Coexistence-curve singularities in isotropic ferromagnets

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The critical behavior of a classical isotropic ferromagnet ($n \geq 2$) is studied to first order in $\epsilon = 4 - d$. Using renormalization-group recursion relations, we obtain expressions for the equation of state and longitudinal susceptibility which describe singular behavior both at the critical point and on the coexistence curve. It is found that, although the diverging susceptibility on the coexistence curve should not be expected in real magnetic crystals, it can produce a large effect on the initial susceptibility measured below T_c in hexagonal layered crystals such as $CrCl_3$. Specifically, we find that the susceptibility should diverge as $T \to T_c$ from below as $\chi_L \sim |t|^{-\gamma'}$, with $\gamma' \simeq \gamma + 0.79 \simeq 2.12$.

I. INTRODUCTION

Isotropic magnetic systems with continuous rotational symmetry behave rather differently than simple Ising-like systems below the Curie temperature.[1] In particular, it is expected that the initial susceptibility is infinite everywhere on the coexistence curve. As the magnetic field h tends to zero for $T < T_c$, spin-wave calculations,[2,3] for example, indicate that the longitudinal susceptibility χ_L diverges as $h^{-1/2}$ in three dimensions. More generally, it is expected from renormalization-group arguments[4,5] and the spin-wave results[1] that

$$\chi_L \sim h^{-\epsilon/2}, \tag{1.1}$$

as $h \to 0^+$ with $T < T_c$, where $\epsilon = 4 - d$ (d is the dimensionality of space). Of course, *at* T_c the longitudinal susceptibility diverges according to

$$\chi_L \sim h^{-(\delta-1)/\delta} \tag{1.2}$$

as $h \to 0^+$, a result which follows from the usual scaling description of a critical point.[6,7] This state of affairs is summarized in Fig. 1. We present here a calculation of χ_L for an isotropic, continuous spin model of Heisenberg critical behavior. Our calculation gives a complete description of the vicinity of the critical point to $O(\epsilon)$, and exhibits both the divergences (1.1) and (1.2) discussed above. Results are also obtained for the equation of state.

The pioneering calculations of thermodynamic functions by renormalization-group methods were done by Brézin, Wallace, and Wilson.[4,5,8] These authors used a direct Feynman-graph approach[9] to calculate the equation of state of an Ising ($n = 1$) ferromagnet to $O(\epsilon^2)$,[8] and went on to calculate the corresponding quantity for systems with continuous symmetry ($n \geq 2$).[4,5] Work along similar lines has been reported by Avdeeva and Migdal.[10] However, an infrared divergence led to difficulties in calculating the longitudinal susceptibility.[4,5] It was then argued that this anomalous infrared behavior led to the expected $h^{-\epsilon/2}$ divergence on the coexistence curve.[4,5] The diverging susceptibility along the coexistence curve and at the critical point can be thought of as an example of crossover critical behavior, which is usually associated with the competition between two fixed points.[11] Recently, techniques have been developed which allow a systematic treatment of such problems using renormalization-group recursion relations.[12,13] A similar approach will be taken here.

The usual scaling theory of a Heisenberg ferromagnet can be combined with ideas developed in phenomenological theories of crossover scaling.[11,15] Given a ferromagnet with magnetization M in a field h, the usual Griffiths[7,16] form for the equation of state near the critical point is

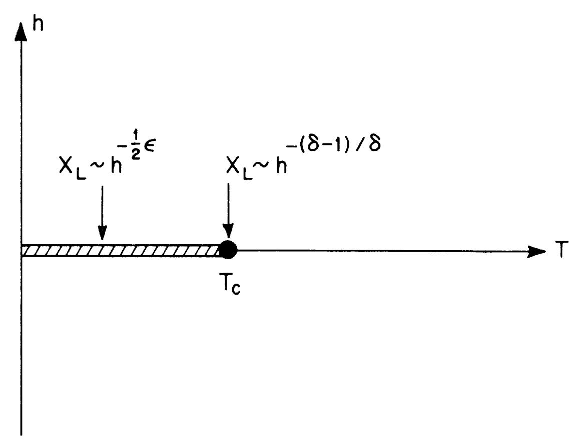

FIG. 1. Different kinds of divergences in a Heisenberg ferromagnet. On the critical isotherm, the longitudinal susceptibility χ_L behaves as $h^{-(\delta-1)/\delta}$, while as $h \to 0$ at fixed $T < T_c$, χ_L goes as $h^{-\epsilon/2}$. The coexistence curve is the shaded line.

$$h/M^{\delta} = f(t/M^{1/\beta}), \tag{1.3}$$

where $t = (T - T_c)/T_c$. The coexistence curve is described by a zero of $f(x)$, at $x = \mathring{x} < 0$. Thus magnetization in zero field is predicted to vary as

$$M \approx \mathring{x}\,|t|^{\beta} \tag{1.4}$$

below T_c. To describe the coexistence-curve behavior of the longitudinal susceptibility for $n \geqslant 2$, we now assume in addition that $f(x)$ is *singular* at $\mathring{x}$, namely,

$$f(x) \sim (x - \mathring{x})^{2-\iota} \quad \text{as } x \to \mathring{x}, \tag{1.5}$$

where the exponent ι is expected to be less than unity.[1] It follows that the susceptibility $\chi_L = (\partial M/\partial H)_t$ can be expressed in scaling form

