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Maxwell equations in material form

M. Lax
Physics Department, City College of the City University of New York, New York, New York 10031 and Bell Laboratories, Murray Hill, New Jersey 07974*

D. F. Nelson
Bell Laboratories, Murray Hill, New Jersey 07974
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The Maxwell electromagnetic equations are obtained expressed in the material coordinate system of elasticity theory for a moving, deforming body. They are shown to be form-invariant to the deformation transformation. The transformation laws for the electric field, the electric displacement, the magnetic induction, the magnetic intensity, the charge density, the current density, the vector and scalar potentials, the polarization, and the magentization are found. The boundary conditions on the fields are derived in the material coordinate system and the simplicity of the derivation for moving, deforming bodies is emphasized. The boundary conditions are then transformed to the familiar spatial coordinate system. A Lagrangian density capable of giving the Lorentz form of the electromagnetic equations in the material coordinate system is found. The Lorentz form of the equations is shown not to be form-invariant to the deformation transformation.

I. INTRODUCTION

Two descriptions of matter are commonly employed in the study of deformation of solids. One, called the material or Lagrangian description, regards the position $\vec{x}$ of a point of matter in a continuum to be a function of the designation $\vec{X}$ of the matter point and the time t. The other, called the spatial or Eulerian description, regards the designation $\vec{X}$ of the matter point to be a function of its position $\vec{x}$ and time t. The designation $\vec{X}$ of the matter point is usually, but not necessarily, taken as the position of the matter point in the absence of of deformation.

In studies of the interaction of the electromagnetic fields with matter[1,2] the electromagnetic fields have been regarded as functions of position and time and thus in the spatial description. This is quite natural since the electromagnetic fields can exist in a vacuum as well as within matter. Nevertheless, one wonders whether in a region occupied by matter a material description of the electromagnetic fields might lead to new insights into the field-matter interaction or simpler modes of expressing or manipulating the interaction. Our immediate motivation for obtaining a material form of the Maxwell equations was a need for the electromagnetic boundary conditions at the surface of a *moving*, *deforming* body. Since such a surface is fixed when viewed from the material frame of reference, boundary-condition derivations are conceptually simpler in this frame. Illustrative of the difficulty of boundary-condition calculations for moving bodies in the spatial frame is the incorrect derivation of the boundary condition on the magnetic intensity $\vec{H}$ by Sommerfeld in his textbook on electrodynamics.[3] Use of the material form of the Maxwell equations for boundary-condition calculations has proven useful to us; we suspect other uses will be found by others.

In this paper we will first summarize various formulas relating the material and spatial description that we will need in later sections. We then find the material form of the Maxwell equations and the definitions of the material measures of the $\vec{E}$, $\vec{B}$, $\vec{D}$, and $\vec{H}$ fields from a consideration of the integral form of the equations. The Maxwell form of the electromagnetic equations is shown to be form invariant to the deformation transformation. We then derive the electromagnetic boundary conditions at a moving, deforming surface in the material frame and transform them to the spatial frame. Next we transform the Lagrangian of the electromagnetic fields to the material description and obtain from it the Lorentz form of the electromagnetic equations in this description. The Lorentz form of the electromagnetic equations is shown not to be form invariant to the deformation transformation. A multipole expansion of the charge and current densities then introduces the polarization and magnetization and leads to constitutive relations for the electric displacement and magnetic intensity needed to yield the Maxwell form of the equations.

II. RELATIONS BETWEEN MATERIAL AND SPATIAL DESCRIPTIONS

As defined in Sec. I the deformation of the medium in the material description is specified by

$$x_i = x_i(\vec{X}, t), \tag{2.1}$$

while in the spatial description it is specified by

$$X_I = X_I(\vec{x}, t). \tag{2.2}$$

We assume Eqs. (21) and (22) can be continued through the body surface a short distance.

Note that lower case letters are used to denote components of vectors such as $\vec{x}$ in a spatial coordinate system, while upper case letters are used to denote components of vectors such as $\vec{X}$ in a material coordinate system. This convention is needed since the two coordinate systems need not be identical; we will, however, choose them both to be Cartesian. The convention allows a compact notation for the deformation gradients

$$x_{i,J} \equiv \frac{\partial x_i}{\partial X_J}, \quad X_{I,j} \equiv \frac{\partial X_I}{\partial x_j} \quad . \tag{2.3}$$

By the chain rule of differentiation the deformation gradients are inverses,

$$x_{i,J} X_{J,j} = \delta_{ij}, \quad X_{I,j} x_{j,J} = \delta_{IJ}. \tag{2.4}$$

The Jacobian of the transformation from $\vec{X}$ to $\vec{x}$ is denoted by

$$J \equiv J(\vec{x}/\vec{X}) \equiv \det(x_{i,J}). \tag{2.5}$$

Time derivatives in the two descriptions have different meanings and notations. The spatial time derivative is

$$\frac{\partial F}{\partial t} \equiv \frac{\partial F(\vec{x}(\vec{X},t),t)}{\partial t}\bigg|_{\vec{x}\text{ fixed}} , \tag{2.6}$$

while the material time derivative is

$$\dot{F} \equiv \frac{dF}{dt}\bigg|_{\vec{X}\text{ fixed}} \equiv \frac{\partial F(\vec{x}(\vec{X},t),t)}{\partial t}\bigg|_{\vec{x}\text{ fixed}} + \dot{x}_i \frac{\partial F(\vec{x}(\vec{X},t),t)}{\partial x_i} \quad . \tag{2.7}$$

