Equations of state for bicritical points. I. Calculations in the disordered phase

David R. Nelson*
Baker Laboratory and Materials Science Center, Cornell University, Ithaca, New York 14853

Eytan Domany
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853

(Received 18 August 1975)

Equations of state are calculated for anisotropic spin Hamiltonians suitable for describing the spin-flop transition in materials such as $GdAlO_3$ and MnF_2. Crossover scaling functions for the specific heat, nonordering susceptibility, and parallel and transverse susceptibilities are derived to first order in $\epsilon = 4 - d$ directly from renormalization-group recursion relations.

I. INTRODUCTION

Consider a continuous spin Hamiltonian for two coupled spin fields, $s = s(R) = [s_i(R)]$ and $\sigma = \sigma(R) = [\sigma_i(R)]$, of the form

$$\begin{aligned}\bar{\mathcal{H}} &= \frac{-\mathcal{H}}{k_B T} \\ &= -\int d\vec{R}\left(\frac{1}{2}\sum_{i=1}^{m}|\vec{\nabla}s_i|^2 + \frac{1}{2}\sum_{j=1}^{n-m}|\vec{\nabla}\sigma_j|^2\right. \\ &\quad + \tfrac{1}{2}r(|\vec{s}|^2 + |\sigma|^2) - \frac{g}{2n}[(n-m)|\vec{s}|^2 - m|\vec{\sigma}|^2] \\ &\quad \left. + u|\vec{s}|^4 + 2w|\vec{s}|^2|\vec{\sigma}|^2 + v|\vec{\sigma}|^4\right), \qquad (1.1)\end{aligned}$$

where

$$|s|^2 = \sum_{i=1}^{m} s_i^2, \quad |\sigma|^2 = \sum_{i=1}^{n-m} \sigma_i^2,$$

and where the components s_i and σ_i vary between $\pm\infty$. When we Fourier transform (1.1) into momentum space, the momentum integrals will, as usual, be cut off by a spherical Brillouin zone of unit radius. Such Hamiltonians (including fixed-spin-length versions) have long been of interest in theoretical studies of critical phenomena.[1-5] Assume for concreteness that the quartic couplings are isotropic ($u = w = v$), and consider the case $n = 3$ and $m = 1$; then, for $g = 0$, Eq. (1.1) should have a critical point with Heisenberg critical exponents. For g positive, (1.1) should display Ising-like critical behavior, and with g negative an XY-like transition should result.[3,4] If we take g to be a variable parameter, (1.1) displays three distinct types of critical behavior as g varies, and provides an interesting example of *crossover* from one variety of critical behavior to another. A phase diagram in the (T, g) plane is shown Fig. 1(a). Note that there is a line of first-order transitions separating the XY and Ising ordered phases. This connects to a unique *bicritical point*, at the juncture of two critical lines.

Recently Hamiltonians such as (1.1) have become of much more immediate *experimental* interest. Renormalization-group arguments have shown[6] that the Hamiltonian for a uniaxial *anti*ferromagnet with a magnetic field applied along the direction of anisotropy can be renormalized into the form (1.1). Such systems are known[7] to exhibit a first-order transition from an Ising antiferromagnetic state to a "spin-flopped" ordering in strong enough magnetic fields [see Fig. 1(b)]. It is now expected[6] that the phase diagrams shown in Fig. 1 are quite closely related. In fact, knowledge of the critical properties of (1.1) together with a phenomenological scaling treatment of spin-flop transitions lead to a number of concrete predictions[8] about the spin-flop bicritical point. One of the most striking of these is that the critical lines should enter the bicritical point with slopes *tangent* to the first-order line[8] rather than at some angle as predicted by mean-field theory.[9] We note that work by Aharony and Bruce[10] indicates that (1.1) may also be an appropriate description of systems with displacive phase transitions.

Although the existence of spin-flop transitions has been known for a long time,[11] the detailed critical behavior of such systems has only recently come under close scrutiny. The experiments of Rohrer,[12] for example, indicate that the critical line does indeed come into the bicritical point tangentially, and also test other predictions of the scaling analysis.[8] In view of the experimental interest in spin-flop transitions, and the existence of theories[6,8] which give concrete predictions for the critical exponents, it seems appropriate to attempt a calculation of the *scaling functions* associated with bicritical points.

A phenomenological theory of bicritical points[8] is simply expressed in terms of the variables g and

$$t = (T - T_b)/T_b \ , \tag{1.2}$$

where T_b is the bicritical temperature [the variable r plays the role of temperature in (1.1)]. It is expected that the singular part of the free energy should behave asymptotically close to the bicritical point as[8]

$$F_s(t, g) \approx t^{2-\alpha}\Phi(g/t^{\phi}) \ , \tag{1.3}$$

where $\alpha = \alpha(n)$ is the bicritical specific heat index and ϕ is the crossover exponent.[2] Equation (1.3) differs from a scaling description of an ordinary critical point in that $\Phi(z)$ is expected to have singularities describing the critical-*line* behavior.

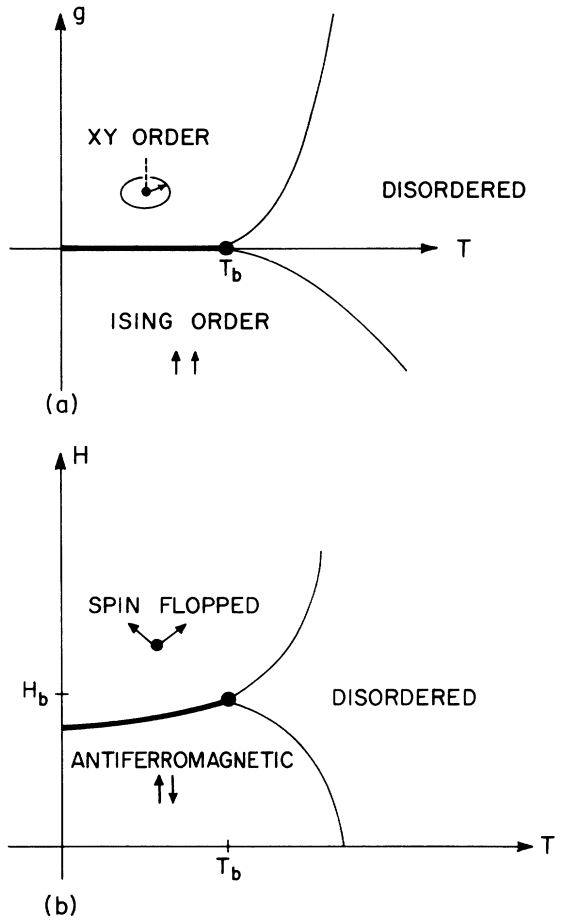

FIG. 1. (a) Phase diagram for the Hamiltonian (1.1). A (bold) line of first-order transitions separates an Ising-like ordered phase from an XY-like ordered phase. Two lines of continuous phase transitions meet the first-order line at the bicritical point ($T_b, g=0$). (b) Schematic phase diagram for a uniaxial antiferromagnet in a magnetic field H along the easy axis. A (bold) line of first-order transitions separates an ordered antiferromagnetic phase from an ordered spin-flop phase. Two lines of continuous phase transitions meet at the bicritical point.

A similar scaling hypothesis applies to the susceptibility and other thermodynamic functions as well.[8] The purpose of this paper is to present calculations of functions like $\Phi(z)$ in the disordered region (see Fig. 1) of the bicritical phase diagram.

We will discuss here the Hamiltonian (1.1) only under the simplifying assumption that $u = v = w$. This equality is generated asymptotically by the initial iterations of the recursion relations close to the bicritical fixed point, provided the total number of spin components n is less than a critical number $n_\times(d) \simeq 3.1$ in three dimensions. This initial relaxation occurs [to $O(\epsilon)$] even when $g \neq 0$ because g does not enter the recursion relations for u, v, and w to $O(\epsilon)$.[3] When g becomes large, it *does* effect u, v, and w, but we will stop integrating the recursion relations before this happens (see below). By setting $u = v = w$, and considering $n \leq 3$, we ignore only transient effects which do not contribute to the universal part of the scaling function in the disordered phase. Note, however, that this restriction neglects entirely the interesting possibility of a *tetracritical point*, which arises when $w^2 < uv$, and which involves a third, intermediate ordered phase.[8,9,14] This will be considered in a separate paper.[15]

Our procedure is to work directly with renormalization-group recursion relations[16] to $O(\epsilon)$ ($\epsilon = 4 - d$, where d is the dimensionality of space). A naive perturbation theory (which assumes u, w, and v are small) will not work for (1.1) in the critical region because of problems at small momenta.[17] A rather direct and intuitive way to overcome this difficulty is to use the recursion relations to integrate the Hamiltonian out of the critical regime. Perturbation theory on the partially renormalized noncritical Hamiltonian presents no divergence problem. The behavior of quantities calculated with the renormalized Hamiltonian is then related to the behavior of quantities deep within the critical region.

Although this program has previously been carried out for (1.1) with g set to zero (and $u = w = v$),[18] its implementation for nonzero g involves some novel features. The recursion relations map the unrenormalized Hamiltonian $\mathcal{H}$ into a Hamiltonian $\mathcal{H}(l)$. The basic idea behind this approach is to choose $l = l^*(t, g)$ such that $\mathcal{H}(l^*)$ is noncritical. We note, however, that there are both longitudinal and transverse correlation lengths associated with (1.1) for nonzero g. Consider for definiteness the case $g > 0$, so that the m-component $\vec{s}$ spin field will eventually order as the temperature is lowered. Integrating the recursion relations until *both* the longitudinal and transverse

correlation lengths are of order unity requires a complete solution of the full nonlinear differential recursion relations. We circumvent the analytic difficulties inherent in such an approach as follows: The recursion relations are first integrated until the transverse correlation length is of order unity. It is straightforward to solve the recursion relations in this regime. At this point, the $\vec{\sigma}$ spin field can be systematically integrated out of the problem for fixed values of the $\vec{s}$ spins treated as parameters entering the potential. One is left with an m-component *isotropic* Hamiltonian in the spins $\vec{s}$ which could still be close to criticality. However, the isotropic recursion relations can then be integrated as in Ref. 18 until the Hamiltonian is fully noncritical. The repeated use of the recursion relations to force first the transverse and second the longitudinal correlation lengths to be order unity in renormalized Hamiltonians leads to the crossover scaling functions in a rather straightforward fashion.

