New singularities in the critical behavior of random Ising models at marginal dimensionalities

Amnon Aharony*
Bell Laboratories, Murray Hill, New Jersey 07974
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Renormalization-group equations are exactly solved for the random Ising model with (i) short-range interaction at $d = 4$, and (ii) dipolar interactions at $d = 3$. In both cases, the leading singularities of the susceptibility χ and of the specific heat C are found to be $\chi \propto t^{-1}\exp[(D|\ln t|)^{1/2}]$ and $C \propto -|\ln t|^{1/2}\exp[-2(D|\ln t|)^{1/2}]$ as $t = (T-T_c)/T_c \to 0$. D is a universal constant, equal to 6/53 in case (i) and to $9/[81\ln(4/3)+53]$ in case (ii). Relations between amplitudes of C and of the correlation length, corrections to the leading singularities, crossover effects from the nonrandom region or from the mean-field region to the asymptotic critical region and possible experiments are also discussed.

I. INTRODUCTION

The critical behavior of magnetic systems with (quenched) random impurities has recently been studied by several authors. A general argument, due to Harris,[1] shows that one should expect a new type of critical behavior for the random system, distinct from that of the pure one, whenever the specific heat of the pure system diverges at the transition temperature. This argument was recently corroborated by renormalization-group (RG) studies near four dimensions.[2-5] For m-component spin systems, with isotropic short-range exchange interactions, these studies reveal an instability of the usual Heisenberg-like ("nonrandom") fixed point with respect to the impurities whenever α_m, the pure m-component specific-heat exponent, is positive. For $m>1$, at $d=3$, the Hamiltonian flows under recursion-relation iterations to a new ("random") fixed point, which has a negative α of order $\epsilon=4-d$ (d is the dimensionality). For $m=1$ it was recently shown by Khmel'nitzkii[6] that the random fixed point still exists, but its α is of order $\epsilon^{1/2}$.

In a previous paper,[7] we studied the critical behavior of random magnets with dipolar interactions, for $m>1$. In that case, we found that the "pure" dipolar fixed point is unstable for all m's (even though $\alpha_m<0$), and that the RG flow probably leads to none of the eight fixed points found, which have coupling constants of order ϵ. The case $m=1$ was not treated, since it is expected to have a different behavior.

Studies of the pure dipolar Ising model[8,9] show that it has mean-field-like critical behavior for $d>3$, and logarithmic corrections to this behavior at $d=3$. In fact, the leading logarithmic corrections at $d=3$ are the same as those of the short-range Ising model at $d=4$.[10] More recently, Brézin and Zinn-Justin[11] showed that the corrections to the leading behavior are different in these two cases. Numerically, however, the differences are very small.

The specific heat of the pure dipolar Ising model at $d=3$ diverges as $|\ln t|^{1/3}$, where $t=(T-T_c)/T_c\to 0$.[8] This has recently been confirmed experimentally for $LiTbF_4$ by Ahlers *et al.*[12] Following the general argument by Harris,[1] we would thus expect a new type of critical behavior to occur for the random dipolar Ising model at $d=3$ (and similarly for the random short-range Ising model at $d=4$). This new behavior, for $T>T_c$, is the subject of the present paper. We shall first concentrate on the problem of the random short-range Ising model at $d=4$, and then point out the expected differences for the random dipolar Ising model at $d=3$. The case of the random short-range Ising model at $d=4$ was previously considered, using a different approach, by Khmel'nitzkii.[6] However, he only carried out some of the mathematics involved in solving the RG recursion relations, and did not go on to study the physical consequences of these solutions as regards the singularities of thermodynamic measurable quantities.

There are several reasons for which the present study is of interest. First, the RG recursion relations can be solved *exactly* for marginal dimensionalities ($d=4$ for the short-range case, $d=3$ for the dipolar Ising case, $d=3$ for tricritical points,[10] etc.). Second, experiments can be performed on three-dimensional dipolar Ising systems,[12] so that the exact results can be carefully checked and verified. Such experiments may serve as direct checks of the RG theory. Finally, the nature of the results is quite new and interesting: Until now, all known critical singularities were described by powers either of t or of $\ln t$. For the first time we now find singularities which also involve $\exp[(D|\ln t|)^{1/2}]$. Such factors were mentioned as complex possibilities in a classification of singularities by Fisher,[13] but they have never before been associated with actual experimentally accessible situations.

The general formalism and the RG recursion relations will be described in Sec. II. These recursion relations will be solved for the random short-range Ising model at $d=4$ in Sec. III and for the random dipolar Ising model at $d=3$ in Sec. IV. The experimental consequences will be discussed in Sec. V.