Here $d\vec{x}/dt$ is the velocity of the matter point. It is related to the flow of matter $\partial\vec{X}/\partial t$ at a fixed position $\vec{x}$ by

$$\dot{x}_i = -x_{i,K}\frac{\partial X_K}{\partial t}, \quad \frac{\partial X_K}{\partial t} = -X_{K,i}\,\dot{x}_i. \tag{2.8}$$

A very useful relation between the material and spatial derivatives, called the spatial equation of continuity,[4] is

$$\frac{1}{J}\frac{d(J\gamma)}{dt} = \frac{\partial\gamma}{\partial t} + \frac{\partial(\gamma\dot{x}_j)}{\partial x_j} \quad , \tag{2.9}$$

where γ may be a scalar, vector, or tensor.

Several other identities will be needed. One, called the Euler-Piola-Jacobi identity, has dual forms,

$$(JX_{K,j})_{,K} = 0, \quad (J^{-1}x_{k,J})_{,k} = 0. \tag{2.10}$$

Another involves the transformation of the permutation symbol from the spatial to the material coordinate system. It may be expressed as either

$$\epsilon_{IJK} = J^{-1}\epsilon_{ijk}\,x_{i,I}\,x_{j,J}\,x_{k,K} \tag{2.11}$$

or

$$\epsilon_{IJK} = J\epsilon_{ijk}\,X_{I,i}\,X_{J,j}\,X_{K,k}. \tag{2.12}$$

Finally, for reference we record here the transformations of arc, area, and volume. An element of arc $d\vec{X}$ in the material coordinate system is related to the element of arc $d\vec{x}$ joining the same two neighboring matter points in the spatial coordinate system as

$$dx_i = x_{i,J}\,dX_J. \tag{2.13}$$

Similarly, oriented elements of area in the two systems are related by Nanson's formula

$$da_i = JX_{J,i}\,dA_J\,. \tag{2.14}$$

If the oriented area elements are expressed in terms of unit normals $\vec{n}$ and $\vec{N}$,

$$da_i = n_i\,da, \quad dA_J = N_J\,dA, \tag{2.15}$$

then Nanson's formula yields

$$n_i = JX_{J,i}\,N_J\,\frac{dA}{da}. \tag{2.16}$$

Finally, volume elements are related by

$$dv = J\,dV. \tag{2.17}$$

For further discussion of deformation the reader is referred to Truesdell and Toupin.[5]

III. MATERIAL FORM OF MAXWELL EQUATIONS

The Maxwell equations in the conventional form are

$$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{j}^{\,f}, \tag{3.1}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} , \tag{3.2}$$

$$\vec{\nabla}\cdot\vec{D} = q^f, \tag{3.3}$$

$$\vec{\nabla}\cdot\vec{B} = 0 , \tag{3.4}$$

where the electric field $\vec{E}$, electric displacement $\vec{D}$, magnetic induction $\vec{B}$, and magnetic intensity $\vec{H}$ are functions of $\vec{x}$ and t and their components are referred to the spatial coordinate system. The free charge density is denoted by q^f. The free charge current $\vec{j}^{\,f}$ is composed of a conduction current part $\vec{j}^{\,c}$ and a convection current part $q^f(d\vec{x}/dt)$

$$\vec{j}^{\,f} = \vec{j}^{\,c} + q^f\,\frac{d\vec{x}}{dt} \quad . \tag{3.5}$$

We consider these equations in a volume of space occupied by matter.

In order to determine the transformation of these equations to material form we consider their integral forms. Consider Eq. (3.1) first. Integrating over a spatial frame area yields

$$\int(\vec{\nabla}\times\vec{H})\cdot d\vec{a} = \int\left(\frac{\partial\vec{D}}{\partial t} + \vec{j}^{\,f}\right)\cdot\,d\vec{a}. \tag{3.6}$$

Use of Eqs. (A1a) and (A4) from the Appendix to reexpress $\partial\vec{D}/\partial t$ and the Stokes theorem to reexpress the curl term leads to

$$\int\left(\vec{H}-\frac{d\vec{x}}{dt}\times\vec{D}\right)\cdot d\vec{x}=\frac{d}{dt}\int\vec{D}\cdot d\vec{a}+\int\left(\vec{j}^{f}-\vec{\nabla}\cdot\vec{D}\frac{d\vec{x}}{dt}\right)\cdot d\vec{a}. \quad (3.7)$$

We now transform the integration variables to the material frame via Eqs. (2.13) and (2.14) and use Eq. (3.3) to eliminate $\vec{\nabla}\cdot\vec{D}$ with the result

$$\int\left(\vec{H}-\frac{d\vec{x}}{dt}\times\vec{D}\right)_i x_{i,J}\,dX_J=\frac{d}{dt}\int D_i J X_{J,i}\,dA_J+\int(j_i^f-q^f\dot{x}_i)JX_{J,i}dA_J. \quad (3.8)$$