$$\chi_L/M^{1-\delta} = \Phi(t/M^{1/\beta}). \tag{1.6}$$

where the scaling function $\Phi(x)$ now *diverges* at $\mathring{x}$, i.e.,

$$\Phi(x) \sim (x - \mathring{x})^{\iota - 1}. \tag{1.7}$$

The original Feynman-graph work[4] was unable to produce the exponentiated singular behavior displayed in Eqs. (1.5) and (1.7). Meaningful results were nevertheless obtained for $f(x)$, because $f(x)$ does not actually diverge at $\mathring{x}$. However, the scaling function thus obtained turned out to be negative in a small region of $x > \mathring{x}$,[4] which is physically nonsensical. We shall see that exponentiating the singularity at $\mathring{x}$ properly will ensure a positive $f(x)$, and determine the complete crossover scaling function $\Phi(x)$ for the susceptibility.

In Sec. II we review previous work which has produced recursion relations applicable for $n \geqslant 2$ below T_c.[13] These recursion relations can be used to map a Hamiltonian in small field just beneath the critical point away from T_c. If one tries to calculate quantities with the resulting partially renormalized Hamiltonian, one finds the same infrared difficulties on the coexistence curve that plagued the Feynman-graph work. We summarize the results which can be obtained in this way.

Section III shows how a simple resummation procedure, in the spirit of the parquet-graph approach,[17] removes the infrared difficulties on the coexistence curve and leads to results for the longitudinal susceptibility and equation of state.

In Sec. IV we discuss the applicability of these results to real magnetic crystals. Real crystals, of course, represent some space group and do not have complete rotational invariance. It is pointed out that symmetry-breaking perturbations in real crystals should, in fact, clamp the fluctuations which lead to the diverging susceptibility. However, we discover that, if the symmetry-breaking perturbations are irrelevant variables, they can lead to an enhanced critical exponent γ' describing the divergence of χ_L as T_c is approached from below. For hexagonal layered crystals, such as $CrCl_3$, we find $\gamma' \simeq \gamma + 0.79 \simeq 2.12$.

II. RECURSION RELATIONS FOR $n \geqslant 2$

Consider an isotropic Landau-Ginzburg-Wilson Hamiltonian in a magnetic field $\vec{h}$, namely,

$$\overline{\mathcal{H}} = -\frac{\mathcal{H}}{k_B T} = -\int d\vec{R}\,[\tfrac{1}{2}(\vec{\nabla}\vec{s})^2 + \tfrac{1}{2}r|\vec{s}|^2 + u|\vec{s}|^4 - \vec{h}\cdot\vec{s}], \tag{2.1}$$

where $\vec{s} = \vec{s}(\vec{R})$ is an n-component continuous spin variable,

$$|s|^2 = \sum_{i=1}^{n} s_i^2 \quad \text{and} \quad (\vec{\nabla}\vec{s})^2 = \sum_{i,j=1}^{d,n} (\nabla_i s_j)^2.$$

When Eq. (2.1) is transformed into momentum space, the Fourier integrals are restricted, as usual,[11] to a Brillouin zone of unit radius. The recursion relations appropriate to Eq. (2.1) for $\vec{h} \neq 0$ and $T < T_c$ were discussed in an Appendix to Ref. 13 (henceforth referred to as I). It is convenient to take $\vec{h}$ along the nth component of the spin field, and to shift s_n by the exact magnetization.[4]

$$s_n = M + \sigma, \tag{2.2}$$

whereupon Eq. (2.1) becomes

$$\overline{\mathcal{H}} = -\int d\vec{R}[\tfrac{1}{2}r_L\sigma^2 + \tfrac{1}{2}r_T|\vec{s}_\perp|^2 + w_1\sigma|\vec{s}_\perp|^2 + w_2\sigma^3 + u(\sigma^2 + |\vec{s}_\perp|^2)^2 - \tilde{h}\sigma]. \tag{2.3}$$

In Eq. (2.3), we have deleted a spin-independent term, set $\vec{s}_\perp \equiv \{s_i,\ i = 1, \ldots, n-1\}$, and defined

$$r_L = r + 12uM^2, \quad r_T = r + 4uM^2, \qquad w_1 = w_2 = 4uM^2, \quad \tilde{h} = h - rM - 4uM^3. \tag{2.4}$$

The magnetization M is determined by the condition

$$\langle\sigma\rangle = 0. \tag{2.5}$$

Recursion relations for $r_L, r_T, w_1, w_2, u,$ and $\tilde{h}$ can readily be derived by integrating out Fourier components of $\overline{\mathcal{H}}$ in a small shell of momentum space, as described by Wilson and Kogut.[11] To $O(\epsilon)$, these are[13]

$$\frac{dr_L}{dl} = 2r_L + 12K_4ug_L + 4(n-1)K_4ug_T - 18K_4w_2^2g_T^2 - 2(n-1)K_4w_1^2g_2^2, \tag{2.6}$$

$$\frac{dr_T}{dl} = 2r_T + 4(n+1)K_4 u g_T + 4K_4 u g_L - 4K_4 w_1^2 g_L g_T, \tag{2.7}$$

$$\frac{dw_1}{dl} = (1+\tfrac{1}{2}\epsilon)w_1 - 12K_4 u w_2 g_T^2 - 4(n+1)K_4 u w_1 g_L^2, \tag{2.8}$$

$$\frac{dw_2}{dl} = (1+\tfrac{1}{2}\epsilon)w_2 - 36K_4 u w_2 g_T^2 - 4(n-1)K_4 u w_1 g_L^2, \tag{2.9}$$