II. HAMILTONIAN AND RECURSION RELATIONS (SHORT-RANGE CASE)

An easy way to study the random quenched problem has been recently suggested by Emery.[14] He proved that the free energy per degree of freedom of the random m-component spin system is equal, in the limit $n\to 0$, to that of an nm-component system with the effective Hamiltonian

$$\bar{\mathcal{H}}_{nm}=\sum_{j=1}^{n}\mathcal{H}_0[\vec{S}_j]-\int d^dx\, g(\vec{\sigma}(\vec{x})^2). \quad (1)$$

Here, $\mathcal{H}_0[\vec{S}]$ is the Hamiltonian of the pure m-component system. For short-range interactions, this may be represented by the Ginzburg-Landen-Wilson Hamiltonian,

$$\mathcal{H}_0[\vec{S}]=-\frac{1}{2}\int d^dx\,[r_0\vec{S}(\vec{x})^2+(\vec{\nabla}\vec{S})^2+2v\,|\vec{S}\,|^4], \quad (2)$$

where r_0 is linear in the temperature T. The function g depends on the nm-component spin vector

$$\vec{\sigma}\equiv\{\vec{S}_1,\ldots,\vec{S}_n\}\equiv(S_{11},\ldots,S_{1m},S_{21},\ldots,S_{nm}), \quad (3)$$

and has the expansion[15]

$$g(y)=\langle\Psi\rangle y+\sum_{j=2}^{\infty}u_{2j}y^j, \quad (4)$$

where $\langle\Psi\rangle$ is the average of the random local shift in the parameter $\frac{1}{2}r$ (or in the local-mean-field critical temperature), and

$$u_{2j}=(-1)^{j-1}(1/j!)\langle\Psi^j\rangle_c, \quad (5)$$

$\langle\Psi^j\rangle_c$ being the jth cumulant of the distribution of random Ψ's.

We can thus apply the standard RG iterations,[16] integrating over the Fourier components of the spins with large momenta and rescaling spin and space coordinates. The coefficients u_{2j}, for $j>2$, are irrelevant,[17] and it is sufficient for our purposes to consider the parameters r [including $\langle\Psi\rangle$ from Eq. (4)], v, and $u\equiv u_4$. In the following discussion we shall be interested only in the Ising case, $m=1$. For this case, the Hamiltonian (1) is the same as that of a ferromagnet with cubic symmetry. The recursion relations for this Hamiltonian were first studied some time ago.[18] Expanding in powers of u and v, these recursion relations are[19]

$$u'=b^{\epsilon-2\eta}\{u-4I_1(b)[(n+8)u^2+6uv]+16I_1(b)^2\times[(n^2+6n+20)u^3+9(n+4)u^2v+27uv^2]+32I_2(b)[(5n+22)u^3+36u^2v+9uv^2]+\cdots\}, \quad (6)$$

$$v'=b^{\epsilon-2\eta}\{v-4I_1(b)(12uv+9v^2)+16I_1(b)^2(36u^2v+54uv^2+27v^3)+32I_2(b)[3(n+14)u^2v+72uv^2+27v^3]+\cdots\}, \quad (7)$$

and

$$r'=b^{2-\eta}\{r+4I_3(b,r)[(n+2)u+3v]+\cdots\}, \quad (8)$$

where b is the space rescaling factor,

$$b^\eta=1+8I_4(b)[(n+2)u^2+6uv+3v^2]+\cdots, \quad (9)$$

and (for short-range interactions, large b and small $\epsilon=4-d$)

$$I_1(b)=\int_{\vec{q}}^{>}q^{-2}\approx K_d\ln b, \quad (10a)$$

$$I_2(b)=2\int_{\vec{q}_1}^{>}\int_{\vec{q}_2}^{>}q_1^{-2}q_2^{-2}(\vec{q}_1+\vec{q}_2)^{-2}\approx K_d^2\ln b(1+\ln b), \quad (10b)$$

$$I_3(b,r)=\int_{\vec{q}}^{>}(q^2+r)^{-1}\approx K_d[\tfrac{1}{2}(1-b^{-2})-r\ln b+O(r^2)], \quad (10c)$$

$$I_4(b)=4b^2\frac{\partial}{\partial q^2}\int_{\vec{q}_1}^{>}\int_{\vec{q}_2}^{>}q_1^{-2}q_2^{-2}(\vec{q}_1+\vec{q}_2+\vec{q})^{-2}\Big|_{\vec{q}=0}\approx K_d^2\ln b, \quad (10d)$$

$$K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(\tfrac{1}{2}d). \quad (10e)$$

The integrals $\int_{\vec{q}}^{>}$ mean $(2\pi)^{-d}\int d^dq$, with $b^{-1}<|\vec{q}|<1$.