In Sec. II we first discuss the anisotropic spin recursion relations and their solution. We review results appropriate to the isotropic case ($g=0$), and discuss expectations from phenomenological crossover scaling theories. Section III presents explicit calculations of the longitudinal susceptibility and the free energy for the Hamiltonian (1.1). From the free energy expressions for the specific heat and nonordering susceptibility (two derivatives of F with respect to g) are obtained. Our results are then compared with Feynman-graph and series-expansion work, and are finally extended to allow for the mean-field behavior far from the critical point. In Sec. IV we summarize what has been accomplished. Appendix B shows how the techniques developed here can be used to calculate the transverse or perpendicular susceptibility for bicritical systems.

II. RECURSION RELATIONS, ISOTROPIC RESULTS, AND CROSSOVER SCALING

A. Anisotropic spin recursion relations

Consider a more general version of the Hamiltonian (1.1), namely,

$$\mathcal{H} = -\frac{1}{2}\int d\vec{R}\Big(\sum_{i=1}^{n}\left[r_i s_i^2 + (\nabla s_i)^2\right] + 2\sum_{i,j=1}^{n} u_{ij} s_i^2 s_j^2\Big) . \tag{2.1}$$

Renormalization-group recursion relations were first constructed for (2.1) by Fisher and Pfeuty[3] to $O(\epsilon)$ using the approximate recursion formula. Their results can be redrived using the exact momentum integration method of Wilson[16] which gives[6]

$$r_i' = b^2\Big(r_i + 8u_{ii}A(r_i) + 4\sum_j u_{ij}A(r_i)\Big) \tag{2.2}$$

$$u_{ij}' = b^{\epsilon}\Big(u_{ij} - 8u_{ij}u_{ii}B(r_i, r_i) - 16u_{ij}^2 B(r_i, r_j) - 8u_{ij}u_{jj}B(r_j, r_j) - 4\sum_m u_{im}u_{mj}B(r_m, r_m)\Big), \tag{2.3}$$

where b is the spatial rescaling factor, and

$$A(r_i) = \int_q^> (r_i + q^2)^{-1},$$
$$B(r_i, r_j') = \int_q^> (r_i + q^2)^{-1}(r_j + q^2)^{-1} . \tag{2.4}$$

The symbol $\int_q^>$ denotes a d-dimensional momentum integration over the shell $b^{-1} < |q| < 1$.

Writing $b = e^{\delta}$, with $\delta \ll 1$, (2.2)–(2.4) can be expanded in δ and rewritten in differential form,

$$\frac{dr_i}{dl} = 2r_i + 4K_4\Big(2u_{ii}q_i + \sum_j u_{ij}q_j\Big) \tag{2.5}$$

$$\frac{du_{ij}}{dl} = \epsilon u_{ij} - 4K_4\Big(2u_{ij}u_{ii}q_i^2 + 4u_{ij}^2 q_i q_j + 2u_{ij}u_{jj}q_j^2 + \sum_m u_{im}u_{mj}q_m^2\Big) , \tag{2.6}$$

where $r_i = r_i(l)$ and $u_{ij} = u_{ij}(l)$ are couplings appropriate to renormalizations involving a spatial rescaling e^l, $q_i = (1 + r_i)^{-1}$, and $K_4 = 1/8\pi^2$.

Consider first the isotropic case $r_i = r$ (all i) and $u_{ij} = u$ (all i and j), when (2.5) and (2.6) reduce to differential equations derived by Wegner and Houghton,[19] namely,

$$\frac{dr}{dl} = 2r + \frac{Au}{1+r} , \tag{2.7}$$

$$\frac{du}{dl} = \epsilon u - \frac{Bu^2}{(1+r)^2} \tag{2.8}$$

with

$$A = 4K_4(n+2), \quad B = 4K_4(n+8) . \tag{2.9}$$

These equations were solved approximately to leading order in ϵ and $u(l)$ by Nelson and Rudnick, who found that[20]

$$t(l) \equiv r(l) + \tfrac{1}{2}Au(l) - \tfrac{1}{2}Au(l)r(l)\ln[1 + r(l)] = t(0)e^{2l}/Q(l)^{A/B}, \tag{2.10}$$

$$u(l) = u(0)e^{\epsilon l}/Q(l) , \tag{2.11}$$

where

$$Q(l) = 1 + Bu(e^{\epsilon l} - 1)/\epsilon . \tag{2.12}$$

Note that (2.10) can be rewritten as

$$r(l) = t(l) - \tfrac{1}{2}Au(l) + \tfrac{1}{2}Au(l)t(l)\ln[1 + t(l)] \quad (2.13)$$

to leading order in $u(l) = O(\epsilon)$. The solutions (2.10) and (2.11) are valid provided $r(l)$ does not get larger than order unity.[18]

We will not solve (2.5) and (2.6) here for the case of general quadratic coupling u_{ij}. For all r_i of order ϵ and approximately equal, and $n < 4 - O(\epsilon)$,[13] however, the recursion relations drive the u_{ij} toward an isotropic "Heisenberg" fixed point,[3,4]

$$u_{ij}(l) \to u_n^* = \epsilon/B + O(\epsilon^2) , \quad (2.14)$$

so equality is generated asymptotically. Setting all $u_{ij} = u_H^*$ from the start eliminates these transient effects, which do not affect the universal part of the scaling functions.[16,22] It will be instructive to relax this condition and put $u_{ij} = u$, where u is an arbitrary starting parameter. To lowest order in ϵ, this isotropy at quartic order is preserved under iteration, even if the initial r_i are unequal, $u(l)$ being given by (2.11).

Various forms of *quadratic* symmetry breaking in (2.1) were discussed by Fisher and Pfeuty.[3] Here we concentrate on one particular case, that of m-dominant anisotropy (extension to other forms of symmetry breaking is straightforward). We will take

$$\begin{aligned} r_i &= r_\parallel = O(\epsilon), \quad i = 1, \ldots, m \\ r_i &= r_\perp = O(\epsilon), \quad i = m+1, \ldots, n \end{aligned} \quad (2.15)$$

which together with the assumption $u_{ij} = u$, reduces (2.1) to a special case of the Hamiltonian (1.1). For this case, Eqs. (2.5) take the form

$$\frac{dr_\parallel}{dl} = 2r_\parallel + \frac{4K_4(m+2)u}{1+r_\parallel} + \frac{4K_4(n-m)u}{1+r_\perp} , \quad (2.16)$$

$$\frac{dr_\perp}{dl} = 2r_\perp + \frac{4K_4(n-m+2)u}{1+r_\perp} + \frac{4K_4 mu}{1+r_\parallel} . \quad (2.17)$$

It is convenient to define new variables

$$r_s = [mr_\parallel + (n-m)r_\perp]/n, \quad r_d = r_\perp - r_\parallel \quad (2.18)$$

in terms of which (2.16) and (2.17) can be rewritten

$$\frac{dr_s}{dl} = 2r_s + Au - Aur_s + Au\left(\frac{m}{n}\frac{r_\parallel^2}{1+r_\parallel} + \frac{n-m}{n}\frac{r_\perp^2}{1+r_\perp}\right), \quad (2.19)$$

$$\frac{dr_d}{dl} = 2r_d - 8K_4 ur_d + 8K_4 u\left(\frac{r_\perp^2}{1+r_\perp} - \frac{r_\parallel^2}{1+r_\parallel}\right) . \quad (2.20)$$

Using the isotropic solution (2.11) for u, these equations may be solved immediately by techniques developed in Ref. 18, provided the terms in large parentheses are neglected. These preliminary solutions can then be used as the starting point for an iterative solution of (2.19) and (2.20). To leading order we find

$$t(l) \equiv r_s(l) + \tfrac{1}{2}Au(l) - \tfrac{1}{2}Au(l)\left(\frac{m}{n}r_\parallel(l)\ln[1 + r_\parallel(l)] + \frac{n-m}{n}r_\perp(l)\ln[1 + r_\perp(l)]\right)$$
$$= t(0)e^{2l}/Q(l)^{(n+2)/(n+8)} , \quad (2.21)$$

$$g(l) \equiv r_d(l) - 4K_4 u(l)\{r_\perp(l)\ln[1 + r_\perp(l)] - r_\parallel\ln[1 + r_\parallel(l)]\}$$
$$= g(0)e^{2l}/Q(l)^{2/(n+8)}, \quad (2.22)$$

where $Q(l)$ is given by (2.12). Remembering that $r_\parallel(0)$, $r_\perp(0)$, and $u(0)$ are $O(\epsilon)$, we see that

$$t(0) \approx r_s + \tfrac{1}{2}Au, \quad g(0) \approx r_d . \quad (2.23)$$

These variables play the role of the t and g variables used in the scaling description of crossover discussed in Sec. I. The equations (2.21) and (2.22) may be inverted to give implicit expressions for $r_\perp(l)$ and $r_\parallel(l)$, namely,

$$r_\perp(l) = t(l) + (m/n)g(l) - \tfrac{1}{2}Au(l) + 2mK_4 u(l)r_\parallel(l)\ln[1 + r_\parallel(l)] + 2(n-m+2)K_4 u(l)r_\perp(l)\ln[1 + r_\perp(l)], \quad (2.24)$$

$$r_\parallel(l) = t(l) - [(n-m)/n]g(l) - \tfrac{1}{2}Au(l) + 2(m+2)K_4 u(l)r_\parallel(l)\ln[1 + r_\parallel(l)] + 2(n-m)K_4 u(l)r_\perp(l)\ln[1 + r_\perp(l)] . \quad (2.25)$$

These purely algebraic equations can, of course, be solved to leading order in $u(l)$ to give $r_\parallel(l)$ and $r_\perp(l)$ explicitly in terms of the functions $u(l)$, $t(l)$, and $g(l)$.

