{M}\,\frac{F_+(t)}{R^{1/4}}\,, \tag{7.2}$$

where we have used (2.22). Recalling that $t^2=\bar{M}^2+\bar{N}^2$ we can write (7.2) as a single integral [use (2.5) also] and obtain

$$\beta^{-1}\chi(T)\sim 2\pi\,|z_1z_2+z_1+z_2-1|^{-7/4}(\tfrac{1}{4}\gamma_1\gamma_2)^{7/16}\times\int_0^\infty dt\,t^{3/4}F_+(t)\,. \tag{7.3}$$

Expanding the factors, multiplying the integral in (7.3) (using the various identities in Appendix C) we obtain, for $T\to T_c^+$,

$$\beta^{-1}\chi(T)\sim|1-T_c/T|^{-7/4}D'\int_0^\infty dt\,t^{3/4}F_+(t)=C_{0+}|1-T_c/T|^{-7/4}\,, \tag{7.4}$$

where

$$D'=2^{-13/8}\pi\{\beta_c[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}]^{-7/4}(z_{1c}z_{2c})^{-7/8}\,. \tag{7.5}$$

If we use (2.39) we can write the integral appearing in (7.4) as

$$\int_0^\infty dt\,t^{3/4}F_+(t)=(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}2^{5/4}\int_0^\infty d\theta\,\theta[1-\eta(\theta)]\exp\left(\int_\theta^\infty dx\,x\ln x(1-\eta^2)-h(\theta)\right)\,. \tag{7.6}$$

Combining (7.4)–(7.6) we obtain (2.46a). To obtain (2.48) from (7.5) and (7.6) the various identities in Appendix C must be used. Using (2.23) in (7.4) we also obtain for C_{0+} the expression

$$C_{0+}=D'2^{1/4}(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}\times\int_0^\infty dt\,t\hat{F}_+(t)\,. \tag{7.7}$$

If we use (4.154) we can compute C_{0+} perturbatively as

$$C_{0+}=\sum_{n=1}^{\infty}C_{0+}^{(2n-1)}\,, \tag{7.8}$$

with

$$C_{0+}^{(2n-1)}=D'2^{1/4}(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}\times\int_0^\infty dt\,t\hat{F}_+^{(2n-1)}(t) \tag{7.9a}$$

$$=2^{-3/2}(z_{1c}+z_{2c})^{1/4}(z_{1c}z_{2c})^{-1}\times\{\beta_c[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}]\}^{-7/4}I_{2n-1}^+\,, \tag{7.9b}$$

where we define

$$I_{2n-1}^+=\pi\int_0^\infty dt\,t\hat{F}_+^{(2n-1)}(t)\,dt\,, \tag{7.10}$$

and $\hat{F}_+^{(2n-1)}(t)$ is given by (4.156). From the analysis of the large-t behavior of $\hat{F}_+^{(2n-1)}(t)$ in Sec. IV we can conclude that $C_{0+}^{(2n-1)}$ represents the contribution to C_{0+} coming from that part of $\hat{F}_+(t)$ that behaves as $e^{-(2n-1)t}$ (times some power of t) for $t\to\infty$. Stated slightly differently, we have $C_{0+}^{(1)}$ being the contribution to C_{0+} coming from the Ornstein-Zernike pole, and $C_{0+}^{(2n-1)}$, $n\geq 2$, the contribution to C_{0+} coming from the $(2n-1)$-particle branch cut. This latter interpretation will become more evident when we examine the k-dependent susceptibility $\chi(\vec{k},T)$ in the scaling limit $k\to 0$, $T\to T_c$ such that $k|T-T_c|^{-1}$ is fixed.[42]

From (4.156a) and (7.10) for $n=1$ we can obtain[43]

$$I_1^+=\int_0^\infty dt\,tK_0(t)=1\,. \tag{7.11}$$

Using (4.159) we obtain for I_3^+ the expression

$$I_3^+ = 2^{-1}\pi^{-2}\int_1^\infty dy_1\int_1^\infty dy_2\int_1^\infty dy_3\left(\frac{y_2^2-1}{(y_1^2-1)(y_3^2-1)}\right)^{1/2}\times\left(\frac{y_1-y_3}{(y_1+y_2)(y_2+y_3)(y_1+y_2+y_3)}\right)^2 . \tag{7.12}$$

Numerically evaluating this we find

$$I_3^+ = 0.814\,462\,565\,5\ldots\times10^{-3} . \tag{7.13}$$

Using (4.156c) we also numerically study I_5^+ to obtain

$$I_5^+ = 0.7969\times10^{-6} . \tag{7.14}$$

For the symmetric lattice these results have already been summarized in (2.59S)–(2.61S) and Table I.

We now consider (7.1) for $T\to T_c^-$. In this case instead of (7.1) we have

$$\beta^{-1}\chi(T)\sim 4(\tfrac14\gamma_1\gamma_2)^{1/2}\,|z_1z_2+z_1+z_2-1|^{-2}\times\int_0^\infty d\bar N\int_0^\infty d\bar M\,[F_-(t)/R^{1/4}-(z_{1c}z_{2c})^{-1/8}\times 2^{1/8}(z_{1c}+z_{2c})^{1/4}t^{1/4}/R^{1/4}] , \tag{7.15}$$

where we used

$$S_\infty^< \sim (z_{1c}z_{2c})^{-1/8}[\sqrt2\,(z_{1c}+z_{2c})\,t/R]^{1/4}$$

(see Appendix C). Expanding the factors multiplying the integral, and using (2.5) we obtain

$$\beta^{-1}\chi(T)\sim |1-T_c/T|^{-7/4}D'\int_0^\infty dt\,[t^{3/4}F_-(t) - (\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}\,2^{1/4}t] , \tag{7.16}$$

with D' given by (7.5). Using (2.39) we can write the integral appearing in (7.16) as

$$\int_0^\infty dt\,[t^{3/4}F_-(t)-(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}2^{1/4}t]$$
$$=(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}2^{5/4}\int_0^\infty d\theta\,\theta\left[[1+\eta(\theta)]\times\exp\left(\int_\theta^\infty dx\,x\ln x(1-\eta^2)-h(\theta)\right)-2\right] . \tag{7.17}$$

From (7.16) and (7.17) follows (2.46b).

Using (2.23) in (7.16) we obtain for C_{0-} the expression

$$C_{0-}=D'2^{1/4}(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}\times\int_0^\infty dt\,t[\hat F_-(t)-1] . \tag{7.18}$$

If we use (3.151) we can compute C_{0-} perturbatively as

$$C_{0-}=\sum_{n=1}^\infty C_{0-}^{(2n)} , \tag{7.19}$$

with

$$C_{0-}^{(2n)}=D'2^{1/4}(\sinh 2\beta_cE_1+\sinh 2\beta_cE_2)^{1/8}\times\int_0^\infty dt\,t\hat F_-^{(2n)}(t)=2^{1/2}(z_{1c}+z_{2c})^{1/4}(z_{1c}z_{2c})^{-1}\times\{\beta_c[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}]\}^{7/4}I_{2n}^- , \tag{7.20}$$

where we define

$$I_{2n}^-=\pi\int_0^\infty dt\,t\hat F_-^{(2n)}(t) , \tag{7.21}$$

and $\hat F_-^{(2n)}(t)$ is given by (3.152).

From (3.150), (3.152), and (7.21) for $n=1$ we obtain

$$I_2^-=\frac1\pi\int_0^\infty dt\,\{t^3[K_1^2(t)-K_0^2(t)] - t^2K_0(t)K_1(t)+\tfrac12 tK_0^2(t)\} = (12\pi)^{-1}=0.026\,525\,823\,8\ldots , \tag{7.22}$$

where we have used[44]

$$\int_0^\infty K_\mu(t)K_\nu(t)t^\rho\,dt=\frac{2^{\rho-2}}{\Gamma(1+\rho)}\Gamma[\tfrac12(1+\nu+\mu+\rho)]\Gamma[\tfrac12(1+\nu-\mu+\rho)]\Gamma[\tfrac12(1-\nu+\mu+\rho)]\Gamma[\tfrac12(1-\nu-\mu+\rho)].$$

Using (3.153) and (7.21) for $n=2$ gives

$$I_4^-=(4\pi^3)^{-1}\int_1^\infty dy_1\cdots\int_1^\infty dy_4\left(\frac{(y_2^2-1)(y_4^2-1)}{(y_1^2-1)(y_3^2-1)}\right)^{1/2}\left(\frac{(y_2-y_4)(y_1-y_3)}{(y_1+y_2)(y_2+y_3)(y_3+y_4)(y_4+y_1)(y_1+y_2+y_3+y_4)}\right)^2 . \tag{7.23}$$