To obtain the random Ising model we must eventually let $n\to 0$.[14] In this limit, the combinations of terms of order u^2, uv, and v^2 appearing in Eqs. (6) and (7) become proportional to each other. For $\epsilon>0$ ($d<4$), this leads to a new "random" fixed point with u and v of order $\epsilon^{1/2}$.[6] Note that the terms of order u^3, u^2v, etc., given by Khmel'nitzkii[6] are different from ours. This difference seems to result from the absence of the exponent η in the prefactors of the equations equivalent to (6) and (7) in Ref. 6. This leads to different coefficients in the expansions of the critical exponents in powers of $\epsilon^{1/2}$. Explicitly, at the random fixed point, we find[20]

$$4K_4u^* \simeq -(3\epsilon/106)^{1/2}\,, \quad 4K_4v^* \approx \tfrac{4}{3}(3\epsilon/106)^{1/2}\,, \tag{11}$$

and hence

$$2\nu = 1 + (3\epsilon/106)^{1/2} + O(\epsilon)\,, \quad \eta = -\epsilon/106 + O(\epsilon^{3/2})\,. \tag{12}$$

The nonrandom Ising-like fixed point ($u^* = 0$, $4K_4v^* \approx \epsilon/9$) is unstable with respect to the randomness, and the Hamiltonian always flows to this random fixed point.[5]

III. RANDOM SHORT-RANGE ISING MODEL AT $d = 4$

We now concentrate on the critical behavior of the random Ising model at $d = 4$ ($\epsilon = 0$). In this case, all the three nontrivial fixed points found from Eqs. (6) and (7) at $d < 4^{2-4}$ become degenerate with the Gaussian fixed point, $u^* = v^* = 0$. Both u and v are now marginal operators, since the linear terms in Eqs. (6) and (7) vanish. In the nonrandom case, where $u \equiv 0$, this fact leads to logarithmic corrections to the otherwise mean-field-like behavior.[10] Equation (7) becomes

$$v' = v - 36K_4 \ln b\, v^2 + O(v^3)\,, \tag{13}$$

yielding, for large l,

$$v(l) = [36K_4 \ln b (l + l_0)]^{-1}\,, \quad l_0^{-1} = 36K_4 \ln b\, v(0)\,. \tag{14}$$

Substituting into Eq. (8), this gives

$$\begin{aligned} t(l) &\approx r(l) + 6K_4(1 - b^{-2})v(l) \\ &\approx t(0)b^{2l}(1 + l/l_0)^{-1/3}\,. \end{aligned} \tag{15}$$

If we iterate until $t(l^*) = 1$, and note that the effective correlation length $\xi(l^*) = \xi(0)b^{-l^*}$ also becomes equal to $1 + O(l_0/l)$, then we finally find

$$\chi \propto \xi^2 \propto t^{-1} |\ln t|^{1/3}\,, \tag{16}$$

where χ is the susceptibility and $t = t(0) \propto T - T_c \to 0$.

We now "switch on" a small amount of randomness, i.e., a small value of u [< 0, see Eq. (5)]. To leading order, for $|u| \ll v$, Eq. (6) now reads

$$u' \approx u[1 - 2/3(l + l_0)]\,, \tag{17}$$

which leads to

$$u(l) \approx u(0)(1 + l/l_0)^{-2/3}\,. \tag{18}$$

Thus, $|u(l)|$ decreases with l more slowly than $v(l)$. Eventually, $|u(l)|$ and $v(l)$ will become comparable in magnitude. The value of l at which this happens, l_x, corresponds to a temperature t_x below which one should expect a new, random, type of critical behavior. Comparing (18) with (14), we find that $(1 + l_x/l_0)^{1/3} \simeq v(0)/|u(0)|$. Thus, for large l_x,

$$|\ln t_x| \simeq \tfrac{1}{2} l_x \ln b \simeq \pi^2 v(0)^2/9\, |u(0)|^3\,. \tag{19}$$

From (5), $u(0) = -\frac{1}{2}\langle\Psi^2\rangle_c$. For an impurity with a shift 2Δ in the local value of the mean-field critical temperature and concentration p, this gives[21]

$$u(0) = -\tfrac{1}{2} p(1 - p)\Delta^2\,. \tag{20}$$

Thus,

$$|\ln t_x| \approx 8\pi^2 v(0)^2/9[p(1 - p)\Delta^2]^3\,. \tag{21}$$

For small values of p, this gives a large value of $|\ln t_x|$, or a very small value of t_x. However, the value of $v(0)$ is unknown, and Δ may be large enough so that t_x may still happen to be in a reasonable range for measurements (this, of course, refers to the three-dimensional dipolar Ising case, to be discussed later).

