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Induced torque in an ellipsoid. Torque anisotropy in potassium

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We present the theory of the induced torque in a conducting ellipsoid rotating in a magnetic field. The results for compensated metals and for metals with open orbits are not strikingly different from those found in a case of a sphere. However, in a noncompensated metal with closed orbits only, large nonsaturating anisotropies appear. These can fully explain some of the torque anomalies observed in potassium. Applicable expressions for the torque are given, and the magnetic moment and charge density are discussed.

I. INTRODUCTION.

The study of the induced torque on conducting spheres rotating in a magnetic field is used as an experimental tool in unraveling the galvanomagnetic properties of pure metals. Early experimental work on spheres of gallium was reported by Datars and Cook.[1] Details of the technique and an exact theory were first given by Lass and Pippard,[2] and a more complete theoretical discussion was presented by Visscher and Falicov.[3] Delaney and Pippard[4] have provided the most complete discussion of the various facets of the induced-torque method, including a comparison of the difficulties of electrodeless methods with the problems encountered in conventional high-field magnetoresistance techniques.

The experimental work to date has concentrated on the following magnetoresistance problems: the mapping of open orbits,[1,5,6] the detection of magnetic breakdown,[7,8] anisotropic relaxation times,[8–10] and the study of high-field magnetoresistance of potassium and aluminum.[8,11–14] In the process some very interesting new effects were described: coupled mechanical and helicon oscillations,[15] helicon resonances in spheres,[16] and nonsaturating anisotropy of the induced torque owing to sample shape.[13,17] This last effect, the influence of sample shape on the induced torque and its importance for the measuring technique, is the subject of this paper.

The induced-torque experiment which we will treat involves a spherical or an ellipsoidal sample of conducting material rotating about a fixed axis in a stationary magnetic field perpendicular to the axis of rotation. Equivalently, the sample is stationary and the magnetic field rotates in a plane. Currents are induced in the sample by the rotation, these interact with the magnetic field to produce a torque on the sample, and this torque is what has been measured in most variants of the experiment. It is also possible to measure the magnetic moment of the induced currents with a pick-up coil.[18] The field magnitude can vary from zero to values at which high-field galvanomagnetic effects occur, while the frequency of rotation must be low enough so that perfect penetration of the magnetic field is possible. The requirements on the symmetry of the sample are not so easily stated, but some guidance in the matter of sample shape can be obtained from the striking difference between the solutions for the torque of a sphere and that of an ellipsoid.

The goal of this paper is to present a generalization of the theory published so far to include an ellipsoidal sample, and to discuss the influence of the ellipsoidal shape on the torque in a number of specific experimental situations. In one case we will be able to compare our results with detailed measurements made on samples known not to be spheres.

II. THEORY

A. Linear transformation of coordinates

It is possible to solve the problem for an ellipsoid within the framework of the theory for a sphere, by using a suitable coordinate transformation. Let $\tilde{S}$ be real 3×3 tensor representing a linear coordinate transformation such that if $\vec{r}_s'$ is a radius vector of a spherical surface then $\vec{r}_e = \tilde{S}^{-1}\vec{r}_s'$ is a radius vector of an ellipsoidal surface, and

$$\vec{r}_s' \cdot \vec{r}_s' = \vec{r}_e \tilde{S}^\dagger \tilde{S} \vec{r}_e = R^2, \tag{1}$$

where R is the radius of the sphere, and $\tilde{S}^\dagger$ is the transposed tensor to $\tilde{S}$.

By taking the gradient in undashed coordinates of R^2 it follows that the vector

$$\vec{n} = \tilde{S}^\dagger \tilde{S} \vec{r}_e = \tilde{\epsilon} \vec{r}_e \tag{2}$$

is normal to the surface of the ellipsoid at $\vec{r}_e$, where we have denoted $\tilde{\epsilon} = \tilde{S}^\dagger \tilde{S}$. For elements of volume in the two coordinate systems the following holds:

$$d^3r' = \det\tilde{S}\, d^3r = (\det\tilde{\epsilon})^{1/2} d^3r. \tag{3}$$

Let us consider a specific ellipsoid of rotation which is developed from a sphere by the following sequence of transformations: First the sphere is compressed along the y axis by a factor $b<1$ (or elongated, $b>1$), then tilted through an angle φ around the z axis, and finally rotated around the y axis by an angle θ. The total transformation is then

$$\tilde{S}^{-1}=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\times\begin{pmatrix}1 & 0 & 0\\ 0 & b & 0\\ 0 & 0 & 1\end{pmatrix} \quad . \tag{4}$$