The functions $t(l)$ and $g(l)$ correspond to eigen perturbations about the isotropic fixed point, which is given by $u = u_n^*$ and $t^* = g^* = 0$. In the vicinity of this fixed point we have

$$t(l) \approx t(0)e^{\lambda_1 l}, \quad g(l) \approx g(0)e^{\lambda_2 l} , \quad (2.26)$$

where

$$\lambda_1 = 2 - [(n+2)/(n+8)]\epsilon, \quad \lambda_2 = 2 - [2/(n+8)]\epsilon \quad (2.27)$$

are eigenvalues first found to $O(\epsilon)$ by Fisher and Pfeuty[3] and by Wegner.[4] The solutions presented above are valid provided neither $r_\parallel(l)$ nor $r_\perp(l)$ becomes much larger than order unity. When either $r_\parallel(l)$ or $r_\perp(l)$ becomes of order unity, we will cut off the renormalizations and apply a direct graphical technique (see below). That (2.24) and (2.25) are indeed solutions of the basic equations (2.16) and (2.17) may be checked by direct substitution. We note that the crucial feature which allows

systematic ϵ expansion solutions of (2.24) and (2.25) is that $u(l)$ is slowly varying, i.e., $du/dl = O(u^2(l), \epsilon u(l))$.

B. Results for the isotropic case

Once the recursion relations have been solved, we can proceed to calculate various thermodynamic functions. First, however, we review the solutions for the isotropic case,[18] to which our results must reduce when $g=0$. Furthermore, as explained in Sec. I, we will eventually relate the Hamiltonian (2.1) (with $g>0$) to an *isotropic* m-component spin Hamiltonian. At this point, it will be convenient to use the results tabulated in this section directly.

Essential to the idea of using recursion relations to calculate equations of state are relations between quantities calculated with the initial Hamiltonian $\mathcal{H}$ and those calculated with the renormalized Hamiltonian $\mathcal{H}(l^*)$. The susceptibility obeys a straightforward relation to $O(\epsilon)$, namely,[16,18]

$$\chi(r,u) = e^{2l}\chi(r(l),u(l)) \ . \tag{2.28}$$

The analogous expression for the free energy[18,21,22] is more complicated, namely,

$$F(r,u) = \int_0^l e^{-dl'} G_0(l')\,dl' + e^{-dl} F(r(l),u(l)), \tag{2.29}$$

where

$$G_0(l) = \tfrac{1}{2} nK_4\{\ln[1+r(l)] - \tfrac{1}{2}\} \ . \tag{2.30}$$

In Ref. 18, these relations, together with the isotropic recursion relation solutions (2.10) and (2.11), were used to calculate the free energy and susceptibility in the disordered phase. Equations (2.28) and (2.29) were evaluated[18] at a special value of $l = l^*(r,u)$ chosen so that $r(l^*)$ was of order unity. The noncritical functions $\chi(r(l),u(l))$ and $F(r(l),u(l))$ were then evaluated by standard graphical techniques. The results for χ and the singular part of the free energy F_s were, to $O(\epsilon)$,

$$\chi = t^{-\gamma} R^{(n+2)/(n+8)} \tag{2.31}$$

and

$$F_s = \frac{-n}{16(4-n)} \frac{t^{2-\alpha}}{u} R^{(4-n)/(n+8)} + \frac{1}{16}\frac{n}{4-n}\frac{t^2}{u} \ , \tag{2.32}$$

where

$$R = (1 - Bu/\epsilon)t^{\epsilon/2} + Bu/\epsilon, \quad B = 4K_4(n+8) \ , \tag{2.33}$$

and where

$$\gamma = \gamma(n) = 1 + \frac{1}{2}\frac{n+2}{n+8}\epsilon \ ,$$
$$\alpha = \alpha(n) = \frac{1}{2}\frac{4-n}{n+8}\epsilon \tag{2.34}$$

are susceptibility and specific heat indices appropriate to an n-component isotropic system.

C. Review of crossover scaling

Before plunging into the actual calculations of the crossover scaling functions, it seems advisable to review briefly expectations from phenomenological scaling theories.[2,5,8] As discussed in Sec. I for the free energy, thermodynamic functions (we consider the susceptibility as an example) should assume a homogeneous form near the bicritical point, $(t,g) = (0,0)$, namely,

$$\chi(t,g) \approx t^{-\gamma}\Psi(g/t^{\phi}) \ . \tag{2.35}$$

At fixed positive g, the susceptibility should be singular as t approaches some critical point $t_c(g)$, with exponents appropriate to an m-component isotropic system. To describe this behavior,[5] the scaling function $\psi(z)$ should have a singularity at $z^+ > 0$ of the form

$$\Psi(z) \sim (z^+ - z)^{-\dot{\gamma}} \tag{2.36}$$

as $z \to z^+$, where $\dot{\gamma} = \gamma(n)$. To obtain the analogous $(n-m)$-component divergence for negative g, another singularity is required at $z^- < 0$ of the form

$$\Psi(z) \sim (z^- - z)^{-\ddot{\gamma}} \tag{2.37}$$

as $z \to z^-$ with $\ddot{\gamma} = \gamma(n-m)$. It follows that the two critical lines are given by the relation[8]

$$g = z^{\pm} t^{\phi} \ . \tag{2.38}$$

It is often convenient to display the singularity in $\Psi(z)$, explicitly replacing, for example, (2.35) with

$$\chi(t,g) \approx t^{-\gamma}(z^+ - z)^{-\dot{\gamma}} P(z) \ . \tag{2.39}$$

III. BICRITICAL CROSSOVER SCALING FUNCTIONS

A. Ordering susceptibility

As discussed in Secs. I and II, we will calculate quantities such as the ordering susceptibility (i.e., the susceptibility corresponding to fluctuations of the *parallel* spins, assuming $g>0$) by first integrating the anisotropic recursion relations (2.16) and (2.17) until $r_\perp(l)$ is of order unity. Choosing $l = l_1^*(r,g,u)$ such that $r_\perp(l_1^*) \simeq 1$, the susceptibility we wish to calculate is related to the susceptibility of the Hamiltonian $\mathcal{H}(l_1^*)$ through a relation like (2.28), namely,[16]

$$\chi(r_\parallel, r_\perp, u) = e^{2l_1^*}\chi(r_\parallel(l_1^*), r_\perp(l_1^*), u(l_1^*)) \ . \tag{3.1}$$

The susceptibility on the right-hand side of this equation can be calculated by first integrating the $\vec{\sigma}$ spin field out of the problem, treating $\vec{s}$ spins as constant parameters of the potential acting on the $\vec{\sigma}$ spins. We are then left with an m-component *isotropic* effective Hamiltonian, which can then serve as the starting point for the calculation sketched in Sec. IIB. In principle one then integrates the recursion relations for the isotropic system until l reaches l_2^* chosen such that the *isotropic* system is noncritical. In practice it will be easier to simply use known isotropic results,[18] such as (2.31), directly.

The first task clearly is to calculate $l_1^*(t, g, u)$. Consider Eq. (2.24) for $r_\perp(l)$. To leading order in u and ϵ the condition for l_1^* can be written as

$$t(l_1^*) + (m/n)g(l_1^*) = 1 \ . \tag{3.2}$$

For simplicity, we now take $u = u_n^*$ and note that (3.2) takes the simple form

$$te^{\lambda_1 l_1^*(t,g)} + (m/n)ge^{\lambda_2 l_1^*(t,g)} = 1 \ . \tag{3.3}$$

We will generalize our results to an arbitrary $[O(\epsilon)]$ initial value u at the end of this section. For every initial value t and g, (3.3) determines a function

$$e^{l_1^*} = L_1(t, g) \ . \tag{3.4}$$

It is straightforward to show that $L_1(t, g)$ satisfies a homogeneity relation

$$\begin{aligned} L_1(t, g) &= bL_1(b^{\lambda_1}t, b^{\lambda_2}g) \\ &= t^{-1/\lambda_1}\psi(g/t^{\phi}) \ , \end{aligned} \tag{3.5}$$

where $\psi(z) = L(1, z)$ and ϕ is the anisotropic spin crossover exponent given by[3,4]

$$\phi = \lambda_2/\lambda_1 = 1 + \tfrac{1}{2}\epsilon n/(n+8) + O(\epsilon^2) \ . \tag{3.6}$$

Substituting (3.5) into (3.3) gives an equation for $\psi(z)$,

$$\psi^{\lambda_1}(z) + (m/n)z\psi^{\lambda_2}(z) = 1 \ , \tag{3.7}$$

where $z = g/t^{\phi}$ is the crossover scaling parameter. Developing $\psi(z)$ as

$$\psi(z) = \psi_0(z) + \epsilon\psi_1(z) + \ldots \ , \tag{3.8}$$

it is easy to see that

$$\psi_0(z) = [1 + (m/n)z]^{-1/2} \ . \tag{3.9}$$

The function $\psi_1(z)$ can also be calculated straightforwardly, but will not be needed in this order-ϵ calculation.

