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Asymmetric total stress tensor

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From arguments based on momentum conservation, the stress boundary condition, the allowed functional dependence of a stress tensor, the gauge invariance, and the vacuum form of the Maxwell stress tensor, a proper identification of the total stress (or momentum flow) tensor for a closed system consisting of an arbitrary dielectric crystal in interaction with the electromagnetic field is found. This tensor is shown to be *asymmetric* even though the system conserves angular momentum. Jump conditions on the total stress tensor are found both for surfaces fixed in the spatial or laboratory coordinate system, and for surfaces fixed in the material or body coordinate system, and thus moving and deforming with respect to the laboratory coordinate system. The ideas developed are also applied to the flow of energy and the flow of angular momentum.

I. INTRODUCTION

The proper identification of the total stress tensor representing total momentum flow in a closed system consisting of an arbitrary dielectric crystal in interaction with the electromagnetic field is complicated by many considerations. It initially arises within a divergence in a conservation law and so may be altered by the addition of divergenceless quantities and by other types of transformations. Also, the first form that is naturally obtained in a Lagrangian based theory, the canonical stress tensor, is not gauge invariant. Furthermore, both fixed surfaces and moving body surfaces, across which the momentum flows, must be considered. Last, the question of the interchange symmetry of the total stress tensor needs reexamination since the theory here contains mechanical variables that account for all long-wavelength modes of motion of the crystal, not just center-of-mass motion.

The purpose of this paper will be to develop a procedure for proper identification of the total stress tensor and to show that the one determined is asymmetric under rather general circumstances. The treatment will be based on our long-wavelength theory of electrodynamics of elastic dielectrics.[1,2] The total stress tensor found will represent a closed system consisting of the electromagnetic field interacting with a dielectric crystal which can have any symmetry and anisotropy, any nonlinearity, and any structural complexity and which may have any elastic deformation or internal motion. We will use arguments based on the momentum conservation law, the stress boundary condition, the allowed functional dependence of a stress tensor, gauge invariance and the vacuum forms of the stress tensor and momentum density to arrive at a total stress tensor which represents true momentum flow. The expression for the momentum density will be obtained at the same time.

We will show that there are two forms of the total stress tensor that have important and somewhat differing meanings. One represents the momentum flow relative to the laboratory or spatial coordinate system. Its flow across a surface fixed in the spatial coordinate system is continuous. The second represents the momentum flow relative to the material coordinate system of a body. Its flow across a surface fixed in the material coordinate system, such as the body surface itself, is continuous. Thus, the latter tensor is particularly important to considerations of momentum transfer between bodies or between the electromagnetic field and a moving body.

The interchange symmetry of the tensor representing momentum flow across a surface in the laboratory coordinate system is explored. We find that it is in general *asymmetric*, though at sufficiently low frequencies it becomes symmetric. The asymmetry of the total stress tensor is present even though the system obeys angular momentum conservation. This situation arises since this theory includes all internal modes of motion (optic modes) of the crystal which must carry angular momentum.

The reasoning applied to the flow of momentum in this paper can also be applied to the flow of energy and angular momentum. A brief summary of these applications is given in Secs. VII and VIII.

II. CANONICAL STRESS TENSOR

The homogeneity of free space causes the equations of motion of physics to be form-invariant to spatial displacements. In a theory, such as ours, based on a Lagrangian this requirement will be met if the Lagrangian is not an explicit function of the position $\vec{x}$.[1,3] Momentum conservation results from this spatial displacement invariance.[3] It may be expressed as

$$\frac{\partial g_i^C}{\partial t} - \frac{\partial t_{ij}^C}{\partial z_j} = 0, \tag{2.1}$$

where g_i^C is the canonical momentum density given by

$$g_i^C \equiv -\sum_\alpha \left[\partial \mathfrak{L}^S / \partial \left(\frac{\partial \psi^\alpha}{\partial t}\right)\right] \psi^\alpha_{,i} \tag{2.2}$$

and t_{ij}^C is the canonical stress tensor given by

$$t_{ij}^C \equiv \sum_\alpha \frac{\partial \mathfrak{L}^S}{\partial \psi^\alpha_{,j}} \psi^\alpha_{,i} - \mathfrak{L}^S \delta_{ij}. \tag{2.3}$$

Here $\mathfrak{L}^S$ is the spatial frame Lagrangian density. It is regarded as a function of the N fields ψ^α ($\alpha = 1, 2, \ldots, N$), their spatial time derivatives $\partial\psi^\alpha/\partial t$ ($\vec{z}$ held fixed), and their space derivatives $\psi^\alpha_{,j} \equiv \partial\psi^\alpha/\partial z_j$. The independent variables in the spatial description used here are $\vec{z}$ and t. Within matter $\vec{z}$ and $\vec{x}$, the position of the center of mass, are synonomous. The minus sign has been introduced into the conservation law (2.1) in order to give the stress tensor the conventional sign.