Evaluating this numerically we find

$$I_4^-=0.254\,48\times10^{-4} . \tag{7.24}$$

B. $\chi(T)$ for $T<T_c$

Using (2.9) and (2.11) in (2.43) we see that for $T<T_c$ we can expand $\chi(T)$ as

$$\beta^{-1}\chi(T)=\sum_{n=1}^\infty \chi_<^{(2n)}(T)\quad (T<T_c) , \tag{7.25}$$

where

$$\chi_<^{(2)}(T)=-S_\infty^<\sum_{M=-\infty}^{+\infty}\sum_{N=-\infty}^{+\infty}F_{<MN}^{(2)} , \tag{7.26a}$$

$$\chi^{(4)}(T)=-S_\infty^< \sum_{M=-\infty}^{+\infty}\sum_{N=-\infty}^{+\infty}[F_{<MN}^{(4)}-\tfrac{1}{2}(F_{<MN}^{(2)})^2]\,, \tag{7.26b}$$

$$\chi^{(6)}(T)=-S_\infty^< \sum_{M=-\infty}^{+\infty}\sum_{N=-\infty}^{+\infty}[F_{<MN}^{(6)} - F_{<MN}^{(2)}F_{<MN}^{(4)}+(1/3!)(F_{<MN}^{(2)})^3]\,, \tag{7.26c}$$

etc. and with $F_{<MN}^{(2n)}$ given by (2.12). In the Sec. VII A we analyzed the leading divergence of $\chi(T)$ as $T\to T_c^\pm$. Here we analyze the next diverging term, viz., $|1-T_c/T|^{-3/4}$. A general term in the expansion (7.26) of $\chi_<^{(2n)}(T)$ is of the form

$$\sum_{M=-\infty}^{+\infty}\sum_{N=-\infty}^{+\infty}F_{<MN}^{(2\alpha_1)}\cdots F_{<MN}^{(2\alpha_p)}\,, \tag{7.27a}$$

with

$$\alpha_1+\alpha_2+\cdots+\alpha_p=n\,,\quad p\le n\,. \tag{7.27b}$$

For $\mathrm{Im}\phi<0$ we have

$$\sum_{l=-\infty}^{+\infty}e^{-i|l|\phi}=\frac{1+e^{-i\phi}}{1-e^{-i\phi}}=\cot\tfrac{1}{2}\phi\,. \tag{7.28}$$

Using (7.28) when we substitute (2.12) into (7.27a) we see that (7.3a) is of the form

$$(-1)^n[2z_2(1-z_1^2)]^{2n}(2^p\alpha_1\cdots\alpha_p)^{-1}(2\pi)^{-4n}\times\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_{4n}\cot\tfrac{1}{2}(\phi_1+\phi_3+\cdots+\phi_{4n-1})\times\cot\tfrac{1}{2}(\phi_2+\phi_4+\cdots+\phi_{4n})S_{\alpha_1}S_{\alpha_2}\cdots S_{\alpha_p}\,, \tag{7.29}$$

where

$$S_{\alpha_1}=S_{\alpha_1}(\phi_1\cdots\phi_{4\alpha_1})$$

$$=\prod_{j=1}^{2\alpha_1}\frac{1}{\Delta(\phi_{2j-1},\phi_{2j})}\frac{\sin\frac{1}{2}(\phi_{2j-1}-\phi_{2j+1})}{\sin\frac{1}{2}(\phi_{2j}+\phi_{2j+2})}\,. \tag{7.30}$$

We now analyze (7.29) for $T\to T_c^-$. From arguments identical to those in Sec. III G we see that the factor $S_{\alpha_1}\cdots S_{\alpha_p}d\phi_1\cdots d\phi_{4n}$ leads to an $O(1)$ quantity with a correction term of order $(\beta-\beta_c)^2$. The quantities $\cot\frac{1}{2}(\phi_1+\cdots+\phi_{4n-1})$ and $\cot\frac{1}{2}(\phi_2+\cdots+\phi_{4n})$ in (7.29) diverge as $(\beta-\beta_c)^{-2}$ with a correction term of order $(\beta-\beta_c)^2$. Hence as $T\to T_c^-$ we can expect (7.27a) to have a divergence of order $(\beta-\beta_c)^{-2}$ with a correction term smaller than order $(\beta-\beta_c)^{-1}$. That is (7.27a) will diverge like $|1-T_c/T|^{-7/4}$ but will not have a term of the form $|1-T_c/T|^{-3/4}$. The quantity $S_\infty^<$ behaves as

$$S_\infty^<\sim\mathrm{const}(\beta-\beta_c)^{1/4}[1+R_0\beta_c(\beta/\beta_c-1)+\cdots](\beta-\beta_c)\,, \tag{7.31}$$

where the constant R_0 is given by (2.49). Using (7.31) and the fact that (7.27a) has no divergence of order $(\beta-\beta_c)^{-1}$ we must conclude that $\chi_<^{(2n)}(T)$ has two diverging terms, viz., $(\beta-\beta_c)^{-7/4}$ and $(\beta-\beta_c)^{-3/4}$, where the second diverging term, i.e., $(\beta-\beta_c)^{-3/4}$, comes from the *leading* divergence of (7.27a) times the second term in (7.31). Thus if we write, as $T\to T_c^-$,

$$\chi_<^{(2n)}(T)=C_{0-}^{(2n)}|1-T_c/T|^{-7/4}+C_{1-}^{(2n)}|1-T/T_c|^{-3/4}+O(1)\,, \tag{7.32}$$

then $C_{1-}^{(2n)}$ is just $C_{0-}^{(2n)}$ times the constant $R_0\beta_c$. From (7.25) we can therefore conclude that C_{1-} is related to C_{0-} by (2.47). In the previous section we derived an *exact* formula [see (2.46b)] for C_{0-}; and hence, by (2.47) we have an exact formula for C_{1-}. For the symmetric lattice these results have been summarized in Table I.

Using (2.12) for $n=1$ in (7.26a) we obtain

$$\beta^{-1}\chi_<^{(2)}(T)=S_\infty^<\frac{\gamma_1\gamma_2}{2(2\pi)^4}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_4[\Delta(\phi_1,\phi_2)\Delta(\phi_3,\phi_4)]^{-1}\frac{1+e^{-i(\phi_1+\phi_3)}}{1-e^{-i(\phi_1+\phi_3)}}\frac{1+e^{-i(\phi_2+\phi_4)}}{1-e^{-i(\phi_2+\phi_4)}}\times\frac{(e^{i\phi_1}-e^{i\phi_3})(e^{i\phi_2}-e^{i\phi_4})}{(e^{i(\phi_1+\phi_3)}-1)(e^{i(\phi_2+\phi_4)}-1)}, \tag{7.33}$$

where in (7.33) we have expanded the contours of integration to the unit circle except that $\phi_1(\phi_2)$ is to be indented inward at $-\phi_3(-\phi_4)$. Noting that the integrand in (7.33) is odd under the transformation $\phi_{1,3}\to-\phi_{1,3}$ $(\phi_{2,4}\to-\phi_{2,4})$, we see that the contribution from the $\phi_1(\phi_2)$ integration comes from the residue of the pole at $-\phi_3(-\phi_4)$. Hence

$$\beta^{-1}\chi_<^{(2)}(T)=\tfrac{1}{8}S_\infty^<\gamma_1\gamma_2(2\pi i)^{-2}\oint\frac{d\xi_3}{\xi_3}\oint\frac{d\xi_2}{\xi_2}\frac{\partial}{\partial\xi_1}\left(\frac{(\xi_1+\xi_3^{-1})(\xi_3^{-1}-\xi_1^{-1})}{\Delta(\xi_1,\xi_2)}\right)\bigg|_{\xi_1=\xi_3^{-1}}\frac{\partial}{\partial\xi_4}\left(\frac{(\xi_4+\xi_2^{-1})(\xi_4^{-1}-\xi_2^{-1})}{\Delta(\xi_3,\xi_4)}\right)\bigg|_{\xi_4=\xi_2^{-1}}. \tag{7.34}$$

Performing the indicated differentiations in (7.34) gives

$$\beta^{-1}\chi_<^{(2)}(T)=-\tfrac{1}{8}S_\infty^<\frac{\gamma_1\gamma_2}{(2\pi)^2}\int_{-\pi}^{\pi}d\phi_1\int_{-\pi}^{\pi}d\phi_2\left(\frac{2\cos\phi_1}{\Delta(\phi_1,\phi_2)}-\frac{4\gamma_1\sin^2\phi_1}{\Delta^2(\phi_1,\phi_2)}\right)\left(\frac{2\cos\phi_2}{\Delta(\phi_1,\phi_2)}-\frac{4\gamma_2\sin^2\phi_2}{\Delta^2(\phi_1,\phi_2)}\right)\,. \tag{7.35}$$

Using the identity

$$\frac{\partial}{\partial\phi_1}\frac{\partial}{\partial\phi_2}\Delta^{-2}(\phi_1,\phi_2)=6\gamma_1\gamma_2\frac{\sin\phi_1\sin\phi_2}{\Delta^4(\phi_1,\phi_2)}\,, \qquad (7.36)$$

we have

$$\gamma_1\gamma_2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\phi_1 d\phi_2\,\frac{\sin^2\phi_1\sin^2\phi_2}{\Delta^4(\phi_1,\phi_2)} =\frac{1}{6}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\phi_1 d\phi_2\,\frac{\cos\phi_1\cos\phi_2}{\Delta^2(\phi_1,\phi_2)}\,. \qquad (7.37)$$

Using this and similar identities for the $\sin^2\phi_1 \cos\phi_2$ and $\sin^2\phi_2\cos\phi_1$ terms, (7.35) becomes

$$\beta^{-1}\chi_<^{(2)}(T)=\tfrac{1}{6}S_\infty^<\,\frac{\gamma_1\gamma_2}{(2\pi)^2}\int_{-\pi}^{\pi}d\phi_1 \times\int_{-\pi}^{\pi}d\phi_2\,\frac{\cos\phi_1\cos\phi_2}{\Delta^2(\phi_1,\phi_2)}\,. \qquad (7.38)$$

In Fig. 10 we plot $\beta^{-1}\chi_<^{(2)}(T)$ for two values of E_1/E_2 as a function of $(1-T/T_c)$.