$$\frac{du}{dl} = \epsilon u - 4(n+7)K_4 u^2 g_L^2 - 4K_4 u^2 g_T^2 + O(uw_1^2, uw_2^2, w_1^4, w_2^4), \tag{2.10}$$

$$\frac{d\tilde{h}}{dl} = (3-\tfrac{1}{2}\epsilon)\tilde{h} + 3K_4 w_2 g_L + (n-1)K_4 w_1 g_T, \tag{2.11}$$

where

$$g_L = (1+r_L)^{-1}, \quad g_T = (1+r_T)^{-1}. \tag{2.12}$$

The initial values (2.4) entering these recursion relations suggest we look for solutions of the form $r_L = r(l) + 12u(l)M^2(l)$, $r_T = r(l) + 4u(l)M^2(l)$, $w_1(l) = 4u(l)M(l)$, etc. This parametrization of the solutions works, provided we take

$$M(l) = e^{(1-\epsilon/2)l}M. \tag{2.13}$$

As in I, the recursion relations will be integrated from $l=0$ to $l=l^*$ such that $r_L(l^*) = O(1)$. If the equations are integrated further, the neglected terms in Eq. (2.10) break the isotropy at quartic order. Over the restricted range indicated above, however, $u(l)$ remains constant (and the quartic coupling remain isotropic) up to corrections of $O(\epsilon^2)$,[13] provided we set $u = u_c = 2\pi^2\epsilon/(n+8)$, the critical value of the four-spin coupling constant.[11] With this simplifying assumption, the order ϵ solutions given in I reduce to

$$\begin{aligned} r_L(l) = {} & T_L(l)2(n+2)K_4 u_c + 6K_4 u_c T_L(l)\ln[1+T_L(l)] \\ & + 2(n-1)K_4 u_c T_T(l)\ln[1+T_T(l)] \\ & + 144K_4 u_c^2 M^2(l)\{\ln[1+T_L(l)] + T_L(l)/[1+T_L(l)]\} \\ & + 16(n-1)K_4 u_c^2 M^2(l)\{\ln[1+T_T(l)] \\ & + T_T(l)/[1+T_T(l)]\}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} r_T(l) = {} & T_T(l) - 2(n+2)K_4 u_c + 2(n-1)K_4 u_c T_T(l) \\ & \times \ln[1+T_T(l)] + 6K_4 u_c T_L(l)\ln[1+T_L(l)], \end{aligned} \tag{2.15}$$

$$w_1(l) = 4u_c M(l) + O(u_c^2 M(l)), \tag{2.16}$$

$$w_2(l) = 4u_c M(l) + O(u_c^2 M(l)), \tag{2.17}$$

where

$$\begin{aligned} T_L(l) &= te^{\lambda_t l} + 12u_c M^2(l), \\ T_T(l) &= te^{\lambda_t l} + 4u_c M^2(l), \end{aligned} \tag{2.18}$$

$$\lambda_t = 2 - [(n+2)/(n+8)]\epsilon, \tag{2.19}$$

and $t = r + 2(n+2)K_4 u_c$ is a temperaturelike parameter. Results for the evolution of $\tilde{h}(l)$, and the correction terms to Eqs. (2.16) and (2.17) will not be needed here. Although M may be taken arbitrarily small by taking the initial Hamiltonian parameters sufficiently close to the critical point, it turns out that $M(l^*)$ is $O(\epsilon^{-1/2})$,[13] so the corrections to Eqs. (2.16) and (2.17) are $O(\epsilon^{3/2})$.

The idea behind calculating thermodynamic functions with recursion relations is that quantities of interest in the critical region can be related to quantities calculated with renormalized parameters such as $r_L(l^*)$ and $r_T(l^*)$.[12,13] Chosing l^* such that $r_L(l^*) = O(1)$ insures that integrations over the longitudinal propagator can be safely carried out in a graphical expansion of quantities calculated with the renormalized Hamiltonian $\overline{\mathcal{H}}(l^*)$. This choice leads to difficulties near the coexistence curve, however, because $r_T(l^*)$ tends to zero in this region.[13] In particular, there is an infrared divergent contribution to χ_L from the graph shown in Fig. 2.

In spite of these difficulties, it is nevertheless instructive to ignore the infrared problems and calculate the equation of state and susceptibility to $O(\epsilon)$. This was done in I with the results

$$\frac{h}{M^\delta} = f(x) = x + 4u_c + \frac{3\epsilon}{2(n+8)}(x+12u_c)\ln(x+12u_c) + \frac{1}{2}\epsilon\frac{n-1}{n+8}(x+4u_c)\ln(x+4u_c), \tag{2.20}$$

$$\frac{\chi_L}{M^{1-\delta}} = \Phi(x) = (x+12u_c)^{-1} \times \left(1 - \frac{3\epsilon}{2(n+8)}\frac{x+36u_c}{x+12u_c}\ln(x+12u_c) - \frac{\epsilon}{2}\frac{n-1}{n+8}\ln(x+4u_c) - \frac{4\epsilon u_c}{x+12u_c}\right), \tag{2.21}$$

where $x = t/M^{1/\beta}$. The coexistence curve is given by the zero of Eq. (2.20), which, to leading order is

$$x = \mathring{x} = -4u_c. \tag{2.22}$$

Evidently, the infrared problems near the coexistence curve show up in Eqs. (2.20) and (2.21) as logarithms in $x + 4u_c$. The term logarithmic in $x + 4u_c$ in Eq. (2.20) tends to zero as $x \to -4u_c$, a feature which allowed the original construction of