In the accompanying paper[2] the matter Lagrangian density was expressed in the material description, that is, with $\vec{X}$, the material coordinate that names and rides with a matter point, and t as the independent variables. The transformation of the material frame Lagrangian density $\mathfrak{L}^M$ to $\mathfrak{L}^S$ must satisfy

$$\int \mathfrak{L}^S \, dv = \int \mathfrak{L}^M \, dV, \tag{2.4}$$

where $dv = dz_1 \, dz_2 \, dz_3$ and $dV = dX_1 \, dX_2 \, dX_3$. Thus

$$\mathfrak{L}^S = [J(\vec{x}/\vec{X})]^{-1} \mathfrak{L}^M, \tag{2.5}$$

where $J(\vec{x}/\vec{X})$ is the Jacobian of the transformation from the material to spatial coordinates. We use Eq. (2.5) to transform the matter Lagrangian density of Eq. (3.17) of the accompanying paper[2] to a spatial form. Then, using the electromagnetic field and interaction Lagrangian densities of Eqs. (3.3) and (3.13) of the same paper we obtain

$$\mathfrak{L}^S = \tfrac{1}{2}\rho \frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt} + \frac{1}{2} \sum_\nu{}' \rho^\nu \frac{d\vec{y}^{T\nu}}{dt} \cdot \frac{d\vec{y}^{T\nu}}{dt} - \rho\Sigma + \vec{P} \cdot \left(\vec{E} + \frac{d\vec{x}}{dt} \times \vec{B}\right) + \tfrac{1}{2}\epsilon_0 (\vec{E}\cdot\vec{E} - c^2 \vec{B}\cdot\vec{B}). \tag{2.6}$$

Here $\vec{P}$ is the polarization given by

$$\vec{P} = J^{-1} \sum_\nu{}' q^\nu \vec{y}^{T\nu} \tag{2.7}$$

(the primed sum indicating ν ranges from 1 to $N-1$), $\vec{E}$ and $\vec{B}$ are the electric field and magnetic induction field that are given in terms of the vector and scalar potentials, $\vec{A}$ and Φ, by

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}, \tag{2.8}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{2.9}$$

The quantities $d\vec{x}/dt$ and $d\vec{y}/dt$ are material time derivatives ($\vec{X}$ held fixed), used for compactness to denote

$$\dot{x}_i = -x_{i,K} \frac{\partial X_K}{\partial t}, \tag{2.10}$$

$$\mathring{y}_i^{T\nu} = \frac{\partial y_i^{T\nu}}{\partial t} - y_{i,j}^{T\nu} x_{j,K} \frac{\partial X_K}{\partial t}. \tag{2.11}$$

The spatial frame mass densities ρ and ρ^ν associated with the center of mass and the νth internal coordinate $\vec{y}^{T\nu}$ are related to the corresponding material frame mass densities ρ^0 and m^ν by

$$\rho = J^{-1}\rho^0, \quad \rho^\nu = J^{-1} m^\nu. \tag{2.12}$$

The stored energy per unit mass is denoted by Σ and q^ν is the charge density associated with the νth internal coordinate in the material description. The electromagnetic fields are expressed in mks units. The deformation gradient $x_{i,K} \equiv \partial x_i/\partial X_K$ in Eqs. (2.10) and (2.11) is to be regarded as a function of $X_{L,j} \equiv \partial X_L/\partial x_j$ in the spatial description. The components of the material coordinate vector $\vec{X}$ are referred to a Cartesian material coordinate system and are denoted by upper-case letters. The components of the spatial position vector $\vec{x}$ are referred to a Cartesian spatial coordinate system and are denoted by lower-case letters.

