PHYSICAL REVIEW B VOLUME 13, NUMBER 5 1 MARCH 1976

Coupling to anisotropic elastic media: Magnetic and liquid-crystal phase transitions*

Marco A. de Moura
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174
and Departamento de Fisica, † *Universidade Federal de Pernambuco, Recife, 50,000, Brasil*

T. C. Lubensky ‡
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174

Yoseph Imry §
Department of Physics and Institute of Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037
and Department of Physics, Brookhaven National Laboratory, Upton, New York 11973

Amnon Aharony ||
Department of Physics and Institute of Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037
and Bell Laboratories, Murray Hill, New Jersey 09794
(Received 2 September 1975)

A generalization of the Larkin-Pikin-Sak model in which an n-component order parameter is coupled to a general anisotropic elastic continuum is studied using the ϵ expansion. It is found that the fixed-point structure is the same as the isotropic model but that all fixed points are unstable with respect to anisotropic perturbations, independent of external boundary conditions. The compressible smectic-A to smectic-C liquid-crystal transition is also studied. It is found to be unaffected by elastic degrees of freedom and is, therefore, expected to have helium exponents, as previously predicted by de Gennes.

I. INTRODUCTION

The question of what happens to a usual second-order transition when the order parameter is coupled to the elastic degrees of freedom, e.g., when the magnetic exchange interactions depend on separation, has attracted a considerable amount of theoretical interest. This is due to the fact that real systems have finite elastic constants and the modulation of the couplings, e.g., the exchange integrals, by the lattice vibrations may lead to qualitative effects on the phase transition. Thus, Rice[1] and Domb[2] found that whenever the specific heat of the ideal incompressible system diverges at T_c, the second-order transition becomes first order. Mattis and Schultz[3] reached the same conclusion using a different approximation. All of these calculations were, however, open to criticism since most approximations break down close to T_c, i.e., exactly where the new behavior was predicted. Therefore, Fisher[4] formulated a theory based on plausible thermodynamic assumptions of renormalization of critical exponents describing the second-order transition by constrained "hidden variables." Wagner,[5] using a droplet-model picture, advanced arguments for the appearance of a smeared transition. The nonexistence of a first-order transition in the magnetoelastic problem was also featured in the work of Wagner and Swift[6] and others. In 1970, Baker and Essam[7] introduced an exactly soluble mode, which had a vanishing shear elastic constant μ, and found that for the cases of a constant volume and constant positive pressure, the transition remained second order but the critical indices were renormalized in the sense of Fisher.[4] Gunther, Bergman, and Imry[8] later showed that the Baker-Essam model in fact gave a first-order transition at negative pressures and that Fisher's theory could be generalized to yield either a first-order or a renormalized second-order transition for various types of external constraints. The point where the first-order transition becomes second order is a tricritical point, whose critical behavior turned out to be identical to that of the incompressible system (i.e., nonrenormalized). Different conclusions were drawn by Larkin and Pikin, in 1969.[9] They considered a Ginzburg-Landau-like free-energy density for a structural phase transition, coupled to an elastic continuum having finite bulk and shear elastic moduli. In the harmonic approximation, elastic modes could be integrated over exactly. This analysis suggested that the transition would be first order at any finite pressure. It is interesting to note that this result requires a positive value for the shear rigidity modulus μ.

The advent of modern renormalization-group (RG) techniques[10-12] gave one a better handle on phase-transition problems. The ϵ-expansion, i.e., expansion about four dimensions ($\epsilon = 4 - d$), appears to be useful, at least for answering qualitative questions such as the one relating to the order of the transition. Following the work of Wilson,[10] it is now believed that the Ginzburg-Landau-like Hamiltonian considered by Larkin and Pikin is equivalent, as far as its critical behavior is concerned, to that of short-range magnetic Ising-like systems. These can easily be generalized to Heisenberg n-component systems. Using this, Sak[13] studied the renormalization-group recursion relations for the effective spin Hamiltonian obtained from that of Larkin-Pikin after integrating out the elastic modes. This Hamiltonian has an additional pairing term, which was also considered by Rudnick, Bergman, and Imry.[14] The RG treatment of this Hamiltonian[14] for n-component "spins"[13] led to four fixed points: Gaussian, constrained Gaussian (spherical), n component, and constrained n component. The last fixed point, which would lead to Fisher-renormalized critical exponents, is the most stable one for $n=1$, while the n-component fixed point is the most stable one for $n=2, 3, \ldots,$ at $d=3$. However, a Fisher-renormalized transition does not follow at constant pressure for $n=1$, because for $\mu > 0$ the initial value of the new pairing parameter is outside the region of attraction of the aforementioned fixed point. In fact, the RG recursion relations lead to a "runaway" which was interpreted[13] as a first-order transition.

These results appeared to contradict the exact mathematical results of the Baker-Essam model. Although the latter model is admittedly not physical, this apparent contradiction was disconcerting. By using higher-order terms in the definition of the volume, a procedure which is justified only in the harmonic approximation, it has been shown by one of us[15] that the transition in the Larkin-Pikin model (when it is assumed to be harmonic even for large uniform deformation), became second order for pressures $P > P_t = \mu$. It is also second order when the experiment is performed at constant volume. This agreed with the Baker-Essam-model results. However, Bergman and Halperin[16] correctly remarked that for $P > P_t$, both the Larkin-Pikin and the Baker-Essam models are unstable against shear deformations. Thus the second-order transitions for $P > P_t$ pertain in these cases to models which are also mechanically unstable and which thus appear to be physically irrelevant. As long as the harmonic approximation is made around the volume corresponding to the external given pressure and the system has a positive μ, the transition will be first order in the above models. It should be noted, however, that decreasing the effective value of μ will decrease the size of the first-order transition. If the effective μ would decrease to zero as a function of some external parameter, as it does in the harmonic models with increasing P, then the size of the first-order transition would tend to zero at that point. However, around the same point the system becomes unstable with respect to shear deformations, which start to play the role of an additional critical order parameter (together with the original n-component one). Thus, higher-order terms in the shear deformation have to be considered and more work is needed on this two-order-parameter problem. For the case of constant volume, the second-order transition may still be maintained, although the more complete treatment of Bergman and Halperin[16] using Wegner's[17] analysis emphasizes the relevance of surface pinning in this case.

