PHYSICAL REVIEW B VOLUME 13, NUMBER 5 1 MARCH 1976

Some cluster-size and percolation problems for interacting spins

Antonio Coniglio

Istituto di Fisica Teorica dell'Università, Unità Gruppo Nazionale di Struttura della Materia del Consiglio Nazionale delle Ricerche, Mostra d'Oltremare padiglione 19, Napoli, Italy

(Received 25 August 1975)

The problem of cluster-size distribution and percolation for interacting spins on a regular lattice is briefly discussed. Exact solutions are given for Bethe lattices and other more complex branching media. It is found that the critical behavior is not changed with respect to the noninteracting case. For a ferromagnetic interaction the critical density p_c has been found to be always less than the corresponding critical density in the random distribution. Moreover, at zero external magnetic field p_c has been always $\leq 1/2$, which means that an infinite cluster of overturned spins appears before the Curie temperature is reached. The pair connectedness is also calculated for the simple Bethe lattices and it is found to satisfy homogeneity conditions.

I. INTRODUCTION

The percolation problem (for review articles see Refs. 1–3) has been studied mostly for noninteracting systems. It plays an important role in the theory of dilute ferromagnets[3–12] and inhomogeneous conductors.[13] Connection of the bond percolation problem to the Ising model has been done by Kastleyin and Fortuin.[14] A suggestion has been advanced by Bishop[15] to relate the Curie temperature T_C to the critical probability of the bond and the site percolation problem. The knowledge of the cluster distribution in the Ising model can be used to shed light on the investigation of metastable states.[16] Kikuchi[17] has developed a method of approximations which enable one to study the site percolation problem for noninteracting and interacting systems. The relation between the Askin-Teller-Potts model and the percolation has been used by Harris, Lubensky, Holcomb, and Dasgupta[18] in order to apply to the percolation problem the usual techniques valid for a Hamiltonian formulation, including the renormalization-group approach.

More recently Müller-Krumbhaar[19] has calculated, by means of the Monte Carlo method, the percolation probability for the cubic lattice with ferromagnetic interaction, showing that for zero external field an infinite cluster of overturned spins appears at a temperature T_p below the Curie temperature T_C. The author[20] has given some arguments which suggest that this should be the case for every three-dimensional system, while for two-dimensional systems T_p and T_C should coincide.

The interest in studying the site percolation problem with interaction is due to the fact that such a problem is equivalent to studying the cluster distribution of overturned spins in an Ising model. It is interesting to investigate whether there is or is not a connection between the percolation problem and phase transitions.

The percolation problem has not been solved in closed form for the lattices of main interest: the three-dimensional and the two-dimensional lattices. One can obtain appreciable insight by studying this problem for a class of models such as Bethe lattices (examples of such lattices are given in Fig. 1).

The percolation problem in the random case has already been solved by Fisher and Essam[21] and in a different way by Essam.[3] In this paper we want to solve the percolation problem with interaction for the same class of models. In Sec. II we define the quantities of interest for the percolation problem and pave the way to the introduction of the interaction. In Sec. III we sketch briefly the known main results of the percolation problem without interaction on the simple Bethe lattices. In Sec. IV the solution of the generating function for the percolation problem on simple Bethe lattices with ferrogmagnetic interaction is found. The pair connectedness is treated in Sec. V, and it is found to satisfy homogeneity conditions. In Sec. VI general solutions for decorated Bethe lattices are given, while in Sec. VII the explicit solutions of Bethe lattices decorated with an extra site on each bond are discussed, to show that in this case also, where the critical probability for the corresponding noninteracting case is above $\frac{1}{2}$, the interaction lowers its value below $\frac{1}{2}$ at zero external field. This result is in agreement with what has been conjectured in Ref. 20: For any real lattice which exhibits spontaneous magnetization, $p_c \leq \frac{1}{2}$ at zero external field. In particular for the two-dimensional lattices $p_c = \frac{1}{2}$. In the Appendix an alternative way for a more direct calculation of the percolation probability is given which is a generalization of Essam's[3] procedure for the noninteracting case.

II. GENERALITY ON THE PERCOLATION PROBLEM

The site percolation problem in its easiest form consists of studying the distribution of clusters of particles which occupy at random the sites of a lattice for a given density of particles. In order to facilitate the introduction of the interaction let us formulate the percolation problem in a slightly different way. Consider a lattice of N spins interacting with an external magnetic field H. The Hamiltonian of such a system is given by

$$\mathcal{H}_{0N} = -mH \sum_{i=1}^{N} \sigma_i , \tag{1}$$

where σ_i are the usual variables of spin, which take on the values +1 and −1 corresponding to the spin "up" and "down," m is the magnetic moment of the spin.

For convenience let us introduce the following variables:

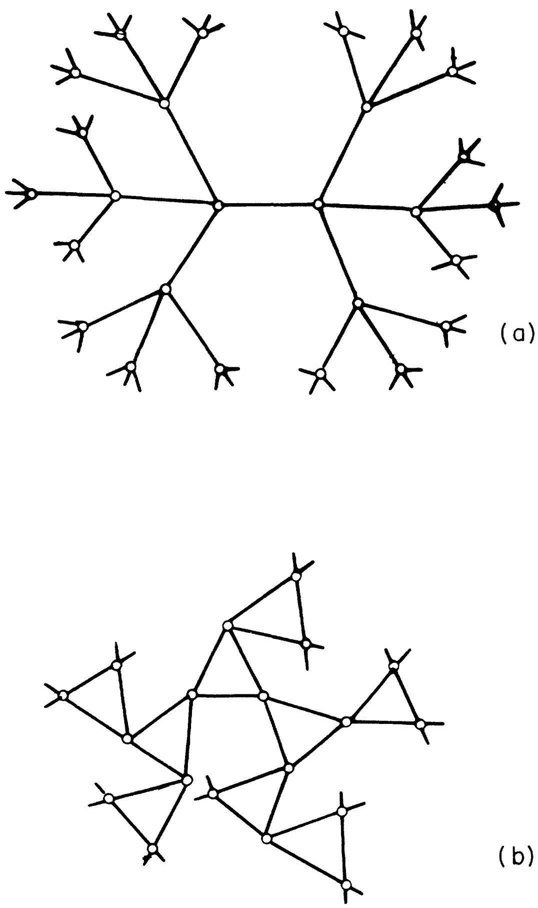

FIG. 1. Examples of Bethe lattices: (a) simple Bethe lattice of coordination number $\sigma+1=4$; (b) triangular cactus.

$$\bar{\pi}_i = \tfrac{1}{2}(1+\sigma_i) , \quad \pi_i = \tfrac{1}{2}(1-\sigma_i) ,$$

which are, respectively, the projectors on the state "up" and "down" of the ith spin. In the following, using the lattice-gas terminology, we shall also say that a vertex is empty or occupied by a particle if the spin in that vertex is correspondingly "up" or "down." The density of overturned spins is given by

$$p = \langle \pi_i \rangle_0 , \tag{2}$$

where $\langle \cdots \rangle_0$ is $\lim_{N\to\infty} \langle \cdots \rangle_{0N}$, and $\langle \cdots \rangle_{0N}$ is the thermal average, i.e.,

$$\langle \cdots \rangle_{0N} = \sum_{\{\sigma\}} \cdots e^{-\beta \mathcal{H}_{0N}} \Big/ \sum_{\{\sigma\}} e^{-\beta \mathcal{H}_{0N}} ; \tag{3}$$

$\beta = 1/KT$, where K is the Boltzman constant and T is the temperature. $\sum_{\{\sigma\}}$ is the sum over all the configurations of spins. The relation between the reduced magnetization M and p is given by

$$M = 1 - 2p . \tag{4}$$

For such a system of spins, the number of clusters of s overturned spins per spin, in the limit $N\to\infty$, will be called n_s. Two spins "down" belong to the same cluster if there is at least one chain of nearest-neighbor reversed spins connecting the two spins.

The functions of main interest in the percolation problem are

$$K(p) = \sum_{s=1}^{\infty} n_s(p) ,$$

$$P(p) = 1 - \frac{1}{p} \sum_{s=1}^{\infty} s n_s(p) , \tag{5}$$

$$S(p) = \sum_{s=1}^{\infty} s^2 n_s(p) \Big/ \sum_{s=1}^{\infty} s n_s(p) ,$$

where the sum is over all possible clusters of finite size, $K(p)$ is the mean number of clusters per sites, $P(p)$ is the probability that a given spin down belongs to a cluster of infinite spins, and $S(p)$ is the mean size of finite clusters containing a randomly chosen spin "down."

One can define the generating function[22]

$$K(x,p) = \sum_{s=1}^{\infty} x^s n_s(p) . \tag{6}$$

From this function it is easily derived that

$$K(p) = K(1,p) , \tag{7}$$

$$P(p) = 1 - x \left.\frac{\partial K(x,p)}{\partial x}\right|_{x=1} \Big/ p , \tag{8}$$

$$S(p) = 1 + x\,\frac{\partial^2 K(x,p)}{\partial x^2}\Big/\frac{\partial K(x,p)}{\partial x}\bigg|_{x=1} . \tag{9}$$

For convenience let us introduce the following notation: For any subset A of the index set R, representative of the coordinates of the spins, define

$$\pi^A = \prod_{i\in A}\pi_i\ ;\ \bar{\pi}^A = \prod_{i\in A}\bar{\pi}_i .$$

With this notation,

$$n_s = \lim_{N\to\infty}\frac{1}{N}\sum_{A_s}\langle \pi^{A_s}\bar{\pi}^{\partial A_s}\rangle_{0N} , \tag{10}$$

where A_s is a subset of coordinates representative of a cluster of s particles and ∂A_s is the subset corresponding to the coordinates of the perimeter of such a cluster. The sum is over all possible clusters of s particles.