The fields ψ^α of Eqs. (2.2) and (2.3) now refer to X_A, $y_i^{T\nu}$, A_i, and Φ. Substitution of $\mathfrak{L}^S$ from Eq. (2.6) into Eqs. (2.2) and (2.3) leads to

$$g_i^C = \rho\mathring{x}_i - (\vec{P}\times\vec{B})_i + D_j A_{j,i}, \tag{2.13}$$

$$t_{ij}^C = t_{ij}^y - \rho\dot{x}_i\dot{x}_j - \tfrac{1}{2}\epsilon_0 E_k E_k \delta_{ij} + (1/2\mu_0) B_k B_k \delta_{ij} + (\vec{P}\times\vec{B})_i \dot{x}_j + \left(E_i + \frac{\partial A_i}{\partial t}\right) D_j - H_l \epsilon_{ljk} A_{k,i}, \tag{2.14}$$

where $\vec{D}$ and $\vec{H}$, the electric displacement and magnetic intensity, are given by

$$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}, \tag{2.15}$$

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \left(\vec{P} \times \frac{d\vec{x}}{dt}\right), \tag{2.16}$$

and t_{ij}^y is defined by

$$t_{ij}^{y} \equiv \rho x_{j,B} \frac{\partial \Sigma}{\partial x_{i,B}}\bigg|_{\mathfrak{y}\,T\,\nu\ \mathrm{fixed}} . \qquad (2.17)$$

Though we began from a gauge-invariant Lagrangian density, Eq. (2.6), we have obtained a canonical momentum density, Eq. (2.13), and a canonical stress tensor, Eq. (2.14), which are not gauge invariant. Clearly, they cannot represent the physical quantities we are considering. However, since they do obey the momentum-conservation law, Eq. (2.1), they must be closely related to the quantities we seek. Thus we must explore alternate expressions of the momentum-conservation law.

III. TOTAL STRESS TENSOR

In order to find gauge-invariant measures of the momentum density and the total stress tensor we first rearrange the canonical forms of these quantities. With the use of Eqs. (2.9) and (2.15) the canonical momentum density can be expressed as

$$g_i^C = \rho\dot{x}_i + \epsilon_0(\vec{E}\times\vec{B})_i + A_{i,j}D_j, \qquad (3.1)$$

and the canonical stress tensor can be expressed as

$$t_{ij}^C = t_{ij}^y + m_{ij} + \mathcal{E}_i P_j - \rho\dot{x}_i\dot{x}_j + \frac{\partial A_i}{\partial t} D_j + A_{i,k}\epsilon_{kjl}H_l \ , \qquad (3.2)$$

where

$$\vec{\mathcal{E}} \equiv \vec{E} + \frac{d\vec{x}}{dt}\times\vec{B} \qquad (3.3)$$

and

$$m_{ij} \equiv \epsilon_0 E_i E_j + (1/\mu_0)B_i B_j - \tfrac{1}{2}\epsilon_0 E_k E_k \delta_{ij} - (1/2\mu_0)B_k B_k \delta_{ij} = m_{ji} \qquad (3.4)$$

is the Maxwell vacuum-field stress tensor. The non-gauge-invariant parts of g_i^C and t_{ij}^C can now be seen to cancel in the momentum-conservation law,

$$\frac{\partial}{\partial t}(A_{i,j}D_j) - \frac{\partial}{\partial z_j}\left(\frac{\partial A_i}{\partial t}D_j + A_{i,k}\epsilon_{kjl}H_l\right) = 0, \qquad (3.5)$$

with the use of the two Maxwell equations for a dielectric

$$\vec{\nabla}\cdot\vec{D} = 0, \qquad (3.6)$$

$$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} . \qquad (3.7)$$

The cancellation exhibited in Eq. (3.5) leads us to identify

$$g_i \equiv \rho\dot{x}_i + \epsilon_0(\vec{E}\times\vec{B})_i \qquad (3.8)$$

as the *total momentum density* in the spatial or laboratory frame and

$$t_{ij}^L \equiv t_{ij}^y + \mathcal{E}_i P_j + m_{ij} - \rho\dot{x}_i\dot{x}_j \qquad (3.9)$$

as the *total stress tensor* in the laboratory frame. The justification for such assignments must now be given. First, it is clear from the foregoing that these quantities satisfy the spatial frame momentum-conservation law,

$$\frac{\partial g_i}{\partial t} - t_{ij,j}^L = 0. \qquad (3.10)$$

It is also clear from an examination of the expressions for g_i and t_{ij}^L that application of the conventional "pill box" argument[4] to Eq. (3.10) leads to a jump condition of t_{ij}^L,

$$[t_{ij}^L]n_j = 0, \qquad (3.11)$$

where $[t_{ij}^L]$ denotes the jump in the quantity enclosed in the brackets across the surface considered,

$$[t_{ij}^L] \equiv (t_{ij}^L)^{\mathrm{out}} - (t_{ij}^L)^{\mathrm{in}}, \qquad (3.12)$$

and $\vec{n}$ is the unit normal of the surface pointing in the outward direction. It is essential to the proof of the boundary condition (3.11) that the boundary surface considered be stationary in the spatial or laboratory coordinate system.