After introducing the conduction current through Eq. (3.5) into the last integral on the right-hand side we are led to define the material form of the magnetic intensity $\vec{\mathcal{H}}$, the electric displacement $\vec{\mathfrak{D}}$, and the conduction current $\vec{\mathcal{J}}^c$ by

$$\mathcal{H}_J(\vec{X},t)\equiv\left(\vec{H}(\vec{x}(\vec{X},t),t)-\frac{d\vec{x}(\vec{X},t)}{dt}\times\vec{D}(\vec{x}(\vec{X},t),t)\right)_i x_{i,J}(\vec{X},t), \quad (3.9)$$

$$\mathfrak{D}_J\equiv JX_{J,i}D_i, \quad (3.10)$$

$$\mathcal{J}_K^c\equiv JX_{K,i}j_i^c\ , \quad (3.11)$$

where the functional dependence typical of all the field transformation equations is shown in Eq. (3.9). The transformed electric displacement field of Eq. (3.10) has recently been used by Baumhauer and Tiersten.[6] With these definitions, Eq. (3.8) becomes

$$\int(\vec{\nabla}_X\times\vec{\mathcal{H}})\cdot d\vec{A}=\int\left(\frac{d\vec{\mathfrak{D}}}{dt}+\vec{\mathcal{J}}^c\right)\cdot d\vec{A}. \quad (3.12)$$

Here Stokes theorem has been used on the left-hand side and the gradient operator with respect to $\vec{X}$ has been denoted by $\vec{\nabla}_X$. On the right-hand side the commutativity of material time differentiation and material space integration has been used. Since the area of integration is arbitrary, we must have

$$\vec{\nabla}_X\times\vec{\mathcal{H}}=\frac{d\vec{\mathfrak{D}}}{dt}+\vec{\mathcal{J}}^c \quad (3.13)$$

for the first transformed Maxwell equation.

Next consider Eq. (3.2). Following a procedure similar to that just used we integrate over an area in the spatial frame, use Stokes theorem, and Eqs. (A1a) and (A4) to obtain

$$\int\vec{E}\cdot d\vec{x}=-\frac{d}{dt}\int\vec{B}\cdot d\vec{a}+\int\left(\vec{B}\times\frac{d\vec{x}}{dt}\right)\cdot d\vec{x}+\int\vec{\nabla}\cdot\vec{B}\frac{d\vec{x}}{dt}\cdot d\vec{a}. \quad (3.14)$$

The last term vanishes because of Eq. (3.4). Transformation of the integration variables to the material frame now yields

$$\int\left(\vec{E}+\frac{d\vec{x}}{dt}\times\vec{B}\right)_i x_{i,J}dX_J=-\frac{d}{dt}\int B_iJX_{J,i}dA_J. \quad (3.15)$$

This leads us to define

$$\mathcal{E}_J\equiv\left(\vec{E}+\frac{d\vec{x}}{dt}\times\vec{B}\right)_i x_{i,J}, \quad (3.16)$$

$$\mathcal{B}_J\equiv JX_{J,i}B_i \quad (3.17)$$

for the material form of the electric field and the magnetic induction. Reuse of the Stokes theorem and the commutativity of the differentiation and integration on the right-hand side gives

$$\int(\vec{\nabla}_X\times\vec{\mathcal{E}})\cdot d\vec{A}=\int-\frac{d\vec{\mathcal{B}}}{dt}\cdot d\vec{A}, \quad (3.18)$$

which in turn gives

$$\vec{\nabla}_X\times\vec{\mathcal{E}}=-\frac{d\vec{\mathcal{B}}}{dt} \quad (3.19)$$

for the second transformed Maxwell equation.

Next we integrate Eq. (3.3) over a spatial frame volume and use Gauss's theorem,

$$\int\vec{\nabla}\cdot\vec{D}dv=\int\vec{D}\cdot d\vec{a}=\int q^f dv. \quad (3.20)$$

Transformation to material frame integration variables leads to

$$\int\vec{\mathfrak{D}}\cdot d\vec{A}=\int\vec{\nabla}_X\cdot\vec{\mathfrak{D}}dV=\int\mathcal{Q}^f dV, \quad (3.21)$$

where the definition of $\vec{\mathfrak{D}}$ [Eq. (3.10)] and the material form of the charge density

$$\mathcal{Q}^f\equiv q^f J \quad (3.22)$$

have been used. Since the integration volume is arbitrary, we obtain

$$\vec{\nabla}_X\cdot\vec{\mathfrak{D}}=\mathcal{Q}^f \quad (3.23)$$

for the third transformed Maxwell equation. By a completely analogous procedure we also obtain

$$\vec{\nabla}_X\cdot\vec{\mathcal{B}}=0 \quad (3.24)$$

for the last transformed Maxwell equation.

We conclude that the Maxwell equations are *form invariant* to the deformation transformation.

IV. FIELD TRANSFORMATIONS

It is worth exploring the field transformation equations a bit further. First, the inverse solutions of Eqs. (3.10) and (3.17) are found to be

$$D_i=J^{-1}x_{i,K}\mathfrak{D}_K, \quad (4.1)$$

$$B_i = J^{-1} x_{i,K} \mathcal{B}_K \tag{4.2}$$

with the use of Eq. (2.4). The inverse of Eq. (3.9) becomes

$$\begin{aligned} H_j &= \mathcal{H}_J X_{J,j} + \epsilon_{jkl} \left(- x_{k,K} \frac{\partial X_K}{\partial t} \right) J^{-1} x_{l,L} \mathcal{D}_L \\ &= \mathcal{H}_J X_{J,j} - X_{J,j} (J^{-1} \epsilon_{ikl} x_{i,J} x_{k,K} x_{l,L}) \frac{\partial X_K}{\partial t} \mathcal{D}_L \\ &= X_{J,j} \left(\mathcal{H}_J - \epsilon_{JKL} \frac{\partial X_K}{\partial t} \mathcal{D}_L \right) \end{aligned} \tag{4.3}$$

with the successive use of Eqs. (2.4), (2.8), (4.1), and (2.11). By an analogous procedure we obtain

$$E_J = X_{J,j} \left(\mathcal{E}_J + \epsilon_{JKL} \frac{\partial X_K}{\partial t} \mathcal{B}_L \right) . \tag{4.4}$$