Since the spins are not interacting, Eq. (10) can also be written in the usual form[23,24]

$$n_s(p) = \sum_t \kappa_{st}\,p^s q^t ,$$

where κ_{st} is the number of cluster configurations of size s and perimeter t per site of the lattice and $q = 1 - p$.

If we introduce a ferromagnetic interaction among the spins, the Hamiltonian of N spins is

$$\mathcal{H}_N = -Hm\sum_{i=1}^{N}\sigma_i - J\sum_{\langle ij\rangle}\sigma_i\sigma_j , \tag{11}$$

where $\sum_{\langle ij\rangle}$ is the sum over all pairs of nearest-neighbor spins. The percolation problem is now formally identical to the noninteracting case. It is enough to substitute in Eqs. (2) and (3) the thermal average $\langle\cdots\rangle_{0N}$, with

$$\langle\cdots\rangle_N = \sum_{\{\sigma\}}\cdots e^{-\beta\mathcal{H}_N}\Big/\sum_{\{\sigma\}}e^{-\beta\mathcal{H}_N} ,$$

and define

$$\lim_{N\to\infty}\langle\cdots\rangle_N = \langle\cdots\rangle .$$

Let us consider that (11) is the Hamiltonian of the Ising model in a magnetic field. Nevertheless, the knowledge of the partition function of the Ising model does not give information to the percolation problem, for which one needs to calculate the generating function

$$K(x,\mu,z) = \sum_{s=1}^{\infty} x^s n_s(\mu,z) \tag{12}$$

with

$$n_s(\mu,z) = \lim_{N\to\infty}\frac{1}{N}\sum_{A_s}\langle \pi^{A_s}\bar{\pi}^{\partial A_s}\rangle_N , \tag{13}$$

where the following variables[25] have been introduced:

$$\mu = e^{-2Hm/KT} ,\ z = e^{-2J/KT} .$$

We note that all the quantities of interest depend now on two variables, i.e., the external magnetic field and the temperature.

III. BETHE LATTICES WITHOUT INTERACTION

The percolation problem with zero interaction has already been solved by Fisher and Essam[21] and by Essam[3] for a class of models such as Bethe lattices. In solving these models the usual approximation of neglecting the surface effects[21,25] is made. This does not give the exact solution[26] of the Bethe lattices, but is an attempt to better reproduce the behavior of real lattices.

For convenience we shall report here the results of the simple Bethe lattice of coordination number $\sigma+1$. For the details we refer to the original paper.[21]

The perimeter of a cluster of s occupied sites is given by

$$t = (\sigma - 1)s + 2 , \tag{14}$$

From (6) and (11) the generating function $K^0(x,p)$ (from now on we shall label with a superscript 0 all the quantities relative to the system without interaction) is given by

$$K^0(x,p) = \sum_{s=1}^{\infty} b_s x^s p^s q^{(\sigma-1)s+2} , \tag{15}$$

where

$$b_s = \kappa_{s,(\sigma-1)s+2} . \tag{16}$$

If we define

$$B_\sigma(Z) = \frac{1}{\sigma+1}\sum_{s=1}^{\infty} b_s Z^s , \tag{17}$$

then

$$K^0(x,p) = \tfrac{1}{2}(\sigma+1)\,xpq^{\sigma+1}B_\sigma(Z(x,p)) , \tag{18}$$

where

$$Z(x,p) = xpq^{\sigma-1} .$$

It is found that[21]

$$B_\sigma(Z) = \frac{1}{\sigma+1}\,\frac{2-(\sigma+1)X(Z)}{[1-X(Z)]^{\sigma+1}} , \tag{19}$$

where $X(Z) = X(x,p)$ is the root of the equation

$$X(1-X)^{\sigma-1} = xpq^{\sigma-1} = Z \tag{20}$$

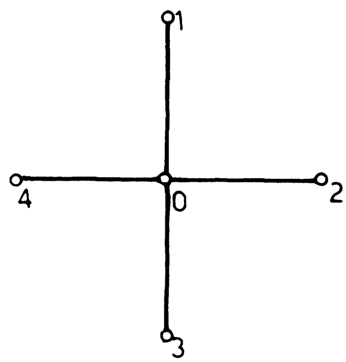

FIG. 2. Elementary cell of a simple Bethe lattice of coordination number $\sigma+1=4$.

which vanishes with Z.

Let us define

$$X(1,p)=p^*(p). \tag{21}$$

From (20) $p^*(p)$ is the root of the equation

$$p^*(1-p^*)^{\sigma-1}=p(1-p)^{\sigma-1}=z, \tag{22}$$

which vanishes continuously with z.

It has been shown by Fisher and Essam[21] that

$$\begin{aligned} &p^*(p)=p \text{ for } p\leq p_c^0=1/\sigma, \\ &p^*(p)\simeq p_c^0-|p-p_c^0| \end{aligned} \tag{23}$$

for p near p_c^0; thereafter $p^*(p)$ decreases monotonically and vanishes at $p=1$ as $(1-p)^{\sigma-1}$.

From Eqs. (8), (9), (18), and (19) we have

$$P^0(p)=\begin{cases} 0 \quad \text{for } p<p_c^0 \\ 1-\dfrac{p^*}{p}\dfrac{(1-p)^2}{(1-p^*)^2}\sim\dfrac{2(\sigma+1)}{\sigma-1}\dfrac{p-p_c^0}{p} \\ \text{for } p>p_c^0 \end{cases} \tag{24}$$

$$S^0(p)=\frac{1+p^*}{1-\sigma p^*}\sim\frac{\sigma(\sigma+1)}{|(p-p_c^0)/p_c^0|}. \tag{25}$$

From (24) and (25) it is easy to see that $p_c^0=1/\sigma$ is the critical probability which is defined by $\sup_{P^0(p)=0}p=p_c^0$ and $S^0(p_c^0)=\infty$.

IV. BETHE LATTICES WITH FERROMAGNETIC INTERACTION

Now we want to solve the percolation problem for the simple Bethe lattices of coordination number $\sigma+1$ with the Hamiltonian of the system given by (11). Let us write the Hamiltonian (11) in terms of the variables

$$\pi_i=\tfrac{1}{2}(1-\sigma_i),$$

$$\mathcal{H}_N=-HmN-\tfrac{1}{2}NJ(\sigma+1)+[2Hm+2J(\sigma+1)]\sum_{i=1}^{N}\pi_i-4J\sum_{\langle ij\rangle}\pi_i\pi_j\,; \tag{26}$$

the partition function is

$$Z_N=e^{\beta[HmN+NJ(\sigma+1)/2]}\Lambda_N, \tag{27}$$

where

$$\Lambda_N=\sum_{\{\pi_i\}}\exp\left(-h'\sum_{i=1}^{N}\pi_i+J'\sum_{\langle ij\rangle}\pi_i\pi_j\right) \tag{28}$$

and

$$h'=2[Hm+J(\sigma+1)]\beta,\; J'=4J\beta. \tag{29}$$

The sum is over all possible values of the set of variables $\{\pi_i\}_{i\in R}$.

Let us first calculate the density of overturned spins. Consider an elementary cell of a simple Bethe lattice of coordination number $\sigma+1$. In Fig. 2 an example is given for $\sigma+1=4$. The center is labeled with O and the nearest neighbors with $1,\ldots,\sigma+1$. Say N_k is the number of spins in the kth branch, which starts from the site k ($k=1,\ldots,\sigma+1$), and R_k the subset of the integer set R representative of the branch k. The probability of having the spin at the site O overturned is given by

$$\langle\pi_0\rangle_N=\frac{1}{\Lambda_N}e^{-h'}\prod_{k=1}^{\sigma+1}[e^{J'}\Lambda_{N_k}(1)+\Lambda_{N_k}(0)], \tag{30}$$

where

$$\Lambda_{N_k}(\pi_k)=\sum_{\substack{\{\pi_i\}_{i\in R_k}\\ i\neq k}}\exp\left(-h'\sum_{i\in R_k}\pi_i+J'\sum_{\langle ij\rangle\in R_k\times R_k}\pi_i\pi_j\right), \tag{31}$$

where the first sum is over all possible values of $\{\pi_i\}_{i\in R_k,\, i\neq k}$ relative to the kth branch except π_k. In the same way we find

$$\Lambda_N=e^{-h'}\prod_{k=1}^{\sigma+1}[e^{J'}\Lambda_{N_k}(1)+\Lambda_{N_k}(0)]+\prod_{k=1}^{\sigma+1}[\Lambda_{N_k}(1)+\Lambda_{N_k}(0)]. \tag{32}$$

From (30) and (32) after performing $\lim N_k\to\infty$ for $k=1,\ldots,\sigma+1$ we have

$$\langle\pi_0\rangle=e^{-h'}(e^{J'}y+1)/e^{-h'}(e^{J'}y+1)^{\sigma+1}, \tag{33}$$

where

$$y=\lim_{N_k\to\infty}\frac{\Lambda_{N_k}(1)}{\Lambda_{N_k}(0)}$$

independent of k.