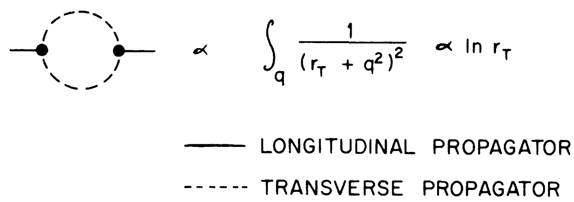

FIG. 2. Feynman graph which leads to the divergence in the longitudinal susceptibility. In four dimensions, this graph goes as $\ln r_T$, where r_T is the transverse mass.

meaningful equations of state for $n \geqslant 2$.[4] However, Eq. (2.21) is unacceptable as it stands—some procedure must be found to exponentiate the term proportional $\ln(x+4u_c)$. A method for calculating the equation of state and longitudinal susceptibility which resolves this problem is described in Sec. III.

III. LONGITUDINAL SUSCEPTIBILITY AND EQUATION OF STATE FOR $n \geqslant 2$

As mentioned in Sec. II, renormalization-group theory can be used to relate quantities in the critical region to quantities far from T_c. In particular, we will be interested in the longitudinal and transverse susceptibilities,

$$\chi_L \equiv r_L^{-1} = \int d\vec{R}\ \langle \sigma(\vec{0})\sigma(\vec{R})\rangle_{\mathcal{K}} \tag{3.1}$$

and

$$\chi_T \equiv r_T^{-1} = \int d\vec{R}\langle \vec{s}_\perp(\vec{0})\cdot\vec{s}_\perp(\vec{R})\rangle_{\mathcal{K}}\ , \tag{3.2}$$

where the brackets denote a thermodynamic average defined by a functional integral,[4]

$$\langle A\rangle_{\mathcal{K}} = \frac{\int\int \mathfrak{D}\sigma(\vec{R})\mathfrak{D}\vec{s}_\perp(\vec{R})e^{\overline{\mathcal{K}}}A(\sigma,\vec{s}_\perp)}{\int\int \mathfrak{D}\sigma(\vec{R})\mathfrak{D}\vec{s}_\perp(\vec{R})e^{\overline{\mathcal{K}}}}\ . \tag{3.3}$$

We have introduced the transverse susceptibility because it can be used to obtain the equation of state through the exact relation[4]

$$\chi_T^{-1} = h/M\ . \tag{3.4}$$

To order ϵ, the susceptibilities χ_L and χ_T are related to quantities far from T_c by[11]

$$\chi_L = e^{2l^*}\chi_L(l^*),\quad \chi_T = e^{2l^*}\chi_T(l^*)\ , \tag{3.5}$$

where $\chi_L(l^*)$ and $\chi_T(l^*)$ are susceptibilities calculated via Eqs. (3.1) and (3.2) but using rescaled spins and the renormalized Hamiltonian $\mathcal{K}(l^*)$.[12,13] Henceforth, we will denote the fully renormalized masses of the Hamiltonian $\mathcal{K}(l^*)$ by r_L and r_T. The

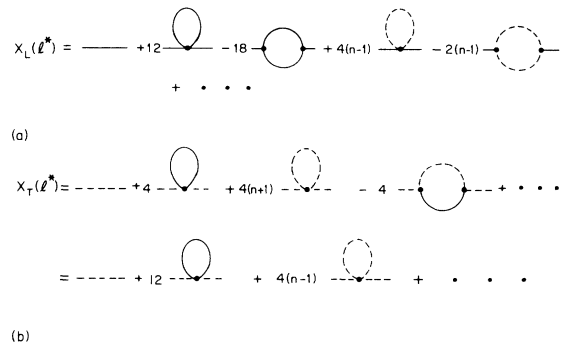

FIG. 3. (a) Leading contributions in the graphical expansion of the longitudinal susceptibility. (b) Leading contributions in the graphical expansion of the transverse susceptibility.

bare masses for this problem are $r_L(l^*)$ and $r_T(l^*)$.

We begin by making a graphical expansion to $O(\epsilon)$ of $\chi_L(l^*)$ [see Fig. 3(a)]. The solid lines in Fig. 3 carry *an unrenormalized* mass $r_L(l^*) = O(1)$, while the dotted lines carry *a fully renormalized* mass r_T which tends to zero on the coexistence curve. Recall that the cubic vertices w_1 and w_2 carry weight $4u_c M(l^*)$, and are thus of $O(\epsilon^{1/2})$. Although the first few graphs displayed in Fig. 3(a) are cut off at low momenta by $r_L(l^*)$, graphs with integrals over the transverse propagators lead to logarithms in r_T (to leading order in ϵ, the integrals in Fig. 3 can be evaluated in four dimensions). Note that the condition (2.5) implies that no "tadpole" insertions of the field σ are required in Fig. 3, since these sum to zero.[4]

We now show that the logarithms appearing in the expansion of $\chi_L(l^*)$ can be exponentiated in a natural way using a parquet-graph[17] summation