To find the new critical behavior we return to Eqs. (6) and (7), for $n = \epsilon = 0$, and study their solutions. As discussed above, the initial flow will be described by (14) and (18), until u and v are of the same order of magnitude. The terms of order u^2, uv, and v^2 in Eqs. (6) and (7) will be larger than the next-order terms as long as $|4u + 3v|$ is not much smaller than v. In this range, we can neglect the higher-order terms, and write

$$\begin{aligned} u' &\approx u - 8K_4 \ln b\, (4u + 3v)u\,, \\ v' &\approx v - 12K_4 \ln b\, (4u + 3v)v\,. \end{aligned} \tag{22}$$

For large l, these equations give

$$\frac{du}{dv} \approx \frac{2u}{3v} \tag{23}$$

or

$$u(l) \approx u(0)[v(l)/v(0)]^{2/3}\,, \tag{24}$$

in agreement with (18). Thus, the flow in the u-v plane will be on the line $u \sim v^{2/3}$, which will eventually approach the line $4u + 3v = 0$. (See Fig. 1.) When this happens, the approximations involved in Eqs. (22) are no longer valid. In fact, the next terms in (6) and (7) must be included as soon as $4u + 3v$ becomes of order u^2. Once this happens, we can try to substitute into (6) and (7)

$$4u + 3v \approx Au^2\,. \tag{25}$$

Thus, (6) and (7) become (note that all terms of order $\ln^2 b$ exactly cancel)

$$\begin{aligned} u' &\approx u - 8K_4 \ln b\, (A + \tfrac{116}{3}K_4)u^3\,, \\ v' &\approx v + 16K_4 \ln b\, (A + \tfrac{136}{9}K_4)u^3\,. \end{aligned} \tag{26}$$

Eliminating A, using (25), we finally find

$$u' \approx u - \tfrac{32\times 53}{3} K_4^2 \ln b\, u^3 . \tag{27}$$

As already noted by Wegner and Riedel,[22] the solution to (27) is (for large l)

$$u(l) = -(1/8K_4)[\tfrac{53}{3} \ln b\,(l+l_0)]^{-1/2} , \tag{28}$$

with l_0 being again determined by $u(0)$. [This $u(0)$ must however be in a region in which (25) holds.] A solution of the type (28) for our problem was also noted by Khmel'nitzkii,[6] but no practical conclusions were drawn.

A typical flow diagram is thus shown[6] in Fig. 1: For $|u| \ll v$, the flow is described by Eq. (24). As the line $4u+3v=0$ is approached, the flow becomes slower, and turns towards the origin along this line. Note that the same will happen even if $|u(0)| \gg v(0)$: The flow first goes away from the origin, along a line described by (24), and then turns back along $4u+3v=0$. Only for the random Gaussian model, when $v\equiv 0$, will the origin never be reached, and the flow will "run away" along the negative u axis. This situation was discussed in Ref. 5, and may be relevant to tricritical points in random Ising models.

We can now go back to Eq. (8), and draw some physical consequences from the flow behavior.

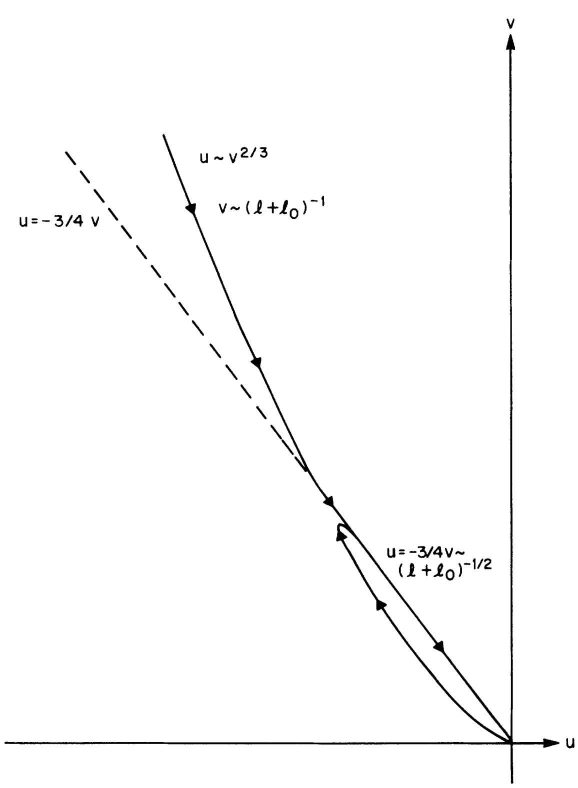

FIG. 1. Examples of the RG iteration flow in the u-v plane for the random case.