We must still find a relation for y. For this reason we evaluate the probability of having the spin at the site 1 overturned. This is given by

$$\langle \pi_1 \rangle_N = \frac{1}{\Lambda_N} \sum_{\pi_0,\, \pi_1 \cdots \pi_{\sigma+1}} \exp[(-h'+J')\pi_0 + J'\pi_0(\pi_2 + \cdots \pi_{\sigma+1})]\Lambda_{N_1}(1)\Lambda_{N_2}(\pi_2)\cdots\Lambda_{N_{\sigma+1}}(\pi_{\sigma+1})$$

$$= \frac{1}{\Lambda_N}\Lambda_{N_1}(1)\left(e^{-h'+J'}\prod_{k=2}^{\sigma+1}[e^{J'}\Lambda_{N_k}(1)+\Lambda_{N_k}(0)] + \prod_{k=2}^{\sigma+1}(\Lambda_{N_k}(1)+\Lambda_{N_k}(0)]\right).$$

From (32) by performing $\lim N_k \to \infty$ for $k=1,\ldots,\sigma+1$ it follows

$$\langle \pi_1 \rangle = \frac{e^{-h'+J'}y(e^{J'}y+1)^\sigma + (y+1)^\sigma}{e^{-h'}(e^{J'}y+1)^{\sigma+1} + (e^{J'}y+1)^{\sigma+1}}. \tag{34}$$

We make the assumption that in the limit $N_k \to \infty$ for $K=1,\ldots,\sigma+1$ the lattice is translational invariant. This is equivalent to assuming that the surface effect can be neglected as we always do for the aforesaid reasons. This assumption leads to the equality

$$\langle \pi_0 \rangle = \langle \pi_1 \rangle$$

and from (33) and (34)

$$e^{-h'}y(e^{J'}y+1)^{\sigma+1} = e^{-h'+J'}(e^{J'}y+1)^\sigma + (1+y)^\sigma. \tag{35}$$

If we introduce the usual variables[25]

$$\mu = e^{-2Hm/KT},$$
$$\mu_1 = ye^{2J/KT},$$
$$z = e^{-2J/KT}, \tag{36}$$

Eq. (35) becomes

$$\frac{\mu_1}{\mu} = \frac{(\mu_1+z)^\sigma}{(1+\mu_1 z)^\sigma} \tag{37}$$

and Eq. (29)

$$\langle \pi_0 \rangle = p = \mu_1(\mu_1+z)/\mu_1^2 + 2\mu_1 z + 1); \tag{38}$$

Eqs. (37) and (38) coincide with the result derived by Domb[25] who introduced a self-consistent field H_1 applied to the perimeter of the unit cell. H_1 is related to μ_1 by the relation $\mu_1 = e^{-2H_1/KT}$. As was shown by Domb,[25] at low temperature and for zero external field ($\mu=1$), there is spontaneous magnetization which goes to zero at the critical temperature corresponding to

$$z_c = \frac{\sigma-1}{\sigma+1}. \tag{39}$$

Let us calculate now the mean number of clusters of s particles per vertex. In the Bethe lattices, because of the absence of loops, the probability that a given cluster occurs depends only on the dimension s, i.e., if A_s and B_s are the subsets representative of two clusters of s particles then

$$\langle \pi^{A_s}\bar{\pi}^{\partial A_s}\rangle = \langle \pi^{B_s}\bar{\pi}^{\partial B_s}\rangle.$$

Therefore, from (13),

$$n_s(\mu, z) = b_s\langle \pi^{A_s}\bar{\pi}^{\partial A_s}\rangle, \tag{40}$$

and from (12)

$$K(x,\mu,z) = \sum_{s=1}^{\infty} b_s x^s \langle \pi^{A_s}\bar{\pi}^{\partial A_s}\rangle, \tag{41}$$

where b_s is given by (16) and the dimension s of the cluster and the perimeter t are related by (14). If we consider that $\bar{\pi}_i = 1-\pi_i$, the angular bracket in (40) can also be written in the following way:

$$\langle \pi^{A_s}\bar{\pi}^{\partial A_s}\rangle = \sum_{k=0}^{t}\binom{t}{k}(-1)^k\langle \pi^{A_{s+k}}\rangle. \tag{42}$$

A_{s+k} is any one of the $\binom{t}{k}$ subset representative of the clusters of $s+k$ particles obtained by developing

$$\bar{\pi}^{\partial A_s} = \prod_{i\in\partial A_s}(1-\pi_i).$$

In order to calculate (40) we need then to evaluate expressions of the kind $\langle \pi^{A_r}\rangle$ where A_r is a cluster of r particles. Letting $1,2,\ldots,r$ be the elements of A_r and r be one of the peripherical sites,[27] we have

$$\langle \pi^{A_r}\rangle_N = \langle \pi_1\cdots\pi_r\rangle_N = \langle \pi_1\cdots\pi_{r-1}\rangle_N - \langle \pi_1\cdots\pi_{r-1}\bar{\pi}_r\rangle_N. \tag{43}$$

And by using the same procedure adopted to calculate the density of overturned spins it is easy to find

$$\langle \pi_1\cdots\pi_r\rangle_N = \frac{1}{\Lambda_N}e^{J'}A_{1,2,\ldots,r-1}\Lambda_{N_r}(1), \tag{44}$$

$$\langle \pi_1\cdots\bar{\pi}_r\rangle_N = \frac{1}{\Lambda_N}A_{1,\ldots,r-1}\Lambda_{N_r}(0), \tag{45}$$

where $\Lambda_{N_r}(\pi_r)$ is defined in Eq. (31) and

$$A_{1,2,\ldots,r-1} = \sum_{\{\pi_i\}_{i\in\tilde{R}_r}} \pi_1\cdots\pi_{r-1}$$
$$\times \exp\left(-h'\sum_{i\in\tilde{R}_r}\pi_i + J'\sum_{\langle ij\rangle\in\tilde{R}_r\times\tilde{R}_r}\pi_i\pi_j\right),$$

where $\tilde{R}_r = R - R_r$ and R_r is the subset representative of the branch leaving the cluster from the site r. From (43)–(45) after performing $\lim N\to\infty$

$$\frac{\langle \pi_1 \cdots \pi_r\rangle}{\langle \pi_1 \cdots \pi_{r-1}\rangle} = \frac{ye^{J'}}{ye^{J'}+1} = a(\mu_1 z)\,, \tag{46}$$

which from (36) gives

$$a(\mu, z) = \mu_1/(\mu_1 + z). \tag{47}$$

In other words the probability that the sites $1, 2, \ldots, r$ are occupied is given by the product of the probability that $1, 2, \ldots, r-1$ are occupied, times $a(\mu_1 z)$, which is the probability that r is occupied being $r-1$ occupied.[28]

By repeated applications of (46)

$$\langle \pi^{A_r}\rangle = \langle \pi_1 \cdots \pi_r\rangle = \langle \pi_1\rangle [a(\mu_1 z)]^{r-1}.$$

It is easy to prove that Eq. (42) becomes

$$\langle \pi^{A_s}\bar{\pi}^{\partial A_s}\rangle = [\langle \pi_1\rangle / a(\mu, z)] \times [a(\mu, z)]^s [1 - a(\mu, z)]^{(\sigma-1)s+2}, \tag{48}$$

and from (40) and (41) the generating function is given by

$$K(x, \mu, z) = \frac{\langle \pi_1\rangle}{a(\mu, z)} \sum_{s=1}^{\infty} b_s x^s a(\mu, z) \times [1 - a(\mu, z)]^{(\sigma-1)s+2}. \tag{49}$$

A comparison with (15) leads to the simple result

$$K(x, \mu, z) = \frac{p(\mu, z)}{a(\mu, z)} K^0(x, a(\mu, z)), \tag{50}$$

and from (8) and (9),

$$P(\mu, z) = P^0(a(\mu, z))\,, \tag{51}$$

$$S(\mu, z) = S^0(a(\mu, z))\,; \tag{52}$$

the condition for percolation is given by

$$a(\mu, z) = p_c^0 = 1/\sigma\,, \tag{53}$$

which from (47) gives a line of critical points

$$\mu_{1c}(z) = \mu_{1c}(\mu_c(z), z) = z/(\sigma - 1) \tag{54}$$

which substituted in (38) gives

$$p_c(z) = \frac{1}{\sigma} \frac{z^2\sigma^2}{(\sigma-1)^2 + z^2(2\sigma - 1)}\,, \tag{55}$$

coinciding with the result found by Kikuchy[17] in a different way.