Inserting (3.5) into (3.1) gives

$$\chi(r_\parallel, r_\perp, u_n^*) = t^{-\gamma}\psi^2(z)\chi(r_\parallel(l_1^*), r_\perp(l_1^*)) \ , \tag{3.10}$$

where we have made the identification

$$\gamma = 2/\lambda_1 + O(\epsilon^2) \ . \tag{3.11}$$

The susceptibility entering the right-hand side of (3.10) is by definition (in the disordered phase)

$$\chi(r_\parallel(l_1^*), r_\perp(l_1^*)) = \int d\vec{R}\langle \vec{s}(\vec{0}) \cdot \vec{s}(\vec{R})\rangle_{\mathcal{H}(l_1^*)} \ , \tag{3.12}$$

where the expectation is meant to be taken with respect to the Hamiltonian $\mathcal{H}(l_1^*)$ (see Appendix A). Since $r_\perp(l_1^*) \simeq 1$, the transverse correlation length is now of order unity and the $\vec{\sigma}$ spins can be integrated out of (3.12) with impunity. This is done to leading order in Appendix A, with the result that $\chi(r(l_1^*), r_\perp(l_1^*))$ is given in terms of an expectation taken with respect to an *isotropic* effective Hamiltonian

$$\mathcal{H}_{\text{eff}} = -\int d\vec{R}\,\frac{1}{2}\sum_{i=1}^{m} |\vec{\nabla}s_i|^2 + \tfrac{1}{2}r_\parallel^{\text{eff}}|\vec{s}|^2 + u(l_1^*)|\vec{s}|^4 \ , \tag{3.13}$$

where

$$\begin{aligned} r_\parallel^{\text{eff}} &= r_\parallel(l_1^*) + 4(n-m)K_4u(l_1^*) \\ &\quad \times \int_0^1 \frac{q^3\,dq}{r_\perp(l_1^*) + q^2} \ . \end{aligned} \tag{3.14}$$

Setting $u = u_n^*$, evaluating the integral in this expression, and using (2.25), various cancellations occur and we obtain

$$\begin{aligned} r_\parallel^{\text{eff}} &= t(l_1^*) - [(n - m/n)]g(l_1^*) - 2(m+2)K_4u_n^* \\ &\quad + 2(m+2)K_4u_n^*r_\parallel(l_1^*)\ln[1 + r_\parallel(l_1^*)] \ . \end{aligned} \tag{3.15}$$

[We have neglected a term $\sim u_n^*r_\perp(l_1^*)\ln r_\perp(l_1^*)$ which, since $r_\perp(l_1^*) = 1 + O(u_n^*)$, is $O(\epsilon^2)$.]

The couplings $r_\parallel^{\text{eff}}$ and u_n^* in (3.13) form the starting parameters for an isotropic m-component Hamiltonian. For that system, the temperature-like parameter entering the solutions is[18] (see Sec. IIB)

$$\begin{aligned} t^{\text{eff}} &= r_\parallel^{\text{eff}} + 2(m+2)K_4u_n^* \\ &\quad - (m+2)K_4u_n^*r_\parallel^{\text{eff}}\ln(1 + r_\parallel^{\text{eff}}) \ . \end{aligned} \tag{3.16}$$

From (3.16) and (3.15), we easily see that to first order in ϵ,

$$\begin{aligned} t^{\text{eff}} &= t(l_1^*) - [(n-m)/n]g(l_1^*) + O(\epsilon^2) \\ &= \psi^{\lambda_1} - [(n-m)/n]z\psi^{\lambda_2} + O(\epsilon^2) \ . \end{aligned} \tag{3.17}$$

With λ_1 and λ_2 given by (2.27) and $\psi(z)$ by (3.8) and (3.9), this can be rewritten to $O(\epsilon)$

$$t^{\text{eff}} = \psi^2 \left[\left(1+\frac{m}{n}z\right)^{\epsilon(n+2)/2(n+8)} - \frac{n-m}{n} z \left(1+\frac{m}{n}z\right)^{\epsilon/(n+8)} \right] \simeq \psi^2(z)\Delta(x)(1-x) , \tag{3.18}$$

where

$$x = z/\dot{z} , \quad \dot{z} = \frac{n}{n-m}\left(1+\frac{\epsilon}{2}\frac{n}{n+8}\ln\frac{n}{n-m}\right) \approx \left(\frac{n}{n-m}\right)^{\phi} , \tag{3.19}$$

and

$$\Delta(x) = 1 - \frac{\epsilon}{2}\left[\left(\frac{2x-n-2}{n+8}\right)\ln\left(1+\frac{m}{n-m}x\right) + \frac{n}{n+8}x\ln\left(\frac{n}{n-m}\right)\right] \Big/ (1-x) \equiv 1 + \epsilon\Delta_1(x) . \tag{3.20}$$

Note that $\lim_{x\to 1}\Delta_1(x)$ is finite. We recall from Sec. II B that the m-component isotropic system will be critical when $t^{\text{eff}} = 0$. Since t^{eff} vanishes when $z = \dot{z}$, we discover a *line* of critical points in the (t, g) plane, given by

$$t_c(g) = \dot{z}^{1/\phi} g^{1/\phi} . \tag{3.21}$$

This result, with $\dot{z}$ given by (3.19), is in agreement with direct Feynman-graph calculations.[23,24]

Inserting (3.18) into the basic isotropic result (2.31), recalling our choice $u = u_n^*$, and using (3.10), the ordering susceptibility may be written

$$\chi = t^{-\gamma}(1-x)^{-\dot{\gamma}}P(x) , \tag{3.22}$$

where

$$P(x) = R_A^{(m+2)/(m+8)}\left[1 - \epsilon\Delta_1(x) + \frac{\epsilon}{2}\frac{m+2}{n+8}\frac{\ln(1+[m/(n-m)]x)}{R_A}\right] , \tag{3.23}$$

$$R_A(x) = \frac{n-m}{n+8}(1-x)^{\epsilon/2} + \frac{m+8}{n+8} , \tag{3.24}$$

and we have normalized χ such that $\chi \simeq t^{-\gamma(n)}$ as $x \to 0$. Note that even though two singularities have been factored out of χ in (3.22), the residual function $P(x)$ still contains a singularity of the form $(1-x)^{\epsilon/2}$ related to the "correction to scaling" exponent first discussed by Wegner.[25] In a Feynman-graph expansion,[23] this singularity would appear as various powers of the logarithm $\ln(1-x)$. Although it is possible to unambiguously exponentiate the leading singularity displayed in (3.22) using a Feynman-graph approach,[23] more information is required in order to exponetiate factors like $\ln(1-x)$ and obtain the $P(x)$ function displayed in (3.25). Note that expanding factors like $(1-x)^{\epsilon/2}$ as a power series in ϵ,

$$(1-x)^{\epsilon/2} \approx 1 + \tfrac{1}{2}\epsilon\ln(1-x) + O(\epsilon^2) , \tag{3.25}$$

in (3.24) should give agreement with the results of a "naive" graphical calculation.

It is straightforward to compare Eq. (3.22) with Feynman-graph results by expanding for small x. For example, when x or z is small we can write

$$\chi \simeq t^{-\gamma}[1 + Az + O(z^2)] , \tag{3.26}$$

where A is given by (3.22) and (3.23),

$$A = \frac{n-m}{n}\left(1+\frac{\epsilon}{n+8}\right) . \tag{3.27}$$

This is precisely the result obtained to $O(\epsilon)$ from a Feynman-graph approach.[5,23]

The susceptibility for an anisotropic spin system (with *fixed* length classical spins) has been calculated by series-expansion techniques. Pfeuty, Jasnow, and Fisher[5] first produced the anisotropic spin crossover scaling function for the case $n = 3$, $m = 1$, while Singh and Jasnow[26,27] have extended the original work to treat the cases $n = 2$, $m = 1$ and $n = 3$, $m = 2$. One way to compare (3.22) and (3.23) with their results is to plot the effective critical exponent γ_{eff}, defined by

$$\gamma_{\text{eff}} = \frac{-d\ln\chi}{d\ln\dot{t}} \tag{3.28}$$

against $\ln\dot{t}$, where

$$\dot{t} = t/t_c(g) - 1 \tag{3.29}$$

is the distance from the critical line at a fixed value of g. Our result for γ_{eff} is compared with the corresponding plot given by Singh and Jasnow[27] in Fig. 2. Although the asymptotic values $\gamma(3)$ and $\gamma(2)$ for the two plots are different, we note that the over-all shapes of the two functions are quite similar. Our values for $\gamma(3) \approx 1.23$ and $\gamma(2) \approx 1.20$ are, of course, correct only to first order in ϵ; thus it is not surprising that the graphs have different asymptotic behavior. It is interesting, how-

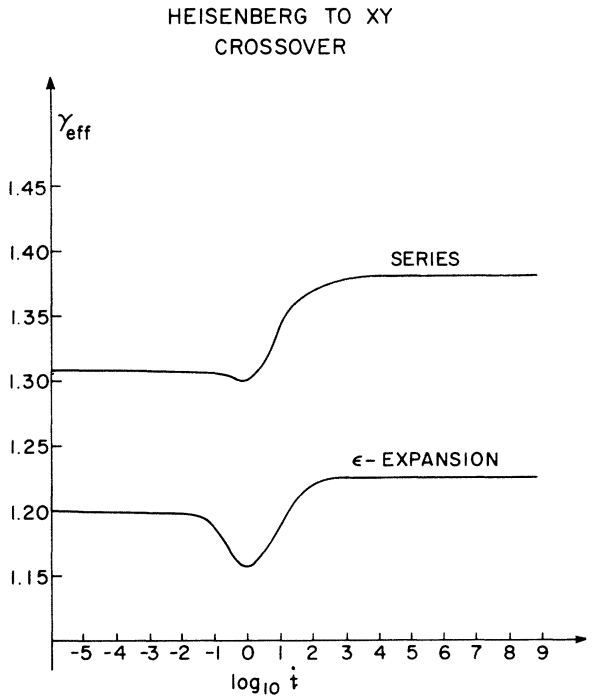

FIG. 2. Comparison $\gamma_{\text{eff}} = -d\ln\chi/d\ln\hat{t}$ for the crossover from Heinsenberg ($n=3$) to XY ($n=2$) critical behavior [calculated here to $O(\epsilon)$] with the results of series expansions (Ref. 26). The ϵ-expansion results are plotted with a value of g chosen to make the crossover regime occur at approximately the same value of $\log_{10}\hat{t}$ as for the series results. The *shape* of the curves should be universal.

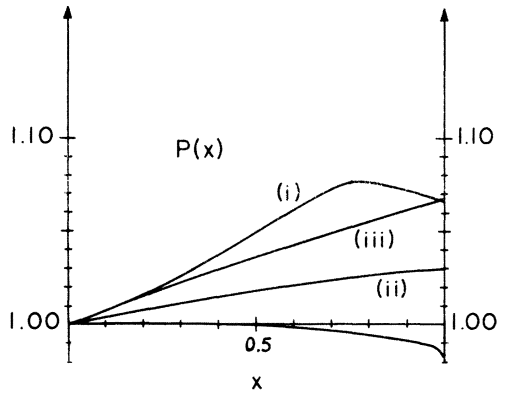

FIG. 3. Comparison of the function $P(x)$ to $O(\epsilon)$ with series-expansion work (Ref. 26). This function describes crossover from Heisenberg ($n=3$) to Ising ($n=1$) critical behavior. Thr result of mean-field theory is given by the straight line $P(x)=1$. Curves (i), (ii), and (iii) are different possibilities for $P(x)$ based on the series work.[26] The remaining curve is the result to $O(\epsilon)$, which comes in with a square-root cusp at $x=1$.

ever, that our $O(\epsilon)$ calculation produces the same dip (although much more pronounced!) in γ_{eff} as was found by the series-expansion work.