It is useful to have the following expressions which are derived from Eq. (4):

$$\tilde{\epsilon}=\tilde{S}^{\dagger}\tilde{S}=1-\eta\tilde{K},$$

$$\tilde{\epsilon}^{-1}=1+b^2\eta\tilde{K},$$

where

$$\eta=(1-b^{-2})\sin^2\varphi, \tag{5a}$$

and $K_{ij}=l_i l_j$, where $\vec{l}=(\cos\theta,\ \cot\varphi,\ -\sin\theta)$. In Eq. (5a) the parameter η has been introduced. It is a measure of the nonsphericity relevant to the induced torque experiment. Further, it can be shown that

$$\det\tilde{S}=b^{-1},$$

$$\det\tilde{\epsilon}=b^{-2}, \tag{5b}$$

$$\det(\tilde{\xi}\tilde{\epsilon})=b^{-2}\det\tilde{\xi}$$

for any tensor $\tilde{\xi}$.

B. Induced torque

For the sake of brevity we shall follow closely the derivation as formulated by Visscher and Falicov,[3] and emphasize only the equations which differ considerably in transformed coordinates from those which apply to a spherical sample.

The ellipsoidal sample, with a homogeneous magnetoresistivity tensor $\tilde{\rho}$, rotating in a magnetic field $\vec{B}$ is effectively being subjected to a variation of the magnetic field $d\vec{B}/dt=\vec{\Omega}\times\vec{B}$. For the low rotational frequencies Ω considered here, the Maxwell's equation to be solved is

$$\vec{\nabla}\times(\tilde{\rho}\vec{j})=-\frac{1}{c}\,\frac{d\vec{B}}{dt} \tag{6}$$

with the boundary condition that the current density $\vec{j}$ is parallel to the surface, that is,

$$\vec{n}\,\vec{j}=\vec{r}_e\,\tilde{\epsilon}\,\vec{j}=0\ , \tag{7}$$

where $\tilde{\epsilon}$ describes the sample, and $\vec{r}_e$ is a radius vector of the surface.

The continuity equation for first-order currents (that is, proportional to the first power of the rotational frequency Ω) requires that the currents be divergence free

$$\operatorname{div}\vec{j}=0. \tag{7a}$$

First-order space charges may also be present, as we shall see later, but vary slowly in time. The uniqueness arguments given in Ref. 3 can be applied directly to solutions of Eqs. (6)–(7a).

We shall look for a solution in the form $\vec{j}=\tilde{T}\vec{r}$. Then Eq. (7) can be satisfied only if $(\tilde{\epsilon}\tilde{T})$ is an antisymmetric tensor, implying equivalently the existence of an axial vector $\vec{t}$, such that $\tilde{\epsilon}\tilde{T}\vec{r}=\vec{t}\times\vec{r}$. Hence

$$\vec{j}=\tilde{\epsilon}^{-1}(\vec{t}\times\vec{r}). \tag{8}$$

To find $\vec{t}$ we insert Eq. (8) into Eq. (6) and after expanding cross products find that

$$\vec{t}=-\frac{1}{2c}\,\tilde{\gamma}\tilde{\epsilon}\,\frac{d\vec{B}}{dt}\ , \tag{9}$$

where

$$\tilde{\gamma}=\{\tfrac{1}{2}[\mathrm{Tr}(\tilde{\rho}\tilde{\epsilon}^{-1})\,\tilde{\epsilon}-\tilde{\rho}^{\dagger}]\}^{-1}\quad . \tag{10}$$

The current is now expressed in terms of known quantities. Its contribution to the induced torque $\vec{N}$ per unit volume is proportional to

$$c\,\frac{d\vec{N}}{d^3r'}\,\det\tilde{S}=\vec{r}\times(\vec{j}\times\vec{B})$$

$$=\vec{j}(\vec{B}\cdot\vec{r})-(\vec{r}\cdot\vec{j})\,\vec{B}. \tag{11}$$

To determine the total torque acting on the sample, we transform the expression into $\vec{r}\,'$ coordinates and integrate over the appropriate sphere

$$\int(\tilde{T}\tilde{S}^{-1}\vec{r}\,')\cdot(\vec{r}\,'\tilde{S}^{-1\dagger}\vec{B})\,d^3r'$$