In order to find the critical behavior of the percolation probability and the mean cluster size as function of p near $p_c(z)$ for a fixed z, we introduce p and z as independent variables and define

$$\bar{P}(p, z) = P(\mu, z);\quad \bar{S}(p, z) = S(\mu, z);\quad \bar{a}(p, z) = a(\mu, z)\,.$$

From (38) and (47), after some manipulations, the expansion of $\bar{a}(p, z)$ near $p_c(z)$ gives

$$\bar{a}(p, z) \simeq \frac{1}{\sigma} + \frac{1}{\sigma} \frac{(\sigma-1)^2 + (2\sigma-1)z^2}{\sigma^2 - 1 + z^2} \frac{p - p_c(z)}{p_c(z)}\,,$$

which from (24), (25), (51), and (52) leads to

$$\bar{P}(p, z) \simeq \begin{cases} 0 \text{ for } p < p_c(z) \\ \dfrac{2\sigma+1}{\sigma-1} \dfrac{(\sigma-1)^2 + (2\sigma-1)z^2}{\sigma^2 - 1 + z^2} \dfrac{p - p_c(z)}{p_c(z)} \text{ for } p > p_c^0\,, \end{cases}$$

$$\bar{S}(p, z) \simeq \sigma(\sigma+1) \frac{\sigma^2 - 1 + z^2}{(\sigma-1)^2 + (2\sigma-1)z^2} \frac{1}{|[p - p_c(z)]/p_c(z)|}\,.$$

From (55) we note that $p_c(z) \leq p_c^0,\ \forall z$, as the ferromagnetic interaction facilitates clustering. It must be pointed out that the values of $p_c(z)$ given by (55) correspond to stability only if $\mu_c(z) \leq 1$. Let us say z_p is the value of z satisfying Eq. (54) for $\mu_c = 1^-$ ($H_c = 0^+$). From (54)

$$\mu_{1c}(1^-, z_p) = z_p/(\sigma - 1)\,. \tag{56}$$

Since μ_1 is a decreasing function of μ and z, Eq. (54) can hold for $z < z_p$ only if $\mu_c > 1$, which leads to instability. This means that the minimum value of p_c which can be reached is given by

$$p_{c,\min} = p_c(z_p)\,.$$

This value of z_p corresponds to the percolation point for zero external field, and since $p_c(z_p) < \frac{1}{2}$, $\forall \sigma \geq 2$, this means that $z_p < z_c$ for all the simple Bethe lattices of coordination number $\sigma + 1 = 3$. In other words an infinite cluster of spins "down" already appears before the critical point is reached. For $\sigma = 1$ (linear chain) $p_c = 1$, $\forall z$. Recently Müller-Krumbhaar[19] has calculated, at zero external field, the critical density for percolation by means of the Monte Carlo technique with the following result:

$$p_c(z_p) \simeq 0.19\,,$$

while for the same lattice without interaction series expansion[29]

$$p_c^0 \simeq 0.307\,;$$

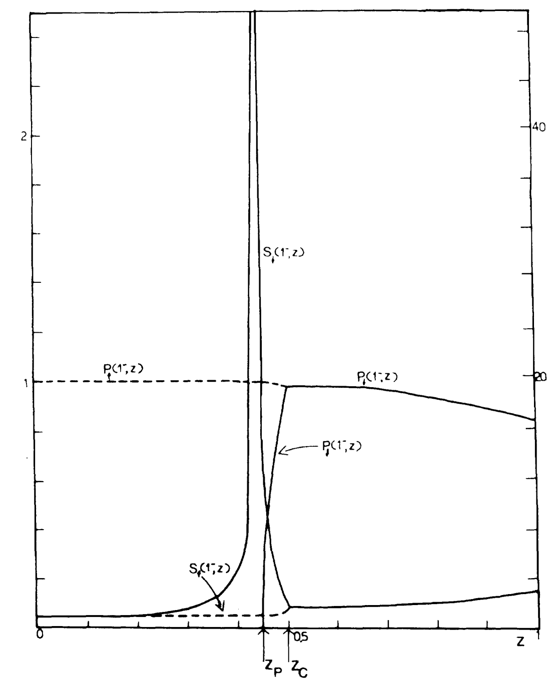

FIG. 3. Broken curves are $P_\uparrow(1^-, z)$ and $S_\uparrow(1^-, z)$, respectively, the probability that a given spin "up" belongs to an infinite cluster of spins up and the mean cluster size of spins "up," for external magnetic field $H = 0^+$ vs $z = e^{-2J/KT}$, T is the absolute temperature, J is the nearest-neighbor interaction, for the $\sigma = 4$ Bethe lattice. The solid curves are $P_\downarrow(1^-, z)$ and $S_\downarrow(1^-, z)$ the same quantities relative to spins "down." It has been reported also z_C corresponding to the Curie temperature. For $z \geq z_C$, $P_\uparrow(1^-, z) = P_\downarrow(1^-, z)$ and $S_\uparrow(1^-, z) = S_\downarrow(1^-, z)$. Note the percolation temperature $z_p < z_C$. On the left is the scale of $P_\uparrow(1^-, z)$ and $P_\downarrow(1^-, z)$, on the right the scale of $S_\uparrow(1^-, z)$ and $S_\downarrow(1^-, z)$.

combining these results we have

$$p_c(z_p)/p_c^0 \simeq 0.61 .$$

The coordination number of the simple-cubic lattice is 6 while the "connectivity"[30] is ~ 4.68. In the Bethe lattice the coordination number and the "connectivity" coincide. In order to compare the above numerical results with the Bethe lattice we have calculated for the simple Bethe lattice of coordination number $\sigma + 1 = 6$

$$p_c(z_p) = 0.092 , \quad p_c^0 = 0.2 , \quad p_c(z_p)/p_c^0 = 0.460 ,$$

while for the simple Bethe lattice with "connectivity" $\sigma + 1 = 4.68$, $p_c(z_p) = 0.108$, $p_c^0 = 0.272$, $p_c(z_p)/p_c^0 = 0.396$.

A complete solution for $\sigma = 3$ is given in Fig. 3 where we have reported for $H \to 0^+$ $(\mu \to 1^-)$ the percolation probability and the mean cluster size, relative to clusters of reversed spins, which we have called here $P_\downarrow(1^-, z)$ and $S_\downarrow(1^-, z)$ to distinguish them from $P_\uparrow(1^-, z)$ and $S_\uparrow(1^-, z)$, which also have been reported in the same figure, and by obvious notations are referred to the same quantities relative to clusters of spins "up."

Because of symmetry we also have

$$P_\downarrow(1^+, z) = P_\uparrow(1^-, z) ,$$

$$S_\downarrow(1^+, z) = S_\uparrow(1^-, z) .$$

A general argument has been given by the author[20] which leads to the conclusion that for three-dimensional systems at zero external field the percolation point should occur below the critical temperature. This result is supported by series expansion[31] and by means of the Monte Carlo technique.[19] For two dimensions the conclusion is that the critical temperature and the percolation point coincide, which seems to be verified by series expansion[31] and by an exact result on the triangular lattice.[3,32]

Strong evidence for this conclusion is also given on the grounds of the cluster distribution of the square Ising model, found by Stoll, Binder, and Schneider[33] by means of the Monte Carlo technique. Their result also support the cluster model proposed by Fisher[34,35] in which the large (divergent) probability of very large clusters indicates that condensation has taken place.

In conclusion we should expect that, for any lattice for which there is spontaneous magnetization at zero magnetic field $p_c \leq \frac{1}{2}$. This has been proved not only for the simple Bethe lattice for which already for zero interaction $p_c^0 \leq \frac{1}{2}$ but, as it will be shown in Sec. VII, this is true also for other pseudolattices for which $p_c^0 > \frac{1}{2}$.

V. PAIR CONNECTEDNESS

The pair connectedness in the percolation problem[3,36] plays a similar role to the pair correlation in critical phenomena.[35] Let us define first

$$P_{ij}^{(k)} = \langle \gamma_{ij}^{(k)} \rangle ,$$

where i and j are two sites of the lattice and

$$\gamma_{ij}^{(k)} = \begin{cases} 1 \text{ if } i,j \text{ belong to the same cluster} \\ \quad \text{whose dimension is not larger than } k \\ 0 \text{ otherwise.} \end{cases}$$

The pair connectedness is defined by

$$P_{ij} = \lim_{k \to \infty} P_{ij}^{(k)} .$$

In other words P_{ij} is the probability that i and j belong to the same finite cluster. With this definition it is easy to show[3]

$$\sum_{s=1}^{\infty} s^2 n_s - \sum_{s=1}^{\infty} s n_s = \sum_{i \neq j} P_{ij} ,$$

where in the sum on the right-hand side j is taken as fixed. As usually we assume translational invariance. From the third of Eqs. (5)

$$S=1+\sum_{i\neq j}P_{ij}\Big/\sum_{s=1}^{\infty}sn_s\,. \tag{57}$$

In the simple Bethe lattice of coordination number $\sigma+1$ with zero interaction, call C_{ij} the walk going from i to j and r the number of steps, we have[3,37]

$$P_{ij}^0=p^r[Q^0(p)]^{(\sigma-1)r+2}, \tag{58}$$

where $Q^0(p)$ is the probability that the branch leaving from a given occupied site of the perimeter of C_{ij} is closed, i.e., all open walks of the branch are finite. A walk is said to be open when all its sites are occupied. It will be shown in the Appendix that

$$Q^0(p)=[1-P^0(p)]^{1/(\sigma+1)}\,. \tag{59}$$

In order to prove relation (57), consider that on the simple Bethe lattice there are $(\sigma+1)\sigma^{r-2}$ vertices which are r steps "distant" from a given site J so that

$$1+\frac{\sum_{i\neq j}P_{ij}^0}{\sum_{s=1}^{\infty}sn_s}=1+\frac{(\sigma+1)\sum_{r\geq 2}\sigma^{r-2}p^r[Q^0(p)]^{(\sigma-1)r+2}}{p[1-P^0(p)]}$$
$$=\frac{1+p[Q^0(p)]^{\sigma-1}}{1-p[Q^0(p)]^{\sigma-1}}\,.$$

Since from (22), (24), and (59)

$$p[Q^0(p)]^{\sigma-1}=p^*(p)\,,$$

Eq. (57) easily follows.