It thus appears that the RG method is extremely useful in attempting to resolve this long-standing problem of magnetoelastic effects on the order of the transition. For example, it helps to clarify that the correct physical result to be drawn from the Baker-Essam model at constant pressure is the appearance of the first-order transition. In this paper we would like to carry the RG treatment one step further and consider more realistic cases where the assumption of elastic isotropy is not justified. We shall thus generalize the Larkin-Pikin-Sak treatment to the case of an anisotropic elastic medium (a typical case is that of cubic symmetry, but lower lattice symmetries are also included). This generalization is relevant because no real crystal is exactly isotropic elastically and because symmetry is well known[12] to be an important relevant variable in critical phenomena. Furthermore, results on the anisotropic Baker-Essam model[18] suggest that lattice anisotropy may be of qualitative importance, even in determining the order of the transition.

We find that for the Ising case, $n=1$, all of the four fixed points mentioned above are unstable against anisotropic elastic perturbations, and that the renormalization-group recursion relations yield a runaway, out of the region in which our calculations can be carried out. This runaway may correspond to a first-order transition, but could also mean other instabilities.[5] It is found for *both* constant pressure and constant volume and is completely independent of the subtleties in the treatment of the volume and pressure which exist in the isotropic case. The understanding of this instability is thus extremely pertinent.

A more complete treatment of these questions in the case of cubic symmetry was independently carried out by Bergman and Halperin,[16] who employ the RG method on the full Hamiltonian, without integrating over the elastic degrees of freedom.

This method enables them to discuss the anomalous elastic properties and to achieve a better physical understanding of the instabilities.[19] In particular, they show the transition is first order in the Ising cubic case. A similar method was used, for a single nonordering parameter, by Halperin, Hohenberg, and Ma[20] and Achiam and Imry.[21] We believe that although our method is less illuminating on the physics of the instabilities, its advantage is in the simplicity with which the instability is obtained for a general elastic symmetry.

We have also applied these considerations to the liquid-crystalline smectic-A-to-smectic-C transition. De Gennes[22] has argued that this transition has a two-component order parameter and that the critical exponents should be the same as for the λ transition in He^4. The smectic-A state has uniaxial symmetry but is unable to support shear stresses. Its elastic Hamiltonian, therefore, differs from that of a uniaxial solid. We show that the coupling of the smectic-C order parameter to elastic degrees of freedom does not affect the transition. In other words, the compressible smectic-A-to-smectic-C transition remains second order with helium exponents for all external constraints (constant volume, pressure, uniaxial stress, etc.).

In Sec. II we shall present the effective Hamiltonian. The elastic degrees of freedom are integrated out, leading to new four-spin terms in the Hamiltonian. These terms are nonanalytic in the wave vector $\vec{k}$ for $k \to 0$, and they are similar to the anisotropic long-range four-spin terms found, for example, in Refs. 5 and 6. In Sec. III, we show that the part which does not have spherical symmetry makes all the fixed points unstable. In Sec. IV, the specific results for the particular case of cubic symmetry are given. In Sec. V, we consider the smectic $A \to C$ transition coupled to the elastic field. Section VI summarizes our results.

II. EFFECTIVE HAMILTONIAN

We start with the most general harmonic elastic Hamiltonian, i.e.,[23]

$$\beta\mathcal{H}_{\rm el} = \frac{1}{2}\int d^dx \sum_{\alpha,\beta,\gamma,\delta=1}^{d} \lambda_{\alpha\beta\gamma\delta}\, e_{\alpha\beta}(\vec{x}) e_{\gamma\delta}(\vec{x}), \qquad (1)$$

where $\{\lambda_{\alpha\beta\gamma\delta}\}$ are the components of the elastic tensor (in units which include β), and where $\{e_{\alpha\beta}\}$ are the components of the strain tensor. These are related to the local displacement vector $\vec{u}(\vec{x})$ via[23]

$$e_{\alpha\beta}(\vec{x}) = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} + \sum_\gamma \frac{\partial u_\gamma}{\partial x_\alpha}\frac{\partial u_\gamma}{\partial x_\beta}\right). \qquad (2)$$

In Sec. III we shall discuss particular forms of the tensor $\lambda_{\alpha\beta\gamma\delta}$.

For simplicity, we start our discussion assuming that the order parameter is an n-component vector $\vec{S}$, whose rigid-lattice critical behavior may be described by an isotropic Ginzburg-Landau-Wilson effective Hamiltonian[10]

$$\beta\mathcal{H}_m = \int d^dx\left[\tfrac{1}{2}(r|\vec{S}|^2 + |\vec{\nabla}\vec{S}|^2) + u|\vec{S}|^4 + O(|\vec{S}|^6)\right], \qquad (3)$$

where r is linear in the temperature and u is a constant. This is justified in cases in which the symmetry of the n-dimensional order-parameter space is uncorrelated to that of the d-dimensional real space. We shall discuss more general cases[24] in Sec. IV.

Accepting the rotational invariance of the order-parameter space, the most general magnetoelastic coupling (to lowest order) is

$$\beta\mathcal{H}_{\rm int} = \int d^dx\, |\vec{S}(\vec{x})|^2 \sum_{\alpha,\beta=1}^{d} g_{\alpha\beta} e_{\alpha\beta}. \qquad (4)$$

We now follow Larkin and Pikin,[9] Sak,[13] and Imry,[15] and separate the homogeneous deformations from the phonon parts of the displacement,

$$\frac{\partial u_\alpha}{\partial x_\beta} = u^0_{\alpha\beta} + \frac{1}{V}\sum_{\vec{k}\neq 0} ik_\beta u_\alpha(\vec{k}) e^{i\vec{k}\cdot\vec{x}}, \qquad (5)$$