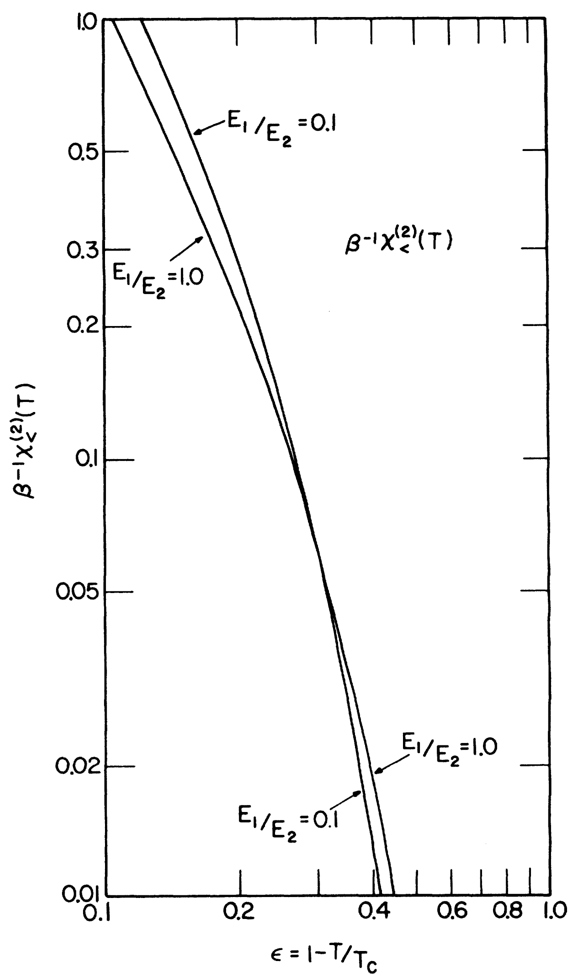

FIG. 10. Function $\beta^{-1}\chi_<^{(2)}(T)$ as a function of $\epsilon=1-T/T_c$ for $E_1/E_2=1.0$ and $E_1/E_2=0.1$.

TABLE IV. Low-temperature series expansion of $\beta^{-1}\chi(T)$ [see (7.40)] is compared with the low-temperature series expansion $\beta^{-1}\chi_<^{(2)}(T)$ [see (7.42)]. The coefficients h_n are taken from Ref. 45.

n	h_n	$h_n^{(2)}$
1	8	8
2	60	60
3	416	416
4	2791	2791
5	18 296	18 296
6	118 016	118 012
7	752 008	751 944
8	4 746 341	4 745 661
9	29 727 472	29 721 472
10		184 968 932

For the symmetric lattice the integrals in (7.38) can be expressed in terms of the complete elliptic integrals $E(k)$ and $K(k)$. The result is

$$\beta^{-1}\chi_<^{(2)}(T)=\frac{S_\infty^<}{6\pi}\left[\frac{a^2-2\gamma^2}{a^2-4\gamma^2}\,E\!\left(\frac{2\gamma}{a}\right)-K\!\left(\frac{2\gamma}{a}\right)\right]\,, \qquad (7.39)$$

where $\gamma=\gamma_1=\gamma_2$, and so as $T\to T_c$ we have $2\gamma/a\to 1$. Expanding this about $T=T_c$ we see that after the $|1-T_c/T|^{-7/4}$ and $|1-T_c/T|^{-3/4}$ terms, there is a term proportional to $|1-T_c/T|^{1/4}\ln|1-T_c/T|$.

It is instructive to compare the low-temperature expansion of $\beta^{-1}\chi_<^{(2)}(T)$ with the low-temperature expansion of $\beta^{-1}\chi(T)$. Following Essam and Fisher[45] we write (see also Sykes, Gaunt, Martin, Mattingly, and Essam[45b]) for the isotropic lattice

$$\beta^{-1}\chi(T)=4u^2\left(1+\sum_{n=1}^{\infty}h_n u^n\right)\,, \qquad (7.40)$$

with

$$u=e^{-4\beta E}\,. \qquad (7.41)$$

In Ref. 45b the quantities h_n are computed for $n=1, 2, \ldots, 9$. We now make an analogous low-temperature expansion of $\beta^{-1}\chi_<^{(2)}(T)$:

$$\beta^{-1}\chi_<^{(2)}(T)=4u^2\left(1+\sum_{n=1}^{\infty}h_n^{(2)}u^n\right)\,. \qquad (7.42)$$

In computing $h_n^{(2)}$ we expanded the denominator in (7.38) and integrated term by term. Our results are displayed in Table IV. One sees that $h_n=h_n^{(2)}$ for $n=1, 2, \ldots, 5$; $h_6-h_6^{(2)}=4$, and $h_7-h_7^{(2)}=64$. Thus the low-temperature expansion of $\beta^{-1}\chi(T)$ is dominated in the early terms by contributions to $\beta^{-1}\chi(T)$ coming from the two-particle cut, i.e., $\beta^{-1}\chi_<^{(2)}(T)$. From (2.12) we see that the earliest the $2n$th branch cut can contribute to the coefficients h_k is for $k\geq n$.

C. $\chi(T)$ for $T>T_c$

Using (2.14)–(2.16) in (2.43) we see that for $T>T_c$ we can expand $\chi(T)$ as

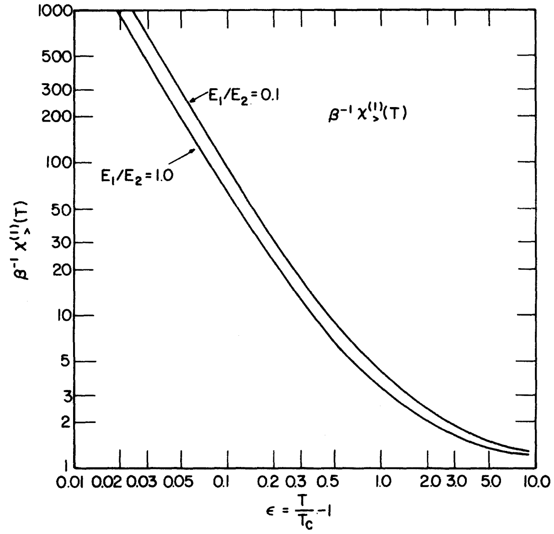

FIG. 11. Function $\beta^{-1}\chi_>^{(1)}(T)$ as a function of $\epsilon = T/T_c - 1$ for $E_1/E_2=1.0$ and $E_1/E_2=0.1$.

$$\beta^{-1}\chi(T)=\sum_{n=1}^{\infty}\chi_>^{(2n-1)}(T)\ , \tag{7.43}$$

where

$$\chi_>^{(1)}(T)=S_\infty^>\sum_{M=-\infty}^{\infty}\sum_{N=-\infty}^{+\infty}x_{>MN}^{(1)}\ , \tag{7.44a}$$

$$\chi_>^{(3)}(T)=S_\infty^>\sum_{M=-\infty}^{\infty}\sum_{N=-\infty}^{\infty}\left[x_{>MN}^{(3)}-x_{>MN}^{(1)}F_{>MN}^{(2)}\right]\ , \tag{7.44b}$$

$$\chi_>^{(5)}(T)=S_\infty^>\sum_{M=-\infty}^{\infty}\sum_{N=-\infty}^{\infty}\left[x_{>MN}^{(5)}-F_{>MN}^{(2)}x_{>MN}^{(3)}-x_{>MN}^{(1)}F_{>MN}^{(4)}+\tfrac{1}{2}x_{>MN}^{(1)}(F_{>MN}^{(2)})^2\right]\ , \tag{7.44c}$$

etc.

The arguments needed to relate C_{0+} and C_{1+} [see (2.47)] are similar to those given in Sec. VII B relating C_{0-} and C_{1-}. Hence we consider (2.47) established.

Using (2.15) for $k=1$ and (7.44a) we have

$$\beta^{-1}\chi_>^{(1)}(T)=S_\infty^>(a-\gamma_1-\gamma_2)^{-1}\ , \tag{7.45}$$

which for the symmetric lattice is

$$\beta^{-1}\chi_>^{(1)}(T)=(1-z^2)(1-4z^2-10z^4-4z^6+z^8)^{1/4}\times(1-2z-z^2)^{-2}\ , \tag{7.45a}$$

with $z=\tanh\beta E$. In Fig. 11 we plot $\beta^{-1}\chi_>^{(1)}(T)$ for two values of E_1/E_2 as a function of (T/T_c-1).

It is instructive to compare the high-temperature expansion of $\beta^{-1}\chi(T)$ with the high-temperature expansion of $\beta^{-1}\chi_>^{(1)}(T)$. We write for the symmetric lattice

$$\beta^{-1}\chi(T)=\sum_{n=0}^{\infty}\chi_n z^n \tag{7.46}$$

and

$$\beta^{-1}\chi_>^{(1)}(T)=\sum_{n=0}^{\infty}\chi_n^{(1)}z^n\ . \tag{7.47}$$

The coefficients χ_n in (7.46) have been computed for $0\le n\le 21$ by Sykes *et al.*[14] In Table V we compare χ_n and $\chi_n^{(1)}$. We see that the first seven terms and the ninth term are identical. Clearly the lowest-lying excitation dominates the low-order coefficients in the high-temperature series expansion of $\beta^{-1}\chi(T)$. From the fact that $x_{>MN}^{(k)}$ in (2.15) is proportional to z^{k-1}, we see that the earliest the effect of the kth branch cut can have on the expansion (7.46) is at the $(k-1)$th order. From Table V we see that this is a poor bound.