The boundary condition (3.11) involves the stress tensor directly while the conservation law (3.10) involves the divergence of the stress tensor. Because of this the boundary condition is much more restrictive in determining what the total stress tensor is. This, we will see, is true because a variety of transformations can alter the quantity appearing in the divergence term of the conservation law without affecting the fact that the stress tensor t_{ij}^L appears in the boundary condition. Besides transformations that involve the equations of motion like that leading from the canonical forms of the momentum density and stress tensor to those of Eqs. (3.8) and (3.9), the addition of a curl-like quantity inside the divergence of the conservation law yields a new stress tensor,

$$t_{ij}^{L'} = t_{ij}^L + \epsilon_{ikl}f_{il,k} \ , \qquad (3.13)$$

without affecting the fact that the boundary condition found from this altered form of the conservation law will still involve t_{ij}^L. In showing this care must be taken in applying the pill-box argument since contributions from the sides of the pill box, which usually vanish in the limit of vanishing side area, then arise and cancel the top and bottom contributions of the curl term of Eq. (3.13). Another type of transformation,

$$t^{L\prime}_{ij}=t^L_{ij}+\frac{\partial h_{ij}}{\partial t}\ , \qquad (3.14)$$

$$g'_i=g_i+h_{ij,j}\ , \qquad (3.15)$$

affects the form of both the stress tensor and the momentum density in the conservation law while clearly not affecting the boundary condition (3.11). Last, the transformation

$$g'_i=g_i+c_i\ , \qquad (3.16)$$

where c_i is independent of time, does not affect the stress tensor in either the conservation law or the boundary condition. A corollary to this discussion is that the first three types of transformations mentioned can be used to alter the interchange symmetry of the stress tensor appearing in the conservation law from asymmetric to symmetric or vice versa without affecting the interchange symmetry of the stress tensor entering the boundary condition (3.11).

From the above discussion it is clear that we should focus our attention on the boundary condition (3.11) to determine the total stress tensor. The first question that arises is whether the scalar product of the stress tensor with the unit surface normal in that relation leads to an indeterminance in the stress tensor. The answer here is negative because t^L_{ij} must be a function of only the fields and the bulk material properties and not of the orientation of a body surface and because the orientation of the body surface (and hence $\vec{n}$) is arbitrary. The second question is whether an addition to the stress tensor via one of the transformations discussed in the preceding paragraph could be continuous at every surface inside, at, and outside a body surface and so produce no affect on the boundary condition (3.11) while still being a part of the stress tensor. It is clear that such an addition to the stress tensor would be unobservable and so should be excluded from the definition. The way to eliminate these null stresses, as they may be called, is to define the total stress tensor in some simple reference state. We choose a vacuum for this purpose and define the total stress and momentum density of a vacuum to be, respectively,

$$t^L_{ij}\equiv m_{ij}\quad \text{(vacuum)}, \qquad (3.17)$$

$$g_i\equiv\epsilon_0(\vec{E}\times\vec{B})_i\quad \text{(vacuum)}. \qquad (3.18)$$

Since our general definitions (3.8) and (3.9) reduce to these for a vacuum, we conclude that the quantity in Eq. (3.8) is the momentum density and the quantity in Eq. (3.9) is the total stress tensor.

We may now interpret the total spatial frame stress tensor of Eq. (3.9) as the total force per unit area acting in a direction determined by the first tensor index on an imaginary surface, fixed in the laboratory frame, whose normal is determined by the second tensor index. Alternatively, we may interpret the total stress tensor as the total true momentum in a direction determined by the first index flowing per unit time across a unit area of a surface fixed in the laboratory frame whose normal is determined by the second index. This leaves one severe problem. If momentum transfer from an electromagnetic field to a material body or from one material body is being considered, the surface of most interest for the transfer of momentum is the body surface which will, in general, be a moving deforming surface when viewed from the laboratory frame of reference. Clearly, there must be another total stress tensor related to momentum transfer across a body surface which will be of equal importance to the one defined by Eq. (3.9).