V being the volume of the system at equilibrium. The elastic Hamiltonian and the interaction Hamiltonian thus become

$$\beta\mathcal{H}_{\rm el} = \tfrac{1}{2}V \sum_{\alpha,\beta,\gamma,\delta} \lambda_{\alpha\beta\gamma\delta} e^0_{\alpha\beta} e^0_{\gamma\delta} + \frac{1}{2V}\sum_{\vec{k}\neq 0}\sum_{\alpha,\beta} A_{\alpha\beta}(\vec{k}) u_\alpha(\vec{k}) u_\beta(-\vec{k}) \qquad (6)$$

and

$$\beta\mathcal{H}_{\rm int} = \sum_{\alpha,\beta} g_{\alpha\beta} e^0_{\alpha\beta} \int d^dx\, |\vec{S}(\vec{x})|^2 + \frac{1}{V}\sum_{\vec{k}\neq 0}\sum_\alpha B_\alpha(\vec{k}) u_\alpha(-\vec{k}), \qquad (7)$$

where

$$A_{\alpha\beta}(\vec{k}) = \sum_{\gamma,\delta} \lambda_{\alpha\gamma\beta\delta} k_\gamma k_\delta \qquad (8)$$

and

$$B_\alpha(\vec{k}) = -\frac{i}{V}\sum_{\vec{q}} (\vec{S}_{\vec{q}}\cdot\vec{S}_{\vec{k}-\vec{q}}) \sum_\beta g_{\alpha\beta} k_\beta, \qquad (9)$$

with $\vec{S}_{\vec{q}}$ being the Fourier transform of $\vec{S}(\vec{x})$.

The calculation of the partition function now involves integrating over all possible configurations in spin space and in the displacement space. We first leave all $e^0_{\alpha\beta}$ constant, which means keeping the volume and the shape of the sample fixed.

Since we used the harmonic approximation, the integrals over $u_\alpha(\vec{k})$ are Gaussian, yielding

$$\int d^d u(\vec{k}) \exp\left[-\frac{1}{2V}\left(\sum_{\alpha,\beta} A_{\alpha\beta} u_\alpha u_\beta + B_\alpha u_\alpha\right)\right] \sim \exp\left[\frac{1}{2V}\sum_{\alpha,\beta}(A^{-1})_{\alpha\beta} B_\alpha(\vec{k}) B_\beta(-\vec{k})\right]. \quad (10)$$

This may, in turn, be rewritten as a new effective spin Hamiltonian, of the form

$$\beta\mathcal{H}_{\rm eff} = \frac{1}{V^3}\sum_{\vec{k}\neq 0}\sum_{\vec{q},\vec{p}} v(\hat{k})(\vec{S}_{\vec{q}}\cdot\vec{S}_{\vec{k}-\vec{q}})(\vec{S}_{\vec{p}}\cdot\vec{S}_{\vec{k}-\vec{p}}), \quad (11)$$

with $\hat{k} = \vec{k}/|\vec{k}|$, and

$$v(\hat{k}) = -\frac{1}{2}\sum_{\alpha,\beta;\gamma,\delta}[A^{-1}(\vec{k})]_{\alpha\beta} g_{\alpha\gamma} g_{\beta\delta} k_\gamma k_\delta. \quad (12)$$

This is a function only of the direction of the vector $\vec{k}$, and not of its length, because $A_{\alpha\beta}(\vec{k})$ is of order k^2 [see (8)], and thus A^{-1} is of order k^{-2}. As discussed by Sak,[13] $v(\hat{k})$ reduces to an angle-independent constant for the isotropic case. We shall come back to this case later.

In addition to (11), we have a spin-dependent contribution in the first term of Eq. (7). However, this is simply a shift in the temperature variable r of Eq. (3).

In summary, our effective spin Hamiltonian at this stage is

$$\beta\mathcal{H} = \frac{1}{2}\int_{\vec{q}} v_2(\vec{q})(\vec{S}_{\vec{q}}\cdot\vec{S}_{-\vec{q}}) + \int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3}\int_{\vec{q}_4}\delta(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4) \times[u+\tilde{v}(\vec{q}_1+\vec{q}_2)](\vec{S}_{\vec{q}_1}\cdot\vec{S}_{\vec{q}_2})(\vec{S}_{\vec{q}_3}\cdot\vec{S}_{\vec{q}_4}), \quad (13)$$

where $\int_{\vec{q}} \equiv (2\pi)^{-d}\int d^d q$, with $|\vec{q}| < \Lambda$,

$$\tilde{v}(\vec{k}) = \begin{cases} 0 & \text{if } \vec{k} = 0 \\ v(\hat{k}) & \text{if } \vec{k}\neq 0, \end{cases} \quad (14)$$

and

$$v_2(\vec{q}) = r + q^2. \quad (15)$$

At this point we can also integrate over all possible values of $e^0_{\alpha\beta}$. This would correspond to the constraint of constant stress. Again, the integrals are Gaussian, and the result is simply a shift of the parameters u and $\tilde{v}$ in Eq. (13) by q-independent constants.[15] As we shall see, our results for the anisotropic cases will turn out to be independent of these constants. Thus, our discussion will apply to all types of constraints.

III. RENORMALIZATION GROUP

We are now ready to perform renormalization-group iterations on the effective "spin" Hamiltonian (13). This involves integrating (in the partition function) over $\vec{S}_{\vec{q}}$, $\Lambda/b < \vec{q} < \Lambda$ $(b > 1)$, and rescaling the remaining spin and space variables so that the new Hamiltonian, as a function of these remaining variables, has the form (13). For momenta in the range $\Lambda/b < |\vec{q}|$, the function $\tilde{v}(\vec{q})$ is analytic. If we have a term with $\tilde{v}(\vec{q}+\vec{q}_1)$, where $|\vec{q}_1| < \Lambda/b$ is the momentum of a spin variable which is not integrated over, then we may assume that $|\vec{q}_1| \ll |\vec{q}|$ (we are only interested in the leading behavior for very small momenta, or long waves) and expand

$$\tilde{v}(\vec{q}+\vec{q}_1) \simeq v(\hat{q}) + O(\vec{q}_1/|\vec{q}|), \quad |\vec{q}_1| \ll |\vec{q}|. \quad (16)$$