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APPENDIX A: NUMERICAL WORK

We solved (2.36)–(2.38) by a standard Runge-Kutta method. The technique consisted in using both the large-θ and small-θ expansions of $\eta(\theta)$ and $\eta'(\theta)$ to obtain the initial conditions. For instance, by using all terms through order θ^{13} in the small-θ expansion of $\eta(\theta)$ [and the correspond-

TABLE V. High-temperature series expansion of $\beta^{-1}\chi(T)$ [see (7.46)] is compared with the high-temperature series expansion of $\beta^{-1}\chi_>^{(1)}(T)$ [see (7.47)]. The coefficients χ_n are taken from Ref. 14a.

n	χ_n	$\chi_n^{(1)}$	$\chi_n-\chi_n^{(1)}$
0	1	1	0
1	4	4	0
2	12	12	0
3	36	36	0
4	100	100	0
5	276	276	0
6	740	740	0
7	1 972	1 972	0
8	5 172	5 168	4
9	13 492	13 492	0
10	34 876	34 844	32
11	89 764	89 748	16
12	229 628	229 420	208
13	585 508	585 316	192
14	1 486 308	1 484 980	1328
15	3 763 460	3 761 860	1600
16	9 497 380	9 488 960	8420
17	23 918 708	23 906 884	11 824
18	60 080 156	60 027 340	52 816
19	150 660 388	150 577 892	82 568
20	377 009 300	376 680 820	328 480
21	942 105 604	941 551 252	554 352

ing expansion for $\eta'(\theta)$] we started the forward integration at $\theta_{start}=0.013$ with the initial values obtained by evaluating these small-θ expansions at $\theta=\theta_{start}$. For large we solved for

$$\epsilon(\theta)=1-\eta(\theta) \tag{A1}$$

by starting the backward integration at $\theta=10$. For initial conditions we used

$$\epsilon(\theta)=(2/\pi)K_0(2\theta)-(2/\pi^2)K_0^2(2\theta), \tag{A2}$$

$$\epsilon'(\theta)=-(4/\pi)K_1(2\theta)+(8/\pi^2)K_0(2\theta)K_1(2\theta) \tag{A3}$$

for $\theta=10$. We found that the forward and backward integration programs had an optimal overlap at $\theta=0.8$. The two values of $\eta(\eta')$ agreed to 1(3) digit(s) in the 13th place. In Table VI we present our numerical solution to (2.36)–(2.38). For $\theta \geq 0.9$ we display $\epsilon(\theta)$ rather than $\eta(\theta)$. In Fig. 12 we plot $\eta(\theta)$ and $\eta'(\theta)$ as a function of θ.

Once we obtained accurate $\eta(\theta)$ and $\eta'(\theta)$, (2.39) could be numerically integrated to give the scaling functions $F_\pm(t)$. In Table VII we give for the symmetric lattice the scaling functions $F_\pm(t)$. To obtain $F_\pm(t)$ for arbitrary E_1 and E_2 the entry in Table VII should be multiplied by $2^{-1/8}(\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8}$. For $E_1=E_2$ this quantity is one, as it must be. The susceptibility coefficients $C_{0\pm}$ were obtained by numerically integrating (2.36).

In Fig. 13 we plot the function $I(\theta)$ defined by

$$I(\theta)=\int_\theta^\infty dx\, x\ln x[1-\eta^2(x)]. \tag{A4}$$

APPENDIX B: INTEGRAL EQUATION AND PAINLEVÉ FUNCTION $\eta(\theta)$

In this appendix we discuss integral equations of the form

$$\int_a^b K_0(\theta|x-t|)\,\psi(t)\,dt=\phi(x), \tag{B1}$$

where a and b are finite real numbers, $a<x<b$, and $K_0(x)$ is the modified Bessel function of zeroth order. The cases of one or both of the end points infinite are solved by standard methods.

Equation (B1) has been the subject of considerable investigation as it arises in the context of scattering from a thin strip. Since the geometry of the strip is a degenerate case of an elliptic cylinder, it is not surprising that the solution to (B1) can be expressed as infinite series of Mathieu functions. Unfortunately the representation of a function as an infinite series of Mathieu functions is cumbersome at best.

Progress toward the solution of (B1) was made by Latta[46] when he showed that solutions to

$$\Gamma f=\cosh\theta t, \tag{B2a}$$

$$\Gamma g=\sinh\theta t, \tag{B2b}$$

where we define the operator Γ by

$$(\Gamma\phi)(t)=\int_{-1}^1 K_0(\theta|t-s|)\,\phi(s)\,ds, \tag{B3}$$

satisfy a pair of linear first order coupled differential equations. Latta did not determine the coefficients in this set of differential equations (these coefficients will be a function of θ). This Myers[34,35] did by imposing the boundary conditions that are implicit in (B2) on the set of differential equations. These results can be summarized by a sequence of theorems.[47]

Theorem 1 (Latta-Myers)

If we define the operator Γ by (B3) and if $f(t)$ and $g(t)$ are the solution to (B2), then we have

$$(1-t^2)\frac{df}{dt}-(\tfrac12-\rho)tf=[\eta^{-1}(\rho+\tfrac12)+\theta(1-t^2)]g, \tag{B4a}$$

$$(1-t^2)\frac{dg}{dt}-(\tfrac12+\rho)tg=[\eta(\tfrac12-\rho)+\theta(1-t^2)]f, \tag{B4b}$$

where

$$\rho=\theta[2\eta(\theta)]^{-1}\left(1-\eta^2(\theta)-\frac{d\eta}{d\theta}\right) \tag{B5a}$$

and

$$\eta(\theta)=\lim_{t\to1}\frac{g(t)}{f(t)}. \tag{B5b}$$

Theorem 2 (Myers)

The function $\eta(\theta)$, (B5b), satisfies a Painlevé equation of third kind[33]

$$\frac{d^2\eta}{d\theta^2}=\eta^{-1}(\theta)\left(\frac{d\eta}{d\theta}\right)^2-\eta^{-1}(\theta)+\eta^3(\theta)-\theta^{-1}\frac{d\eta}{d\theta}. \tag{B6a}$$

Equation (B6a) is subject to the boundary conditions

$$\eta(\theta)=-\theta(\ln\tfrac14\theta+\gamma_E)+O(\theta^5\ln^3\theta) \tag{B6b}$$

as $\theta\to0$, and

$$\eta(\theta)=1-(2/\pi)K_0(2\theta)+O(e^{-4\theta}) \tag{B6c}$$

as $\theta\to\infty$. In (B6b) γ_E is Euler's constant.

In Sec VI we will need expansions of $f(t)$ and $g(t)$ about the singular point $t=1$. Since this point is a regular singular point, we can easily find from (B4)

$$f(t,\theta)=\frac{f_0(\theta)}{(1-t)^{1/2}}\left(1+\sum_{n=1}^\infty f_n(\theta)(1-t)^n\right), \tag{B7a}$$

$$g(t,\theta)=\frac{f_0(\theta)\,\eta(\theta)}{(1-t)^{1/2}}\left(1+\sum_{n=1}^\infty g_n(\theta)(1-t)^n\right), \tag{B7b}$$

with

$$\begin{aligned} f_1&=-\tfrac12(\tfrac32+\rho)(\rho+2\theta\eta)-\tfrac12(\tfrac12+\rho)(\rho-2\theta/\eta),\\ f_2&=\tfrac1{24}f_1[(\tfrac72+\rho)(\tfrac12-\rho)+(\tfrac14-\rho^2)+(4\theta/\eta)(\tfrac12+\rho)] \end{aligned} \tag{B7c}$$

TABLE VI. Painlevé function $\eta(\theta)$ and its first derivative $\eta'(\theta)$ are given for $0 \le \theta \le 8$. For $0 \le \theta \le 0.8$ the entry in the second column is $\eta(\theta)$, and for $\theta \ge 0.9$ the entry in the second column is $\epsilon(\theta) = 1 - \eta(\theta)$. The entry in the third column is for $0 \le \theta \le 8$. If an entry is followed by a number enclosed in a parenthesis, e.g., $(-n)$, then this entry should be multiplied by 10^{-n}. Thus, for instance, the value of $\eta(\theta)$ at $\theta = 0.001$ is $0.771\,683\,397\,5 \times 10^{-2}$ and the value of $\eta'(\theta)$ at $\theta = 0.001$ is $0.671\,683\,397\,5 \times 10^{1}$.