We note that because of the peculiarity of the model, the number of sites which are r steps distant from a given site is $(\sigma+1)\sigma^{r-2}$ while for a real d-dimensional lattice the number of sites which are at a distant R from a given site goes as R^{d-1}. We define then a renormalized pair connectedness function which better reproduces the behavior of a real lattice, when r is interpreted as a real distance,

$$\tilde{P}_{ij}^0=\frac{(\sigma+1)\sigma^{r-2}}{r^{d-1}}P_{ij}^0=\frac{\sigma+1}{\sigma^2}[Q^0(p)]^2\frac{e^{-r/\xi^0(p)}}{r^{d-1}}\,, \tag{60}$$

where we have defined the connectedness length

$$\xi^0(p)=\frac{1}{\ln\sigma p[Q^0(p)]^{\sigma-1}}\,, \tag{61}$$

which diverges at $p=p_c^0=1/\sigma$. The scaling homogeneity (60) which the pair connectedness obeys has also been argued from a droplet picture by Stauffer.[38] The pair connectedness function in the interacting case is given by

$$P_{ij}=\langle\pi^{C_{ij}}\rangle[Q(\mu,z)]^{(\sigma-1)r+2}\,, \tag{62}$$

where $Q(\mu,z)$ is the probability that the branch leaving from a given occupied site of the perimeter of C_{ij} is closed. As in the noninteracting case

$$Q(\mu,z)=[1-P(\mu,z)]^{1/(\sigma+1)} \tag{63}$$

while

$$\langle\pi^{C_{ij}}\rangle=\frac{p(\mu,z)}{a(\mu,z)}a^r(\mu,z)\,, \tag{64}$$

hence

$$P_{ij}=\frac{p(\mu,z)}{a(\mu,z)}a^r(\mu,z)[Q(\mu,z)]^{(\sigma-1)r+2}\,. \tag{65}$$

In the same way as before Eq. (57) can be verified, and a renormalized pair connectedness function can be defined:

$$\tilde{P}_{ij}=\frac{(\sigma+1)\sigma^{r-2}}{r^{d-1}}P_{ij}$$
$$=\frac{\sigma+1}{\sigma^2}\frac{p(\mu,z)}{a(\mu,z)}[Q(\mu,z)]^2\frac{e^{-r/\xi(\mu,z)}}{r^{d-1}}\,, \tag{66}$$

where

$$\xi(\mu,z)=\frac{1}{\ln\sigma a(\mu,z)[Q(\mu,z)]^{\sigma-1}}$$

which is divergent for $a(\mu,z)=1/\sigma$, which is the equation for the critical line of percolation points.

VI. DECORATED BETHE LATTICES

We now consider a class of lattices which can be derived from the simple Bethe lattices by substituting a bond with a finite graph of sites and bonds, usually called the bond graph. Examples of such lattices are given in Fig. 4, along with the bond graph.

To simplify the general treatment we consider only bond graphs which are symmetric with respect to the two terminals. Starting from the center O of the lattice we label the bond graph by the coordinate of its terminal further from the origin. The coordinate of such a terminal will be the same as in the simple Bethe lattice, from which the decorated lattice has been derived.

Let us define the following operators:

$$\pi_i^*(x)=\sum_{C_k^i}x^k\pi^{C_k^i}\bar{\pi}^{\partial C_k^i}\,, \tag{67}$$

where C_k^i is a configuration of k occupied sites going from one terminal to the other of the ith bond and ∂C_k^i is the perimeter;

$$\tilde{\pi}_i^*(x)=\sum_{D_k^i}x^k\pi^{D_k^i}\bar{\pi}^{\partial D_k^i}\,, \tag{68}$$

where D_k^i is a configuration of k occupied sites which are connected to the first terminal but not

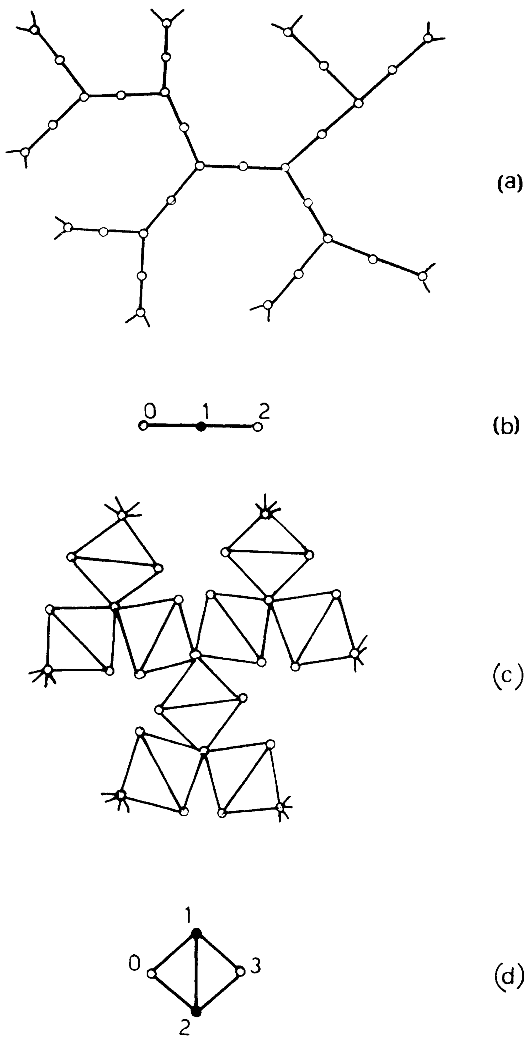

FIG. 4. Decorated Bethe lattices (a) and (c) derived from the $\sigma=2$ Bethe lattice by replacing bonds by the bond graphs (b) and (d).

to the other on the ith bond graph. In the definition of π_i^* and $\bar{\pi}_i^*$, the first terminal i_0 is supposed to be open but the corresponding operator π_{i_0} should not be included for it. From the definition, $\pi_i^*(1)$ is the projector on the configurations which connect one terminal of the ith bond graph to the other, the first terminal being supposedly occupied. Conversely, $\bar{\pi}_i^*(1)$ is the projector on the configurations, which being connected to the first terminal of the ith bond, does not reach the second one. The first terminal being supposedly occupied. Consequently $\pi_i^*(1)+\bar{\pi}_i^*(1)=1$.

Following the procedure adopted for the random case by Fisher and Essam[21] we define three bond generating functions

$$C(x,\mu,z)=\frac{\langle \pi_{i_0}\pi_i^*(x)\rangle}{\langle \pi_{i_0}\rangle}\,, \tag{69}$$

$$D(x,\mu,z)=\frac{\langle \pi_{i_0}\bar{\pi}_i^*(x)\rangle}{\langle \pi_{i_0}\rangle}\,, \tag{70}$$

$$E(x,\mu,z)=\sum_{E_k^i} x^k\langle \pi^{E_k^i}\bar{\pi}^{\partial E_k^i}\rangle\,, \tag{71}$$

where E_k^i is a configuration of k occupied sites which are not connected to either terminals in the ith bond. Of course $C(x,\mu,z)$, $D(x,\mu,z)$, and $E(x,\mu,z)$ are independent of the coordinate i of the bond graph because of the supposed translational invariance on the undecorated Bethe lattice. For example, for the two decorated lattices of Fig. 4 we have

$$\begin{aligned} C(x,\mu,z)&=x^2\frac{\langle \pi_0\pi_1\pi_2\rangle}{\langle \pi_0\rangle}\,,\\ D(x,\mu,z)&=x\frac{\langle \pi_0\pi_1\bar{\pi}_2\rangle}{\langle \pi_0\rangle}+\frac{\langle \pi_0\bar{\pi}_1\rangle}{\langle \pi_0\rangle}\,,\\ E(x,\mu,z)&=x\langle \bar{\pi}_0\pi_1\bar{\pi}_2\rangle\,, \end{aligned} \tag{72}$$

and

$$\begin{aligned} C(x,\mu,z)&=x^3\frac{\langle \pi_0\pi_1\pi_2\pi_3\rangle}{\langle \pi_0\rangle}+x^2\frac{2\langle \pi_0\pi_1\bar{\pi}_2\pi_3\rangle}{\langle \pi_0\rangle}\,,\\ D(x,\mu,z)&=x^2\frac{\langle \pi_0\pi_1\pi_2\bar{\pi}_3\rangle}{\langle \pi_0\rangle}+x\frac{\langle \pi_0\pi_1\bar{\pi}_2\bar{\pi}_3\rangle}{\langle \pi_0\rangle}+\frac{\langle \pi_0\bar{\pi}_1\bar{\pi}_2\rangle}{\langle \pi_0\rangle}\,,\\ E(x,\mu,z)&=x^2\langle \bar{\pi}_0\pi_1\pi_2\bar{\pi}_3\rangle+2x\langle \bar{\pi}_0\bar{\pi}_1\pi_2\bar{\pi}_3\rangle\,. \end{aligned} \tag{73}$$