Defining

$$t(l) = r(l) + 2K_4(1-b^{-2})[2u(l)+3v(l)] , \tag{29}$$

we now find

$$t' = b^2[t - 4K_4 \ln b\,(2u+3v)t + \cdots]$$
$$\approx b^2\{1 - [3 \ln b/53(l+l_0)]^{1/2}\} t , \tag{30}$$

with the solution

$$t(l) \approx t(0) b^{2l - 2[3(l+l_0)/53 \ln b]^{1/2} + 2(3l_0/53 \ln b)^{1/2}} . \tag{31}$$

Iterating until $t(l^*) = 1$, and using $\xi(l^*) \simeq 1$ we thus find for $t\to 0$

$$\chi \propto \xi^2 \propto t^{-1} \exp[(\tfrac{6}{53} |\ln t|)^{1/2}] . \tag{32}$$

This is to be compared to the nonrandom behavior, Eq. (16). In both cases we find small corrections to the mean-field behavior. However, the correction in the random case increases much faster as $t\to 0$. Note that the coefficient in the exponent in (32), i.e., $\frac{6}{53}$, is universal. It originates directly from the coefficients of the terms of order u^3 in Eqs. (6) and (7).

There are several ways to calculate the singular part of the free energy. The most straightforward one has recently been suggested by Nelson and Rudnick.[23] For $d=4$, their result for the free energy per spin component (in units of k_BT) is

$$F(r,u,v) \approx e^{-4l} F(r(l), u(l), v(l))$$
$$+ \tfrac{1}{2} K_4 \int_0^l \{\ln[1+r(l')] - \tfrac{1}{2}\} e^{-4l'}\, dl' . \tag{33}$$

To write this, one replaces b by $e^{\delta l}$, and transforms to a continuous sequence of renormalization iterations. This procedure gives the correct answer to leading order, even if one used the large-b limit in deriving $r(l)$. For large $l=l^*$, the leading singular term in (33) comes from

$$F_{\text{sing}} = -\tfrac{1}{4} K_4 \int_0^{l^*} r(l)^2 e^{-4l}\, dl$$
$$\propto t(0)^2 (l^*+l_0)^{1/2}$$
$$\times \exp\{-4[\tfrac{3}{53}(l^*+l_0)]^{1/2}\} . \tag{34}$$

Thus, the singular part of the specific heat per spin component behaves as

$$C_{\text{sing}} \propto -|\ln t|^{1/2} \exp[-2(\tfrac{6}{53} |\ln t|)^{1/2}] . \tag{35}$$

The same procedure using the nonrandom result (15) yields the known result[8,10]

$$C_{\text{sing}}^{\text{pure}} \propto |\ln t|^{1/3} . \tag{36}$$

Thus, the specific heat diverges for the nonrandom case, but does not diverge for the random behavior: Equation (35) gives a specific heat which approaches a constant (from below), with a cusp.

Again, this is consistent with the Harris argument.[1]

IV. RANDOM DIPOLAR ISING MODEL AT $d=3$

As mentioned in the Introduction, the leading singularities of the pure dipolar Ising model at $d=3$ are the same as those of the short-range Ising model at $d=4$, but the correction terms are slightly different.[11] We thus turn to formulate the Hamiltonian for the random dipolar Ising case, and to point out its relations to the one discussed above.

The pure Hamiltonian (2) must now be replaced by[9]

$$\mathcal{H}_0[\vec{S}] = -\tfrac{1}{2}\int_{\vec{q}} G(\vec{q})^{-1} S_{\vec{q}} S_{-\vec{q}} - v\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''} S_{\vec{q}} S_{\vec{q}'} S_{\vec{q}''} S_{-\vec{q}-\vec{q}'-\vec{q}''}\,, \tag{37}$$

where $S_{\vec{q}}$ is the Fourier transform of $S(\vec{x})$, $\int_{\vec{q}} \equiv (2\pi)^{-d}\int d^dq$ ($|\vec{q}|<1$), and

$$[G(\vec{q})]^{-1} = r_0 + q^2 + g(q_z/q)^2 \tag{38}$$

[r_0 is again linear in the temperature T]. Combining (37) with (1) and (4) we now have to consider recursion relations for the four parameters r, g, u, and v. The recursion relation for g is exactly[9]

$$g' = b^{2-\eta} g\,, \tag{39}$$

where again $\eta = O(u^2, uv, v^2)$. The parameter g now enters into the recursion relations for r, u, and v through the momentum integrals I_1 to I_4, which were given for the short-range case in Eq. (10). Now, the short-range propagator $(r+q^2)^{-1}$ must be replaced by $G(\vec{q})$ of Eq. (38). Thus, for example,

$$I_1(b) = \frac{K_{d-1}}{2\pi}\int_{b^{-1}}^{1} q^{d-1}dq \int_0^{\pi} \frac{\sin^{d-1}\theta\, d\theta}{(q^2+g\cos^2\theta)^2}\,. \tag{40}$$