θ	$\eta(\theta)$	$\eta'(\theta)$
0.0	0	∞
0.001	0.771 683 397 5 (−2)	0.671 683 397 5 (1)
0.002	0.140 473 735 9 (−1)	0.602 368 679 6 (1)
0.003	0.198 546 650 6 (−1)	0.561 822 169 4 (1)
0.004	0.253 221 584 8 (−1)	0.533 053 963 6 (1)
0.005	0.305 369 803 7 (−1)	0.510 739 611 0 (1)
0.006	0.355 504 471 6 (−1)	0.492 507 459 6 (1)
0.007	0.403 964 670 2 (−1)	0.477 092 398 1 (1)
0.008	0.450 991 399 1 (−1)	0.463 739 268 0 (1)
0.009	0.496 764 853 2 (−1)	0.451 960 976 9 (1)
0.010	0.541 424 900 1 (−1)	0.441 424 941 7 (1)
0.011	0.585 083 275 5 (−1)	0.431 893 944 5 (1)
0.012	0.627 831 306 0 (−1)	0.423 192 832 8 (1)
0.013	0.669 745 038 8 (−1)	0.415 188 593 8 (1)
0.014	0.710 888 784 0 (−1)	0.407 777 834 8 (1)
0.015	0.751 317 638 9 (−1)	0.400 878 592 9 (1)
0.016	0.791 079 337 4 (−1)	0.394 424 793 9 (1)
0.017	0.830 215 634 1 (−1)	0.388 362 393 4 (1)
0.018	0.868 763 362 4 (−1)	0.382 646 622 8 (1)
0.019	0.906 755 256 6 (−1)	0.377 239 981 4 (1)
0.02	0.944 220 600 3 (−1)	0.372 110 743 4 (1)
0.03	0.129 469 251 6	0.331 565 894 0 (1)
0.04	0.161 118 716 6	0.302 801 233 6 (1)
0.05	0.190 241 946 6	0.280 493 065 3 (1)
0.06	0.217 352 406 9	0.262 270 476 5 (1)
0.07	0.242 789 553 0	0.246 869 068 7 (1)
0.08	0.266 794 827 5	0.233 534 360 9 (1)
0.09	0.289 548 934 9	0.221 779 899 8 (1)
0.10	0.311 192 313 4	0.211 273 700 4 (1)
0.11	0.331 837 327 7	0.201 779 101 5 (1)
0.12	0.351 575 986 6	0.193 121 465 1 (1)
0.13	0.370 485 069 2	0.185 168 252 4 (1)
0.14	0.388 629 662 5	0.177 816 503 9 (1)
0.15	0.406 065 682 5	0.170 984 651 0 (1)
0.16	0.422 841 718 8	0.164 606 975 8 (1)
0.17	0.439 000 415 9	0.158 629 753 6 (1)
0.18	0.454 579 527 2	0.153 008 501 0 (1)
0.19	0.469 612 733 7	0.147 705 968 5 (1)
0.20	0.484 130 289 6	0.142 690 649 9 (1)
0.21	0.498 159 537 0	0.137 935 655 4 (1)
0.22	0.511 725 323 5	0.133 417 848 0 (1)
0.23	0.524 850 341 5	0.129 117 171 6 (1)
0.24	0.537 555 409 7	0.125 016 123 9 (1)
0.25	0.549 859 706 3	0.121 099 335 3 (1)
0.26	0.561 780 964 6	0.117 353 232 4 (1)
0.27	0.573 335 639 2	0.113 765 764 5 (1)
0.28	0.584 539 045 8	0.110 326 179 9 (1)
0.29	0.595 405 482 3	0.107 024 841 9 (1)
0.30	0.605 948 332 3	0.103 853 076 6 (1)
0.31	0.616 180 154 6	0.100 803 044 9 (1)

TABLE VI. (*Continued*)

θ	$\eta(\theta)$ [$\epsilon(\theta)$]	$\eta'(\theta)$
0.32	0.626 112 761 3	0.978 676 356 8
0.33	0.635 757 286 1	0.950 403 755 5
0.34	0.645 124 243 7	0.923 153 515 9
0.35	0.654 223 583 2	0.896 871 460 6
0.36	0.663 064 733 8	0.871 507 800 5
0.37	0.671 656 647 2	0.847 016 650 6
0.38	0.680 007 833 5	0.823 355 610 6
0.39	0.688 126 395 2	0.800 485 401 0
0.40	0.696 020 055 8	0.778 369 546 2
0.41	0.703 696 187 2	0.756 974 096 4
0.42	0.711 161 833 3	0.736 267 384 6
0.43	0.718 423 731 8	0.716 219 812 1
0.44	0.725 488 334 1	0.696 803 659 0
0.45	0.732 361 822 9	0.677 992 916 7
0.46	0.739 050 128 8	0.659 763 138 8
0.47	0.745 558 945 3	0.642 091 308 7
0.48	0.751 893 742 3	0.624 955 721 0
0.49	0.758 059 778 7	0.608 335 875 5
0.50	0.764 062 114 2	0.592 212 382 7
0.52	0.775 594 986 9	0.561 381 942 2
0.54	0.786 529 235 3	0.532 328 441 8
0.56	0.796 899 149 2	0.504 929 014 3
0.58	0.806 736 677 3	0.479 072 130 5
0.60	0.816 071 638 5	0.454 656 146 0
0.62	0.824 931 907 6	0.431 588 090 3
0.64	0.833 343 578 0	0.409 782 647 8
0.66	0.841 331 106 1	0.389 161 295 9
0.68	0.848 917 439 5	0.369 651 571 6
0.70	0.856 124 131 9	0.351 186 442 4
0.72	0.862 971 445 9	0.333 703 764 2
0.74	0.869 478 446 4	0.317 145 811 6
0.76	0.875 663 084 3	0.301 458 867 9
0.80	0.887 131 959 7	0.272 501 075 4
0.90	0.887 641 364 4 (−1)	0.212 220 235 0
1.0	0.699 637 109 1 (−1)	0.165 787 696 0
1.1	0.552 571 023 1 (−1)	0.129 868 027 9
1.2	0.437 230 351 6 (−1)	0.101 980 718 6
1.3	0.346 558 553 8 (−1)	0.802 614 032 9 (−1)
1.4	0.275 125 892 1 (−1)	0.632 981 360 9 (−1)
1.5	0.218 738 225 1 (−1)	0.500 153 640 0 (−1)
1.6	0.174 144 827 7 (−1)	0.395 898 923 7 (−1)
1.7	0.138 818 467 9 (−1)	0.313 891 168 8 (−1)
1.8	0.110 788 873 6 (−1)	0.249 251 486 7 (−1)
1.9	0.885 160 017 3 (−2)	0.198 204 800 8 (−1)
2.0	0.707 931 889 0 (−2)	0.157 821 321 1 (−1)
2.2	0.454 093 067 1 (−2)	0.100 424 345 9 (−1)
2.4	0.292 242 112 9 (−2)	0.641 765 253 0 (−2)
2.6	0.188 627 153 6 (−2)	0.411 658 230 3 (−2)
2.8	0.122 060 623 8 (−2)	0.264 920 767 5 (−2)
3.0	0.791 637 912 4 (−3)	0.170 977 737 3 (−2)
3.2	0.514 453 726 3 (−3)	0.110 626 523 1 (−2)
3.4	0.334 919 280 2 (−3)	0.717 379 331 8 (−3)
3.6	0.218 387 124 9 (−3)	0.466 123 080 2 (−3)
3.8	0.142 605 824 2 (−3)	0.303 404 672 3 (−3)
4.0	0.932 417 997 9 (−4)	0.197 803 779 6 (−3)
4.2	0.610 372 497 6 (−4)	0.129 142 509 9 (−3)
4.4	0.399 985 887 5 (−4)	0.844 242 044 1 (−4)
4.6	0.262 373 474 2 (−4)	0.552 556 064 1 (−4)
4.8	0.172 260 488 2 (−4)	0.362 036 489 6 (−4)
5.0	0.113 190 751 6 (−4)	0.237 440 870 6 (−4)

TABLE VI. (*Continued*)

θ	$\epsilon(\theta)$	$\eta'(\theta)$
5.2	0.744 334 697 2 (−5)	0.155 865 837 5 (−4)
5.4	0.489 816 220 4 (−5)	0.102 401 878 6 (−4)
5.6	0.322 540 129 8 (−5)	0.673 285 185 8 (−5)
5.8	0.212 520 291 2 (−5)	0.442 996 113 4 (−5)
6.0	0.140 108 798 2 (−5)	0.291 667 890 5 (−5)
6.2	0.924 193 229 1 (−6)	0.192 152 267 1 (−5)
6.4	0.609 927 237 4 (−6)	0.126 663 901 9 (−5)
6.6	0.402 715 142 8 (−6)	0.835 400 302 5 (−6)
6.8	0.266 017 712 7 (−6)	0.551 259 710 0 (−6)
7.0	0.175 794 311 1 (−6)	0.363 935 553 2 (−6)
7.2	0.116 217 268 8 (−6)	0.240 373 856 2 (−6)
7.4	0.768 597 265 2 (−7)	0.158 830 325 0 (−6)
7.6	0.508 488 099 3 (−7)	0.104 991 203 4 (−6)
7.8	0.336 518 281 0 (−7)	0.694 282 730 5 (−7)
8.0	0.222 779 463 2 (−7)	0.459 277 498 4 (−7)

$$+\tfrac{1}{24}g_1[-(\tfrac{7}{2}+\rho)(\tfrac{1}{2}+\rho)$$

$$-4\eta\theta(\tfrac{7}{2}+\rho)-(\tfrac{1}{2}+\rho)^2]-\tfrac{1}{12}\rho(2+\rho), \tag{B7d}$$

$$g_1=\tfrac{1}{2}(\tfrac{3}{2}-\rho)(\rho-2\theta/\eta)+\tfrac{1}{2}(\rho+2\theta\eta)(\tfrac{1}{2}-\rho), \tag{B7e}$$

$$g_2=-\tfrac{1}{24}(2-5\rho+2\rho^2)f_1+\tfrac{1}{24}(2+3\rho-2\rho^2)g_1$$

$$-(\theta/6\eta)(\tfrac{7}{2}-\rho)f_1+\tfrac{1}{6}\eta\theta(\tfrac{1}{2}-\rho)g_1+\tfrac{1}{12}\rho(2-\rho), \tag{B7f}$$

and ρ given by (B5c). Since $f(t)[g(t)]$ is an even (odd) function of t, this fact could be incorporated into the differential Eqs. (B5a) and (B5b). This will simplify things somewhat, but for our purposes (B7) is good enough.

It is useful to have higher-order terms in the expansion (B6b) of $\eta(\theta)$ for small θ. Writing

$$\eta(\theta)=\sum_{n=0}^{\infty}\theta^{4n+1}\rho_n, \tag{B8a}$$

with

$$\rho_n=\sum_{k=0}^{2n+1}b_{n,k}\Omega^k \tag{B8b}$$

and

$$\Omega=\ln\tfrac{1}{4}\theta+\gamma_E, \tag{B8c}$$

and substituting (B8a) into (B6a) determines the coefficients $b_{n,k}$. Myers[34] did this through order θ^5. We have computed through order θ^{13}. Our results are given in Table VIII.

The large-θ expansion of $\eta(\theta)$ is given by

$$\eta(\theta)=1-(2/\pi)K_0(2\theta)+(2/\pi^2)K_0^2(2\theta)+O(e^{-6\theta}). \tag{B9}$$

To compute the higher-order terms makes use of (6.95).