The inverse current and charge equations become

$$j_i^c = J^{-1} x_{i,J} \mathcal{J}_J^c , \tag{4.5}$$

$$q^f = J^{-1} \mathcal{Q}^f . \tag{4.6}$$

An examination of the origins of these field transformations shows that

$$\vec{\mathcal{H}} \cdot d\vec{X} = \left(\vec{H} - \frac{d\vec{x}}{dt} \times \vec{D} \right) \cdot d\vec{x}, \tag{4.7}$$

$$\vec{\mathcal{E}} \cdot d\vec{X} = \left(\vec{E} + \frac{d\vec{x}}{dt} \times \vec{B} \right) \cdot d\vec{x} , \tag{4.8}$$

$$\vec{\mathcal{D}} \cdot d\vec{A} = \vec{D} \cdot d\vec{a} , \tag{4.9}$$

$$\vec{\mathcal{B}} \cdot d\vec{A} = \vec{B} \cdot d\vec{a} , \tag{4.10}$$

$$\vec{\mathcal{J}}^c \cdot d\vec{A} = \vec{j}^c \cdot d\vec{a} , \tag{4.11}$$

$$\mathcal{Q}^f dV = q^f dv \tag{4.12}$$

are invariants with respect to the deformation transformation.

V. BOUNDARY CONDITIONS IN THE MATERIAL FRAME

Since the Maxwell equations have now been transformed to the material coordinate frame of the moving deforming body, the boundaries of the body appear at rest. This permits the derivation of the boundary conditions at a body surface by the conventional arguments.[7] Because of this we will simply quote the boundary conditions in this frame. They are

$$[\vec{\mathcal{H}}] \times \vec{N} = \vec{K}^c , \tag{5.1}$$

$$[\vec{\mathcal{E}}] \times \vec{N} = 0, \tag{5.2}$$

$$[\vec{\mathcal{D}}] \cdot \vec{N} = \Sigma^f, \tag{5.3}$$

$$[\vec{\mathcal{B}}] \cdot \vec{N} = 0, \tag{5.4}$$

where $\vec{N}$ is a unit surface normal defined by Eq. (2.15) and the surface conduction current density $\vec{K}^c$ and surface charge density Σ^f consisting of free charge and extrinsic immobile charge are given by

$$\vec{\mathcal{J}}^c \equiv \vec{K}^c \delta(S), \tag{5.5}$$

$$\mathcal{Q}^f \equiv \Sigma^f \delta(S). \tag{5.6}$$

Here $\delta(S)$ is the one-dimensional Dirac δ function and S is a material coordinate measured normal to the surface. For a plane boundary surface, we can write

$$S = \vec{N} \cdot (\vec{X} - \vec{X}^0), \tag{5.7}$$

so that $S = 0$ is the equation of the surface. If the surface is not planar, Eq. (5.7) is valid locally in the vicinity of any point $\vec{X}^0$ on the surface. The bracket notation in Eqs. (5.1)–(5,4) indicates the jump in the enclosed quantity that is,

$$[\vec{\mathcal{H}}] \equiv \vec{\mathcal{H}}^{\text{out}} - \vec{\mathcal{H}}^{\text{in}} . \tag{5.8}$$

The unit normal to the surface in the material frame,

$$\vec{N} \equiv \frac{d\vec{A}}{|d\vec{A}|} , \tag{5.9}$$

has been removed from the brackets since it is not discontinuous at the surface.

VI. TRANSFORMATION OF BOUNDARY CONDITIONS TO THE SPATIAL FRAME

The boundary conditions in the material frame were obtained by the simple conventional argument because the body surface was at rest in that frame and because questions of continuity of various material fields, such as the deformation gradients, did not arise. Both these difficulties arise if the boundary conditions are derived in the spatial frame and the latter difficulty also appears in the transformation of the material frame boundary conditions to the spatial frame. Thus, it is important to discuss the continuity of material fields at a body surface.

Consider two different material media in intimate contact. It is apparent that the position vector $\vec{x}(\vec{X}, t)$ of a matter point is continuous across such a surface provided no fracture or slippage occurs. Imagine all the material properties of one medium such as the mass density, the polarization, the stiffness, etc., to approach zero, that is, the value characteristic of a vacuum. Since throughout the limiting process continuity of $\vec{x}(\vec{X}, t)$ holds, such continuity must hold in in the limit when one medium becomes a vacuum. This leads immediately to the conclusion that gradients of the position *tangential* to the surface must also be continuous even when one medium is a vacuum. Gradients of the position normal to the surface need not be continuous. The foregoing reasoning also indicates that the velocity $d\vec{x}/dt$ is continuous across a body surface even when the second medium is a vacuum. The subtlety of this conclusion is best illustrated by the fact that Sommerfeld in his textbook on electrodynamics[3] assumed that $d\vec{x}/dt$ was discontinuous across a body surface and obtained an incorrect boundary condition for the spatial frame $\vec{H}$ field at a moving body surface. Lastly, we remark that the surface normal and scalar element of area of a body surface have the same values when viewed from either side of the surface. This is true whether they are expressed in the material or spatial frames.