A much more sensitive test of our results is to compare the function $P(x)$ with the analogous quantity gound from series work. In Fig. 3, we have plotted the $P(x)$ function found by Singh and Jasnow[26] together with the function given by (3.23) for $n=2$, $m=1$. The mean-field-theory result[5] is $P(x)=1$, and our result differs almost negligibly from this. In fact, it is clear that our expression for $P(x)$ gives a correction to mean-field theory which goes in the *wrong* direction. (This is also true for $n=3$, $m=1$, but not for $n=3$, $m=2$.) The situation here is evidently similar to the first-order-in-epsilon result for $\alpha(n)$,

$$\alpha(n) = \tfrac{1}{2}\epsilon[(4-n)/(n+8)]\,, \tag{3.30}$$

which for $n=3$ gives a *positive* correction to the mean-field-theory result, although results from series expansions, etc.[16] give $\alpha \approx -0.10$. It is hoped that the agreement will improve when the calculations are extended to $O(\epsilon^2)$. We note that quite good agreement with series estimates of crossover scaling amplitudes has been obtained by $O(\epsilon^2)$ Feynman-graph techniques.[23]

The *transverse* susceptibility, that is, the susceptibility corresponding to fluctuations of the perpendicular spin field $\vec{\sigma}$ (when $g>0$), can also be calculated with the methods employed in the subsection. The calculation is sketched in Appendix B.

B. Free energy, specific heat, and nonordering susceptibility

The free energy may be straightforwardly determined using the result for l_1^* derived in the calculation of the susceptibility. The relation analogous to (3.1) relating the free energies of the initial Hamiltonian and the Hamiltonian $\mathcal{H}(l_1^*)$ is[21,22]

$$F(r_{\parallel}, r_{\perp}, u) = \int_0^{l_1^*} e^{-dl} G_A(l) + e^{-dl_1^*} F(r_{\parallel}(l_1^*), r_{\perp}(l_1^*), u(l_1^*))\,, \tag{3.31}$$

where the first term can be thought of as a line integral along a renormalization-group trajectory.[22] The kernel of the anisotropic spin trajectory integral is

$$G_A(l) = \tfrac{1}{2}mK_4\{\ln[1+r_{\parallel}(l)] - \tfrac{1}{2}\} + \tfrac{1}{2}(n-m)K_4\{\ln[1+r_{\perp}(l)] - \tfrac{1}{2}\}\,. \tag{3.32}$$

Making the same choice for l_1^* as in Sec. III A, the free energy on the right-hand side of (3.31) can be evaluated by first integrating out the $\vec{\sigma}$ spin field just as in the case of the susceptibility. We will be left with the free energy of the m-com-

ponent *isotropic* Hamiltonian (3.13) which is given by (2.32).

In Appendix C we integrate out the $\vec{\sigma}$ spins in (3.31) and evaluate the trajectory integral to obtain

$$F(r_{\parallel}, r_{\perp}, u) = -\tfrac{1}{4}mK_4 \int_0^{l_1^*} e^{-dl} r_{\parallel}^2(l)dl - \tfrac{1}{4}(n-m)K_4 \int_0^{l_1^*} e^{-dl} r_{\perp}^2(l)dl$$
$$+ e^{-dl_1^*} \tfrac{1}{8}(n-m)K_4 r_{\perp}^2(l_1^*) \ln r_{\perp}(l_1^*) + e^{-dl_1^*} F_m^{(s)}(r_{\parallel}^{\mathrm{eff}}, u(l_1^*)) , \tag{3.33}$$

where $F_m^{(s)}$ is the singular part of an m-component isotropic free energy, $r_{\parallel}^{\mathrm{eff}}$ is given by (3.14), and we have suppressed certain nonsingular terms. Again, making the simplifying assumption $u = u_n^*$, we use the expressions (3.4), (3.5), (3.8), and (3.9) determining l_1^*. Since $r_{\perp}(l_1^*) = 1 + O(\epsilon)$, the term proportional to $r_{\perp}^2(l_1^*)\ln r_{\perp}(l_1^*)$ in (3.33) drops out entirely to leading order. The trajectory integral contribution may be evaluated exactly to give

$$-\tfrac{1}{4}K_4 \int_0^{l_1^*} e^{-dl}[m r_{\parallel}^2(l) + (n-m) r_{\perp}^2(l)]dl$$
$$= -\frac{1}{4\epsilon} K_4(n+8) t^{2-\alpha} \left[\frac{n}{4-n} + \frac{(n-m)m}{n(4+n)} z^2 - \frac{\epsilon}{2} \left(\frac{n}{n+8} + \frac{(n-m)m}{n(n+8)} z^2 \right) \ln[1 + (m/n)z] \right] , \tag{3.34}$$

where we have dropped analytic terms proportional to t^2 and g^2. These terms allow the free energy to go smoothly into logarithmic behavior where n or m approaches 4. Since these values are outside the range of physical interest, the neglected terms need not concern us.

The final term entering in (3.33) is easily evaluated using the known expression[18] for an isotropic free energy. The effective temperaturelike parameter entering the results is again given by (3.18). Combining the various terms entering (3.33) we obtain our basic result for the free energy,

$$F(r_{\parallel}, r_{\perp}, u_n^*) = -t^{2-\alpha}[Q_1(x) + (1-x)^{2-\delta} Q_2(x)] , \tag{3.35}$$

with

$$Q_1(x) = \frac{n}{4-n} + \frac{nm}{(n-m)(4+n)} x^2 - \frac{m}{4-m} - \frac{\epsilon}{2} \left(\frac{n}{n+8} + \frac{nm}{(n-m)(n+8)} x^2 \right) \ln \left(1 + \frac{m}{n-m} x \right)$$
$$- \epsilon \frac{m}{4-m} (1-x)^2 \left[2\Delta_1(x) - \tfrac{1}{2} \ln \left(1 + \frac{m}{n-m} x \right) \right] , \tag{3.36}$$

$$Q_2(x) = \frac{m}{4-m} R_A^{(4-m)(m+8)} \left[1 + 2\epsilon\Delta_1(x) + \frac{\epsilon}{2} \frac{4-m}{n+8} \ln \left(1 + \frac{m}{n-m} x \right) \Big/ R_A - \frac{\epsilon}{2} \ln \left(1 + \frac{m}{n-m} x \right) \right] . \tag{3.37}$$

We recall that $x = z/\dot{z} = g/t^{\phi}\dot{z}$, and that $R_A(x)$ was defined in Eq. (3.24). No particular normalization has been used here, except to suppress for simplicity an overall factor $\frac{1}{4}K_4(n+8)/\epsilon$.

Other quantities of interest may be obtained using the formalism developed in this subsection for the free energy. Two experimentally measurable quantities are the specific heat, defined by

$$C = -\frac{\partial^2 F}{\partial t^2} , \tag{3.38}$$

and the nonordering susceptibility

$$\chi_g = -\frac{\partial^2 F}{\partial g^2} . \tag{3.39}$$

Although C and χ_g may be obtained from (3.34) by straightforward differentiation, the results to $O(\epsilon)$ are more easily obtained from (3.33). When differentiating this expression, the g and t dependence of l_1^* can be ignored, since the right-hand side must be independent of the precise value of l_1^* (it is tedious, but straightforward, to verify this explicitly). We find that the specific heat and nonordering susceptibility can be written as (with $u = u_n^*$)

$$C = \tfrac{1}{2}nK_4 \int_0^{l_1^*} e^{(2\lambda_1 - d)l} dl - \tfrac{1}{4}(n-m)K_4 e^{(2\lambda_1 - d)l_1^*} \ln r_{\perp}(l_1^*) + e^{(2\lambda_1 - d)l_1^*} C_m^{(s)}(r_{\parallel}^{\mathrm{eff}}, u(l_1^*)) , \tag{3.40}$$

$$\chi_g = \frac{1}{2}\frac{m(n-m)}{n} K_4 \int_0^{l_1^*} e^{(2\lambda_2 - d)l}\, dl - \frac{1}{4}\frac{(n-m)m^2}{n^2} K_4 e^{(2\lambda_2-d)l_1^*} \ln r_\perp(l_1^*)$$

$$+\left(\frac{n-m}{n}\right)^2 e^{(2\lambda_2-d)l_1^*} C_m^{(s)}(r_\parallel^{\text{eff}}, u_n^*), \tag{3.41}$$

where $C_n^{(s)}(r,u)$ is the specific heat of an n-component isotropic system[18]

$$C_n^{(s)}(r,u) = \frac{n}{8(4-n)}\frac{t^{-\dot\alpha}}{u} R^{(4-n)/(n+8)} - \frac{1}{8}\frac{n}{4-n}\frac{1}{u}. \tag{3.42}$$