Myers considers another function $\chi(t;z)$ that is defined by the equation

$$\Gamma\chi=e^{-Z\theta t}. \tag{B10}$$

His results are summarized in the following.

Theorem 3 (Myers)

If we denote by $\chi(t;Z)$ the solution to (B10), then

$$\chi(t;Z)=\theta(Z^2-1)^{1/2}\psi(t;Z)$$
$$-\lambda_1(\theta,Z)f(t,\theta)-\lambda_2(\theta,Z)g(t,\theta), \tag{B11}$$

where f and g are solutions to (B2a) and (B2b), respectively. The function $\psi(t;Z)$ is the solution to the differential equation

$$\frac{\partial\psi}{\partial t}+Z\theta\psi(t,Z)$$
$$=(Z^2-1)^{1/2}[(\lambda_2+Z\lambda_1)f+(\lambda_1+Z\lambda_2)g], \tag{B12}$$

subject to the boundary conditions $\psi(\pm 1)=0$. The functions $\lambda_1(\theta,Z)$ and $\lambda_2(\theta,Z)$ are determined by

$$\lambda_1(\theta,Z)=M(\theta)[\bar f(Z,\theta)-Z\bar g(Z,\theta)], \tag{B13a}$$

$$\lambda_2(\theta,Z)=M(\theta)[\bar g(Z,\theta)-Z\bar f(Z,\theta)], \tag{B13b}$$

where the bar operation is defined by

$$\bar h(\xi,\theta)=\int_{-1}^{1}e^{-t\xi\theta}h(t)\,dt. \tag{B13c}$$

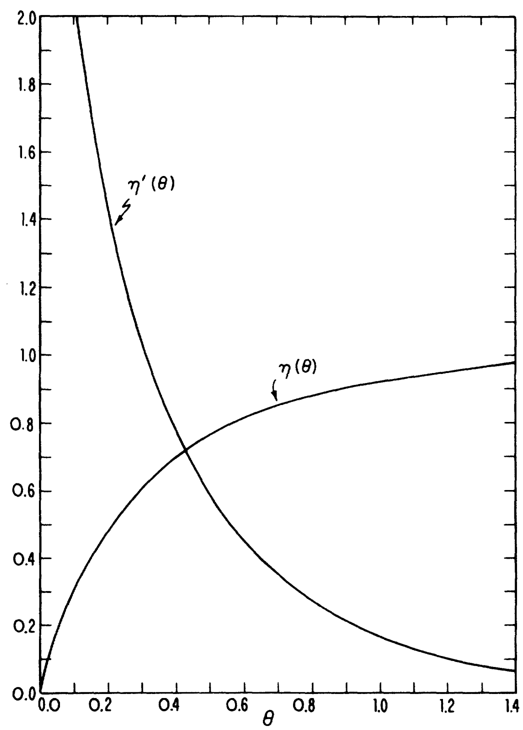

FIG. 12. Painlevé function $\eta(\theta)$ and its first derivative $\eta'(\theta)$.

TABLE VII. Scaling functions $F_{\pm}(t)$ for the symmetric lattice are given for $0 \leq t \leq 8.0$. To obtain the scaling functions $F_{\pm}(t)$ for arbitrary E_1 and E_2 the entry for the symmetric lattice should be multiplied by $(\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8} 2^{-1/8}$. If an entry is followed by a number enclosed in parenthesis, e.g., $(-n)$, then this entry should be multiplied by 10^{-n}. Thus, for instance, the value of $F_+(t)$ at $t=2$ is 0.055 910 659 65.

t	$F_+(t)$	$F_-(t)$
0.0	0.703 380 157 7	0.703 380 157 7
0.002	0.697 952 464 3	0.708 808 202 8
0.004	0.693 500 207 3	0.713 261 514 8
0.006	0.689 416 331 3	0.717 347 149 0
0.008	0.685 571 795 6	0.721 194 145 7
0.010	0.681 905 312 4	0.724 863 792 9
0.012	0.678 380 781 7	0.728 392 189 9
0.014	0.674 974 348 7	0.731 803 191 2
0.016	0.671 669 057 7	0.735 113 752 4
0.018	0.668 452 229 7	0.738 336 551 9
0.020	0.665 314 022 6	0.741 481 431 4
0.022	0.662 246 573 1	0.744 556 253 7
0.024	0.659 243 453 5	0.747 567 446 0
0.026	0.656 299 311 1	0.750 520 360 6
0.028	0.653 409 618 5	0.754 319 524 3
0.030	0.650 570 496 6	0.756 268 815 6
0.032	0.647 778 584 3	0.759 071 595 1
0.034	0.645 030 941 1	0.761 830 802 6
0.036	0.642 324 972 7	0.764 549 031 8
0.038	0.639 658 373 2	0.767 228 588 0
0.040	0.637 029 079 5	0.769 871 533 6
0.042	0.634 435 234 6	0.772 479 725 0
0.044	0.631 875 158 6	0.775 054 841 3
0.046	0.629 347 323 9	0.777 598 409 4
0.048	0.626 850 335 8	0.780 111 823 3
0.050	0.624 382 915 7	0.782 596 360 8
0.06	0.612 451 118 3	0.794 624 092 9
0.07	0.601 114 045 8	0.806 074 271 5
0.08	0.590 286 569 4	0.817 031 918 0
0.09	0.579 904 960 0	0.827 560 647 1
0.10	0.569 919 685 0	0.837 709 872 0
0.11	0.560 291 117 3	0.847 519 095 0
0.12	0.550 986 818 7	0.857 020 625 6
0.13	0.541 979 733 9	0.866 241 386 3
0.14	0.533 246 945 0	0.875 204 159 0
0.15	0.524 768 782 4	0.883 928 474 7
0.16	0.516 528 173 8	0.892 431 264 1
0.17	0.508 510 157 5	0.900 727 345 5
0.18	0.500 701 509 1	0.908 829 797 6
0.19	0.493 090 452 5	0.916 750 249 3
0.20	0.485 666 431 6	0.924 499 108 3
0.21	0.478 419 927 1	0.932 085 743 8
0.22	0.471 342 309 2	0.939 518 634 6
0.23	0.464 425 716 5	0.946 805 490 6
0.24	0.457 662 955 8	0.953 953 352 0
0.25	0.451 047 419 7	0.960 968 673 5
0.26	0.444 573 015 7	0.967 857 393 8
0.27	0.438 234 108 0	0.974 624 995 0
0.28	0.432 025 466 3	0.981 276 553 1
0.29	0.425 942 223 2	0.987 816 781 5
0.30	0.419 979 836 6	0.994 250 068 1
0.31	0.414 134 057 8	1.000 580 507 4

TABLE VII. (*Continued*)

t	$F_+(t)$	$F_-(t)$
0.32	0.408 400 902 9	1.006 811 929 4
0.33	0.402 776 628 6	1.012 947 923 6
0.34	0.397 257 710 4	1.018 991 861 4
0.35	0.391 840 823 4	1.024 946 914 2
0.36	0.386 522 825 8	1.030 816 071 7
0.37	0.381 300 743 3	1.036 602 155 7
0.38	0.376 171 756 2	1.042 307 834 6
0.39	0.371 133 187 0	1.047 935 634 8
0.40	0.366 182 489 8	1.053 487 951 9
0.41	0.361 317 240 8	1.058 967 060 2
0.42	0.356 535 129 0	1.064 375 121 5
0.43	0.351 833 948 7	1.069 714 193 5
0.44	0.347 211 592 3	1.074 986 236 2
0.45	0.342 666 043 6	1.080 193 119 4
0.46	0.338 195 371 8	1.085 336 628 2
0.47	0.333 797 726 2	1.090 418 468 2
0.48	0.329 471 331 0	1.095 440 271 3
0.49	0.325 214 481 2	1.100 403 599 5
0.50	0.321 025 537 6	1.105 309 949 8
0.52	0.312 845 122 0	1.114 957 400 1
0.54	0.304 918 157 7	1.124 393 430 9
0.56	0.297 233 559 8	1.133 628 025 4
0.58	0.289 780 997 1	1.142 670 429 8
0.60	0.282 550 816 7	1.151 529 228 3
0.62	0.275 533 978 4	1.160 212 409 8
0.64	0.268 721 997 6	1.168 727 424 6
0.66	0.262 106 895 2	1.177 081 235 1
0.68	0.255 681 153 3	1.185 280 359 7
0.70	0.249 437 676 4	1.193 330 912 2
0.72	0.243 369 756 7	1.201 238 636 3
0.74	0.237 471 043 1	1.209 008 936 4
0.76	0.231 735 514 1	1.216 646 905 0
0.78	0.226 157 452 4	1.224 157 347 8
0.80	0.220 731 423 5	1.231 544 805 3
0.82	0.215 452 254 8	1.238 813 573 1
0.84	0.210 315 018 1	1.245 967 720 0
0.86	0.205 315 012 9	1.253 011 103 9
0.88	0.200 447 751 2	1.259 947 387 1
0.90	0.195 708 944 2	1.266 780 049 4
0.92	0.191 094 489 7	1.273 512 400 9
0.94	0.186 600 460 8	1.280 147 592 6
0.96	0.182 223 095 1	1.286 688 627 0
0.98	0.177 958 785 4	1.293 138 367 8
1.0	0.173 804 070 4	1.299 499 548 1
1.1	0.154 562 530 2	1.330 064 530 4
1.2	0.137 612 980 7	1.358 762 886 4
1.3	0.122 647 434 5	1.385 828 041 9
1.4	0.109 407 490 3	1.411 452 008 3
1.5	0.976 743 235 9 (−1)	1.435 794 682 2
1.6	0.872 612 080 4 (−1)	1.458 990 641 1
1.8	0.697 756 688 9 (−1)	1.502 383 351 4
2.0	0.559 106 596 5 (−1)	1.542 365 330 2
2.2	0.448 789 760 4 (−1)	1.579 490 531 5
2.4	0.360 772 704 3 (−1)	1.614 186 500 3
2.6	0.290 386 372 2 (−1)	1.646 789 986 5
2.8	0.233 990 436 6 (−1)	1.677 570 899 7
3.0	0.188 729 732 3 (−1)	1.706 748 894 3
3.2	0.152 354 133 3 (−1)	1.734 505 137 3