This approximation will be used whenever $|\vec{q}| < \Lambda/b$ and $|\vec{q}| > \Lambda/b$. After integration over $\vec{q}$, terms of order $\vec{q}_1/|\vec{q}|$ vanish leaving only correction terms of order q_1^2. The contribution of these terms to the four-spin couplings u and $\tilde{v}$ are irrelevant and will, therefore, be ignored in Eqs. (18) and (19) below. The actual recursion relations near four dimensions involve a perturbative expansion in u and $\tilde{v}$, which leads to diagrammatic integrals contributing to the new values of $v_2(\vec{q})$, u, and $\tilde{v}(\vec{k})$. Using the standard choice of the spin rescaling factor,[10-12] these recursion relations evaluated to the order necessary to give exponents to second order in ϵ are

$$v_2'(\vec{q}) = b^{2-\eta}\Big(v_2(b^{-1}\vec{q}) + \int_p^> [4(n+2)u + 8\tilde{v}(\vec{p}+b^{-1}\vec{q})]v_2(\vec{p})^{-1} - 32\int_{\vec{p},\vec{k}}^> \{(n+2)[u^2+2u\tilde{v}(\vec{p}+\vec{k})] + n\tilde{v}^2(\vec{p}+\vec{k}) + 2\tilde{v}(\vec{p}+\vec{k})\tilde{v}(\vec{p}+b^{-1}\vec{q})\}v_2(\vec{p})^{-1}v_2(\vec{k})^{-1}v_2(\vec{p}+\vec{k}+b^{-1}\vec{q})^{-1}\Big), \quad (17)$$

$$u' = b^{\epsilon-2\eta}\Big(u - 4\int_{\vec{p}}^> [(n+8)u^2 + 12uv(\hat{p}) + 4v^2(\hat{p})]v_2(\vec{p})^{-2} + 64\int_{\vec{p},\vec{k}}^> [(5n+22)u^3 + 3(4n+11)u^2 v(\hat{k}) + 27u^2 v(\hat{p}) + (7n+8)uv^2(\hat{k}) + 36uv(\hat{k})v(\hat{p}) + 2(n+5)uv^2(\hat{p}) + 2nv^3(\hat{k}) + 8v^2(\hat{k})v(\hat{p}) + 8v(\hat{k})v^2(\hat{p})]v_2(\vec{p})^{-1}v_2(\vec{k})^{-2}v_2(\vec{p}+\vec{k})^{-1}\Big), \quad (18)$$

and

$$v'(\hat{q}) = b^{\epsilon-2\eta}\Big(v(\hat{q}) - 4\int_{\vec{p}}^{>} [2(n+2)uv(\hat{q}) + 4v(\hat{q})v(\hat{p}) + nv^2(\hat{q})]v_2(\vec{p})^{-2}$$
$$+64v(\hat{q})\int_{\vec{p},\vec{k}}^{>} [3(n+2)u^2 + 2(n+2)uv(\hat{k}) + 4(n+2)uv(\hat{p}) + nv^2(\hat{k})$$
$$+4v(\hat{k})v(\hat{p}) + 2(n+1)v^2(\hat{p})]v_2(\vec{p})^{-1}v_2(\vec{k})^{-2}v_2(\vec{p}+\vec{k})^{-1}\Big), \quad (19)$$

where $\int_{\vec{p}}^{>}$ means an integration over $\Lambda/b<|\vec{p}|\Lambda$. In the last two equations, we have used the simplifying assumption (16), since parts in the quartic coefficient which vanish for vanishing momentum are irrelevant. For simplicity, we have written Eqs. (17)–(19) for the case in which there is a center of inversion. Our final results, however, do not depend on this simplification. We have also left out contributions to these recursion relations which are not proportional to $\ln b$.[25] η is the usual exponent describing deviations from Ornstein-Zernicke behavior. For isotropic fixed points, $v(\hat{k})$ is a constant, v_0^*, and $u=u^*$, and η follows from Eq. (17),

$$\eta = 8K_d^2(n+2)(u^*+v_0^*)^2, \quad (20)$$

where

$$K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2).$$

In the isotropic case, Eqs. (18) and (19) are easily solved to yield four fixed points which were found previously by Sak[13] and by Rudnick, Bergman, and Imry[14]:

Gaussian: $u_G^* = v_{0G}^* = 0;$ (21)

spherical: $u_S^* = -v_{0S}^* = \epsilon\Lambda^\epsilon/4nK_d;$ (22)

n-component Heisenberg:

$$u_H^* = \frac{\epsilon\Lambda^\epsilon}{4K_d(n+8)}\left(1+\frac{3(3n+14)}{(n+8)^2}\epsilon\right), \quad v_{0H}^* = 0; \quad (23)$$

renormalized: $u_R^* = u_H^* + v_{0R}^*$,

$$v_{0R}^* = \frac{\epsilon\Lambda^\epsilon}{4K_d n(n+8)}\left((n-4)+\frac{(n+2)(13n+44)}{(n+8)^2}\epsilon\right). \quad (24)$$

(Note that our notation differs from that of Sak, via $u_{\rm Sak} = u + v_0$, $v_{\rm Sak} = -v_0$.)

Our main interest in this paper has to do with the effects of anisotropy or with the $\hat{q}$-dependent part of $v(\hat{q})$. We first note that Eq. (19) may be written in the form

$$v'(\hat{q}) = b^{\epsilon-2\eta}\{v(\hat{q}) - 4A(b)[2(n+2)uv(\hat{q}) + 4\langle v\rangle v(\hat{q}) + nv^2(\hat{q})] + 64C(b)v(\hat{q})\langle L(u,v)\rangle\}, \quad (25)$$

where

$$A(b) = \int_{\vec{k}}^{>} v_2(\vec{k})^{-2}, \quad (26a)$$

$$C(b) = \int_{\vec{p},\vec{k}}^{>} v_2(\vec{p})^{-1}v_2(\vec{k})^{-2}v_2(\vec{p}+\vec{k})^{-1}, \quad (26b)$$

$$\langle v\rangle = A^{-1}\int_{\vec{k}}^{>} v(\hat{k})v_2(\vec{k})^{-2}, \quad (26c)$$

and $\langle L(u,v)\rangle$, which can be obtained easily from Eqs. (19), (25), and (26) are $\hat{k}$-independent functions. Since all the coefficients in Eq. (25) are $\hat{q}$ independent, it follows *that the only fixed-point values of $v(\hat{q})$ are those given in Eqs. (21)–(24), namely, $\hat{k}$-independent ones*. This result is quite general and is independent of the symmetry of the elastic tensor or of the explicit structure of the functions $v(\hat{q})$ and $v_2(\vec{q})$.