From the definition of $C(x,\mu,z)$ it follows that $C(1,\mu,z)$ is the probability of reaching the second terminal of the bond graph when the first one is occupied. Conversely, $D(1,\mu,z)$ is the probability of failing to reach the second terminal starting from the first one, which is supposed to be occupied. Consequently, it follows that

$$C(1,\mu,z)+D(1,\mu,z)=1\,. \tag{74}$$

From Eq. (41) we remember that the generating function for the simple Bethe lattice is given by

$$\begin{aligned} K(x,\mu,z)&=\sum_{s=1}^{\infty} b_s x^s\langle \pi^{A_s}\bar{\pi}^{\partial A_s}\rangle\\ &=\sum_{s=1}^{\infty} b_s x^s\langle \pi_0\pi^{A_{s-1}}\bar{\pi}^{\partial A_s}\rangle\,, \end{aligned} \tag{75}$$

where we have isolated π_0 relative to the origin of the simple Bethe lattice. A_{s-1} is defined by

$$\pi_0\pi^{A_{s-1}}=\pi^{A_s}\,.$$

The configurational generating function for the decorated lattice, indicated by an asterisk, can be obtained by making the following transformations:

$$x^{s-1}\pi^{A_s-1} \to [\pi^*(x)]^{A_s-1}$$

and

$$\bar{\pi}^{\partial A_s} \to [\bar{\pi}^*(x)]^{\partial A_s},$$

and adding a correction for the clusters which do not span a bond graph.

Thus the configurational generating function per site is given by

$$K^*(x,\mu,z) = \frac{1}{\frac{1}{2}(\sigma+1)g_s-\sigma}\left[\sum_{s=1}^{\infty} b_s x\langle \pi_0[\pi^*(x)]^{A_s-1}[\bar{\pi}^*(x)]^{\partial A_s}\rangle + \tfrac{1}{2}(\sigma+1)E(x,\mu,z)\right]; \tag{76}$$

g_s is the number of sites in the bond graph including the terminals, and $1/[\frac{1}{2}(\sigma+1)g_s-\sigma]$ is the ratio between the sites in the simple Bethe lattice and the number of sites in the decorated one. $\frac{1}{2}(\sigma+1)$ is the number of bonds per sites in the simple Bethe lattice.

In the same way as it was obtained [Eq. (48)] it is possible to show that

$$\langle \pi_0[\pi^*(x)]^{A_s-1}[\bar{\pi}^*(x)]^{\partial A_s}\rangle = \langle \pi_0\rangle [C(x,\mu,z)]^{s-1}[D(x,\mu,z)]^{(\sigma-1)s+2}. \tag{77}$$

Hence from (17) and (18)

$$K^*(x,\mu,z) = \frac{\sigma+1}{(\sigma+1)g_s-2\sigma}\{x\langle\pi_0\rangle[D(x,\mu,z)]^{\sigma+1}B_\sigma[Z^*(x,\mu,z)] + E(x,\mu,z)\}, \tag{78}$$

where

$$Z^*(x,\mu,z) = C(x,\mu,z)[D(x,\mu,z)]^{\sigma-1}. \tag{79}$$

The percolation points in the μ, z plane are determined by those values for which $B_\sigma(Z^*)$ become singular, which happens for $Z^* = \sigma^{-\sigma}(\sigma-1)^{\sigma-1}$. Consequently, the critical line of percolation points is given by

$$C(1,\mu,z) = 1/\sigma, \tag{80}$$

which for a fixed μ might also have more than one solution in z. The critical behaviors of $P(\mu,z)$ and $S(\mu,z)$ near every critical point also have the same form as that one obtained for the simple Bethe lattice without interaction.

The results of the percolation problem for the random case can be derived by putting everywhere $z=1$. It is easy to verify that in this limit $z=1$, they coincide with those ones obtained by Fisher and Essam.[21]

VII. EXPLICIT SOLUTION FOR SOME DECORATED LATTICES

In this section we are interested in giving the explicit solution of the simple Bethe lattices decorated with only one extra site on each bond [see Fig. 4(a)] for which the critical probability is given by[3,21]

$$p_c^0 = 1/\sqrt{\sigma},$$

which for $\sigma=2$ and 3 is larger than $\frac{1}{2}$. As was said in Sec. IV, it will be proved that even in this case, for zero external field $p_c \le \frac{1}{2}$. The Hamiltonian for a decorated lattice of N sites is given by

$$\frac{\mathcal{H}_N}{kT} = -\frac{HmN}{kT} - \frac{2\sigma+1}{2\sigma+3}\frac{NJ}{kT} + h'\sum_{i\in R_1}\pi_1 + h''\sum_{j\in R_2}\pi_j - J'\sum_{\langle ij\rangle}\pi_i\pi_j$$

and the partition function is

$$Z_N = \exp\left[\left(HmN + \frac{2\sigma+1}{2\sigma+3}NJ\right)\Big/KT\right]\Lambda_N,$$

where

$$\Lambda_N = \sum_{\{\pi_i\}} \exp -h'\sum_{i\in R_1}\pi_i - h''\sum_{j\in R_2}\pi_j + J'\sum_{\langle ij\rangle}\pi_i\pi_j$$

and

$$\begin{aligned} h' &= [2Hm + 2J(\sigma+1)]/KT, \\ h'' &= (2Hm+4J)/KT, \\ J' &= 4J/KT. \end{aligned} \tag{81}$$

R_1 is the subset of R corresponding to the sites of original Bethe lattice, while R_2 is the subset of R corresponding to the decorating sites.

Following the same procedure adopted for the simple Bethe lattices, let us consider an elementary cell of the decorated Bethe lattice of coordination number $\sigma+1$ (see Fig. 5 for the particular case $\sigma+1=3$). Then we have

$$\begin{aligned} \langle\pi_0\rangle_N &= \frac{1}{\Lambda_N}\sum_{\pi_i,\ldots,\pi_{\sigma+1}} e^{-h'}\prod_{k=1}^{\sigma+1} e^{J'\pi_k}\Lambda_{N_k}(\pi_k) \\ &= \frac{1}{\Lambda_N} e^{-h'}\prod_{k=1}^{\sigma+1}[e^{J'}\Lambda_{N_k}(1) + \Lambda_{N_k}(0)] \end{aligned} \tag{82}$$

and

$$\Lambda_N = e^{-h'}\prod_{k=1}^{\sigma+1}[e^{J'}\Lambda_{N_k}(1) + \Lambda_{N_k}(0)] + \prod_{k=1}^{\sigma+1}[\Lambda_{N_k}(1) + \Lambda_{N_k}(0)],$$

where $\Lambda_{N_k}(\pi_k)$ has been defined in (31).

$$\langle\pi_{\sigma+2}\rangle = \frac{1}{\Lambda_N}\left((e^{2J'-h''}+1)e^{-h'}\prod_{k=2}^{\sigma+1}[e^{J'}\Lambda_{N_k}(1) + \Lambda_{N_k}(0)]\Lambda_{N_{\sigma+2}}(1)\right) \tag{83}$$

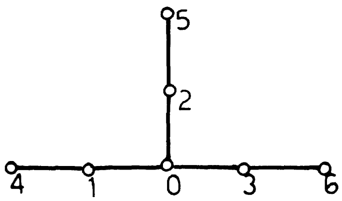

FIG. 5. Elementary cell of the $\sigma = 2$ Bethe lattice decorated by an extra site on each bond.

and

$$\Lambda_{N_{\sigma+2}}(1) = \frac{\Lambda_{N_1}(1) - e^{-h''}\Lambda_{N_1}(0)}{e^{-h''}(e^{J'}-1)} . \tag{84}$$

By equating (82) and (83), from (84) after a few manipulations we have

$$y_2 = \frac{e^{-h''}e^{J'}y_1 + 1}{e^{-h''}y_2 + 1} , \tag{85a}$$

$$e^{-h'}(e^{J'}y_2+1)^{\sigma+1} = [e^{-h'}(e^{2J'-h''}+1)(e^{J'}y_2+1)^{\sigma} + (e^{J'-h''}+1)(y_2+1)^{\sigma}] \times \frac{(y_2 - e^{-h''})}{e^{-h''}(e^{J'}-1)} , \tag{85b}$$

where

$$y_1 = \lim_{N_k \to \infty} \frac{\Lambda_{N_k}(1)}{\Lambda_{N_k}(0)} , \tag{86}$$

k corresponds to a site of the original lattice, and

$$y_2 = \lim_{N_t \to \infty} \frac{\Lambda_{N_t}(1)}{\Lambda_{N_t}(0)} , \tag{87}$$

t corresponds to a decorating site.