With these understandings we proceed to transform the boundary conditions Eqs. (5.1)–(5.4) to the spatial frame. Consider the boundary condition on $\vec{\mathcal{H}}$ first. Inserting the definition of $\vec{\mathcal{H}}$ [Eq. (3.9)] into Eq. (5.1) we obtain

$$\epsilon_{ABC}\left[\left(\vec{H}-\frac{d\vec{x}}{dt}\times\vec{D}\right)_j x_{j,B}\right]N_C=K_A^c\,, \qquad (6.1)$$

where the brackets have the meaning of Eq. (5.8). Since $\vec{N}$ is normal to the surface, the gradient indicated by the subscript B must be tangential to the surface. Hence $x_{j,B}$ may be removed from the brackets. Equation (2.16) may be solved for $\vec{N}$,

$$N_C=J^{-1}x_{k,C}\,n_k\frac{da}{dA}\ , \qquad (6.2)$$

with the aid of Eq. (2.4). If this is substituted into Eq. (6.1) and a scalar product of the equation with $x_{i,A}dA/da$ formed, we obtain

$$J^{-1}\epsilon_{ABC}x_{i,A}x_{j,B}x_{k,C}\left[\vec{H}-\frac{d\vec{x}}{dt}\times\vec{D}\right]_j n_k =\frac{dA}{da}x_{i,A}K_A^c\,. \qquad (6.3)$$

Use of the inverse of Eq. (2.12) and the definition of the spatial frame surface current,

$$k_i^c\equiv x_{i,A}K_A^c\frac{dA}{da} \qquad (6.4)$$

then yields

$$\left[\vec{H}-\frac{d\vec{x}}{dt}\times\vec{D}\right]\times\vec{n}=\vec{k}^c \qquad (6.5)$$

for the boundary condition in the spatial frame of reference.

The correctness of the surface current transformation [Eq. (6.4)] will be demonstrated if in concert with the current transformation equation (4.5) and the material frame surface current definition (5.5) it is shown to be consistent with the proper definition of the spatial frame surface current,

$$j_i^c\equiv k_i^c\delta(s). \qquad (6.6)$$

Here s is a Cartesian coordinate normal to the surface in the spatial frame. If the point $\vec{X}^0$ of the surface and the neighboring point $\vec{X}$ of Eq. (5.7) map into $\vec{x}^0$ and $\vec{x}$, respectively,

$$\vec{x}^0\equiv\vec{x}(\vec{X}^0,t),\quad \vec{x}=\vec{x}(\vec{X},t), \qquad (6.7)$$

then s is defined locally at $\vec{x}^0$ by

$$s=\vec{n}\cdot(\vec{x}-\vec{x}^0). \qquad (6.8)$$

and $s=0$ is the equation of the moving surface in the spatial frame. The differential relation following from this may be written

$$ds=\vec{n}\cdot d\vec{x}=JX_{J,i}N_J\frac{dA}{da}x_{i,K}dX_K =J\frac{dA}{da}\vec{N}\cdot d\vec{X} \qquad (6.9)$$

with the use of Eq. (2.16) for $\vec{n}$ and Eq. (2.13) for $d\vec{x}$. In combination with the differential relation,

$$dS=\vec{N}\cdot d\vec{X} \qquad (6.10)$$

found from Eq. (5.7), Eq. (6.9) implies

$$ds=J\frac{dA}{da}\,dS. \qquad (6.11)$$

Since the Dirac δ function transforms as a density, in this case a one-dimensional density, we obtain

$$\delta(S)=\delta(s)\frac{ds}{dS}=\delta(s)J\frac{dA}{da}\ . \qquad (6.12)$$

With this relation developed we can now reexpress the transformation equation (4.5),

$$j_i^c=J^{-1}x_{i,A}\,\mathcal{J}_A^c=J^{-1}x_{i,A}K_A^c\delta(S) =x_{i,A}K_A^c\frac{dA}{da}\delta(s)=k_i^c\delta(s), \qquad (6.13)$$

with successive use of Eqs. (5.5), (6.12), and (6.4). Because this agrees with Eq. (6.6), consistency is shown.

Though it is reasonable that only a surface conduction current should arise in the material frame $\vec{\mathcal{H}}$ field boundary condition (5.1), it must be regarded as surprising that in the spatial frame $\vec{H}$ field boundary condition (6.5) no convective-type surface current consisting of a product of an immobile extrinsic surface charge and the component of the material velocity $d\vec{x}/dt$ tangential to the surface appears. Its nonappearance may be better understood with reference to Eq. (3.7). If this spatial frame equation is applied to a small rectangular area perpendicular to the surface and within a transition layer representing the surface during the limiting process,[7] the spatial frame $\vec{H}$ field boundary condition can be found from this equation. The line integral on the left-hand side yields the left-hand side of Eq. (6.5). The first area integral on the right-hand side disappears in the limit before the time derivative needs to be performed. The last term on the right-hand side of Eq. (3.7) yields only $\vec{k}^c$ in the limit because through the use of the charge equation (3.3) and the free current definition (3.5) the convective current cancels out of that term.

A procedure completely analogous to the foregoing may be applied to the $\vec{\mathcal{E}}$ field boundary condition (5.2). The resulting boundary condition is

$$\left[\vec{E} + \frac{d\vec{x}}{dt} \times \vec{B}\right] \times \vec{n} = 0 \tag{6.14}$$

in the spatial frame.