We next study the stability of the fixed points (21)–(24) with respect to $\hat{q}$-dependent perturbations to $v(\hat{q})$. Near each of these fixed points, we can write v_2 in the form (15) with $r=O(\epsilon)$. Thus, the leading logarithmic terms are $A(b)=K_d\ln b$ and $C(b)=\frac{1}{2}K_4^2\ln b$, and $\langle v\rangle$ is simply the average of $v(\hat{q})$ over all directions of $\hat{q}$. Equation (25) thus yields

$$\langle v\rangle' = b^{\epsilon-2\eta}\{\langle v\rangle - 4K_d\ln b\,[2(n+2)u\langle v\rangle + 4\langle v\rangle^2 + n\langle v^2\rangle] + 32K_4^2\ln b\,\langle v\rangle\langle L(u,v)\rangle\} \quad (27)$$

and

$$\Delta v' = (v(\hat{q}) - \langle v\rangle)'$$
$$= b^{\epsilon-2\eta}\{\Delta v - 4K_d\ln b\,[2(n+2)(u+\langle v\rangle)\Delta v + n(\Delta v)^2 - n\langle(\Delta v)^2\rangle] + 96(n+2)K_4^2\ln b\,[(u+\langle v\rangle)^2 + O(\Delta v)]\Delta v\}. \quad (28)$$

For small Δv, this may be rewritten as $\Delta v' = b^{\lambda_{\Delta v}}\Delta v$, with

$$\lambda_{\Delta v} = \epsilon - 2\eta - 8K_d(n+2)(u+\langle v\rangle) + 96(n+2)K_4^2(u+\langle v\rangle)^2. \quad (29)$$

Substituting the values $u=u^*$ and $\langle v\rangle = v_0^*$, we find near the Gaussian and spherical fixed points $\lambda_{\Delta v} = \epsilon$, whereas near both the Heisenberg and renormalized fixed points, we find

$$\lambda_{\Delta v}=\frac{4-n}{n+8}\epsilon-\frac{(n+2)(13n+44)}{(n+8)^3}\epsilon^2+O(\epsilon^3)$$
$$=\frac{\alpha_H}{\nu_H}\equiv\frac{|\alpha_R|}{\nu_R} \qquad (30)$$

where $\alpha_H[\alpha_R=-\alpha_H/(1-\alpha_H)]$ and $\nu_H[\nu_R=\nu_H/(1-\alpha_H)]$ are the Heisenberg (renormalized) specific-heat and correlation-length critical exponents. Near the Heisenberg fixed point, this exponent is the same as that of the parameter v_0 itself. Indeed, general arguments show[13] that in that case this exponent should be related to the specific-heat exponent. We thus conclude that the "Heisenberg" fixed point is unstable to magnetoelastic effects whenever $\alpha_H>0$, irrespective of the symmetry of the elastic energy. At three dimensions, one thus concludes that probably only the Ising case ($n=1$) needs further study.

The result is more interesting near the "renormalized" fixed point (24). Although all the thermodynamic exponents at this fixed point are related to those at the Heisenberg fixed point via a Fisher renormalization, this does not apply to $\lambda_{\Delta v}$, which remains the same as for the Heisenberg fixed point. Thus, *both fixed points are unstable with respect to $\hat{k}$-dependent pieces in $v(\hat{k})$, or to asymmetric elastic energies, whenever $\alpha_H>0$.* Note that for $\alpha_H>0$, the renormalized fixed point is stable against $\hat{k}$-independent variations in v_0.

For $\alpha_H>0$ we thus expect any anisotropy, as reflected by Δv, to grow larger, and a crossover to some other type of critical behavior to occur. Since we were unable to find any fixed point with a $\hat{k}$-dependent $v(\hat{k})$, the flow in Hamiltonian space probably exhibits a "runaway," out of the region of validity of our perturbation expansion. To investigate the behavior implied by this runaway, we must integrate the recursion relations until the correlation length is no longer large, and then try to use mean-field-like arguments.[10] As pointed out by Bergman and Halperin, this procedure is rather complicated in our case, owing to the last term in Eq. (13), which may generate complicated forms of $v_2(\vec{q})$ for low symmetries in $v(\hat{k})$. This complication does not occur in Ising cubic case.[16] However, previous experiences with such runaways usually lead to a first-order transition. The actual nature of the transition in the cubic case has recently been investigated by Bergman and Halperin using a different approach.[16] Indeed, they find a microscopic instability which prevents T_c from being reached which thus leads to a first-order transition.

IV. CUBIC CASE

For cubic symmetry, the elastic energy is of the form

$$\mathcal{H}_{el}=\frac{1}{2}\int d^dx\left(C_{11}\sum_\alpha e_{\alpha\alpha}^2+C_{12}\sum_{\alpha\neq\beta}e_{\alpha\alpha}e_{\beta\beta}+\tfrac{1}{2}C_{44}\sum_{\alpha\neq\beta}u_{\alpha\beta}^2\right), \qquad (31)$$

and $g_{\alpha\beta}$ in Eq. (4) must be replaced by $g\delta_{\alpha\beta}$. Thus,

$$A_{\alpha\beta}(\vec{k})=(C_{12}+\tfrac{1}{4}C_{44})k_\alpha k_\beta+(\tfrac{1}{4}C_{44}k^2+fk_\alpha^2)\delta_{\alpha\beta}, \qquad (32)$$

with

$$f=C_{11}-C_{12}-\tfrac{1}{2}C_{44} \qquad (33)$$

measuring the cubic anisotropy. One can now check explicitly that[26]

$$k^2(A^{-1})_{\alpha\beta}=\left(\delta_{\alpha\beta}-\frac{(C_{12}+\frac{1}{4}C_{44})\hat{k}_\alpha\hat{k}_\beta}{(1+Q^2)(\frac{1}{4}C_{44}+f\hat{k}_\beta{}^2)}\right)\Big/(\tfrac{1}{4}C_{44}+f\hat{k}_\alpha^2), \qquad (34)$$

with

$$Q^2=(C_{12}+\tfrac{1}{4}C_{44})\sum_{\alpha=1}^{d}\hat{k}_\alpha^2/(\tfrac{1}{4}C_{44}+f\hat{k}_\alpha^2). \qquad (35)$$

Combining (34) and (12) we thus find

$$v(\hat{k})=-\frac{1}{2}\frac{g^2}{C_{12}+\frac{1}{4}C_{44}}\frac{Q^2}{1+Q^2}. \qquad (36)$$