$$-\int(\vec{r}\,'\tilde{S}^{-1\dagger}\,\tilde{T}\tilde{S}^{-1}\,\vec{r}\,')\,\vec{B}\,d^3r',$$

where we have written $\tilde{T}\tilde{S}^{-1}\vec{r}\,'$ for $\vec{j}$. The first integral is proportional to the vector $\tilde{T}\tilde{\epsilon}^{-1}\vec{B}$ which can be shown to be equal to $\det\epsilon^{-1}\,(\tilde{\epsilon}\,\vec{t})\times\vec{B}$, while the integral of the second term is proportional to $\mathrm{Tr}(\tilde{S}^{-1\dagger}\,\tilde{T}\tilde{S}^{-1})$ which is zero.

Therefore, after substituting from Eq. (9), the resulting torque is

$$\vec{N}=\int d\vec{N}=\frac{1}{c}\,\frac{4\pi R^5}{15}\,(\det\tilde{S})^{-3}(\tilde{\epsilon}\,\vec{t})\times\vec{B}$$

$$=\frac{2\pi R^5}{15c^2}\quad(\det\tilde{S})^{-3}\left(\vec{B}\times\tilde{\epsilon}\tilde{\gamma}\tilde{\epsilon}\,\frac{d\vec{B}}{dt}\right)\ . \tag{12}$$

(Note printer's errors in Ref. 17.)

Equation (10) for $\tilde{\gamma}$ involves an inversion which can be accomplished using the formula, valid for any $\tilde{\xi}$,

$$(\mathrm{Tr}\tilde{\xi}-\tilde{\xi})\,(\tilde{\xi}^{-1}+\tilde{\xi}\,\mathrm{Tr}\tilde{\xi}\,\det\tilde{\xi}^{-1})=\mathrm{Tr}\tilde{\xi}\,\mathrm{Tr}\tilde{\xi}^{-1}-1, \tag{13}$$

where we write $\tilde{\xi}=\tilde{\epsilon}^{-1}\rho^{\dagger}$, providing the right-hand side is nonzero.

To give a result more applicable than the general expressions (12) and (10), we choose the following geometry: Let $\vec{B} \parallel \vec{z}$, $\vec{\Omega} \parallel \vec{y}$, and $d\vec{B}/dt \parallel \vec{x}$, and let the transformation of coordinates which describes the sample be given by Eq. (4). Then the component N_y of the torque in the direction of the axis of rotation is

$$N_y = \frac{4\pi R^5}{15c^2} B^2 \Omega b^3 \frac{\sigma_{xx} + \rho_{xx} \det\tilde{\sigma}\, b^{-2}\, \mathrm{Tr}\tilde{\rho} - \eta A + \eta^2 E}{\mathrm{Tr}\tilde{\rho}\, \mathrm{Tr}\tilde{\sigma} - 1 - \eta C - \eta^2 D}, \tag{14}$$

where

$$\eta = (1 - b^{-2}) \sin^2\varphi,$$

$$A = (\tilde{\sigma}\tilde{K})_{xx} + (\tilde{K}\tilde{\sigma})_{xx} - \rho_{xx} \det\tilde{\sigma}\, \mathrm{Tr}(\tilde{K}\tilde{\rho}),$$

$$E = \cos^2\theta\, \mathrm{Tr}(\tilde{\sigma}\tilde{K}),$$

$$C = \mathrm{Tr}(\tilde{\sigma}\tilde{K})\, \mathrm{Tr}\tilde{\rho} - b^2\, \mathrm{Tr}(\tilde{K}\tilde{\rho})\, \mathrm{Tr}\tilde{\sigma},$$

$$D = \mathrm{Tr}(\tilde{\sigma}\tilde{K})\, \mathrm{Tr}(\tilde{K}\tilde{\rho})\, b^2.$$