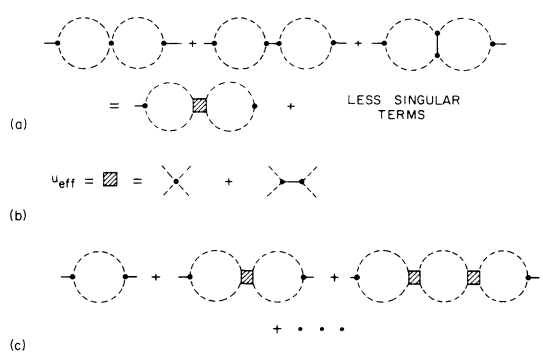

FIG. 4. (a) Dominant contributions to χ_L at $O(\epsilon^2)$. These can be rewritten in terms of a "partially dressed" coupling U_{eff}. (b) Graphical expression for the coupling U_{eff}. (c) Contribution to the longitudinal susceptibility taking into account the dominant terms at each order in ϵ and expressed in terms of U_{eff}.

procedure. Consider first the infrared divergent bubble graph shown in Fig. 3. At higher orders in ϵ, divergent graphs will again appear in the form of repeated bubbles. (Note that no Dyson summation has been made to eliminate these graphs.) The most divergent contributions to the series at $O(\epsilon^2)$ are shown in Fig. 4(a). As suggested by Fig. 4(a) these graphs can be combined into a single expression with two bubbles connected by a coupling constant u_{eff}, where to order ϵ,

$$u_{\text{eff}} = u_c - 8u_c^2 M^2(l^*)/r_L(l^*) = u_c\, r_T(l^*)/r_L(l^*). \tag{3.6}$$

When the second contribution to the renormalized vertex occurs in an orientation which involves an integration over the longitudinal propagator, we rewrite $[r_L(l^*)+q^2]^{-1}$ as

$$1/[r_L(l^*)+q^2] = 1/r_L(l^*) - q^2/[r_L(l^*)+q^2],$$

and observe that the q-dependent terms give rise to a less singular contribution than the leading term. Thus, to a leading approximation, we can take the coupling (3.6) to be q independent. Extracting the most singular contribution at each order in perturbation theory gives rise to the series shown in Fig. 4(b). This series can be summed immediately by simply noting that, to $O(\epsilon)$, it is proportional to the perturbation series for the specific heat $C_{n-1}(r_T(l^*), u_{\text{eff}})$ of an $(n-1)$-component isotropic system with unrenormalized mass $r_T(l^*)$ and bare-four-spin coupling constant u_{eff}. Hence to leading order the series shown in Fig. 4(b) is given by

$$C_{n-1}(r_T(l^*),\ U_{\text{eff}})/r_L^2(l^*). \tag{3.7}$$

The singularity in the one loop graph over the transverse propagator [see Fig. 3(a)] can also be summed in this way. Defining

$$S = \frac{n-1}{r_L^2(l^*)} \int \frac{1}{q^{r_T} + q^2}, \tag{3.8}$$

we consider the perturbation series for[17]

$$S' = \frac{dS}{dr_T(l^*)}, \tag{3.9}$$

where we have taken the derivative with respect to the unrenormalized transverse mass. The most singular contributions to the perturbation series for S' are shown in Fig. 5, where we have been careful to differentiate the tadpole insertions in the series for S. The dominant part of this series consists of streams of bubbles connected by a "partially dressed" coupling u_{eff}, which again can be identified with the specific heat appearing in Eq. (3.7). Integrating this result for S' with respect to $r_L(l^*)$, we discover that S can be rewritten, to leading order as

$$S = E_{n-1}[(r_L(l^*),\ U_{\text{eff}})]\,/r_L^2(l^*), \tag{3.10}$$

where $E_{n-1}(r_L(l^*), U_{\text{eff}})$ is just the *energy* of an isotropic system with $n-1$ components.

Putting together the results (3.7) and (3.10) for the singular contributions of $O(\epsilon)$ to the susceptibility series in Fig. 3(a), we find that

$$\begin{aligned}\chi_L(l^*) = {} & r_L^{-1}(l^*) + \frac{12u_c}{r_L^2(l^*)} K_4 \int_0^1 \frac{q^3 dq}{r_L(l^*)+q^2} + \frac{16(n+8)K_4 u_c^2 M^2(l^*)}{r_L^2(l^*)} \\ & - \frac{288 u_c^2 M^2(l^*)}{r_L^2(l^*)} K_4 \int_0^1 \frac{q^3 dq}{[r_L(l^*)+q^2]^2} + \frac{4u_c}{r_L^2(l^*)} E_{n-1}(r_T(l^*), u_{\text{eff}}) - \frac{32u_c^2 M^2(l^*)}{r_L^2(l^*)} C_{n-1}(r_T(l^*), u_{\text{eff}}) \\ & + O(\epsilon^2),\end{aligned} \tag{3.11}$$

where $E_{n-1}(r_T(l^*), u_{\text{eff}})$ and $C_{n-1}(r_T(l^*), u_{\text{eff}})$ are the energy and specific heat of an isotropic $(n-1)$-component system in zero field with a bare mass $r_T(l^*)$ and four-spin coupling $u_{\text{eff}} = u_c r_T(l^*)/r_L(l^*)$. Note that u_{eff} goes to zero on the coexistence curve. We emphasize that the terms which have been left out of Eq. (3.7) are either higher order in ϵ or display lower powers of $\ln r_T$ than those retained.