Clearly, v is $\hat{k}$ independent for $f=0$. In this case, it reduces to the value found by Sak[13] and by Imry.[15] For small values of f,

$$\Delta v=\frac{2g^2f}{(2C_{12}+C_{44})^2}\left(\sum_\alpha\hat{k}_\alpha^4-\frac{3}{d+2}\right)+O(f^2). \qquad (37)$$

We thus expect deviations from Ising, or renormalized Ising behavior to be felt when

$$\frac{2g^2f}{(2C_{12}+C_{44})^2}\xi^{\lambda_{\Delta v}}\sim 1, \qquad (38)$$

where ξ is the correlation length. As we mentioned in Sec. II, the Hamiltonians (3) and (4) apply only in cases in which the symmetry of the spin variables is independent of that of the space variables, and is of rotational invariance. For realistic structural phase transition of cubic crystals, one must consider more general Hamiltonians, allowing coupling terms of the forms ($n=d$)[21]

$$\sum_\alpha S_\alpha^2e_{\alpha\alpha},\quad \sum_\alpha(\vec{S}^2-S_\alpha^2)e_{\alpha\alpha},\quad \sum_{\alpha\neq\beta}S_\alpha S_\beta e_{\alpha\beta}. \qquad (39)$$

In previous work,[24] such terms were studied with emphasis on the zero-momentum parts [i.e., terms like the first one in Eq. (7)]. Since we find that the phonon-generated $\hat{k}$-dependent terms in our analysis are irrelevant near the isotropic Heisenberg ($n=d=3$) fixed point, it seems reasonable to conjecture that the same will hold for the more general terms of Eq. (39). Thus, the result of Ref. 24 would be retained even if these terms were not ignored.

V. SMECTIC-A-TO-SMECTIC-C TRANSITION

The smectic A liquid-crystalline phase consists of parallel planes of long organic molecules with their long axes perpendicular to the planes. The smectic C phase is obtained from the A phase by tilting the molecules at an angle Ψ relative to the normal to the planes. Since the molecules can tilt in any direction in the $(d-1)$-dimensional smectic plane, the order parameter $\vec{S}$ ($|\vec{S}| \equiv \Psi$) has $d-1$ components. Thus, we can describe the A-to-C transition in an incompressible lattice in terms of the Hamiltonian (3), with $n=d-1$. In three dimensions, $\vec{S}$ is a two-component order parameter leading to the prediction that the critical exponents for the A-to-C transition should be heliumlike.[22]

The harmonic elastic Hamiltonian for the smectic-A liquid crystal differs from that for a crystalline solid. The density of a smectic-A liquid crystal can be written as

$$\rho(z)=\sum_n \rho_n \cos n(q_0 z+u_z),$$

where z is the coordinate perpendicular to the planes and $q_0=2\pi/a$, where a is the interplanar spacing. u_z determines the coordinate-system origin. If u_z varies slowly in space, it gives rise to an elastic[27,28] Hamiltonian

$$\mathcal{H}_{\mathrm{el}}=\frac{1}{2}\int [B(e_{zz})^2+K_1(\nabla_\perp^2 u_z)^2], \tag{40}$$

where $\nabla_\perp$ us a gradient perpendicular to the z axis and e_{zz} is given by Eq. (2). $(\nabla_\perp u_z)^2$ terms are prohibited in this Hamiltonian because they correspond to a rigid rotation of the entire system which must leave the energy invariant. This somewhat unusual elastic energy leads to a fluctuation destruction of long-range order in three-dimensional smectic-A liquid crystals.[28,29] The order parameter is coupled to uniaxial compressions via

$$\mathcal{H}_{\mathrm{int}}=g\int d^dx\,|\vec{S}(x)|^2 e_{zz}. \tag{41}$$

The existence of this coupling has been demonstrated by experiments in which a smectic-A-to-smectic-C transition was induced by uniaxial stress.[29]

We will now analyze the critical properties of an n-component order parameter coupled via Eq. (41) to an elastic continuum with an elastic Hamiltonain given by Eq. (40). We will then discuss the particular case of $n=2$ and $d=3$ describing the A-to-C transition. The strain field $e_{\alpha z}$ can be decomposed into a uniform part and a phonon part as in Eq. (5), and the phonon part can be removed as in Sec. II. If e_{zz}^0 is kept constant (i.e., if the length along the z axis is fixed), we find

$$\tilde{v}(k)=\begin{cases}0 & \text{if } k=0\\ -\frac{1}{2}g^2\dfrac{k_z^2}{Bk_z^2+K_1k_\perp^4}\equiv v(\vec{k}) & \text{if } k\neq 0.\end{cases} \tag{42}$$

Note that $v(\vec{k})$ is now a function of both the direction and magnitude of $\vec{k}$. It is convenient to write $v(\vec{k})$ as follows:

$$v(\vec{k})=-\tfrac{1}{2}g^2/B+w(\vec{k}), \tag{43}$$

with

$$w(\vec{k})=\tilde{w}k^2\frac{\sin^4\theta}{B\cos^2\theta+K_1\sin^4\theta}, \tag{44}$$

where $\tilde{w}=\frac{1}{2}K_1g^2/B$ and θ is the angle between $\vec{k}$ and the z axis. In the Appendix, we show that $w(\vec{k})$ is a distribution in the small-k limit, which behaves like

$$w(\vec{k})=\tilde{w}\frac{\pi k}{(BK_1)^{1/2}}\delta(\cos^2\theta)+O\left(\frac{k^2}{BK_1}\right). \tag{45}$$

The second term in this equation is clearly more irrelevant that the first. The first term rescales under the renormalization group as

$$\tilde{w}'=b^{\epsilon-1}\tilde{w}. \tag{46}$$

Thus $\tilde{w}$ is irrelevant for $\epsilon>1$, and there are no relevant anisotropic quartic potentials above three dimensions. Therefore for $d>3$, phase transitions of an n-component order parameter coupled to smectic-A-like elastic continuum with e_{zz}^0 held constant are identical to phase transitions with the order parameter coupled to an isotropic elastic continuum. In particular, if the specific-heat exponent α for the incompressible lattice is negative, there will be a continuous transition with n-component Heisenberg exponents; if α is positive, there will be a first-order transition.[13] At $d=3$, there may be corrections to this picture arising from the marginally relevant potential $\tilde{w}$. If, on the other hand, e_{zz}^0 is eliminated, i.e., if uniaxial stress rather than strain is held constant, a different situation occurs. In this case, we can replace $\tilde{v}$ by $w(k)$ for $k\neq 0$ and zero for $k=0$, and we can replace u in Eq. (13) by $u-\frac{1}{2}g^2/B$. In other words, there is no new relevant four-point potential coming from interactions with the lattice, and the transition will be sharp with n-component Heisenberg exponents for all n. (This case is in fact equivalent to the Baker-Essam model.[7]) Thus, within this model for $n = 1$ and $d = 3$ there would be a first-order transition at constant strain and a second-order transition at constant stress.