From (85) and (81), after a few manipulations we have

$$\frac{\mu_1}{\mu} = \frac{(\mu_2 + z)^{\sigma}}{(z\mu_2 + 1)^{\sigma}} , \tag{88}$$

$$\frac{\mu_2}{\mu} = \frac{\mu_1 + z}{z\mu_1 + 1} , \tag{89}$$

where we have put

$$\mu_2 = e^{2J/kT}y_2 ; \quad \mu_1 = e^{2J/kT}y_1 . \tag{90}$$

From (82) it follows

$$\langle \pi_0 \rangle = \frac{\mu_1(\mu_2 + z)}{\mu_1\mu_2 + (\mu_1 + \mu_2)z + 1} . \tag{91a}$$

Analogously

$$\langle \pi_1 \rangle = \frac{\mu_2(\mu_1 + z)}{\mu_1\mu_2 + (\mu_1 + \mu_2)z + 1} , \tag{91b}$$

where $\langle \pi_0 \rangle$ is the density of overturned spins of the original lattice and $\langle \pi_1 \rangle$ is the density of overturned spins of the decorating sites. The weighted average density of the overturned spins is given by

$$P(\mu, z) = \frac{2}{\sigma+3}\langle \pi_0 \rangle + \frac{\sigma+1}{\sigma+3}\langle \pi_1 \rangle . \tag{92}$$

In order to solve completely the percolation problem for the decorated Bethe lattice we need to calculate $C(x, \mu, z)$, $D(x, \mu, z)$, and $E(x, \mu, z)$ given by (72). Referring to the elementary cell of Fig. 5 we have

$$\frac{\langle \pi_0\pi_2\pi_5 \rangle}{\langle \pi_0 \rangle} = \frac{e^{2J'-h''}y_1}{y_1 + 1 + e^{J'-h''}(e^{J'}y_1 + 1)} ,$$

from (36) and (90).

$$C(x, \mu, z) = x^2 \frac{\mu\mu_1}{\mu\mu_1 + \mu z + \mu_1 z^2 + z} . \tag{93}$$

In an analogous way we calculate

$$D(x, \mu, z) = \frac{x\mu z + \mu_1 z^2 + z}{\mu\mu_1 + \mu z + \mu_1 z^2 + z} ,$$

$$E(x, \mu, z) = \frac{x\mu_1 z^2}{(\mu_1 + z)[\mu_1\mu_2 + (\mu_1 + \mu_2)z + 1]} .$$

From (80) and (93) the equation for the percolation points is given by

$$\frac{\mu\mu_1}{\mu\mu_1 + \mu z + \mu_1 z^2 + z} = \frac{1}{\sigma} .$$

From Eqs. (88), (89), and (91)–(93) we can find the critical probability $p_c(z)$ as a function of z. In Fig. 6 we have plotted $p_c(z)$ vs z. In the same figure we have plotted $\mu_c(z)$. For $\mu_c = 1$ (zero external field) we find two values z_p and $z_{p'}$ corresponding to $p_c(z_p) < \frac{1}{2}$ and $p_c(z_{p'}) = \frac{1}{2}$.

From the generating function (78) we can derive the percolation probability and the mean cluster size of finite clusters. Because of the asymmetry of the lattice, the sites are not all equivalent, therefore the percolation probability and the mean cluster size depend on the site to which they refer. On the other hand, by using a similar argument given by Broadbent and Hammarsley[39] for the random case it is possible to show that p_c does not depend on it. From the generating function (78) we calculate weighted averages of these different quantities.

In Fig. 7 we give $P_{\downarrow}(1^-, z)$ and $S_{\downarrow}(1^-, z)$ along with $P_{\uparrow}(1^-, z)$ and $S_{\uparrow}(1^-, z)$ for the decorated lattice of coordination number 3. As was pointed out before, there are two percolation points z_p and $z_{p'}$: one below and one above $z_C = e^{-2J/KT_C}$, where T_C is the Curie temperature.

In conclusion, in this paper we have given some methods to solve the site percolation problem for a class of pseudolattices with nearest-neighbors interaction in terms of the solution of the corresponding random case. It seems that there are

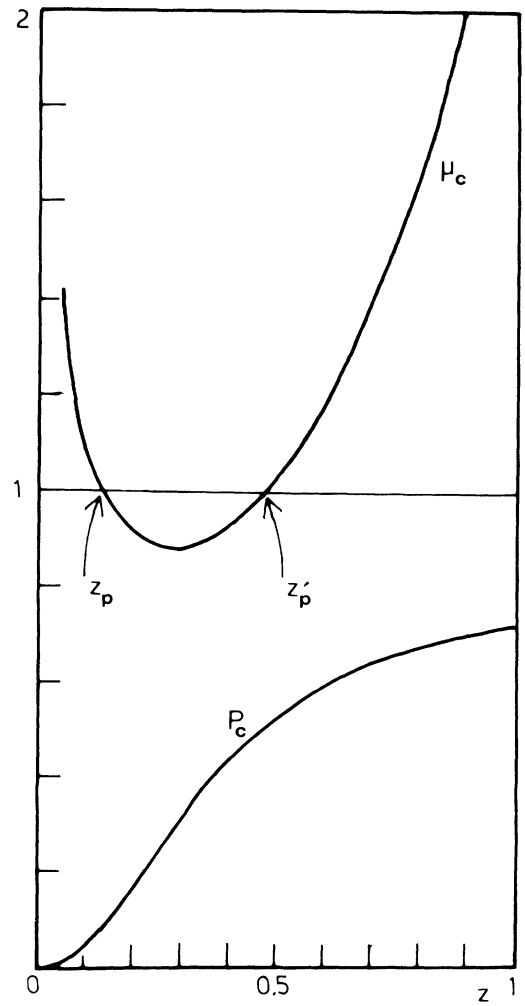

FIG. 6. Upper curve is $\mu_c = e^{-2H_c m/KT}$, where H_c is the critical value of the external magnetic field for percolation versus $z = e^{-2J/KT}$, T is the absolute temperature and J is the nearest-neighbors interaction for the $\sigma = 2$ decorated Bethe lattice [Fig. 4(a)]; z_p and $z_{p'}$ are the percolation temperature for zero external field. The lower curve is p_c, the critical density of overturned spins for percolation. The values of $p_c < p_c(z_p)$ correspond to instability since $\mu_c > 1$.

two common features for all these models: (i) the critical probability is always less than in the corresponding random case; (ii) at zero external magnetic field the critical probability is always less or equal $\frac{1}{2}$. There are arguments[20] which support the idea that in the Ising model these properties are also verified. Further investigations in this direction would be of much interest for a better understanding of phase transitions and the percolation problem.

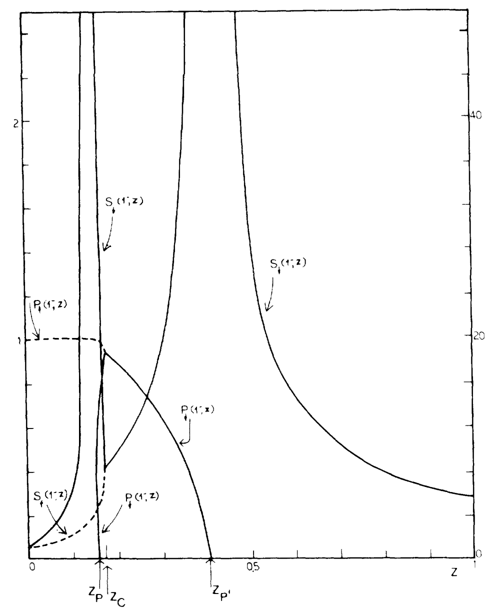

FIG. 7. Broken curves are $P_\uparrow(1^-, z)$ and $S_\uparrow(1^-, z)$, respectively, the percolation probability and the mean cluster size of spins "up" for external magnetic field $H = 0^+$ vs $z = e^{-2J/KT}$, T is the absolute temperature and J the nearest-neighbor interactions, for the $\sigma = 2$ decorated Bethe lattice [Fig. 4(a)]. The solid curves are $P_\downarrow(1^-, z)$ and $S_\downarrow(1^-, z)$, the same quantities relative to spins "down". It has been reported also z_C corresponding to the Curie temperature. For $z \geq z_C$, $P_\uparrow(1^-, z) = P_\downarrow(1^-, z)$ and $S_\uparrow(1^-, z) = S_\downarrow(1^-, z)$. Note two percolation temperatures $z_p < z_c < z_{p'}$. On the left is the scale of $P_\uparrow(1^-, z)$ and $P_\downarrow(1^-, z)$; on the right the scale of $S_\uparrow(1^-, z)$ and $S_\downarrow(1^-, z)$.

Note added in proof. Very recently, A. Coniglio, C. R. Nappi, F. Peruggi, and L. Russo (unpublished) were able to prove rigorously that for a three-dimensional Ising model with ferromagnetic interaction at zero external magnetic field $p_c \leq \frac{1}{2}$, while for a two-dimensional model $p_c = \frac{1}{2}$. This was conjectured in Ref. 20 and is in agreement with the result given here on the Bethe lattice.

ACKNOWLEDGMENTS

I would like to thank Dr. F. M. Sykes for having drawn my attention to these problems and together with Dr. D. S. Gaunt for many helpful discussions. I also would like to thank Professor C. Domb for his encouragement and kind hospitality at King's College, University of London, where part of this work was done.

APPENDIX

Here we want to give an alternative way of calculating the percolation probability for the same class of pseudolattices considered in this paper. We shall generalize the method adopted by Essam[3] for the random case.