The quantity $R = R(r,u)$ was defined in Eq. (2.33). This equation follows from (2.32) by differentiating (and neglecting higher-order terms resulting from the differentiations). Substituting this result into (3.40) and (3.41) and setting $l = t^{\text{eff}}$, we determine the crossover scaling expressions for C and χ_g,

$$C = t^{-\alpha}\left[\frac{n}{4-n} - \frac{\epsilon}{2}\frac{n}{n+8}\ln\left(1+\frac{m}{n-m}x\right) + (1-x)^{-\dot\alpha} Q_3(x)\right], \tag{3.43a}$$

$$\chi_g = t^{2-\alpha-2\phi}\left[\frac{m(n-m)}{(4+n)n} - \frac{\epsilon}{2}\frac{m(n-m)}{(n+8)n}\ln\left(1+\frac{m}{n-m}x\right) + \left(\frac{n-m}{m}\right)^2 (1-x)^{-\dot\alpha} Q_4(x)\right], \tag{3.43b}$$

$$Q_3(x) = \frac{m}{4-m}\left\{R_A^{(4-m)/(m+8)}\left[1 - \frac{\epsilon}{2}\frac{4-n}{n+8}\ln\left(1+\frac{m}{n-m}x\right) + \frac{\epsilon}{2}\frac{4-m}{n+8}\ln\left(1+\frac{m}{n-m}x\right)\Big/R_A\right]\right.$$

$$\left. - (1-x)^{\dot\alpha}\left[1 - \frac{\epsilon}{2}\frac{4-n}{n+8}\ln\left(\frac{m}{n-m}x\right)\right]\right\}, \tag{3.44a}$$

$$Q_4(x) = \frac{m}{4-m}\left\{R_A^{(4-m)/(m+8)}\left[1 - \frac{\epsilon}{2}\frac{4+n}{n+8}\ln\left(1+\frac{m}{n-m}x\right) + \frac{\epsilon}{2}\frac{4-m}{m+8}\ln\left(1+\frac{m}{n-m}x\right)\Big/RA\right]\right.$$

$$\left. - (1-x)^{\dot\alpha}\left[1 - \frac{\epsilon}{2}\frac{4+n}{n+8}\ln\left(1+\frac{m}{n-m}x\right)\right]\right\}. \tag{3.44b}$$

As usual, $\alpha = \alpha(n)$, $\dot\alpha = \alpha(m)$, and $x = g/t^{\phi}\dot z$. We have neglected constant, background contributions to (3.43) and (3.44) and dropped an overall factor $\frac{1}{2}K_4(n+8)/\epsilon$. Note that χ_g crosses over to a specific-heat-like singular behavior on the critical line as predicted by the general bicritical scaling analysis.[8]

C. Results for arbitrary isotropic quartic couplings

For simplicity, calculations in Secs. III A and III B were carried out with the initial quartic coupling constant set to its n-component isotropic fixed-point value $u = u_n^*$. It is, however, both straightforward and instructive to do the calculation for an arbitrary initial coupling of order ϵ.

Using the recursion relation solution (2.24), we see that, in general,

$$r_\perp(l_1^*) = \frac{te^{\lambda_1 l_1^*}}{\left[(1-Bu/\epsilon)e^{-\epsilon l_1^*} + Bu/\epsilon\right]^{(n+2)/(n+8)}} + \frac{(m/n)ge^{\lambda_1 l_2^*}}{\left[(1-Bu/\epsilon)e^{-\epsilon l_1^*} + Bu/\epsilon\right]^{2/(n+8)}} + O(u(l^*),\epsilon). \tag{3.45}$$

We now solve the equation

$$r_\perp(l_1^*) = 1 \tag{3.46}$$

by an iterative procedure. As a first estimate of $l_1^*(t,g,u)$ we use the solution of (3.45) when $u = u_n^*$,

$$e^{l_1^*} = t^{-1/\lambda_1}\psi(z). \tag{3.47}$$

Substituting this into the denominators of Eq. (3.45) leads to the requirement

$$te^{\lambda_1 l_1^*}/R^{(n+2)/(n+8)} + (m/n)ge^{\lambda_2 l_1^*}/R^{2/(n+8)} = 1, \tag{3.48}$$

where R is defined by (2.33). On defining

$$\bar t = t/R^{(n+2)/(n+8)}, \quad \bar g = g/R^{2/(n+8)}, \tag{3.49}$$

we can rewrite (3.48) as

$$\bar t e^{\lambda_1 l_1^*} + (m/n)\bar g e^{\lambda_2 l_1^*} = 1, \tag{3.50}$$

which is identical in form to (3.3). It follows immediately that the solution of (3.45) to leading order is

$$e^{l_1^*} = \bar t^{-1/\lambda_1}\Psi(\bar g/\bar t^{\phi}), \tag{3.51}$$

and that our results for χ, F, C, and χ_g may be extended to arbitrary u [of $O(\epsilon)$] merely by the replacements $(t,g) \to (\bar t, \bar g)$, and $u_n^* \to u(l_1^*)$.

For the susceptibility we find

$$\chi = \bar{t}^{-\gamma}(1-\bar{x})^{-\dot{\gamma}}P(\bar{x}), \tag{3.52}$$

where $\bar{x} = \bar{g}/\bar{t}^{\phi}\dot{z}$, and the function $P(\bar{x})$ is now given by (3.23) with x replaced by $\bar{x}$. For a fixed g, χ diverges at some $t = t_c$; the effective susceptibility exponent, defined by

$$\gamma = \frac{-d\ln\chi}{d\ln\dot{t}}, \tag{3.53}$$

with $\dot{t} = (t - t_c)/t_c$ is shown for $n = 2$, $m = 1$ in Fig. 3, where it is compared to the results of high-temperature series expansions by Singh and Jasnow.[27] Both curves remain initially at an Ising value of γ_{eff}, and then cross over in a range of about one decade to bicritical XY values of γ_{eff}. The curves then exhibit a final nonuniversal crossover to the mean field value $\gamma_{\text{eff}} = 1$. Evidently, extending our calculations to allow for deviations of the irrelevant variable u from its fixed-point value leads to results qualitatively similar to those of series expansions (where a large number of irrelevant variables, of course, deviate from their fixed-point values).

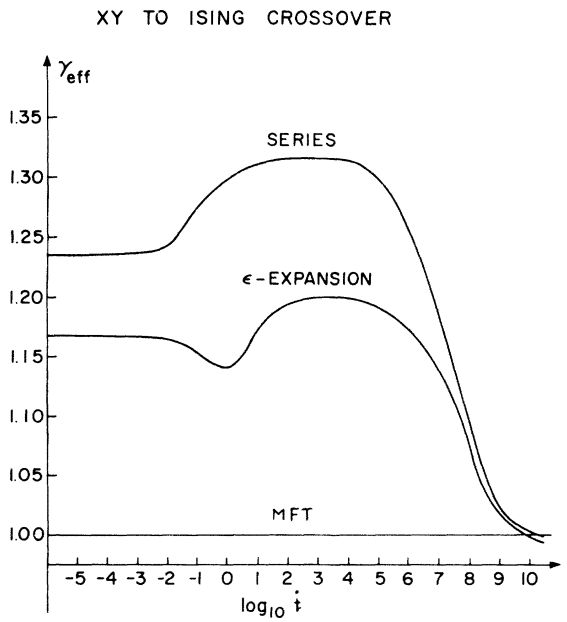

FIG. 4. Plot of $\gamma_{\text{eff}} = -d\ln\chi/d\ln\dot{t}$ to $O(\epsilon)$ compared with the results of series expansions (Ref. 27) showing the effect of introducing an irrelevant variable. The prediction of mean-field theory is given by the horizontal line $\gamma_{\text{eff}} = 1$. The ϵ-expansion result is plotted with g and the deviation of the irrelevant variable u from its fixed-point value chosen to make both crossovers occur at approximately the same place on the $\log_{10}\dot{t}$ axis as for the series result. The shape of the curves before the crossover to mean-field theory sets in should be universal. Because there was no provision in the numerical analysis for expanding in epsilon quantities of the form $1/(1+a\epsilon)$, the large $\dot{t}$ asymptote of the ϵ-expansion result is actually slightly less than 1. The true asymptote, of course, should be precisely unity.

IV. SUMMARY AND CONCLUSIONS

It has recently been recognized that anisotropic spin Hamiltonians such as (1.1) actually occur in nature[6,8,10] with the anisotropy parameter g *experimentally* adjustable by applying a magnetic field[6,8] or pressure.[10] Theoretical calculations of the universal scaling functions are thus of immediate experimental interest. For example, experiments which, in principle, are capable of determining the scaling function for the nonordering susceptibility χ_g have already been carried out for the spin-flopping antiferromagnet $GdAlO_3$.[12]

We have presented here calculations to first order in $\epsilon = 4 - d$ of the longitudinal and transverse susceptibilities, the specific heat, and the nonordering susceptibility for the disordered phase of systems with bicritical points. Although the ϵ-expansion results for the longitudinal susceptibility are clearly inferior to the existing series-expansion calculations,[5,26,27] they do provide a definite improvement over mean-field theory (see Fig. 4). Furthermore, the closed-form expressions obtained here give information about the singularity structure of the crossover scaling functions. For example, the "corrections to scaling" singularity in $P(x)$ [see Eqs. (3.23) and (3.24)] was not detected by the series work. Although series results for the specific heat and nonordering susceptibilities are not presently available, an account of such work is in preparation.[30]

The real utility of the renormalization-group techniques developed here will probably lie in calculations of the bicritical crossover scaling functions appropriate to the ordered phase in Fig. 1. Series expansions have thus far been of little use in this interesting region. We have recently extended the calculations described here to the bicritical ordered phases, and will present the results of our investigation in the future.[31]

After the completion of the investigations described here, we received a report of related work by Kosterlitz.[32] Using a somewhat different technique (which also involves renormalization-group recursion relations), he has obtained results similar to ours for the free energy, specific heat, and longitudinal susceptibility, although there are some minor differences. He has not considered the transverse susceptibility, nor has he treated the nonordering susceptibility χ_g in detail.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge many helpful suggestions and comments from Professor Michael

E. Fisher and numerous conversations with Professor David Jasnow and Dr. Joseph Rudnick. Stimulating correspondance with Dr. J. M. Kosterlitz and Dr. A. D. Bruce is also gratefully acknowledged, as well as the support of the National Science Foundation, in part through the Materials Science Center at Cornell University. We are grateful in addition to support from the National Science Foundation under Grant No. DMR 72-02977 A03 at Harvard University.