For $g \gg 1$, this may be approximated by [9]

$$I_1(b) \approx b_d K_d g^{-1/2}(1-b^{3-d})/(d-3)\,, \tag{41}$$

with

$$b_2 = \tfrac{1}{2}\,,\quad b_3 = \tfrac{1}{4}\pi\,,\quad b_4 = 1\,. \tag{42}$$

Note that this changes only the coefficient I_1, and not the combinatorial combinations

$$(n+8)u^2 + 6uv \quad \text{or} \quad 12uv + 9v^2\,.$$

Near $d=3$ we can replace $(1-b^{3-d})/(d-3)$ by $\ln b$. Similarly,[9]

$$I_3(b,r) \approx b_d K_d[1 - b^{-2} - r\ln b + O(r^2)]\,, \tag{43}$$

$$I_2(b) \approx K_d^2(c_d \ln b + e_d \ln^2 b) g^{-1}\,, \tag{44}$$

and

$$I_4(b) \approx f_d K_d^2 \ln b\, g^{-1}\,. \tag{45}$$

The coefficients c_d, e_d, and f_d are quite difficult to calculate. Fortunately, we can extract them from the calculations of the corrections to the pure-system critical behavior, by Brézin and Zinn-Justin.[11] At $d=3$, their results yield

$$c_3 = (\ln\tfrac{4}{3} + \tfrac{2}{3}) b_3^2\,,\quad e_3 = b_3^2\,,\quad f_3 = \tfrac{8}{9} b_3^2\,. \tag{46}$$

It is important to note that the leading powers of g are always such that we can replace the recursion relations for u, v, and g by new ones, in the variables

$$\tilde{u} = u g^{-1/2}\,,\quad \tilde{v} = v g^{-1/2}\,. \tag{47}$$

The recursion relations for $\tilde{u}$ and $\tilde{v}$ are then exactly similar to (6) and (7), except for the facts that the factor of $b^{\epsilon-2\eta}$ on the right-hand side is replaced by $b^{\epsilon-1-3\eta/2}$ (to leading order), and that the momentum integrals assume new values. Explicitly, at $d=3$ and at $n=0$, these recursion relations are

$$\begin{aligned}\tilde{u}' = \tilde{u} &- 4b_3K_3\ln b\,(8\tilde{u}^2 + 6\tilde{u}\tilde{v}) \\ &+ 16 b_3^2 K_3^2 \ln^2 b\,(20\tilde{u}^3 + 36\tilde{u}^2\tilde{v} + 27\tilde{u}\tilde{v}^2) \\ &+ 32 K_3^2(c_3\ln b + e_3\ln^2 b)(22\tilde{u}^3 + 36\tilde{u}^2\tilde{v} + 9\tilde{u}\tilde{v}^2) \\ &- 12K_3^2 f_3 \ln b\,(2\tilde{u}^3 + 6\tilde{u}^2\tilde{v} + 3\tilde{u}\tilde{v}^2)\end{aligned} \tag{48}$$

and

$$\begin{aligned}\tilde{v}' = \tilde{v} &- 4b_3K_3\ln b\,(12\tilde{u}\tilde{v} + 9\tilde{v}^2) \\ &+ 16 b_3^2 K_3^2 \ln^2 b\,(36\tilde{u}^2\tilde{v} + 54\tilde{u}\tilde{v}^2 + 27\tilde{v}^3) \\ &+ 32K_3^2(c_3\ln b + e_3\ln^2 b)(42\tilde{u}^2\tilde{v} + 72\tilde{u}\tilde{v}^2 + 27\tilde{v}^3) \\ &- 12K_3^2 f_3 \ln b(2\tilde{u}^2\tilde{v} + 6\tilde{u}\tilde{v}^2 + 3\tilde{v}^3)\,,\end{aligned} \tag{49}$$

while the recursion relation for r becomes[9]

$$r' = b^2\{r + 4b_3K_3[1 - b^{-2} - r\ln b + O(r^2)](2\tilde{u} + 3\tilde{v}) + \cdots\}\,. \tag{50}$$