We shall write out the more involved terms in full

$$\begin{aligned}
(\tilde{\sigma}\tilde{K})_{xx} + (\tilde{K}\tilde{\sigma})_{xx} &= 2\sigma_{xx}\cos^2\theta + (\sigma_{xy} + \sigma_{yx})\cos\theta\cot\varphi \\
&\quad - (\sigma_{xz} + \sigma_{zx})\sin\theta\cos\theta, \\
\mathrm{Tr}\,(\tilde{\sigma}\tilde{K}) &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\cot^2\varphi + \sigma_{zz}\sin^2\theta \\
&\quad + (\sigma_{xy} + \sigma_{yx})\cos\theta\cot\varphi - (\sigma_{xz} + \sigma_{zx})\sin\theta\cos\theta \\
&\quad - (\sigma_{yz} + \sigma_{zy})\sin\theta\cot\varphi, \qquad (15) \\
\mathrm{Tr}(\tilde{K}\tilde{\rho}) &= \rho_{xx}\cos^2\theta + \rho_{yy}\cot^2\varphi + \rho_{zz}\sin^2\theta \\
&\quad + (\rho_{xy} + \rho_{yx})\cos\theta\cot\varphi - (\rho_{xz} + \rho_{zx})\sin\theta\cos\theta \\
&\quad - (\rho_{yz} + \rho_{zy})\sin\theta\cot\varphi.
\end{aligned}$$

We see that in all expressions, with the exception of $\det\tilde{\sigma}$, only the symmetrical components of the magnetoconductivity and magnetoresistivity tensors are required. In deriving this result full advantage was taken of the properties of the tensor $\tilde{\epsilon}$ as given in Eq. (5).

The expressions (14) and (15) allow the torque to be calculated for a general magnetoconductivity and for any orientation of the rotational ellipsoid. Unfortunately the result is far from transparent and so we will first consider several specific cases before attempting a more general discussion.

C. Asymptotic behavior

The general result for the torque experienced by an ellipsoidal sample Eqs. (14) and (15), still requires a considerable amount of algebra in order to be applied to a specific case described by a transformation (4), and a conductivity tensor $\tilde{\sigma}$. For the benefit of those who wish to use the induced torque method, it is helpful to point out what can be expected for nonspherical samples.

1. Very low magnetic fields

In low magnetic fields we can consider the magnetoconductivity to be close to that of a nearly-free-electron metal. Then the anisotropy owing to the sample's elliptical shape behaves approximately as

$$1 - \tfrac{1}{2}\eta\cos^2\theta \tag{16}$$

providing $|\eta| \ll 1$, and $\omega_c\tau \ll 1$ (ω_c is the cyclotron frequency, τ is the relaxation time). The torque in very low magnetic fields shows twofold anisotropy as a function of θ, that is, it will have two maxima and two minima per rotation of sample.

2. High magnetic fields

a. Closed orbits only in a noncompensated metal (CONC). We assume that the elements of the conductivity tensor are proportional to powers of the magnetic field

$$\sigma_{xx} \propto \sigma_{yy} \propto B^{-2},\ \sigma_{zz} \propto B^0,\ \sigma_{xy} = -\sigma_{yx} \propto B^{-1}.$$

Then in expression (14) the term $\eta^2 E$ in the numerator contains a term $\eta^2\cos^2\theta\sigma_{zz}$, which at sufficiently high fields will be the leading one. In this case the torque will have a large component quadratic in B, with fourfold variations as a function of θ. This nonsaturating torque will be discussed in greater detail later in Sec. III.

b. Compensated metal. We assume

$$\sigma_{xx} \propto \sigma_{yy} \propto \sigma_{xz} \propto \sigma_{xy} = -\sigma_{yx} \propto B^{-2}, \quad \sigma_{zz} \propto B^0.$$

Then the torque saturates at sufficiently high fields and exhibits an anisotropy with a predominantly twofold symmetry.

c. Open orbits. Let us first consider a set of parallel open orbits in the xy plane such that the average velocity of electrons on these orbits makes an angle α with the y axis. For $\cos\alpha \neq 0$ the torque will increase quadratically in high fields, proportional to $B^2\cos^2\alpha$, and will have a near-twofold θ variation. In general, the torque will no longer be invariant to reversing the magnetic field direction, since the angles $+\alpha$ and $-\alpha$ are not equivalent.

For $\cos\alpha = 0$ the torque will saturate. Saturating terms with twofold and fourfold variations with θ will be present, but their influence will not be as pronounced as in the case of closed orbits only. In this special case ($\cos\alpha = 0$) the electrons in high fields move only in the x direction, and therefore are not excited by the electric fields which are predominantly parallel to the y axis. They can, however, "short circuit" the additional electric fields which would otherwise be present owing to nonsphericity.

A similar special case is that of two open orbits in the xy plane with different directions. In this case the resistivity tensor elements saturate, and so the torque is proportional to B^2. Variations with shape are not dramatic.