The elastic Hamiltonian in Eq. (40) describes the energy due to changes in the single component u_z of the displacement variable $\vec{u}$. The complete Hamiltonian should include the energy resulting from the changes in the components of $\vec{u}$ perpendicular to z as well. The elastic Hamiltonian for uniaxial systems, such as smectic-A liquid crystals, that can support different stresses along the two symmetry directions but cannot support shear has been derived by Martin, Pershan, and Swift.[30] We have considered the effect of coupling an n-component order parameter to an elastic continuum with this Hamiltonian. Our results are essentially the same as for the simpler Hamiltonian Eq. (40). If stress along the z and perpendicular directions is held constant, $\tilde{v}$ has only irrelevant parts above three dimensions, and there is a second-order transition with Heisenberg exponents. For any other external conditions such as length in both the z and perpendicular directions or constant isotropic pressure, the critical behavior is the same as for an isotropic medium above three dimensions. At three dimensions a new marginally relevant variable appears in all cases as before.

The specific-heat exponent for two-component systems in three dimensions is very close to zero and almost certainly negative.[31] We would, therefore, expect that the smectic-A-to-smectic-C transition in three dimensions should by unaffected by lattice compressibility. In other words, we expect the critical exponents to be heliumlike as previously predicted.[27] (This statement must be tempered somewhat because we do not know exactly what effect the marginally relevant operator $\tilde{w}$ will have on exponents, though it will presumably only lead to logarithmic corrections to scaling.) If, however, the smectic-C order parameter could be converted for $n = 2$ to an $n = 1$ order parameter by externally imposing a preferred tilt direction, the transition would be first order or second order depending on boundary conditions. Unfortunately, it is difficult to conceive of an external mechanism for aligning the molecules (such as a magnetic field) that would not also alter the elastic Hamiltonian. Furthermore, the aligning energies that can be produced by external fields in liquid crystals are so small that the crossover between XY and Ising behavior would occur very close to the phase transition.

VI. SUMMARY AND CONCLUSIONS

In this paper, we have considered a generalization of the Larkin-Pikin-Sak model[9,13] in which an n-component order parameter is coupled to a general anisotropic (rather than to an isotropic) elastic continuum. We integrated out the elastic degrees of freedom and obtained an effective spin Hamiltonian with an anisotropic four-spin potential. The anisotropy of this potential is a reflection of the anisotropy of the underlying elastic continuum. We then analyzed this effective Hamiltonian using the ϵ expansion and found that all fixed points are unstable with respect to the anisotropic part of the four-spin potential. This "runaway" is completely independent of external boundary conditions (constant pressure, constant volume, pinned boundaries) and might, therefore, be interpreted to represent some microscopic instability of the system. This interpretation is in agreement with the more complete analysis of Bergman and Halperin,[16] which shows that in cubic systems, a microscopic phonon mode has a sound velocity which vanishes at a temperature above the transition temperature in an ideal incompressible lattice. We have also studied the smectic-A-to-smectic-C transition on a compressible lattice and found that this transition should remain heliumlike regardless of boundary conditions.

Two different approaches have been used to study phase transitions on compressible anisotropic lattices. In this paper, following the work of Larkin and Pikin[9] and of Sak,[13] we consider an effective spin Hamiltonian from which phonon degrees of freedom have been removed. Bergman and Halperin[16] consider the complete coupled spin-elastic Hamiltonian and renormalize elastic constants and spin-phonon coupling constants as well as the usual two- and four-spin potentials. There are certain advantages and disadvantages to both techniques. The advantage of the method presented here is its simplicity. With very little effort, we are able to show that all fixed points show a runaway with respect to anisotropy. Furthermore, we are able to treat general, rather than just cubic, anisotropy. On the other hand, we cannot provide any sound physical interpretation for the runaway, nor can we say anything about the behavior of elastic constants near the phase transition. Bergman and Halperin are able to discuss the behavior of the elastic constants and to show that the instability is microscopic in nature and will lead to a first-order transition, presumably to an inhomogeneous state.

We close with the observation that in practice it may be very difficult to see any evidence of the anisotropic runaway discussed here. This is because the crossover exponent is equal to α, which is always quite small.

ACKNOWLEDGMENTS

We have all benefited from discussions with D. J. Bergman and B. I. Halperin. We would also like to acknowledge the hospitality of the following people and institutions: S. Ma and UCSD (A. A. and Y. I.), B. I. Halperin and Bell Laboratories (A. A.), the physics department of BNL (Y. I.), and the physics department of the University of Pennsylvania (M. A. M.).

APPENDIX

In this appendix, we will derive Eq. (45). Let $f(\theta)$ be any integrable function which satisfies the inversion symmetry of the nematic state: $f(\theta) = f(\pi-\theta)$. We are interested in integrals of the form

$$I_\epsilon = \int_0^\pi d\theta\, \Delta_\epsilon(\theta) f(\theta) \tag{A1}$$

in the limit that ϵ goes to zero, where

$$\Delta_\epsilon(\theta) = \frac{1}{\pi}\frac{\epsilon \sin^4\theta}{\cos^2\theta + \epsilon^2 \sin^4\theta} \equiv \sin^4\theta\left(\frac{1}{\pi}\frac{\epsilon}{\cos^2\theta+\epsilon^2} + R(\theta)\right). \tag{A2}$$

It is straightforward to verify that

$$\int_0^\pi R(\theta) f(\theta)\, d\theta = O(\epsilon^2). \tag{A3}$$

Thus to evaluate I_ϵ to order ϵ, we can replace $\epsilon^2 \sin^4\theta$ by ϵ^2 in the denominator of $\Delta_\epsilon(\theta)$. $\sin^4\theta f(\theta) \equiv g(\theta)$ can be expanded in a Fourier series,

$$g(\theta) = \sum_n [g_{1n}\cos 2n\theta + g_{2n}\sin(2n+1)\theta]. \tag{A4}$$

In what follows, it will be convenient to express $g(\theta)$ in a slightly different form,

$$g(\theta) = \sum_n (g_{1n}\cos 2n\theta + \tilde{g}_{2n}\sin^{2n+1}\theta). \tag{A5}$$