Consider the electric displacement boundary condition (5.3) next. By substituting the transformation of $\vec{\mathfrak{D}}$ [Eq. (3.10)], into Eq. (5.3) we find

$$[JX_{A,i} D_i] N_A = \Sigma^f . \tag{6.15}$$

Next we multiply this by dA/da and put N_A and dA/da inside the brackets since they are continuous at the boundary. This gives

$$\left[JX_{A,i} N_A \frac{dA}{da} D_i\right] = \Sigma^f \frac{dA}{da} . \tag{6.16}$$

Comparison with Eq. (2.16) shows that the combination of factors present in this equation is just $\vec{n}$. Taking $\vec{n}$ outside the brackets since it is continuous at the surface and defining the spatial frame surface charge density by

$$\sigma^f \equiv \Sigma^f \frac{dA}{da} \tag{6.17}$$

we find

$$[\vec{D}] \cdot \vec{n} = \sigma^f \tag{6.18}$$

for the spatial frame boundary condition. The appropriateness of the definition (6.17) can be determined by showing that in conjunction with the transformation of charge density (4.6) and the definition of the material frame surface charge (5.6) it yields the proper spatial frame surface charge definition. Thus Eq. (4.6) becomes

$$q^f = J^{-1} \mathcal{Q}^f = J^{-1} \Sigma^f \delta(S) = \Sigma^f \frac{dA}{da} \delta(s) = \sigma^f \delta(s), \tag{6.19}$$

with the successive use of Eqs. (5.6), (6.12), and (6.17). This demonstrates the correctness of the definition (6.17). By a procedure analogous to the foregoing we obtain the magnetic induction boundary condition

$$[\vec{B}] \cdot \vec{n} = 0 \tag{6.20}$$

in the spatial frame.

VII. MATERIAL FRAME LAGRANGIAN DENSITY FOR MAXWELL-LORENTZ EQUATIONS

The Lagrangian density in the spatial frame which yields the Lorentz form of the electromagnetic equations is

$$\mathcal{L}_S = \tfrac{1}{2}\epsilon_0(\vec{E}\cdot\vec{E} - c^2\vec{B}\cdot\vec{B}) + \vec{j}\cdot\vec{A} - q\Phi, \tag{7.1}$$

where the electric and magnetic induction fields are given in terms of the vector potential $\vec{A}$ and scalar potential Φ by

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}, \tag{7.2}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{7.3}$$

and $\vec{j}$ and q are the current and charge densities that include both bound and free charge. The potentials are the generalized coordinates of the Lagrangian. The Lagrange equation for $\vec{A}$,

$$\frac{d}{dt}\frac{\partial \mathcal{L}_S}{\partial(\partial A_i/\partial t)} = \frac{\partial \mathcal{L}_S}{\partial A_i} - \frac{\partial}{\partial x_j}\frac{\partial \mathcal{L}_S}{\partial A_{i,j}}, \tag{7.4}$$

yields

$$\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{j} \tag{7.5}$$

and the Lagrange equation for Φ,

$$\frac{d}{dt}\frac{\partial \mathcal{L}_S}{\partial(\partial\Phi/\partial t)} = \frac{\partial \mathcal{L}_S}{\partial \Phi} - \frac{\partial}{\partial x_j}\frac{\partial \mathcal{L}_S}{\partial \Phi_{,j}}, \tag{7.6}$$

yields

$$\epsilon_0 \vec{\nabla}\cdot\vec{E} = q. \tag{7.7}$$

The remaining two electromagnetic equations [Eqs. (3.2) and (3.4)] are consequences of the relations (7.2) and (7.3).

In order to transform the Lagrangian to a material form we must first find the transformed vector and scalar potentials. By procedures similar to those used in Sec. VI Eqs. (7.2) and (7.3) may be transformed to

$$\vec{\mathcal{E}} = -\vec{\nabla}_X\varphi - \frac{d\vec{\mathfrak{a}}}{dt}, \tag{7.8}$$

$$\vec{\mathfrak{B}} = \vec{\nabla}_X \times \vec{\mathfrak{a}}, \tag{7.9}$$

where

$$\mathfrak{a}_K \equiv A_I x_{I,K}, \qquad A_J = \mathfrak{a}_K X_{K,J}, \tag{7.10}$$

$$\varphi \equiv \Phi - \frac{d\vec{x}}{dt}\cdot\vec{A}, \quad \Phi = \varphi - \frac{\partial\vec{X}}{\partial t}\cdot\vec{\mathfrak{a}} . \tag{7.11}$$

Note should be taken of the velocity dependent transformation of the scalar potential.

We now wish to transform the Lagrangian density to a function of $\vec{\mathfrak{a}}$ and φ. First, since it is a density, we have

$$\mathcal{L}_M = J\mathcal{L}_S . \tag{7.12}$$

Substitution of Eqs. (7.8)–(7.11) into Eq. (7.1) leads to

$$\mathcal{L}_M = \frac{\epsilon_0}{2}\left[J\left(\vec{\mathcal{E}} + \frac{\partial\vec{X}}{\partial t}\times\vec{\mathfrak{B}}\right)\cdot\overline{C}^{-1}\left(\vec{\mathcal{E}} + \frac{\partial\vec{X}}{\partial t}\times\vec{\mathfrak{B}}\right) - c^2 J^{-1}\vec{\mathfrak{B}}\cdot\overline{C}\cdot\vec{\mathfrak{B}}\right] + \vec{\mathfrak{J}}\cdot\vec{\mathfrak{a}} - \mathfrak{Q}\varphi, \tag{7.13}$$