IV. MATERIAL-FRAME CONSERVATION LAWS

In this section we will transform a general conservation law from the spatial frame to the material frame. In the process we will establish a relation between the flow in the spatial frame of some conserved quantity to the flow of that quantity in the material frame.

Let d be the density in the spatial frame of a conserved quantity and $\vec{f}$ be the corresponding flow of that quantity. They satisfy the conservation law

$$\frac{\partial d}{\partial t}+f_{j,j}=0 \qquad (4.1)$$

and are defined such that application of the pillbox argument[4] to this equation yields the jump condition

$$[f_j]n_j=0 \qquad (4.2)$$

for a surface fixed in the spatial frame whose unit normal is $\vec{n}$.

In order to transform the conservation law to the material frame we use the spatial equation of continuity[1]

$$\frac{1}{J}\frac{d}{dt}(Jd)=\frac{\partial d}{\partial t}+(\dot{x}_j d)_{,j}\ , \qquad (4.3)$$

where d/dt represents the material time derivative ($\vec{X}$ held fixed), to eliminate the spatial time derivative from Eq. (4.1). The result is

$$\frac{1}{J}\frac{d}{dt}(Jd)+(f_j-\mathring{x}_j d)_{,j}=0. \qquad (4.4)$$

Using the chain rule of differentiation leads to

$$\frac{d}{dt}(Jd)+JX_{K,j}(f_j-\mathring{x}_j d)_{,K}=0. \qquad (4.5)$$

Use of the Euler-Piola-Jacobi identity

$$(JX_{K,j})_{,K}=0 \tag{4.6}$$

allows Eq. (4.5) to be rewritten

$$\frac{d}{dt}(Jd)+[JX_{K,j}(f_j-\mathring{x}_j d)]_{,K}=0. \tag{4.7}$$

If we now define the material-frame density of the conserved quantity by

$$D\equiv Jd \tag{4.8}$$

and the material-frame flow of the conserved quantity by

$$F_K\equiv JX_{K,j}(f_j-\dot{x}_j d), \tag{4.9}$$

Eq. (4.7) becomes the corresponding material-frame conservation law

$$\frac{dD}{dt}+F_{K,K}=0. \tag{4.10}$$

If the pill-box argument[4] is applied to the last equation, a jump condition on F_K,

$$[F_K]N_K=0, \tag{4.11}$$

is obtained across a surface fixed in the material frame. The moving body surface itself is the most important example of such a surface. Here $\vec{N}$ is the unit outward normal to an element of area dA of the surface referred to the material frame. It is related to the unit vector $\vec{n}$ normal to an area element da on the surface containing the same matter points referred to the spatial frame by

$$n_j=JX_{K,j}N_K\frac{dA}{da}, \tag{4.12}$$

a form of Nanson's formula.[5] If we define the spatial frame measure of F_K by

$$\mathrm{f}_j\equiv f_j-\dot{x}_j d, \tag{4.13}$$

the formula (4.12) and definition (4.9) may be used to transform the material-frame jump condition (4.11) to a spatial frame jump condition

$$[\mathrm{f}_j]n_j=0 \tag{4.14}$$

applying to a moving deforming surface.

Note that forming the scalar product of Eq. (4.9) with $N_K dA$ and using Eqs. (4.12) and (4.13) leads to

$$F_N N_N\,dA=\mathrm{f}_j n_j\,da, \tag{4.15}$$

a quantity invariant to the deformation transformation. From Eq. (4.8) and the meaning of the Jacobian as a ratio of volume elements we have

$$D\,dV=d\,dV \tag{4.16}$$

as another invariant of the transformation.

V. MOMENTUM CONSERVATION IN MATERIAL FRAME

We now apply the transformation formulas of Sec. IV to the spatial frame momentum law of Sec. III. By Eqs. (4.8)–(4.10) the material-frame momentum density,

$$G_i\equiv Jg_i=\rho^0\mathring{x}_i+\epsilon_0 J(\vec{E}\times\vec{B})_i, \tag{5.1}$$

and the material-frame (or body system) total stress tensor,

$$T^B_{iK}\equiv JX_{K,j}(t^L_{ij}+g_i\dot{x}_j)$$
$$=JX_{K,j}[t^\gamma_{ij}+\mathcal{E}_i P_j+m_{ij}+\epsilon_0(\vec{E}\times\vec{B})_i\mathring{x}_j], \tag{5.2}$$