We see that the induced torque method can be used for finding regions of open orbits, even with samples that are not perfectly spherical. For small values of $|\eta|$ we do not expect the torque anisotropy owing to shape to swamp the variations due to the changing numbers of open orbits excited in various directions.

III. NONCOMPENSATED METAL WITH CLOSED ORBITS ONLY (CONC)

A. Boundary conditions and torque

A close look at dissipation in magnetic fields soon reveals[4,12] that the boundary conditions play just as instrumental a role as the conduction mechanism itself in determining how large a component of the electric field lies in the direction of the electric current, $\vec{E}\cdot\vec{j}$.

In four-probe experiments this is most markedly demonstrated by the nearly-free-electron metal: In the "matchstick" geometry, the boundary conditions allow dissipation that is independent of magnetic fields, while the "Corbino disk" geometry forces dissipation to rise as B^2. In the probeless induced-torque experiment the sphere is to some extent an equivalent of the "matchstick" in the four-probe measurement. The symmetry of the sphere allows the driving electromotoric force and the currents to be perpendicular in the limit of very high fields. An ellipsoid, on the other hand, cannot accommodate this condition without generating extra fields, together with a space charge. Thus the pattern of currents and fields is changed from that of a sphere, and dissipation rises again as B^2. Let us assume a nearly-free-electron-like magnetoresistivity tensor

$$\rho_{xx}=\rho_{yy}=\rho_{zz}=\rho_0 q, \quad \rho_{xy}=-\rho_{yx}=\rho_0\omega_c\tau. \tag{17}$$

The dimensionless coefficient q can describe an isotropic magnetoresistance effect. The expression (14) for the torque N_y reduces to

$$\frac{8\pi R^5}{15}\,\frac{\Omega}{\rho_0}\,\frac{B^2 q}{(\omega_c\tau)^2}\left(\frac{1}{1+4/x^2}\right) \times\frac{1-\frac{1}{2}\eta\cos^2\theta+f\eta^2x^2\cos^2\theta\sin^2\theta}{1-h\eta\sin^2\theta}, \tag{18}$$

where $x=\omega_c\tau/q$ and

$$f=\tfrac{1}{2}b^2(1+b^2)^{-1}, \quad h=(2+b^2)(1+b^2)^{-1}.$$

The expression in large parentheses is approximate, valid for $|\eta|\ll 1$ and $x>2$, but is not of importance in high fields, where the last term in the numerator will be the leading one.

B. Currents and charges in an ellipsoid

The current pattern consists of ellipses in parallel planes, and is most readily described by its magnetic moment. From Eq. (12) we see that

$$\begin{aligned} M_x &\propto -B\Omega\eta^2\cos^2\theta\sin^2\theta,\\ M_y &\propto -B\Omega\eta^2\cot\varphi\,\cos\theta\sin^2\theta,\\ M_z &\propto -B\Omega(\eta 2\cos\theta\sin\theta-\eta^2\cos\theta\sin^3\theta),\end{aligned} \tag{19}$$

where we have neglected the angular variations of the denominator in Eq. (14). When $\eta=0$ in the high-field limit

$$\begin{aligned} M_x &\propto -B\Omega(\rho_{zz}+\rho_{yy})(\sigma_{xx}+\sigma_{yy})\propto B^{-1},\\ M_y &\propto -B\Omega(\rho_{xx}+\rho_{yy})(-\sigma_{xy})\propto B^0,\\ M_z &=0.\end{aligned} \tag{20}$$

From this it follows that for $|\eta|\ll 1$ and for sufficiently high fields B, the largest component of the moment lies parallel to B and varies approximately as $B\Omega\eta\sin\theta\cos\theta$. It appears therefore that the B^2 increase in dissipation compared with a sphere is associated with additional currents lying mostly in the xy plane. These additional currents disappear in the high-symmetry orientations ($\theta=0°$, $90°$) for which the current lines are parallel to a plane slightly tilted from the xz plane. The angle of tilt differs from that in the case of a sphere by corrections of the order η.