where

$$C_{AB} \equiv x_{i,A} x_{i,B}, \tag{7.14}$$

$$(C^{-1})_{AB} = X_{A,i} X_{B,i}, \tag{7.15}$$

$$\mathcal{J}_K = J X_{K,i}(j_i - \dot{x}_i q), \tag{7.16}$$

$$\mathcal{Q} = Jq. \tag{7.17}$$

The Lorentz form of the electromagnetic equations may now be found from Eq. (7.13). The Lagrange equation for $\vec{\mathcal{A}}$,

$$\frac{d}{dt}\frac{\partial \mathcal{L}_M}{\partial \dot{\mathcal{A}}_K} = \frac{\partial \mathcal{L}_M}{\partial \mathcal{A}_K} - \frac{\partial}{\partial X_L}\frac{\partial \mathcal{L}_M}{\partial \mathcal{A}_{K,L}}, \tag{7.18}$$

leads to

$$\epsilon_{KLM}\frac{\partial}{\partial X_L}\left[\frac{1}{\mu_0 J} C_{MN}\mathcal{B}_N + \epsilon_{MNP}\frac{\partial X_N}{\partial t}\epsilon_0 J(C^{-1})_{PQ} \times\left(\vec{\mathcal{E}} + \frac{\partial \vec{X}}{\partial t}\times\vec{\mathcal{B}}\right)_Q\right] = \frac{d}{dt}\left[\epsilon_0 J(C^{-1})_{KJ}\left(\vec{\mathcal{E}} + \frac{\partial \vec{X}}{\partial t}\times\vec{\mathcal{B}}\right)_J\right] + \mathcal{J}_K, \tag{7.19}$$

which is the material frame analog to Eq. (7.5). The Lagrange equation for φ,

$$\frac{d}{dt}\frac{\partial \mathcal{L}_M}{\partial \dot{\varphi}} = \frac{\partial \mathcal{L}_M}{\partial \varphi} - \frac{\partial}{\partial X_L}\frac{\partial \mathcal{L}_M}{\partial \varphi_{,L}}, \tag{7.20}$$

leads to

$$\frac{\partial}{\partial X_L}\left[\epsilon_0 J(C^{-1})_{LM}\left(\vec{\mathcal{E}} + \frac{\partial \vec{X}}{\partial t}\times\vec{\mathcal{B}}\right)_M\right] = \mathcal{Q}, \tag{7.21}$$

which is the material frame analog to Eq. (7.7). The two remaining electromagnetic equations,

$$\vec{\nabla}_X \times \vec{\mathcal{E}} = -\frac{d\vec{\mathcal{B}}}{dt}, \tag{7.22}$$

$$\vec{\nabla}_X \cdot \vec{\mathcal{B}} = 0, \tag{7.23}$$

follow directly from Eqs. (7.8) and (7.9) and agree with Eqs. (3.19) and (3.24). Note that the latter two equations retain the form of Eqs. (3.2) and (3.4), but that Eqs. (7.19) and (7.21) do not retain the form of Eqs. (7.5) and (7.7). Thus the form invariance under a deformation transformation found in Sec. III for the Maxwell form of the electromagnetic equations does not apply to the Lorentz form of the electromagnetic equations.

VIII. POLARIZATION AND MAGNETIZATION

The Maxwell form of the electromagnetic equations is obtained from the Lorentz form in the spatial frame by expanding the charge and current densities in terms of multipole moments. The charge density becomes

$$q = q^f - \vec{\nabla}\cdot\vec{P} \tag{8.1}$$

and the current density becomes

$$\vec{j} = \vec{j}^f + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\times\left(\vec{M} + \vec{P}\times\frac{d\vec{x}}{dt}\right). \tag{8.2}$$

Here $\vec{P}$ is the polarization and $\vec{M}$ is the magnetization. They may be regarded as the electric dipole and magnetic dipole contributions, respectively, to the multipole expansions with higher multipole contributions dropped.[1] Alternatively, they may be regarded as embodying all higher multipole contributions within themselves. By giving no further defining relations for $\vec{P}$ and $\vec{M}$ we allow either of these interpretations and so allow a greater degree of generality to what follows.

The expansions (8.1) and (8.2) may now be inserted into Eqs. (7.5) and (7.7). It can then be seen that the spatial frame Maxwell form of the equations [Eqs. (3.1) and (3.3)] results if we define

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M} - \vec{P}\times\frac{d\vec{x}}{dt}, \tag{8.3}$$

$$\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}. \tag{8.4}$$

We now wish to find the material frame transforms of $\vec{P}$ and $\vec{M}$. Again we omit the algebra since it is similar to that employed in previous sections. Substitution of Eq. (8.1) into Eq. (7.17) leads to

$$\mathcal{Q} = \mathcal{Q}^f - \vec{\nabla}_X\cdot\vec{\mathcal{P}} \tag{8.5}$$

with

$$\mathcal{P}_K \equiv J X_{K,j} P_j, \quad P_i = J^{-1} x_{i,K}\mathcal{P}_K. \tag{8.6}$$