Similar arguments can be used to sum the logarithmic singularities in the series for the *transverse* susceptibility shown in Fig. 3(b), where it has been convenient to use the identity, correct to $O(\epsilon)$,

$$8u_c^2 M^2(l^*) K_4 \int_0^1 \frac{q^3\, dq}{[r_L(l^*)+q^2](r_T+q^2)} = u_c K \int_0^1 \frac{q^3\, dq}{r_L(l^*)+q^2} - u_c K_4 \int_0^1 \frac{q^3 dq}{r_T+q^2}. \tag{3.12}$$

Observing that the dominant contribution of the integral over the transverse propagator is just an $(n-1)$-component energy, we find, in analogy with Eq. (3.11),

$$\chi_T(l^*) = r_T^{-1}(l^*) + \frac{12u_c}{r_T^2(l^*)} K \int_0^1 \frac{q^3\,dq}{[r_L(l^*) + q^2]} + \frac{4u_c}{r_T^2(l^*)} E_{n-1}(r_T(l^*), u_{\text{eff}}) + O(\epsilon^2) . \tag{3.13}$$

The energy and specific heat of an n-component isotropic system (which, for the case we are considering, is effectively *above* T_c) are[18]

$$E_n(r, u) = \frac{n}{8(4-n)u}\, t\left[\left(1 + \frac{(n+8)u(t^{-\epsilon/2} - 1)}{2\pi^2\epsilon}\right)^{(4-n)/(n+8)} - 1\right] + \tfrac{1}{2} n K_4 - \tfrac{1}{2} n K_4 r \ln(1+r) , \tag{3.14}$$

and

$$C_n(r, u) = \frac{n}{8(4-n)u}\left[\left(1 + \frac{(n+8)u(t^{-\epsilon/2} - 1)}{2\pi^2\epsilon}\right)^{(4-n)/(n+8)} - 1\right] - \frac{nK_4}{1+r} + nK_4 \ln(1+r) , \tag{3.15}$$

where $t = r + 2(n+2)K_4 u$, and we have explicitly included regular contributions in Eqs. (3.14) and (3.15). It remains only to determine l^*, which was done in I: The condition determining l^* is taken for convenience to be

$$T_L(l^*) = t e^{\lambda_1 l^*} + 12u_c M^2 e^{(2-\epsilon)l^*} = 1 . \tag{3.16}$$

The solution e^{l^*} can be expressed in terms of a homogeneous function as[13]

$$e^{l^*} = M^{1/2-\delta/2}\, \phi(t/M^{1/\beta}) , \tag{3.17}$$

where δ and β are critical exponents needed here only to $O(\epsilon)$. It follows that $\phi(x)$ solves the equation

$$x\,\phi^{\lambda_1}(x) + 12u_c\,\phi^{2-\epsilon}(x) = 1 , \tag{3.18}$$

which, for small x, yields[13]

$$\phi(x) = (x + 12u_c)^{-1}\left(1 + \frac{\epsilon}{12}\,\frac{x + 36u_c}{x + 12u_c} \ln(x + 12u_c) + O(\epsilon^2)\right). \tag{3.19}$$

Using the solutions of the recursion relations (2.14)–(2.17), and the results (3.14) and (3.15), we evaluate Eqs. (3.11) and (3.13) to find

$$\chi_L(l^*) = T_L^{-1}(l^*) + \frac{n-1}{5-n} T_L^{-2}(l^*)\, T_T^{-1}(l^*)\left[\left(1 + \frac{n+7}{n+8} T_T(l^*)[T_T^{-\epsilon/2}(l^*) - 1]\right)^{(5-n)/(n+7)} - 1\right] + 16(n+8)K_4 u_c^2 M^2(l^*) , \tag{3.20}$$

$$\chi_T(l^*) = \frac{M(l^*)}{h(l^*)} = T_T^{-1}(l^*) + \frac{n-1}{5-n} T_T^{-2}(l^*)\left[\left(1 + \frac{n+7}{n+8} T_T(l^*)[T_T^{-\epsilon/2}(l^*) - 1]\right)^{(5-n)/(n+7)} - 1\right] , \tag{3.21}$$

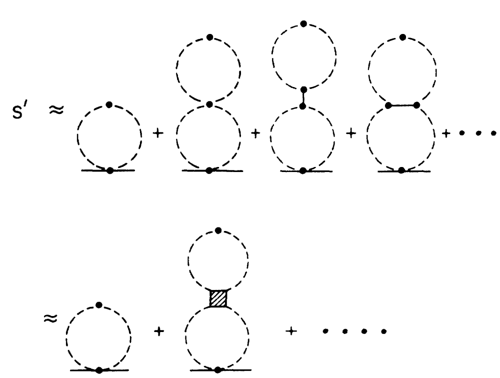

FIG. 5. Dominant contributions for the quantity S' defined in the text. Again, the series is reexpressed, to leading order, in terms of the q-independent coupling U_{eff}.

where various regular contributions have cancelled exactly. The "matching relations" (3.5) together with the parameterization (3.15) for e^{l^*} may now be used to determine the longitudinal susceptibility and equation of state as

$$\chi_L/M^{1-\delta} = \phi^2(x)\chi_L(l^*) , \tag{3.22}$$

$$h/M^{\delta} = 1/\phi^2(x)\chi_T(l^*) , \tag{3.23}$$

where according to Eq. (3.16) we must take

$$T_L(l^*) = 1, \quad T_T(l^*) = x\,\phi^{\lambda_t}(x) + 4u_c\,\phi^{2-\epsilon}(x) . \tag{3.24}$$