The charge density is simply

$$\rho_{\text{charge}}=\frac{1}{4\pi}\vec{\nabla}\cdot\vec{E}=\frac{1}{4\pi}\nabla\tilde{\rho}\tilde{T}\vec{r}=\frac{1}{4\pi}\operatorname{Tr}(\tilde{\rho}\tilde{T})$$

constant over the sample. Only antisymmetric components of the resistivity tensor will contribute to the trace. For a CONC metal with only ρ_{xy}—antisymmetric components, the expression reduces to

$$\rho_{\text{charge}}=(1/4\pi)(1+b^2)\rho_{xy}t_z\propto\rho_{xy}(\tilde{\epsilon}^{-1}\vec{M})_z\,,$$

where the proportionality in the case of our geometry, for high magnetic fields and lowest term in η, becomes

$$\rho_{\text{charge}}\propto-\rho_{xy}B\Omega 2\eta\cos\theta\sin\theta.$$

We see that the additional divergence free currents, present when the ellipsoid is not in a high-symmetry orientation, are in balance with a space charge (and opposite additional surface charge) which varies as

$$\Omega B^2\eta\cos\theta\sin\theta.$$

The discussion so far has been for an ellipsoid of rotation. It can be shown that the same high-field behavior of torque, currents, and charge occurs in a more general ellipsoid, i.e., in one with three different semiaxes. Again, high-symmetry orientations exist in which these nonsaturating effects disappear. These high-symmetry orientations are those for which the central cross section of the sample with the xz plane is an ellipse with one of its semiaxes parallel to B.

The maximum rotational frequency Ω at which our approximation is valid is generally taken to be given by the condition that the classical skin depth $\delta(\Omega) \gg R$. As Delaney and Pippard[4] have shown for a sphere of a CONC metal in high magnetic fields this condition is relaxed to $\delta(\Omega) \gtrsim R$, since the magnetic field is influenced only very slightly by the saturating magnetic moment parallel to the axis of rotation. This will no longer be true for an ellipsoidal sample in which the induced currents have a magnetic moment $M_z \propto -B\Omega\eta \sin 2\theta$ parallel to the applied magnetic field, and will produce a relative correction to the field at the center of the sample of the order of $\eta[R/\delta(\Omega)]^2$. For rotational frequencies where these effects are not negligible, the frequency response of the magnetic moment is difficult to analyze, since it will vary with twice the rotational frequency and involve large second-order terms owing to $\partial\rho_{\text{charge}}/\partial t$. Our results are therefore limited to low frequencies[19] for which $\delta(\Omega) \gg R\,|\eta|^{1/2}$.

C. Torque anisotropy in potassium

Our discussion of the CONC metal case of induced torque was prompted by the measurements of Scheafer and Marcus[13] (SM) of induced torque on potassium, which showed large anisotropy. It is the unexplained magnetoresistance of potassium that has led to suggestions that potassium, despite many observations to the contrary, is not quite such a simple metal is generally assumed. In particular, the anomalies in the induced torque measurements of SM have enabled Overhauser[20,21] to support his hypothesis that "the electronic ground state of Na and K has an almost static charge-density wave structure.[21]"

The experimental results of SM therefore deserve to be both closely scrutinized, and to be repeated. While no reproduction of their results has been reported so far, a suggestion has been made by the present author[17] that the existing results are an experimental artifact, stemming from nonsphericity of their samples. It was hypothesized[17] that the samples were approximately spherical with substantial dimples on one side. This was subsequently supported by Hunt and Werner,[18] who produced photographs of samples prepared in an identical manner as SM showing such dimples. These dimples result from a large (nearly 3%) contraction of potassium at solidification, during which the final contraction of the warmest material at the center produces a collapse of the sphere at a weak point. To demonstrate the influence of such a dimple on the induced torque a model sample was considered: a prolate rotational ellipsoid with one semiaxis 10% shorter than the other two. The induced torque as function of angle of rotation and magnetic field, calculated with the help of the theory presented in Sec. II, reproduced the experimental curves of SM.[17]

In view of the criticism of this explanation voiced by Overhauser,[21] and the profound importance he attaches to the experimental results themselves it seems worthwhile to return once more to a detailed comparison of theory and experiment. It should be stressed that we will be comparing on the one hand measurements made on samples, each of which, we are convinced, had one fairly deep conical dimple, with a theory for an ellipsoidal sample, i.e., one with a highly symmetrical and relatively mild distortion from a sphere, on the other.

As was pointed out in Secs. II and III A, a fourfold component of the torque (i.e., one with four maxima per rotation which do not saturate) dominates the picture at high fields. In low fields the torque shows twofold anisotropy. The torque does, however, saturate in high fields for the minima orientations ($\theta = 0°$, $90°$).