Substitution of Eqs. (8.1) and (8.2) into Eq. (7.16) leads to

$$\vec{\mathcal{J}} = \vec{\mathcal{J}}^c + \vec{\nabla}_X\times\vec{\mathcal{M}} + \frac{d\vec{\mathcal{P}}}{dt} \tag{8.7}$$

with

$$\mathcal{M}_K \equiv x_{i,K} M_i, \quad M_j = X_{K,j}\mathcal{M}_K. \tag{8.8}$$

Last, we must obtain the material forms of the relations (8.3) and (8.4) for $\vec{H}$ and $\vec{D}$. Substitution of the transformations for $\vec{B}$, $\vec{H}$, $\vec{M}$, $\vec{P}$, and $d\vec{x}/dt$ [Eqs. (4.2), (4.3) (8.8), (8.6), and (2.8), respectively] into Eq. (8.3) leads to the definition of the material frame $\vec{\mathcal{H}}$ field

$$\mathcal{H}_A \equiv \frac{1}{\mu_0 J} C_{AB}\mathcal{B}_B + \epsilon_{ABC}\frac{\partial X_B}{\partial t} \times\epsilon_0 J(C^{-1})_{CD}\left(\vec{\mathcal{E}} + \frac{\partial \vec{X}}{\partial t}\times\vec{\mathcal{B}}\right)_D - \mathcal{M}_A. \tag{8.9}$$

Substitution of the transformations for $\vec{D}$, $\vec{E}$, and $\vec{P}$ [Eqs. (4.1), (4.4), and (8.6), respectively] into Eq. (8.4) leads to the definition of the material frame $\vec{\mathcal{D}}$ field

$$\mathfrak{D}_K \equiv \epsilon_0 J(C^{-1})_{KL}\left(\vec{\mathcal{E}} + \frac{\partial \vec{X}}{\partial t}\times\vec{\mathcal{B}}\right)_L + \mathcal{P}_K. \tag{8.10}$$

The definitions (8.9) and (8.10) for $\vec{\mathcal{H}}$ and $\vec{\mathfrak{D}}$ provide the connection between the material Lorentz-form equations and the material Maxwell-form equations. Substitution of those definitions along with the expressions (8.5) and (8.7) for $\mathfrak{Q}$ and $\vec{\mathfrak{J}}$ into the material Lorentz-form equations (7.19) and (7.21) leads to the corresponding material Maxwell-form equations (3.13) and (3.23).

APPENDIX

We give here a short derivation for the material time derivative of a surface integral over a moving deforming body in terms of the convected time derivative.[8] We denote the convected time derivative of F_i by $\overset{*}{F}_i$ and define it by

$$\int \overset{*}{F}_i\, da_i \equiv \frac{d}{dt}\int \vec{F}\cdot da, \tag{A1a}$$

$$= \frac{d}{dt}\int F_j J X_{P,j}\, dA_P, \tag{A1b}$$

$$= \int \frac{d}{dt}(J X_{P,j} F_j) J^{-1} x_{i,P}\, da_i\ , \tag{A1c}$$

with the use of Eq. (2.14). Thus we have

$$\overset{*}{F}_i = J^{-1} x_{i,P}\frac{d}{dt}(J X_{P,j} F_j). \tag{A2}$$

This may be rearranged as

$$\overset{*}{F}_i = J^{-1}x_{i,P}\left(X_{P,j}\frac{d}{dt}(JF_j) + JF_j\frac{dX_{P,j}}{dt}\right) = J^{-1}\frac{d}{dt}(JF_i) - F_j X_{P,j}\frac{dx_{i,P}}{dt} = \frac{\partial F_i}{\partial t} + (F_i\dot{x}_k)_{,k} - F_j X_{P,j}\frac{\partial \dot{x}_i}{\partial X_P}$$

$$= \frac{\partial F_i}{\partial t} + (F_i\dot{x}_k)_{,k} - F_j\dot{x}_{i,j} = \frac{\partial F_i}{\partial t} + (F_i\dot{x}_k - F_k\dot{x}_i)_{,k} + F_{j,j}\dot{x}_i, \tag{A3}$$

where we have used the spatial equation of continuity (2.9), the material time derivative of Eq. (2.4), commutativity of material time and space derivatives, and the chain rule of differentiation. In vector form Eq. (A3) is

$$(\vec{F})^* = \frac{\partial \vec{F}}{\partial t} + \vec{\nabla}\times\left(\vec{F}\times\frac{d\vec{x}}{dt}\right) + \frac{d\vec{x}}{dt}\vec{\nabla}\cdot\vec{F}. \tag{A4}$$

Note added to proof: Thurston has brought to our attention some recent work of his [R. N. Thurston, *Handbuch der Physik*, edited by: S. Flugge (Springer-Verlag, Berlin, 1974), Vol. VIa/4, p. 109] having a number of things in common with this work. He has obtained the field transformations present in Secs. III and IV of this paper. He remarks, however, that these transformations are not unique. We would prefer to say that, given the deformation transformation, the field transformations presented by him and us are the only ones that will leave the Maxwell equations form invariant. Making this a requirement of the procedure makes the transformations unique. Thurston's material representation of the potentials is different from ours. Both are correct; each has a different gauge. We regard Thurston's choice of making the scalar potential invariant to the deformation transformation as awkward. The price is a vector potential transformation which is an integrodifferential relation (not given by him) We might add that in the quasielectrostatic limit (at least in the absence of a static magnetic field) our scalar potential transforms invariantly. Thurston does not discuss boundary conditions, the material form of the electromagnetic Lagrangian, or the Lorentz form of the electromagnetic equations which form the subjects of Secs. V–VIII.

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