The results (3.22) and (3.23), are guaranteed to satisfy the Griffiths' analyticity requirement at large $x = t/M^{1/\beta}$.[4,7] It is customary,[4] however, to present results in terms of an ϵ expansion for $\phi(x)$ which is good for small x, but violates the Griffiths' conditions[7] at large x. Using Eq. (3.19), we find that

$$\frac{\chi_L}{M^{1-\delta}} = \Phi(x) = \frac{1}{x+12u_c} - \frac{3\epsilon}{2(n+8)} \frac{x+36u_c}{(x+12u_c)^2} \ln(x+12u_c) - \frac{4\epsilon u_c}{(x+12u_c)^2} + \frac{n-1}{5-n} \frac{1}{x+4u_c}\left[\left(1+\frac{n+7}{n+8}\frac{x+4u_c}{x+12u_c}\left[(x+4u_c)^{-\epsilon/2}-1\right]\right)^{(5-n)/(n+7)} - 1\right], \tag{3.25}$$

$$\frac{h}{M^\delta} = f(x) = \frac{x+4u_c + \frac{3\epsilon}{2(n+8)}(x+12u_c)\ln(x+12u_c)}{1+\frac{n-1}{5-n}\frac{x+12u_c}{x+4u_c}\left[\left(1+\frac{n+7}{n+8}\frac{x+4u_c}{x+12u_c}\left[(x+4u_c)^{-\epsilon/2}-1\right]\right)^{(5-n)/(n+7)} - 1\right]} \tag{3.26}$$

Expanding the terms $(x+4u_c)^{-\epsilon/2}$ in ϵ, we get agreement with the "naive" graphical results (2.20) and (2.21) quoted in Sec. II. As $x \to \mathring{x} = -4u_c$, $f(x)$ and $\Phi(x)$ are indeed singular in the way prescribed by the crossover scaling in Sec. I.

The equation of state (3.26) (with $\epsilon = 1$) is compared with the result (2.20) originally obtained by Brézin, Wallace, and Wilson[4] in Fig. 6. We have normalized $f(x)$ by requiring that the coexistence curve be given by $x = \mathring{x} = -1$ and that $f(0) = 1$. As noted by the above authors,[4] their result for $f(x)$ is negative over an extremely small range of $x \approx \mathring{x}$. Our result (3.26) repairs this difficulty, and comes into $x = -1$ with zero slope as required by (1.5). The inset to Fig. 6 shows these functions on a scale set by the normalizations—a scale over which they are indistinguishable.

As a final point, we observe that it is straightforward to repeat these calculations for $u \neq u_c$, and, in particular, for $\epsilon = 0$ with u finite. For the behavior of the longitudinal susceptibility as $h \to 0$ for fixed $T < T_c$ we find,

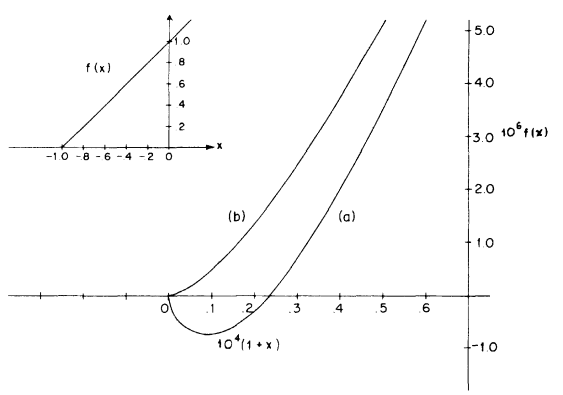

FIG. 6. Comparison of previous Feynman-graph calculations (Ref. 4) [curve (a)] with the current work [curve (b)]. The previous calculations included unexponentiated logarithms which led to spurious behavior near $x = -1$. Note the multiplication factors on the horizontal and vertical scales necessary to actually see the discrepancy. The inset shows to two functions on a scale over which they are indistinguishable.

$$\chi_L \sim \ln h, \tag{3.27}$$

i.e., a pure logarithmic divergence rather than a power of a logarithm.

IV. EXPERIMENTAL CONSEQUENCES

Although the behavior of isotropic ferromagnets on the coexistence curve is certainly of theoretical interest, it seems reasonable to inquire if there are any *experimental* consequences. Real crystals, of course, represent some space group and do not display complete rotational symmetry. There are always easy axes along which the spins prefer to align in the ordered state, even if the number of components of the order parameter exceeds unity.[19] The existence of easy axes is due to symmetry breaking perturbations in the basic isotropic Hamiltonian (2.1).

Consider two specific perturbations to (2.1) which break the $O(n)$ symmetry, namely,

$$\mathcal{H}_v = -v\int d\vec{R}\sum_{i=1}^{n} s_i^4(\vec{R}), \qquad \mathcal{H}_w = -w\int d\vec{R}\sum_{i=1}^{n} s_i^6(\vec{R}). \tag{4.1}$$

Both these perturbations should decay asymptotically to zero (for $n \leq 3$) with repeated iterations of the renormalization-group transformation,[11] and hence are technically irrelevant variables. Nevertheless, it is clear that these perturbations play a vital role in breaking the $O(n)$ symmetry. For example, Bruce and Aharony[20] have shown that nonzero cubic perturbations can significantly alter a phase diagram which has a bicritical point[21] when $v = 0$. We shall see below that the perturbations (4.1) are actually examples of what Fisher[22] has called "dangerous irrelevant variables."