TABLE VII. (*Continued*)

t	$F_+(t)$	$F_-(t)$
3.4	0.123 083 337 1 (−1)	1.760 990 838 3
3.6	0.995 041 383 9 (−2)	1.786 333 548 5
3.8	0.804 916 302 5 (−2)	1.810 641 891 1
4.0	0.651 482 814 1 (−2)	1.834 009 166 4
4.2	0.527 565 272 9 (−2)	1.856 516 137 5
4.4	0.427 416 774 3 (−2)	1.878 233 208 9
4.6	0.346 427 618 9 (−2)	1.899 222 150 6
4.8	0.280 895 291 0 (−2)	1.919 537 475 6
5.0	0.227 842 371 4 (−2)	1.939 227 552 4
5.2	0.184 871 985 9 (−2)	1.958 335 510 2
5.4	0.150 052 769 5 (−2)	1.976 899 982 4
5.6	0.121 827 121 4 (−2)	1.994 955 721 8
5.8	0.989 378 990 (−3)	2.012 534 112 5
6.0	0.803 697 447 (−3)	2.029 663 600 6
6.2	0.653 020 574 (−3)	2.046 370 055 7
6.4	0.530 712 468 (−3)	2.062 677 079 8
6.6	0.431 404 022 (−3)	2.078 606 269 4
6.8	0.350 748 937 (−3)	2.094 177 440 9
7.0	0.285 227 267 (−3)	2.109 408 825 0
7.2	0.231 987 098 (−3)	2.124 317 235 0
7.4	0.188 716 856 (−3)	2.138 918 212 4
7.6	0.153 542 243 (−3)	2.153 226 155 0
7.8	0.124 943 002 (−3)	2.167 254 428 7
8.0	0.101 685 644 (−3)	2.181 015 465 5

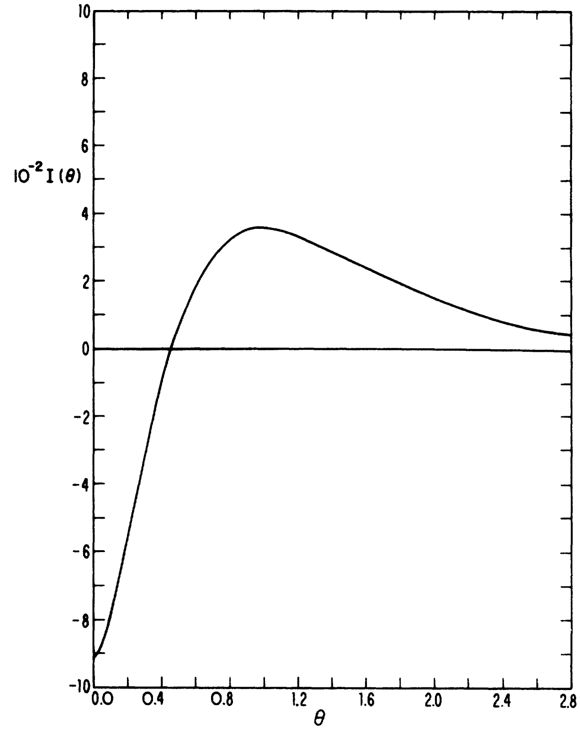

FIG. 13. Quantity $I(\theta)=\int_\theta^\infty x\ln x[1-\eta^2(x)]dx$ as a function of θ.

And finally,

$$\frac{\partial f(t,\theta)}{\partial\theta}-\theta^{-1}(\tfrac{1}{2}+\rho)f(t,\theta)-tg(t,\theta)=0, \tag{B13d}$$

$$\frac{\partial\lambda_1(\theta,Z)}{\partial\theta}+(\lambda_1+Z\lambda_2)\eta(\theta)=0, \tag{B13e}$$

and

$$\frac{\partial\lambda_2(\theta,Z)}{\partial\theta}+(\lambda_2+Z\lambda_1)\eta^{-1}(\theta)=0. \tag{B13f}$$

We have the result

Theorem:

$$f_0^2(\theta)M(\theta)\eta(\theta)=-\theta/(2\pi^2). \tag{B14}$$

To prove this we multiply (B13d) by $(1-t)^{1/2}$ and take the limit $t\to 1$. We obtain

$$\frac{\partial f_0}{\partial\theta}-\frac{(\frac{1}{2}+\rho)}{\theta}f_0-f_0\eta(\theta)=0, \tag{B15}$$

which implies

$$f_0^2(\theta)=(\text{const})\exp\left(2\int^\theta[\eta+(\tfrac{1}{2}+\rho)\theta^{-1}]\,d\theta\right) \tag{B16}$$

for $Z=0$ (B13e) implies

$$\lambda_1(\theta)=(\text{const})\exp\left(-\int^\theta\eta(\theta)\,d\theta\right). \tag{B17}$$

Multiplying (B13d) by $e^{-t\theta t}$ and integrating over t from -1 to $+1$ we obtain for the special case $\xi=0$

$$\frac{\partial}{\partial\theta}\bar{f}(0,\theta)-(\tfrac{1}{2}+\rho)\theta^{-1}\bar{f}(0,\theta)-\int_{-1}^{1}xg(x,\theta)\,dx=0. \tag{B18}$$

One can show, (Myers does)

TABLE VIII. Coefficients $b_{n,k}$ in the small-θ expansion of $\eta(\theta)$. See Eq. (B8) for the definition of these coefficients.

n	k	$b_{n,k}$
0	0	0
0	1	-1
1	0	-2^{-7}
1	1	2^{-5}
1	2	-2^{-4}
1	3	2^{-4}
2	0	-145×2^{-20}
2	1	145×2^{-17}
2	2	-129×2^{-15}
2	3	121×2^{-14}
2	4	-2^{-7}
2	5	2^{-8}
3	0	$-28195\times 3^{-5}\times 2^{-25}$
3	1	$25603\times 3^{-4}\times 2^{-23}$
3	2	$-82729\times 3^{-3}\times 2^{-24}$
3	3	$9539\times 3^{-2}\times 2^{-21}$
3	4	-477×2^{-19}
3	5	275×2^{-18}
3	6	-3×2^{-12}
3	7	2^{-12}

$$\frac{d}{dt}(g-\eta tf)+[\eta(\tfrac{1}{2}+\rho)-\theta]f+\eta\theta tg=0\ . \qquad \text{(B19)}$$

Applying the bar operation to (B19) we have for $\xi=0$

$$[\eta(\tfrac{1}{2}+\rho)-\theta]\bar{f}(0,\theta)=-\theta\eta\int_{-1}^{1}xg(x,\theta)\,dx\ . \qquad \text{(B20)}$$

Then (B18) and (B20) imply

$$\frac{d}{d\theta}\bar{f}(0,\theta)=\eta^{-1}\bar{f}(0,\theta) \qquad \text{(B21)}$$

or

$$\bar{f}(0,\theta)=(\text{const})\exp\left(\int^{\theta}\frac{d\theta}{\eta}\right). \qquad \text{(B22)}$$

From (B13a) for $Z=0$

$$M(\theta)=\lambda_1(0,\theta)/\bar{f}(0,\theta) \qquad \text{(B23)}$$

$$=(\text{const})\exp\left(-\int^{\theta}d\theta\,(\eta+\eta^{-1})\right), \qquad \text{(B24)}$$

where we used (B17) and (B22) in going from (B23) to (B24). Equations (B16) and (B24) imply

$$M(\theta)f_0^2(\theta)=(\text{const})\exp\left[\int^{\theta}\left(\eta-\eta^{-1}+\frac{1+2\rho}{\theta}\right)\right]. \qquad \text{(B25)}$$

Using (B5a) we have

$$\eta-\eta^{-1}+(1+2\rho)\,\theta^{-1}=\theta^{-1}+\frac{d}{d\theta}\ln\eta(\theta)\ , \qquad \text{(B26)}$$

so that (B25) becomes

$$f_0^2(\theta)M(\theta)\,\eta(\theta)/\theta=(\text{const})\ . \qquad \text{(B27)}$$

To obtain the constant in (B27) we evaluate the left-hand side of (B27) for $\theta\to 0$. For small θ (B2) reduces to

$$-\int_{-1}^{1}(\ln\tfrac{1}{2}\theta\,|x-x'|+\gamma_E)\,f(x')\,dx'=\cosh\theta x \qquad \text{(B28a)}$$

and

$$-\int_{-1}^{1}(\ln\tfrac{1}{2}\theta\,|x-x'|+\gamma_E)\,g(x')\,dx'=\sinh\theta x. \qquad \text{(B28b)}$$

Equations (B28) can be solved for it is known that[48] if

$$\int_{-1}^{1}u(x')[\ln 2\,|x-x'|+\Omega]\,dx'=w(x)\ , \qquad \text{(B29)}$$

then

$$u(x)=\frac{1}{\pi^2(1-x^2)^{1/2}}\left(P\int_{-1}^{1}dx'\,\frac{(1-x'^2)^{1/2}}{x'-x}\,\frac{dw(x')}{dx'}+\frac{1}{\Omega}P\int_{-1}^{1}dx'\,\frac{w(x')}{(1-x'^2)^{1/2}}\right) \qquad \text{(B30)}$$

In (B30) the integrals are principal-value integrals. Using (B30) we obtain for $t\to 1$

$$f(t,\theta)\sim-[\Omega\pi 2^{1/2}(1-t)^{1/2}]^{-1} \qquad \text{(B31)}$$

and

$$g(t,\theta)\sim\theta\pi^{-1}2^{-1/2}(1-t)^{-1/2}\ . \qquad \text{(B32)}$$

From the definition (B56) when $\theta\to 0$

$$\eta(\theta)\sim-\theta\Omega\ , \qquad \text{(B33)}$$

[this is how Myers derived (B6b)].