The observed linear increase of the torque in the minima orientations reflects the linear magnetoresistance of potassium. We therefore assume that the magnetoresistivity tensor of potassium has the form (17) with $q = 1 + S\omega_c\tau$. The coefficient S is referred to as the Kohler slope.

A very satisfactory agreement of the theory given by Eq. (18) and the experimental results can be achieved in the following way: The ratio of the torque at $\theta = 0°$ and $90°$ allows η to be determined. The slopes of the torque in high fields for these minima directions determines S, providing $\omega_c\tau$ is known. Then the ratio of the torques at maximum (θ near 45°) and minimum ($\theta = 0°$ or $90°$) directions allows b and φ to be chosen, consistent with the value of η.

Complete data for one particular sample of SM have been presented in Figs. 1 and 5 of Ref. 13. A fit of the theory given by Eq. (18) is shown in Fig. 1. Here $\omega_c\tau$ had to be estimated and therefore φ was assumed to be $\frac{1}{2}\pi$. The values of $b = 0.87$ and $S = 0.031$ followed from the fit consistent with a resistance ratio of approximately 2000. The poor agreement below $\omega_c\tau = 10$ indicates most probably that the expression $q = 1 + S\omega_c\tau$ does not describe the magnetoresistance accurately at low fields.

A Kohler slope value of 0.031 is large, but by no means unusual for samples prepared under poorly controllable conditions (as in an oil bath) and mounted under stress. Similar values have been observed before in induced torque experiments,[12] and values for S of the same order of magnitude would be required if one were to assume that only the longitudinal magnetoresistance is large.[14]

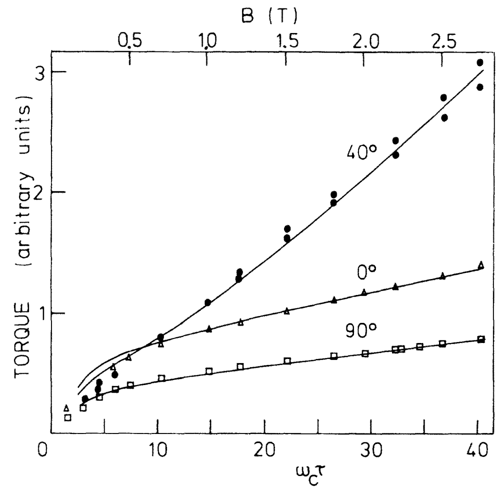

FIG. 1. Fit of the theoretical expression (18) for anisotropic torque (solid line) to the experimental results of Schaefer and Marcus for the high-field minimum directions ($\theta = 0°$ and 90°) and maximum direction ($\theta = 40°$).

The large number of samples and runs used by SM lend weight to their results. Unfortunately they present no statistically averaged values, and therefore we must base our comparison on the good fit of the model predictions to the torque of one sample. We can, however, treat some statistical aspects of the data. Let us fix the adjustable parameter b at an average $b = 0.90$, and assume that φ (the angle of tilt between the shortened semiaxis and the axis of rotation) will follow a random distribution.

The average peak-to-peak anisotropy of the torque at very low fields ($\omega_c\tau \ll 1$) is according to Eq. (16) equal to $0.117 \langle \sin^2\varphi \rangle_{av} = 7.8\%$, considerably more than the observed 2%.[21] However, it is not at all surprising that our model, which was made to fit the perplexing high-field results by choosing $b = 0.90$, fails at very low fields. Obviously the 10% ellipticity required is a measure of the seriousness at high fields of the current distortions caused by the deep dimple.

SM report that of the 200 runs made, only in seven of them was little or no anisotropy observed. Let us take this to mean that less than 5% anisotropy at maximum field was seen in 3.5% of the runs. In our model the anisotropy at high fields is given by Eq. (18). For $S = 0.031$ and $b = 0.90$ only samples for which $\varphi < 19°$ give less than 5% anisotropy at $\omega_c\tau = 40$. The probability of finding $\varphi < 19°$ in a random distribution is $1 - \cos 19° = 5.5\%$ in basic agreement with the observations. The average anisotropy, expressed as the ratio of torques at $\theta = 45°$ and $\theta = 0°$, is about 100%.