Let us calculate

$$\overline{P}(\mu, z) = 1 - P(\mu, z)\,, \tag{A1}$$

the probability that all open walks from a chosen vertex, supposedly occupied, are of finite length. A walk is said to be open if all the vertices are occupied. From a given vertex there are $\sigma+1$ directions. If $Q(\mu,z)$ is the probability that, supposing the vertex is occupied, all open walks in one direction are finite. Then

$$\overline{P}(\mu, z) = [Q(\mu, z)]^{\sigma+1}\,, \tag{A2}$$

$Q(\mu,z)$ satisfies the following equation:

$$Q(\mu, z) = \frac{\langle \pi_0 \bar{\pi}_1 \rangle}{\langle \pi_0 \rangle} + \frac{\langle \pi_0 \pi_1 \rangle}{\langle \pi_0 \rangle}[Q(\mu, z)]^{\sigma}\,, \tag{A3}$$

where π_0 is the projector relative to the chosen site and π_1 is relative to the site after the first step in a chosen direction. For zero interaction Eq. (A3) becomes

$$Q^0(p) = 1 - p + p[Q^0(p)]^{\sigma}\,; \tag{A4}$$

this equation has been discussed by Essam[3] and it has been shown that the physical solution is $Q^0(p)=1$ for $p \leqslant p_c^0 = 1/\sigma$ and then goes to zero as $p \to 1$. When the interaction is different from zero we have shown that

$$\langle \pi_0 \pi_1 \rangle / \langle \pi_0 \rangle = a(\mu, z)\,, \tag{A5}$$

where $a(\mu,z)$ is given by Eq. (47). Consequently, the solution of (A3) is

$$Q(\mu, z) = Q^0(a(\mu, z))\,, \tag{A6}$$

and from (A1)

$$P(\mu, z) = P^0(a(\mu, z))\,. \tag{A7}$$

Following Essam[3] let us consider now more general branching media in which the branches are finite symmetric multiterminal graphs. A particular class of such branching media are the decorated lattices (see, for example, Fig. 4) considered before in which the branches are finite symmetric two-terminal graphs. An example of a branching medium of three-terminal graphs is given in Fig. 1(b). For a graph with n terminals we define the probabilities $\phi_0, \phi_1, \ldots, \phi_{n-1}$, where ϕ_r is the probability that a chosen terminal supposedly occupied is connected to just r other terminals. Since the procedure is the same as for the noninteracting case, we refer for the details to the original paper.[3] It is found that the generalization of (A3) is

$$Q(\mu, z) = \sum_{r=0}^{n-1} \phi_r(\mu, z)[Q(\mu, z)]^{r\sigma}\,, \tag{A8}$$

and the critical line of percolation points is given by

$$\sum_{r=1}^{n-1} r\phi_r(\mu, z) = \frac{1}{\sigma}\,. \tag{A9}$$

The percolation probability for a given terminal vertex is given by (A1) and (A2).

Let us stress here that the percolation probability is referred to a given terminal which is in general different from the percolation probability corresponding to internal vertices of the graph. In the case of decorated Bethe lattices (branches made of two-terminal graphs) $\phi_0(\mu,z)$ and $\phi_1(\mu,z)$ coincide, respectively, with $D(1,\mu,z)$ and $C(1,\mu,z)$ defined by (70) and (69). The condition (A9) is given by (80).

In the example of Fig. 1(b)

$$\begin{aligned} \phi_0 &= \frac{\langle \pi_0 \bar{\pi}_1 \bar{\pi}_2 \rangle}{\langle \pi_0 \rangle}\,, \\ \phi_1 &= 2\frac{\langle \pi_0 \pi_1 \bar{\pi}_2 \rangle}{\langle \pi_0 \rangle}\,, \\ \phi_2 &= \frac{\langle \pi_0 \pi_1 \pi_2 \rangle}{\langle \pi_0 \rangle}\,, \end{aligned} \tag{A10}$$

where 0, 1, 2 are the vertices of the elementary cell. From (A9) and by considering $\bar{\pi}_i = 1 - \pi_i$ and $\sigma = 1$, the critical line of percolation point is given by

$$\frac{\langle \pi_0 \pi_1 \rangle}{\langle \pi_0 \rangle} = \frac{1}{2}. \tag{A11}$$

For zero interaction we have $p_c^0 = \frac{1}{2}$.

From the Fortuin-Kastleyn-Ginibre inequalities[40]

$$\langle \pi_0 \pi_1 \rangle \geqslant \langle \pi_0 \rangle \langle \pi_1 \rangle = p^2\,. \tag{A12}$$

Define

$$g(\pi, 2) = \langle \pi_0 \pi_1 \rangle / \langle \pi_0 \rangle\,, \tag{A13}$$

where p, the density of overturned spins, and z have been used as independent variables. From (A11) and (A12)

$$p_c(z) \leqslant g(p_c(z), z) = \tfrac{1}{2}\,,$$

which leads to the result, already found for other branching media

$$p_c(z) \leqslant p_c^0\,,\ \forall z\,.$$

[1]H. L. Frisch and J. M. Hammarsley, J. Soc. Ind. Appl. Math. 11, 894 (1963).
[2]V. K. S. Shante and S. Kirkpatrick, Adv. Phys. 20, 325 (1971).
[3]J. W. Essam, in *Phase Transitions and Critical Phenomena* edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 2.
[4]H. Sato, A. Arrott, and R. Kikuchi, J. Phys. Chem. Solids 10, 19 (1959).
[5]R. J. Elliott and B. R. Heap, Proc. R. Soc. A 265, 264 (1962).
[6]G. S. Rushbrooke, R. A. Muse, R. L. Stephenson, and K. Pirnie, J. Phys. C 5, 3371 (1972).
[7]G. S. Rushbrooke and D. J. Morgan, Mol. Phys. 4, 1 (1961).
[8]D. C. Rapaport, J. Phys. C 5, 1830 (1972); 5, 2813 (1972).
[9]C. Domb, in Ref. 3, Vol. 3; J. Phys. C 7, 2677 (1974).
[10]D. Stauffer, Phys. Rev. Lett. 35, 394 (1975); Z. Phys. B 22, 161 (1975).
[11]R. B. Griffiths and J. L. Lebovitz, J. Math. Phys. 9, 1284 (1968).
[12]A. R. Bishop, Prog. Theor. Phys. 53, 50 (1975).
[13]S. Kirkpatrick, Rev. Mod. Phys. 45, 574 (1973), and references cited therein.
[14]P. W. Kastleyn and C. M. Fortuin, J. Phys. Soc. Jpn. Suppl. 26, 11 (1969); Physica 57, 536 (1972).
[15]A. R. Bishop, J. Phys. C 6, 2089 (1973); Prog. Theor. Phys. 52, 1789 (1974).
[16]K. Binder and H. Müller-Krumbhaar, Phys. Rev. B 9, 2328 (1974).
[17]R. Kikuchi, J. Chem. Phys. 53, 2713 (1970).
[18]A. B. Harris, T. C. Lubensky, W. K. Holcomb, and C. Dasgupta, Phys. Rev. Lett. 35, 327 (1975).
[19]H. Müller-Krumbhaar, Phys. Lett. A 50, 27 (1974).
[20]A. Coniglio, J. Phys. A 8, 1773 (1975).
[21]M. E. Fisher and J. W. Essam, J. Math. Phys. 2, 609 (1961); see also M. E. Fisher, in Proceedings of the IBM Scientific Computing Symposium on Combinatorial Problems, 1964 (unpublished) p. 179.
[22]Note that here we have slightly modified the definition of the generating function given in Ref. 20, in order to simplify the formalism for the interacting case
[23]C. Domb, Nature 184, 509 (1959).
[24]C. Domb and M. F. Sykes, Phys. Rev. 122, 77 (1962).
[25]C. Domb, Adv. Phys. 9, 149 (1960).
[26]T. P. Eggarter, Phys. Rev. B 9, 2989 (1974); see also E. Müller-Hartmann and J. Zittartz, Phys. Rev. Lett. 33, 893 (1974).
[27]A site of a cluster is said to be peripherical if it is not surrounded by occupied sites.
[28]This is true only because there are no closed loops.
[29]M. F. Sykes and J. W. Essam, Phys. Rev. 133, A310 (1964).
[30]C. Domb, J. Phys. C 3, 256 (1970).
[31]M. F. Sykes and D. S. Gaunt (private communication).
[32]M. F. Sykes and J. W. Essam, J. Math. Phys. 5, 1117 (1964).
[33]E. Stoll, K. Binder, and T. Schneider, Phys. Rev. B 6, 2777 (1972).
[34]M. E. Fisher, Physics 3, 255 (1967); J. Appl. Phys. 38, 981 (1967).
[35]M. E. Fisher, Rep. Prog. Phys. 30, 615 (1967).
[36]J. W. Essam, J. Math. Phys. 12, 874 (1971).
[37]F. Peruggi, thesis (University of Naples, 1975) (unpublished).
[38]D. Stauffer, Z. Phys. B 22, 161 (1975).
[39]S. R. Broadbent and J. M. Hammersley, Proc. Camb. Philos. Soc. 53, 629 (1957).
[40]C. M. Fortuin, P. W. Kastleyn, and J. Ginibre, Commun. Math. Phys. 11, 790 (1970).