$\tilde{g}_{2n}$ can be related to the g_{2k}'s using standard trigonometric identities. Integrals of the individual Fourier components are easily evaluated:

$$\int_0^\pi \frac{1}{\pi}\frac{\epsilon}{\cos^2\theta+\epsilon^2}\cos 2n\theta\, d\theta = \frac{(-1)^n}{(1+\epsilon^2)^{1/2}}[1+2\epsilon^2 - 2\epsilon(1+\epsilon^2)^{1/2}]^n = (-1)^n[1-2n\epsilon + O(\epsilon^2)], \tag{A6}$$

$$\int_0^\pi \frac{1}{\pi}\frac{\epsilon}{\cos^2\theta+\epsilon^2}\sin^{2n+1}\theta\, d\theta = \int_{-1}^1 \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}(1-x^2)^{2n}\, dx = 1 + \frac{2\epsilon}{\pi}\sum_{p=1}^{2n}(-1)^p \binom{2n}{p}\frac{1}{2p-1} + O(\epsilon^2). \tag{A7}$$

We, therefore, have

$$I_\epsilon = \sum_n [(-1)^n g_{1n} + \tilde{g}_{2n}] - \epsilon\sum_n (-1)^n 2n g_{1n} + \frac{2\epsilon}{\pi}\sum_{n=1}^{\infty}\sum_{p=1}^{2n}(-1)^p\binom{2n}{p}\frac{1}{2p-1}\tilde{g}_{2n} + O(\epsilon^2). \tag{A8}$$

Since

$$g(\pi/2) = \sum_n [(-1)^n g_{1n} + \tilde{g}_{2n}] \equiv f(\pi/2),$$

this is equivalent to

$$\Delta_\epsilon(\theta) = \delta(\cos^2\theta) + \epsilon\hat{\Delta} + O(\epsilon^2), \tag{A9}$$

where $\hat{\Delta}$ is the operator which yields the order ϵ term in Eq. (A8). This is the same as Eq. (45) in the text.

---

*Support for this work is as follows: M.A.M.–partial support from Conselho Nacional de Pesquisas and Banco Nacional de Desenvolvimento Economico (Brazilian government); T.C.L.—partial support from the National Science Foundation under Grant DMR-72-03146; Y.I. and A.A.—support for work at the University of California, San Diego, from the National Science Foundation under Grant GP-38627X; Y.I.—work at Brookhaven performed under the auspices of the U.S. Energy Research and Development Administration.

†Permanent address.

‡Alfred P. Sloan Research Fellow.

§On leave of absence from Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel.

‖Present address: Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel.

[1]O. K. Rice, J. Chem. Phys. 22, 1535 (1954).

[2]C. Domb, J. Chem. Phys. 24, 783 (1956).

[3]D. C. Mattis and T. D. Schultz, Phys. Rev. 129, 175 (1963).

[4]M. E. Fisher, Phys. Rev. 176, 257 (1968).

[5]H. Wagner, Phys. Rev. Lett. 25, 31 (1970).

[6]H. Wagner and J. Swift, Z. Phys. 239, 182 (1970).

[7]G. A. Baker, Jr. and J. W. Essam, Phys. Rev. Lett. 24, 447 (1970).

[8]L. Gunther, D. J. Bergman, and Y. Imry, Phys. Rev. Lett. 27, 558 (1971).

[9]A. I. Larkin and S. A. Pikin, Zh. Eksp. Teor. Fiz. 56, 1664 (1969) [Sov. Phys.—JETP 29, 891 (1969)].

[10]K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974).

[11]S. K. Ma, Rev. Mod. Phys. 45, 589 (1973).

[12]A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, to be published), Vol. 6.

[13]J. Sak, Phys. Rev. B 10, 3957 (1974).
[14]J. Rudnick, D. J. Bergman, and Y. Imry, Phys. Lett. A 46, 449 (1974).
[15]Y. Imry, Phys. Rev. Lett. 33, 1304 (1974).
[16]D. J. Bergman and B. I. Halperin (unpublished).
[17]F. J. Wegner, J. Phys. C 7, 2109 (1974).
[18]Y. Imry, O. Entin Wohlman, and D. J. Bergman, J. Phys. C 6, 2846 (1973); Y. Imry, D. J. Bergman, and O. Entin Wohlman, AIP Conf. Proc. 18I, 771 (1974); D. J. Bergman and Y. Imry (unpublished results).
[19]The elastic instabilities were recently considered from the point of view of Le Chatelier's principle by Z. Friedman and L. Gunther, Phys. Rev. B 12, 5123 (1975).
[20]B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. B 10, 139 (1974).
[21]Y. Achiam and Y. Imry, Phys. Rev. B 12, 2768 (1975).
[22]P. G. de Gennes, C. R. Acad. Sci. B 274, 758 (1970); Mol. Cryst. Liq. Cryst. 21, 49 (1973).
[23]L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Addison-Wesley, Reading, Mass., 1959).
[24]A. Aharony and A. D. Bruce, Phys. Rev. Lett. 33, 427 (1974).
[25]A. D. Bruce, M. Droz, and A. Aharony, J. Phys. C 7, 3673 (1974).
[26]For a similar calculation, see A. Aharony and M. E. Fisher, Phys. Rev. B 8, 3323 (1973).
[27]P. G. de Gennes, J. Phys. (Paris) 30, C4, 65 (1969).
[28]A. Caille, C. R. Acad. Sci. B 274, 891 (1972); L. D. Landau and E. M. Lifshitz, *Statistical Physics*, 2nd ed. (Pergamon, Oxford, England, 1969), p. 402.
[29]R. Ribotta, R. B. Meyer, and G. Durand, J. Phys. Lett. (France) 35, L161 (1974).
[30]P. C. Martin, P. S. Pershan, and J. Swift, Phys. Rev. Lett. 25, 844 (1970).
[31]See, e.g., K. H. Mueller, F. Pokell, and G. Ahlers, Phys. Rev. Lett. 34, 513 (1975).