A particularly interesting feature of the observed anomalies is their correlation with the crystallographic axis of the sample. SM report that of the 15 x-ray oriented samples, all had a torque minimum at low fields ($\omega_c\tau \approx 1$) when $\vec{\Omega}$, $\vec{B}$, and a $\langle 110 \rangle$ axis were nearly coplanar. In the context of the present discussion, such a $\langle 110 \rangle$-minimum correlation would require the azimuthal angle of an $\langle 110 \rangle$ axis, and the azimuthal angle of the dimple to differ by ϑ, where for all samples ϑ is significantly smaller than the average difference of aximuthal angles to be expected in a random distribution of orientations of the crystallographic lattice relative to our experimental frame of reference. The large number of $\langle 110 \rangle$ axes makes it quite improbable to have $\vartheta > 20°$, and a careful estimate leads to the conclusion that for the result obtained for 15 samples to be significant, ϑ must be smaller than 14° for all of them.

Unfortunately, the difference in azimuthal angles has been given only for three samples, presumably not atypical, for which the $\langle 110 \rangle$-minimum correlations involved differences of 5°, 5°, and 10°. This would suggest that the dimple had a tendency to appear close to a $\langle 110 \rangle$ axis when viewed from the center of the sample, as was hypothesized in Ref. 17. However, no correlation between dimple and any particular crystallographic axis was found by Hunt and Werner[18] in their x-ray orientation checks, although again exact statistics are not available. The information that is available is insufficient to prove that the torque minima are correlated with the orientation of the dimple. However, in reality this correlation need not be as close as the fit of the theory to the torque amplitude would suggest. This is because the position of the minimum is much more strongly shape dependent than the amplitude of the fourfold component.

A study of the case of a general ellipsoid suggests that when it differs from a rotational ellipsoid by as little as a few percent in radius, the minimum position and the azimuthal angle of the dimple can differ on the average by approximately 10°. The data of SM would then be consistent with an assumption of further smaller distortions from spherical symmetry (such as dimple shape) associated with the $\langle 110 \rangle$ direction nearest to the dimple. Such distortions would cause the enhanced $\langle 110 \rangle$-minimum correlation, but remain nevertheless unnoticed during x-ray orientation checks.

IV. CONCLUSION

It appears that nearly all points of interest in the anomalies of induced torque in potassium as observed by Schaefer and Marcus[13] can be explained with the help of a model sample—an ellipsoid. There is considerable independent evidence that the samples were not spheres. Since an ellipsoid is only an approximation to a sphere with a dimple, one must not attach too much significance to the numerical value of b. The crux of the matter lies in the broken symmetry of the boundary conditions.

The magnetoresistance of potassium still requires a satisfactory explanation. When interpreting results of induced torque measurements, first considerations should be given to samples which are known to be spherical.[22] Sample preparation and its state of strain have proven in the past to be very important, and more attention should be paid to these factors.[23]

The conclusion to be made for induced torque experiments in general will depend very much on the nature of the magnetoconductivity involved, see Sec. II C. For instance, observation of open orbits should not be strongly influenced by shape, and in fact, successful experiments have been made on cylindrical samples.[5] Deviations from sphericity are also tolerable in the case of compensated metals, and this property could be used to distinguish such metals from noncompensated metals with open orbits only (CONC), where the shape-inflicted anomalies can be quite dramatic.

An ellipsoidal sample of a CONC metal, unless it possesses rotational symmetry with respect to its experimental axis of rotation, will have a nonsaturating component of torque at sufficiently high fields, with fourfold symmetry. Corresponding effects will also be present in the magnetic moment and space charge. We thereby conclude that the sample symmetry requirements are a very serious matter in the case of induced torque in CONC metals.

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[19]We see that should the magnetic moment M_x be observed instead of the torque in order to extract information about the magnetoresistance, as in the experiments on potassium by Hunt and Werner (Ref. 18), extra precautions must be taken to eliminate the effect of the larger components M_y and M_z, and of higher-order moments. Apart from such possible complications owing to the relatively high frequencies they used, it would appear that any present fourfold anisotropy would have been hidden in the observed large twofold anisotropy, especially since the measurements were limited to relatively low fields.

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[23]After submission of this manuscript, the author received a preprint of work by Holroyd and Datars [Can. J. Phys. (to be published)]. They conclude from recent measurements on spherical samples of potassium that the torque anisotropy depends more strongly on sample preparation and suspension than on sample shape. Their results could possibly be attributed to the presence of oriented extended inhomogeneities resulting from uniaxial asymmetry in sample treatment. Insight gained from the present analysis of an uniaxial distortion of the sample shape leads one to expect that such inhomogeneities could produce the observed anisotropy.