We can now follow the analysis of Sec. III step by step, and recover exactly the same type of results: The "pure" behavior will be described by Eqs. (16) and (36). The crossover from "pure" to "random" behavior will occur at t_x, given by

$$|\ln t_x| \approx \pi[g(0)]^{1/2}[v(0)]^2/9\,|u(0)|^3\,. \tag{51}$$

The flow in the $\tilde{u}$-$\tilde{v}$ plane will still be similar to that shown in Fig. 1. However, the actual coefficients describing the final flow along the line $4u+3v=0$ [$\frac{116}{3}$ and $\frac{136}{3}$ in Eq. (26)] will be different, depending on the parameters b_3, c_3, e_3, and f_3. Finally, Eq. (28) will be replaced by

$$\tilde{u}(l) = -(1/8b_3K_3)[2D^{-1}\ln b\,(l+l_0)]^{-1/2}\,, \tag{52}$$

where

$$D = 9/(81\ln\tfrac{4}{3} + 53)\,. \tag{53}$$

Substituting in (50) we therefore find

$$t(l) = r(l) + 4b_3 K_3[2\tilde{u}(l) + 3\tilde{v}(l)](1 - b^{-2})$$
$$\approx t(0) b^{2l - 2[D(l+l_0)/2\ln b]^{1/2} + 2(Dl_0/2\ln b)^{1/2}} \quad (54)$$

and hence

$$\chi \propto \xi^2 \propto t^{-1} \exp[(D|\ln t|)^{1/2}] . \quad (55)$$

The derivation of the singular part in the free energy also needs a new modifications. Equation (33) is now replaced by[24]

$$F(r,g,u,v) \approx e^{-3l} F(r(l), g(l), u(l), v(l)) + \tfrac{1}{2} K_3 \int_0^l \{\langle \ln[1 + r(l') + g(l') \cos^2\theta] \rangle - \tfrac{1}{2}\} \times e^{-3l'} dl' , \quad (56)$$

where $\langle \cdots \rangle$ means an average over the angles. The most singular term now is

$$F_{\text{sing}} = -\tfrac{1}{16} \pi K_3 \int_0^{l^*} r(l)^2 g(l)^{-1/2} e^{-3l} dl , \quad (57)$$

which leads to

$$C_{\text{sing}} \propto -|\ln t|^{1/2} \exp[-2(D|\ln t|)^{1/2}] . \quad (58)$$

It is most interesting to compare the actual numerical value of the constant D in this case to that of the previous one. From (53), $D \approx 0.117\,95$, whereas for the short-range case at $d=4$ we found $D = \frac{6}{53} \approx 0.113\,21$. Just as in the case of the corrections to the leading terms in Eqs. (16) and (36), calculated in Ref. 11, the numerical difference between the two cases is very small. For practical purposes, we seem to conclude that calculations for the short-range Ising model at $d=4$ are sufficient for obtaining very good estimates of the behavior of the dipolar Ising model at $d=3$.

V. EXPERIMENTAL CONSEQUENCES

The most interesting results of our study are summarized in Eqs. (55) and (58): In the asymptotic random dipolar Ising critical region, experiments should yield critical singularities which involve the factor $\exp[(D|\ln t|)^{1/2}]$, with $D \simeq 0.118$. As in the pure case,[12] it is reasonable to expect that these singularities will be more easily observable in the specific heat (and not in the susceptibility), since in that case they are not superimposed on powers of t.

Some of the difficulties in the measurement of the singularities in the correlation length may be helped by using universal relations between amplitudes.[24,25] From Eqs. (54) and (57) and from the conditions $t(l^*) = 1$ and $\xi = b^{2l^*}$ we easily find that asymptotically

$$\xi^2 \xi_{\parallel} C/k_B = -(1/32\pi)(|\ln t|/D)^{1/2} , \quad (59)$$

where $\xi_{\parallel} = g(0)^{1/2}\xi^2$ is the "longitudinal" correlation length.[24] Note that the exponential singularities, as well as all the nonuniversal parameters, dropped out to leave the universal coefficient $1/32\pi D^{1/2}$. Thus, the amplitudes in Eqs. (55) and (58) are not independent.

In practice, the main difficulty in seeing the singularities (55) or (58) will have to do with crossover effects. First of all, we note that the crossover region from "nonrandom" to "random" behavior may be very wide, owing to the weak divergence of the nonrandom specific heat. Expressed in a different way, the "crossover" temperature t_x, given by Eq. (51) [with Eq. (20)], may practically be quite small. It is difficult to estimate t_x theoretically, owing to the unknown parameter $v(0)$. However, it is clear that a large value of $u(0)$, i.e., large values of p and of Δ in Eq. (20), will help. One should therefore try to look at random mixtures with a large concentration of ions which have very different exchange and dipolar interactions. An explicit integration of our recursion relations along the trajectories shown in Fig. 1 (and not only in the asymptotic range, when $u \simeq -\frac{3}{4}v$) may give a full crossover function, which will facilitate fitting data in the intermediate crossover region. This remains to be done in the future.