APPENDIX A: LONGITUDINAL SUSCEPTIBILITY

The longitudinal susceptibility associated with a Hamiltonian $\mathcal{H}(l)$ is given by

$$\chi(l) = \int d\vec{R}\, \langle \vec{s}(\vec{0}) \cdot \vec{s}(\vec{R}) \rangle_{\mathcal{H}(l)} . \tag{A1}$$

The angular brackets here denote the thermodynamic average defined by a functional integral,[28] namely

$$\langle A \rangle_{\mathcal{H}(l)} = \frac{\int\int \mathfrak{D}s_i(\vec{R})\mathfrak{D}\sigma_j(\vec{R}')\, e^{-\mathcal{H}(l)/k_BT} A(s_i(\vec{R}), \sigma_j(\vec{R}'))}{\int\int \mathfrak{D}s_i(\vec{R})\mathfrak{D}\sigma_j(\vec{R}')\, e^{-\mathcal{H}(l)/k_BT}} . \tag{A2}$$

On expanding $\vec{s}(\vec{R})$ and $\vec{\sigma}(\vec{R})$ in Fourier components,[16]

$$s_i(\vec{R}) = \int_{\vec{q}} e^{i\vec{q}\cdot\vec{R}} \hat{s}_i(\vec{q}) ,$$
$$\sigma_i(R) = \int_{\vec{q}} e^{i\vec{q}\cdot\vec{R}} \hat{\sigma}_i(\vec{q}) \tag{A3}$$

[$\int_{\vec{q}}$ means $(2\pi)^{-d}\int d\vec{q}$], and (A1) becomes

$$\chi(l) = \sum_i \langle \hat{s}_i(\vec{q})\hat{s}_i(\vec{q}') \rangle_{\mathcal{H}(l)} \Big/ \delta(\vec{q}+\vec{q}') \Big|_{q=q'=0} . \tag{A4}$$

The Hamiltonian $\mathcal{H}(l)$ can be decomposed

$$\mathcal{H}(l) = \mathcal{H}_1(s) + \mathcal{H}_2(\sigma) + \mathcal{H}_\times(s,\sigma) , \tag{A5}$$

with

$$\bar{\mathcal{H}}_1 \equiv \frac{-\mathcal{H}_1(s)}{k_BT} = -\int_{\vec{q}} \tfrac{1}{2}[r_{\parallel}(l_1^*) + q^2] \sum_{i=1}^{m} \hat{s}_i(\vec{q})\hat{s}_i(-\vec{q})$$
$$- u(l_1^*) \sum_{i,j=1}^{m} \int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3} \hat{s}_i(\vec{q})\hat{s}_i(\vec{q}_1)\hat{s}_j(\vec{q}_2)\hat{s}_j(\vec{q}_3)\, \delta(\vec{q}+\vec{q}_1+\vec{q}_2+\vec{q}_3) , \tag{A6}$$

$$\bar{\mathcal{H}}_2 = \frac{-\mathcal{H}_2(\sigma)}{k_BT}$$
$$= -\int_{\vec{q}} \tfrac{1}{2}[r_{\perp}(l_1^*) + q^2] \sum_{i=1}^{n-m} \hat{\sigma}_i(\vec{q})\hat{\sigma}_i(-\vec{q}) - u(l_1^*) \sum_{i,j=1}^{n-m} \int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3} \hat{\sigma}_i(\vec{q})\hat{\sigma}_i(\vec{q}_1)\hat{\sigma}_j(\vec{q}_2)\hat{\sigma}_j(\vec{q}_3)\, \delta(\vec{q}+\vec{q}_1+\vec{q}_2+\vec{q}_3) , \tag{A7}$$

and

$$\bar{\mathcal{H}}_\times = \frac{-\mathcal{H}_\times(s,\sigma)}{k_BT} = -2u(l_1^*) \sum_{i=1}^{m}\sum_{j=1}^{n-m} \int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3} \hat{s}_i(\vec{q})\hat{s}_i(\vec{q}_1)\sigma_j(\vec{q}_2)\sigma_j(\vec{q}_3)\, \delta(\vec{q}+\vec{q}_1+\vec{q}_2+\vec{q}_3) . \tag{A8}$$

Denoting $\int\int \mathfrak{D}s_i(R)\mathfrak{D}s_j(R)$ by $\mathrm{Tr}_s\,\mathrm{Tr}_\sigma$, expectations of the form

$$\mathrm{Tr}_s\,\mathrm{Tr}_\sigma A(s) e^{-\mathcal{H}_l(s,\sigma)} \tag{A9}$$

may be rewritten as

$$\mathrm{Tr}_s A(s) e^{\bar{\mathcal{H}}_1(s)}\, \mathrm{Tr}_\sigma e^{\bar{\mathcal{H}}_2(\sigma)+\bar{\mathcal{H}}_\times(s,\sigma)} . \tag{A10}$$

The $\vec{\sigma}$-spin filed can now be integrated out by evaluating

$$\mathrm{Tr}_\sigma e^{\bar{\mathcal{H}}_2(\sigma)+\bar{\mathcal{H}}_\times(s,\sigma)}$$

diagramatically, expanding in $u(l_1^*)$, to give a function of $\vec{s}$ only. Terms independent of $\vec{s}$ cancel out in averages like (A4), and the $\vec{s}$-dependent terms can be reexponentiated and used to renormalize $\mathcal{H}_1(s)$. Specifically, we find

$$\chi(l) = \sum_i \langle \hat{s}_i(\vec{q})\hat{s}_i(\vec{q}') \rangle_{\mathcal{H}_{\mathrm{eff}}} \Big/ \delta_j(\vec{q}+\vec{q}') \Big|_{\vec{q}=\vec{q}'=0} , \tag{A11}$$

with $\mathcal{H}_{\mathrm{eff}}$ given by Eqs. (3.13) and (3.14) in the text.

APPENDIX B: TRANSVERSE SUSCEPTIBLITY

The transverse susceptibility, which for $g>0$ is

$$\chi_\perp(l) = \int d\vec{R}\, \langle \vec{\sigma}(\vec{0}) \cdot \vec{\sigma}(\vec{R}) \rangle_{\mathcal{H}(l)} , \tag{B1}$$

may be calculated by the method sketched in Appendix A. Expectations like (B1) are of interest because an equation analogous to (3.1) holds for the transverse susceptibility to $O(\epsilon)$, namely

$$\chi_\perp(r_{\parallel}, r_\perp, u) = e^{2l_1^*} \chi_\perp(r_{\parallel}(l_1^*), r_\perp(l_1^*), u(l_1^*)) . \tag{B2}$$

The prefactor $e^{2l_1^*}$ is given by Eq. (3.4) in the text, and we are left only with the task of determining $\chi_\perp(l_1^*)$.

On inserting the Fourier expansions (A3), (B1) becomes

$$\chi_\perp(l) = \sum_i \langle \hat{\sigma}_i(\vec{q})\hat{\sigma}_i(\vec{q}')\rangle_{\mathcal{H}_l(s,\sigma)} \Big/ \delta(\vec{q}+\vec{q}')\Big|_{\vec{q}=\vec{q}'=0} \tag{B3}$$

where $\mathcal{H}_l(s,\sigma)$ is given by (A5)–(A8). The basic quantity entering (B3) may be rewritten in the notation of Appendix A as

$$\langle \hat{\sigma}_i(\vec{q})\hat{\sigma}_i(\vec{q}')\rangle_{\mathcal{H}_l(s,\sigma)} = \frac{\mathrm{Tr}_s e^{\bar{\mathcal{H}}_1(s)}\,\mathrm{Tr}_\sigma \hat{\sigma}_i(q)\hat{\sigma}_i(q')e^{\bar{\mathcal{H}}_2(\sigma)+\bar{\mathcal{H}}_X(s,\sigma)}}{\mathrm{Tr}_s e^{\bar{\mathcal{H}}_1(s)}\,\mathrm{Tr}_\sigma e^{\bar{\mathcal{H}}_2(\sigma)+\bar{\mathcal{H}}_X(s,\sigma)}} . \tag{B4}$$

The denominator can be evaluated exactly as in Appendix A, while the term

$$\mathrm{Tr}_\sigma \sigma_i(q)\sigma_i(q')e^{\bar{\mathcal{H}}_2(\sigma)+\bar{\mathcal{H}}_X(s,\sigma)}$$

in the numerator can be calculated in a similar fashion. Combining these two results, we find that

$$\chi_\perp(l) = \Big\langle r_\perp^{-1}(l)(n-m) + r_\perp^{-2}(l)(n-m+2)u(l)K_4 \times \int_0^1 q^3 dq \Big/ [r_\perp(l)+q^2] + r_\perp^{-2}(l)4(n-m)u(l)\sum_{i=1}^m \int_{\vec{q}} \hat{s}_i(\vec{q})\hat{s}_i(-\vec{q})\Big\rangle_{\mathcal{H}_{\mathrm{eff}}} \tag{B5}$$

to leading order, with $\mathcal{H}_{\mathrm{eff}}$ again given by (3.13) and (3.14).

Setting $l = l_1^*$ and $u = u_n^*$ in the above, this result may be rewritten

$$\chi_\perp(l) = \Big\langle 1 + 2(n-m)u_n^* \sum_{i=1}^m \int_{\vec{q}} \hat{s}_i(\vec{q})\hat{s}_i(-\vec{q})\Big\rangle_{\mathcal{H}_{\mathrm{eff}}} . \tag{B6}$$

Thus our result for the transverse susceptibility is

$$\chi_\perp = e^{2l_1^*}[1 + 2(n-m)u_n^* E_m(r_\parallel^{\mathrm{eff}}, u_n^*)] , \tag{B7}$$

where $E_m(r,u)$ is related to the singular part of the m-component free energy $F_m^{(s)}(r,u)$ by

$$E_m(r,u) = -\frac{\partial F_m^{(s)}(r,u)}{\partial r} . \tag{B8}$$

Thus $E_m(r,u)$ is simply the energy of an m-component isotropic system, which can be obtained immediately from (2.32).

The basic result (B7), when combined with (3.15) and the isotropic result (2.32) for $F_m(r,u)$, shows explicitly that $\chi_\perp$ should have a singularity on approaching the bicritical line of fixed points of the form

$$\chi_\perp t^{-\gamma}[A(x) + B(x)(1-x)^{1-\alpha(m)}] , \tag{B9}$$

as expected.[29] As in the case of the longitudinal susceptibility, (B7) can be evaluated further by substitution of the isotropic result for $E_m(r_\parallel, u_n^*)$ and the expression for l_1^* derived in Sec. II.