Wallace[23] has considered the effect of the cubic perturbation $\mathcal{H}_v$ in the large n limit. He finds that the isotropic result (3.4) for the transverse susceptibility now becomes[23]

$$\chi_T^{-1} = h/M + v\,g(M), \tag{4.2}$$

where $g(M)$ tends to a constant on the coexistence curve. He further argues that the longitudinal and transverse susceptibilities are related by

$$\chi_L \sim \text{const} + \chi_T^{\epsilon/2}, \qquad (4.3)$$

a result which also holds when $v=0$.[4,5] Using Eqs. (4.2) and (4.3) (with $v>0$) shows that

$$\chi_L \sim v^{-\epsilon/2} \qquad (4.4)$$

on the coexistence curve as v tends to zero. The result (4.4) is easily verified using the recursion relation approach described here. Similar arguments lead to an analogous conclusion when $\mathcal{H}_w$ is the leading symmetry-breaking perturbation, namely,

$$\chi_L \sim w^{-\epsilon/2} \qquad (4.5)$$

as $w \to 0^+$ on the coexistence curve. Brézin, Le-Guillou, and Zinn-Justin[24] have also discussed this effect of symmetry-breaking perturbations to the Hamiltonian (2.1).

Wallace[23] has also shown that, although cubic perturbations clamp the divergence of the transverse susceptibility, there are enhanced transverse fluctuations as $T \to T_c$ from below. We now demonstrate that similar effect occurs in the longitudinal susceptibility leading to a huge asymmetry in the behavior above and below T_c, provided *hexagonal* perturbations are the dominant symmetry-breaking terms in the Hamiltonian. Consider hexagonal layered, metamagnetic crystals[25] such as $CrCl_3$,[26,27] $CoBr_2$,[28] and $CoCl_2$.[28] Although these substances have an antiferromagnetic interaction between planes (which display *hexagonal* crystal structure), the spins tend to align ferromagnetically within each plane. Such systems should be describable by an order parameter with $n=2$ components. Furthermore, because of the hexagonal symmetry within each plane, the dominant symmetry-breaking perturbation should be given by $\mathcal{H}_w$; cubic perturbations are not allowed. For these systems then, it is appropriate to treat the longitudinal susceptibility (with $h=0$) as a function of both the reduced temperature $t=(T-T_c)/T_c$ and the strength w of the hexagonal perturbation.

The usual[11] renormalization-group homogeneity arguments applied to $\chi_L(t,w)$ give

$$\chi_L(t,w) = e^{(2-\eta)l}\,\chi(te^{\lambda_t l},\, we^{-|\lambda_w|l}), \qquad (4.6)$$

where e^l is the spatial rescaling factor, $\lambda_t = 1/\nu$, and λ_w, the eigenvalue describing the decay of the hexagonal perturbation, was calculated to $O(\epsilon^2)$ by Wegner and Houghton[29]:

$$\lambda_w = -2 + \frac{2\epsilon(n-7)}{n+8} + \frac{3\epsilon^2}{2(n+8)}(9n^2+110n^2+904). \qquad (4.7)$$

Choosing a particular $l=l^*$ such that the susceptibility on the right-hand side of Eq. (4.6) is evaluated far from T_c (but still on the coexistence curve), we find that

$$\chi_L(t,w) = t^{-\gamma}\chi_L(-1, wt^{|\phi_w|}), \qquad (4.8)$$

where

$$\phi_w = \nu\lambda_w, \qquad (4.9)$$

and

$$\gamma = (2-\eta)\nu \qquad (4.10)$$

is the usual susceptibility exponent which would be measured *above* T_c. Recalling the coexistence curve divergence (4.5), we conclude from Eq. (4.8) that

$$\chi(t,w) \sim w^{-\epsilon/2}t^{-\gamma'} \qquad (4.11)$$

as $t\to 0^+$, with

$$\gamma' = \gamma + \tfrac{1}{2}\epsilon|\phi_w|. \qquad (4.12)$$

Fisher[22] has used similar reasoning to explain the breakdown of hyperscaling above four dimensions. He calls singular irrelevant perturbations such as w "dangerous irrelevant variables."

Similar arguments, of course, apply when cubic interactions dominate, but the effect turns out to be quite small. To estimate γ' for *hexagonal* perturbations, we consider the ϵ expansion of λ_w with $n=2$,[29]

$$\lambda_w = -2 - \epsilon + \tfrac{87}{50}\epsilon^2 - \cdots. \qquad (4.13)$$

Although this series oscillates, it is clear that λ_w is rather large and negative. A concrete estimate may be obtained by forming a Padé approximant

$$\lambda_w \approx (-2 - 4.48\,\epsilon)/(1+1.74\,\epsilon), \qquad (4.14)$$

which gives $\lambda_w = -2.36$ in three dimensions. This result, together with the estimate $\nu \approx 0.67$ for $d=3$ and $n=2$ gives $\phi_w = -1.58$. Taking $\gamma \approx 1.33$ for $d=3$ and $n=2$, we find finally[30]

$$\gamma' \simeq \gamma + 0.79 \approx 2.12. \qquad (4.15)$$

This is a strikingly large susceptibility exponent. In view of the absence of experiments which give a conclusive demonstration of coexistence curve singularities in magnetic systems, an experimental test of the prediction (4.15) for layered hexagonal crystals such as $CrCl_3$ (by neutron scattering or other means) would be invaluable.

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