Furthermore, from (B31) and (B7a) it follows

$$f_0^2(\theta)\sim 1/2\pi^2\Omega^2\ . \qquad \text{(B34)}$$

From the definition of $\bar{f}(0,\theta)$ we have

$$\bar{f}(0,\theta)=\int_{-1}^{1}dx\,f(x)\sim-\frac{1}{\Omega}\ . \qquad \text{(B35)}$$

To obtain $M(\theta)$ we calculate $\lambda_1(0,\theta)$. To compute $\lambda_1(0,\theta)$ for $\theta\to 0$ we examine the equation

$$\Gamma\chi=e^{-Z\theta t}\sim 1\ . \qquad \text{(B36)}$$

Using (B30)

$$\chi(t,\theta)\sim-[\pi\Omega(1-t^2)^{1/2}]^{-1}\ . \qquad \text{(B37)}$$

As $\theta\to 0$, $g(t,\theta)\to 0$, hence

$$\chi(t,Z)\simeq-\lambda_1(\theta,Z)\,f(t,\theta)\ . \qquad \text{(B38)}$$

Using (B31) and (B37) we see from (B38)

$$\lim_{\theta\to 0}\lambda_1(\theta,0)=-1\ . \qquad \text{(B39)}$$

Using (B35) and (B39) we have form (B23)

$$M(\theta)\sim-\Omega\ . \qquad \text{(B40)}$$

Combining (B33), (B34), and (B40) we see that (B14) follows.

APPENDIX C

We record and derive in this appendix various formulas which are used in the text to make some expansions near T_c.

First of all the T_c condition may be written in the equivalent form of

$$\sinh 2\beta_c E_1\,\sinh 2\beta_c E_2=1 \qquad \text{(C1)}$$

or

$$z_{1c}z_{2c}+z_{1c}+z_{2c}-1=0\ , \qquad \text{(C2)}$$

with

$$z_i=\tanh\beta E_i\ ,\quad i=1,\,2\ . \qquad \text{(C3)}$$

Furthermore from (C3)

$$\sinh 2\beta E_i=2(z_i^{-1}-z_i)^{-1},\quad i=1,\,2\ , \qquad \text{(C4)}$$

$$\cosh 2\beta E_i=(1+z_i^2)/(1-z_i^2)\ ,\quad i=1,\,2\ , \qquad \text{(C5)}$$

and

$$\coth 2\beta E_i=\tfrac{1}{2}(z_i^{-1}+z_i)\ ,\quad i=1,\,2\ , \qquad \text{(C6)}$$

and using (C2) we find

$$\sinh 2\beta_c E_1+\sinh 2\beta_c E_2=(z_{1c}+z_{2c})^2/2z_{1c}z_{2c}\ . \qquad \text{(C7)}$$

Moreover, using (C2) we have

$$(z_{2c}^{-1}+z_{2c})(1-z_{1c})=(z_{1c}^{-1}+z_{1c})(1-z_{2c})=2(z_{1c}+z_{2c}). \tag{C8}$$

Therefore as $T\to T_c^-$

$$\begin{aligned}S_\infty^< &\sim 4(\beta-\beta_c)(E_1\coth 2\beta_c E_1+E_2\coth 2\beta_c E_2)\\ &=2(\beta-\beta_c)[E_1(z_{1c}^{-1}+z_{1c})+E_2(z_{2c}^{-1}+z_{2c})]\\ &=4(\beta-\beta_c)(z_{1c}+z_{2c})[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}],\end{aligned} \tag{C9}$$

where we have used (C6) and C7).

Recall the definition of R^2 as

$$\begin{aligned}R^2&=(\gamma_2/\gamma_1)^{1/2}M^2+(\gamma_1/\gamma_2)^{1/2}N^2\\ &=\left(\frac{z_1(1-z_2^2)}{z_2(1-z_1^2)}\right)^{1/2}M^2+\left(\frac{z_2(1-z_1^2)}{z_1(1-z_2^2)}\right)^{1/2}N^2\\ &=\left(\frac{\sinh 2\beta E_1}{\sinh 2\beta E_2}\right)^{1/2}M^2+\left(\frac{\sinh 2\beta E_2}{\sinh 2\beta E_1}\right)^{1/2}N^2\end{aligned} \tag{C10}$$

and

$$\begin{aligned}t^2&=(z_1z_2+z_1+z_2-1)^2\left(\frac{M^2}{z_2(1-z_1^2)}+\frac{N^2}{z_1(1-z_2^2)}\right)\\ &=(z_1z_2+z_1+z_2-1)^2[z_1z_2(1-z_1^2)(1-z_2^2)]^{-1/2}R^2.\end{aligned} \tag{C11}$$

Now at T_c we have the identity

$$(1-z_{1c}^2)(1-z_{2c}^2)=4z_{1c}z_{2c}, \tag{C12}$$

and as $T\to T_c$,

$$z_1z_2+z_1+z_2-1\sim(\beta-\beta_c)4z_{1c}z_{2c}\times[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}]. \tag{C13}$$

Thus we have

$$t/R\sim|\beta-\beta_c|2\sqrt{2}(z_{1c}z_{2c})^{1/2}\times[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}], \tag{C14}$$

and so, using (C8) and (C13) we have, as $T\to T_c^-$,

$$S_\infty^<\sim(z_{1c}z_{2c})^{-1/8}[\sqrt{2}(z_{1c}+z_{2c})t/R]^{1/4}. \tag{C15}$$

Furthermore, since

$$\bar{S}_\infty=[4z_1z_2(1-z_1^2)(1-z_2^2)]^{1/2}\times[(\sinh 2\beta E_1\sinh 2\beta E_2)^{-2}-1]^{1/4}, \tag{C16}$$

we find, as $T\to T_c^+$

$$\begin{aligned}\bar{S}_\infty&\sim 4z_{1c}z_{2c}[4|\beta-\beta_c|\times(E_1\coth 2\beta_c E_1+E_2\coth 2\beta_c E_2)]^{1/4}\\ &=4z_{1c}z_{2c}\{4|\beta-\beta_c|(z_{1c}+z_{2c})\times[E_1(1-z_{2c})^{-1}+E_2(1-z_{1c})^{-1}]\}^{1/4}.\end{aligned} \tag{C17}$$

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[18]The function $S(e^{i\theta})$ is the function $\phi(e^{i\theta})$ in Refs. 5 and 17.

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[20]For numerical work the expression (2.27) for $f^{(2n)}(t)$ is not the most convenient. It is useful (to improve convergence) to change variables $y_j=\cosh\theta_j$ so that (2.27) becomes

$$f^{(2n)}(t)=(-1)^n\pi^{-2n}n^{-1}\int_0^\infty d\theta_1\cdots\int_0^\infty d\theta_{2n}\times\prod_{j=1}^{2n}\frac{e^{-t\cosh\theta_j}}{\cosh\theta_j+\cosh\theta_{j+1}}\prod_{j=1}^{n}\sinh^2\theta_{2j}\ ,$$

where $\theta_{2n+1}=\theta_1$ and n is any positive integer.

[21]See Appendix C.

[22]Our notation differs from CW in that we denote by $\bar{S}$,

$\bar{V}$, $\bar{U}$ what CW denote by $\tilde{S}$, $\tilde{V}$, $\tilde{U}$.

[23]The quantity denoted here by $\bar{S}_\infty$ ($S_\infty^>$) is the same as the quantity denoted by $D(M)$ (D) in Refs. 5 and 17. Note that there is a misprint in both these references. They have $(\gamma_1\gamma_2)^{-1/2}$ in (4.19) instead of the correct value $(\gamma_1\gamma_2)^{1/2}$.

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$$f(\xi) = \sum_{n=-\infty}^{+\infty} f_n \xi^n,$$

then

$$[f(\xi)]_+ = \sum_{n=0}^{\infty} f_n \xi^n$$

and

$$[f(\xi)]_- = \sum_{n=-\infty}^{-1} f_n \xi^n.$$

When considering integral equations on the line, the plus and minus parts of function

$$f(\zeta) = \int_{-\infty}^{+\infty} e^{i\zeta t} \hat{f}(t)\, dt$$

are defined as

$$[f(\zeta)]_+ = \int_0^{\infty} e^{i\zeta t} \hat{f}(t)\, dt$$

and

$$[f(\zeta)]_- = \int_{-\infty}^{0} e^{i\zeta t} \hat{f}(t)\, dt.$$

[37]See, for instance, Chap. 8, p. 257 of Ref. 5.

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[39]See, for example, Appendix B of Ref. 5.

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$$\Gamma f = \cos kx, \quad \Gamma g = \sin kx, \quad \Gamma\phi = e^{-ikx\cos\phi},$$

where

$$(\Gamma\psi)(x) = \int_{-1}^{1} H_0(k\,|x - x'|)\psi(x')\, dx'.$$

It is straightforward to convert his theorems to the theorems stated here.

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