APPENDIX C: FREE ENERGY

In order to evaluate the free energy, we must consider the trajectory integral expression (3.31),

$$F(r_\parallel, r_\perp, u) = \int_0^{l_1^*} e^{-dl}[G_\parallel(l) + G_\perp(l)]\,dl + e^{-dl_1^*}F(r_\parallel(l_1^*), r_\perp(l_1^*), u(l_1^*)), \tag{C1}$$

where, for convenience, the kernel of the trajectory integral has been split into two parts,

$$G_\parallel(l) = \tfrac{1}{2}mK_4\{\ln[1 + r_\parallel(l)] - \tfrac{1}{2}\} \tag{C2}$$

and

$$G_\perp(l) = \tfrac{1}{2}(n-m)K_4\{\ln[1 + r_\perp(l)] - \tfrac{1}{2}\} . \tag{C3}$$

The crucial problem is to evaluate the renormalized free energy $F(r_\parallel(l_1^*), r_\perp(l_1^*), u(l_1^*))$, given in the notation of Appendix A by

$$F(r_\parallel(l_1^*), r_\perp(l_1^*), u(l_1^*)) \equiv \ln(\mathrm{Tr}_s\,\mathrm{Tr}_\sigma e^{-\mathcal{H}(l_1^*)/k_BT}) . \tag{C4}$$

Decomposing $\mathcal{H}(l_1^*)$ as in Appendix A, this can be rewritten as

$$F(r_\parallel(l_1^*), r_\perp(l_1^*), u(l_1^*)) = \ln(\mathrm{Tr}_s e^{\mathcal{H}_1(s)}\,\mathrm{Tr}_\sigma e^{\mathcal{H}_2(\sigma)+\mathcal{H}_X(s,\sigma)}) . \tag{C5}$$

As in the case of the longitudinal and transverse susceptibilities, it is straightforward to calculate $\mathrm{Tr}_\sigma e^{\bar{\mathcal{H}}_2(\sigma)+\bar{\mathcal{H}}_X(s,\sigma)}$ perturbatively to produce a function of s only. The s-independent terms generated by such a procedure do not cancel in this case, and we find

$$F(r_{\parallel}(l_1^*), r_{\perp}(l_1^*), u(l_1^*)) = \tfrac{1}{2}(n-m)K_4 \int_0^1 q^3\, dq \ln[q^2 + r_{\perp}(l_1^*)] + F_m(r_{\parallel}^{\rm eff}, u(l_1^*)), \tag{C6}$$

where

$$F_m(r_{\parallel}^{\rm eff}, u(l_1^*)) = \ln(\mathrm{Tr}_s e^{-\mathcal{H}_{\rm eff}/k_B T}) . \tag{C7}$$

We have retained terms only up to $O(u(l_1^*)^0)$ in (C6) since the free energy itself is $O(u(l_1^*)^{-1})$.

The trajectory integral terms in (C1) were evaluated for the isotropic case in an appendix to Ref. 18. Using techniques identical to the manipulations described there, it is straightforward to show that

$$\begin{aligned}\int_0^{l_1^*} e^{-dl} G_{\perp}(l)\, dl &+ e^{-dl_1^*} \tfrac{1}{2}(n-m)K_4 \int_0^1 q^3\, dq \ln[q^2 + r_{\perp}(l_1^*)] \\ &= -\tfrac{1}{16}(n-m)K_4 + \tfrac{1}{8}(n-m)K_4 \ln(1+r_{\perp}) + \tfrac{1}{8}(n-m) r_{\perp} - \tfrac{1}{8}(n-m)K_4 r_{\perp}^2 \ln(1+r_{\perp}) \\ &\quad - \tfrac{1}{4}(n-m)K_4 \int_0^{l_1^*} e^{-dl} r_{\perp}^2(l)\, dl + \tfrac{1}{8}(n-m)K_4 e^{-dl_1^*} r_{\perp}^2(l_1^*) .\end{aligned} \tag{C8}$$

In addition, it follows that

$$\begin{aligned}\int_0^{l_1^*} e^{-dl} G_{\parallel}(l)\, dl &= \tfrac{1}{16} m K_4 (e^{-dl_1^*} - 1) - \tfrac{1}{8} m K_4 \{e^{-dl_1^*} \ln[1 + r_{\parallel}(l_1^*)] - \ln(1 + r_{\parallel})\} \\ &\quad - \tfrac{1}{8} m K_4 [e^{-dl_1} r_{\parallel}(l_1^*) - r_{\parallel}] - \tfrac{1}{8} m K_4 \{e^{-dl_1^*} r_{\parallel}^2(l_1^*) \ln[1 + r_{\parallel}(l_1^*)] - r_{\parallel}^2 \ln(1 + r_{\parallel})\} \\ &\quad - \tfrac{1}{4} m K_4 \int_0^{l_1^*} e^{-dl} r_{\parallel}^2(l)\, dl ,\end{aligned} \tag{C9}$$

while the relationship between the full free energy $F_m(r_{\parallel}^{\rm eff}, u(l_1^*))$ and the singular part of the free energy (2.32) is

$$F_m(r_{\parallel}^{\rm eff}, u(l_1^*)) = F_m^{(s)}(r_{\parallel}^{\rm eff}, u(l_1^*)) - \tfrac{1}{16} m K_4 + \tfrac{1}{8} m K_4 + \tfrac{1}{8} m K_4 \ln(1 + r_{\parallel}^{\rm eff}) + \tfrac{1}{2} m K_4 r_{\parallel}^{\rm eff} \tfrac{1}{8} m K_4 (r_{\parallel}^{\rm eff})^2 \ln(1 + r_{\parallel}^{\rm eff}) . \tag{C10}$$

Multiplying this expression by $e^{-dl_1^*}$ and then making use of (C6) and (C8)–(C10), we conclude that (C1) is in fact given by Eq. (3.33) quoted in the text. Various *analytic* contributions to the free energy [such as terms like $\frac{1}{8}(n-m)K_4 \ln(1+r_{\perp})$ and $\frac{1}{8} m K_4 r_{\parallel}$ in (C8) and (C9)] have been deleted from (3.33) for simplicity.

Note added in proof. We have recently received a report of work by H. Horner [University of Heidelberg (unpublished)]. Using a different approach, he has calculated to $O(\epsilon)$ the parallel and perpendicular susceptibilities for the case $n=2$, $m=1$.

*Present address: Dept. of Physics, Harvard University, Cambridge, Mass. 02138.

[1]D. Jasnow and M. Wortis, Phys. Rev. 176, 739 (1968).
[2]E. K. Riedel and F. J. Wegner, Z. Phys. 225, 195 (1969).
[3]M. E. Fisher and P. Pfeuty, Phys. Rev. B 6, 1889 (1972).
[4]F. Wegner, Phys. Rev. B 6, 1891 (1972).
[5]P. Pfeuty, D. Jasnow, and M. E. Fisher, Phys. Rev. B 10, 2088 (1974).
[6]D. R. Nelson, J. M. Kosterlitz, and M. E. Fisher, Phys. Rev. Lett. 33, 813 (1974), and unpublished.
[7]For a review, see F. Keffer, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1966), Vol. 18, Pt. 2.
[8]M. E. Fisher and D. R. Nelson, Phys. Rev. Lett. 32, 1350 (1974); see also M. E. Fisher, Phys. Rev. Lett. 34, 1634 (1975).
[9]See, e.g., K.-S. Liu and M. E. Fisher, J. Low Temp. Phys. 10, 655 (1972).
[10]A. Aharony and A. D. Bruce, Phys. Rev. Lett. 33, 427 (1974).
[11]See L. Néel, Ann. Phys. (Paris) 18, 5 (1932); C. R. Acad. Sci. (Paris) 203, 304 (1936).
[12]H. Rohrer, Phys. Rev. Lett. 34, 1638 (1975).
[13]See, e.g., A. Aharony and A. D. Bruce, Phys. Rev. Lett. 33, 427 (1974).
[14]A. D. Bruce and A. Aharony, Phys. Rev. B 11, 478 (1975).
[15]D. R. Nelson and J. Rudnick (unpublished).
[16]K. G. Wilson and J. Kogut, Phys. Rep. C 12, 77 (1974); M. E. Fisher, Rev. Mod. Phys. 42, 597 (1974).
[17]The rules for setting up such a perturbation theory have been discussed, e.g., by D. Thouless [Phys. Rev. 181, 954 (1969)].
[18]D. R. Nelson and J. Rudnick, Phys. Rev. Lett. 35, 178 (1975); J. Rudnick and D. R. Nelson (unpublished).
[19]F. J. Wegner and A. Houghton, Phys. Rev. A 8, 401 (1973).
[20]See Ref. 18. These equations were originally treated by J. F. Nicoll, T. S. Chang, and H. E. Stanley [Phys.

Rev. Lett. 32, 1446 (1974)] who obtained solutions valid in the limit $r(\ell) \to \infty$. However, since we are interested here only in the range $|r(\ell)| \leq 1$, substantial simplifications can be made.
[21]J. Rudnick, Phys. Rev. B 11, 363 (1975).
[22]D. R. Nelson, Phys. Rev. B 11, 3504 (1975).
[23]A. D. Bruce, University of Edinburgh report (unpublished).
[24]D. R. Nelson (unpublished).
[25]F. J. Wegner, Phys. Rev. B 5, 4529 (1972).
[26]S. Singh and D. Jasnow, Phys. Rev. B 11, 3445 (1975).
[27]S. Singh and D. Jasnow, Phys. Rev. B (to be published).
[28]See, for example, E. Brezin, D. J. Wallace, and K. G. Wilson, Phys. Rev. B 7, 232 (1973).
[29]See M. E. Fisher, J. Math. Phys. 4, 124 (1963); and E. K. Riedel and F. Wegner, Z. Phys. 225, 195 (1969).
[30]P. Gerber and M. E. Fisher (unpublished).
[31]D. R. Nelson and E. Domany (unpublished).
[32]J. M. Kosterlitz, University of Birmingham report (unpublished).