Another relevant crossover is the one between the asymptotic critical region and the mean-field region. This crossover is described by correction terms to the leading singularities. Following Larkin and Khmel'nitzkii,[8] we estimate these by not neglecting the terms involving l_0 in Eqs. (54) and (56). Thus, Eq. (55) should really read as

$$\chi \propto \xi^2 \propto t^{-1} \exp\{(2Dl_0)^{1/2}[(1 + |\ln t|/2l_0)^{1/2} - 1]\} . \quad (60)$$

This reduces to (55) for $t \to 0$, and to the mean-field result $\chi \propto \xi^2 \propto t^{-1}$ for $t \simeq 1$. Similarly, apart from analytic terms, Eq. (58) generalizes to become

$$C_{\text{sing}} \propto -\{2[D(|\ln t| + 2l_0)]^{1/2} + 1\} \times \exp\{-2(2Dl_0)^{1/2}[(1 + |\ln t|/2l_0)^{1/2} - 1]\} . \quad (61)$$

It is reasonable to conjecture that similar expressions apply for $T < T_c$. However, the RG treatment of random systems below T_c has yet to be formulated.

In conclusion, it would be very interesting to obtain experimental results on the critical behavior or random dipolar Ising systems, such as $LiTbF_4$ with impurities of nonmagnetic ions (e.g., Y) or ions with a much stronger magnetic interaction. In the former case [$Li(Tb,Y)F_4$], T_c becomes quite low, and experiments become difficult.[26] The latter case may be more promising.

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*Permanent address: Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel.

[1]A. B. Harris, J. Phys. C 7, 1671 (1974).

[2]G. Grinstein and A. H. Luther (unpublished); G. Grinstein, Ph.D. thesis (Harvard University, 1974) (unpublished).

[3]T. C. Lubensky, Phys. Rev. B 11, 3573 (1975).

[4]A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, to be published), Vol. 6.

[5]A. Aharony, Y. Imry, and S. Ma, Phys. Rev. B (to be published).

[6]D. E. Khmel'nitzkii (unpublished).

[7]A. Aharony, Phys. Rev. B 12, 1049 (1975).

[8]A. I. Larkin and D. E. Khmel'nitzkii, Zh. Eksp. Teor. Fiz. 56, 2087 (1966) [Sov. Phys.–JETP 29, 1123 (1969)].

[9]A. Aharony, Phys. Rev. B 8, 3363 (1973); 9, 3946(E) (1974).

[10]F. J. Wegner and E. K. Riedel, Phys. Rev. B 7, 248 (1973).

[11]E. Brézin and J. Zinn-Justin, Saclay report DPh-T/75/27, 1975 (unpublished).

[12]G. Ahlers, A. Kornblit, and H. J. Guggenheim, Phys. Rev. Lett. 34, 1227 (1975).

[13]M. E. Fisher, in *Critical Phenomena*, edited by M. S. Green (Academic, New York, 1971), p. 1, Eq. (2.11).

[14]V. J. Emery, Phys. Rev. B 11, 239 (1975). The same idea was proposed earlier, independently and in another context, by S. F. Edwards, in *4th International Conference on Amorphous Materials*, edited by R. W. Douglas and B. Ellis (Wiley, New York, 1970); and in *Polimer Networks*, edited by A. J. Chompoff and S. Newman (Plenum, New York, 1971). I thank P. W. Anderson for bringing these references to my notice.

[15]See also Ref. 5 and A. Aharony, Phys. Rev. B 12, 1038 (1975).

[16]K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974); S. Ma, Rev. Mod. Phys. 45, 589 (1973); M. E. Fisher, *ibid.* 46, 597 (1974); A. Aharony, Ref. 4.

[17]These coefficients are important, however, in determining the Hamiltonian flow during the first few iterations; see Ref. 5.

[18]A. Aharony, Phys. Rev. B 8, 4270 (1973), and Ref. 4. See also Refs. 2 and 3.

[19]We use here an upper cutoff $|\vec{q}| < 1$, the simplifying assumption of a large rescaling factor b, and related approximations. See A. D. Bruce, M. Droz, and A. Aharony, J. Phys. C 7, 3673 (1974).

[20]Similar results were found by T. C. Lubensky and by V. J. Emery (private communication).

[21]See also Refs. 2 and 5.

[22] Footnote 10 of Ref. 10.

[23]D. R. Nelson and J. Rudnick, Phys. Rev. Lett. 35, 178 (1975), and unpublished. This paper includes references to other papers which used related expressions.

[24]A. Aharony and B. I. Halperin, Phys. Rev. Lett. 35, 1308 (1975).

[25]See D. Stauffer, M. Ferer, and M. Wortis, Phys. Rev. Lett. 29, 345 (1972).

[26]G. Ahlers and A. Kornblit (unpublished).