satisfy the momentum-conservation law in the material frame

$$\frac{dG_i}{dt}-T^B_{iK,K}=0. \tag{5.3}$$

The plus sign in the first form of Eq. (5.2) in comparison to the minus sign in Eq. (4.13) results from the convention of stress tensor signs seen in Eq. (5.3) and mentioned in Sec. II. The general jump condition, Eq. (4.11), yields a total stress tensor jump condition,

$$[T^B_{iK}]N_K=0 \tag{5.4}$$

for a surface, such as the body surface, fixed in the material frame. Such a surface when viewed from the spatial or laboratory frame may be a moving deforming surface.

Following Eq. (4.13) we define the spatial frame measure of T^B_{iK} by

$$t^B_{ij}\equiv t^L_{ij}+g_i\dot{x}_j$$
$$=t^\gamma_{ij}+\mathcal{E}_i P_j+m_{ij}+\epsilon_0(\vec{E}\times\vec{B})_i\dot{x}_j. \tag{5.5}$$

From the general jump condition (4.14) we obtain

$$[t^B_{ij}]n_j=0 \tag{5.6}$$

as the total stress tensor jump condition at a surface moving and deforming in the spatial frame. This is the desired generalization to the boundary condition (3.11) and t^B_{ij} is seen to represent the momentum flow across a moving, deforming surface in the spatial frame. The arguments of Sec. III concerning the uniqueness of t^L_{ij} may now be seen to apply to t^B_{ij} through its definition (5.5).

VI. ASYMMETRY OF TOTAL STRESS TENSOR

The statement that angular momentum conservation requires the total stress tensor to have interchange symmetry on its two tensor subscripts occurs in many works on continuum mechanics. What is left unstated is that this is true if the material continuum is represented only by the center of mass position $\vec{x}(\vec{X},t)$. We will show in this section that the total stress tensor representing a crystal having N particles per unit cell and thus needing N-1 vector internal coordinates

$\vec{y}^{T\nu}$ as well as the center of mass position $\vec{x}$ need not be symmetric even though angular momentum is conserved.

The tensor whose symmetry must be considered is t^L_{ij} since it, in contrast to t^B_{ij}, represents the flow of momentum across a surface fixed with respect to the coordinate system to which the momentum components are referred. Because the last two terms in its definition (3.9) are manifestly symmetric we need consider only the first two terms. We begin by considering t^y_{ij} defined in Eq. (2.17). By arguments similar to those in the preceding paper[2] we may express the stored energy per unit mass, Σ, as

$$\Sigma = \Sigma(\Gamma^\nu_A,\ E_{BC}), \tag{6.1}$$

where

$$E_{BC} \equiv \tfrac{1}{2}(x_{j,B}x_{j,C} - \delta_{BC}) , \tag{6.2}$$

$$\Gamma^\nu_A \equiv x_{j,A}y^{T\nu}_j - Y^\nu_A \tag{6.3}$$

are rotationally invariant measures of the finite strain and the internal coordinates, respectively. Y^ν_A is the value of the νth internal coordinate in the natural state of the crystal and is introduced into Eq. (6.3) to make Γ^ν_A vanish in the natural state. Equation (2.17) now becomes

$$t^y_{ij} = \rho x_{j,B}\left(\frac{\partial E_{AC}}{\partial x_{i,B}}\frac{\partial \Sigma}{\partial E_{AC}}\bigg|_{\Gamma^\nu_D \text{ fixed}} + \sum_\nu{}' \frac{\partial \Gamma^\nu_A}{\partial x_{i,B}}\frac{\partial\Sigma}{\partial\Gamma^\nu_A}\bigg|_{E_{CD}\text{ fixed}}\right)$$

$$= \rho x_{j,B}\left(x_{i,C}\frac{\partial\Sigma}{\partial E_{BC}}\bigg|_{\Gamma_D} + \sum_\nu{}' y^{T\nu}_i \frac{\partial\Sigma}{\partial\Gamma^\nu_B}\bigg|_{E_{CD}}\right) = t^\Gamma_{ij} + \rho x_{j,B}\sum_\nu{}' y^{T\nu}_i\frac{\partial\Sigma}{\partial\Gamma^\nu_B}\bigg|_{E_{CD}}, \tag{6.4}$$

where

$$t^\Gamma_{ij} \equiv \rho x_{j,B}x_{i,C}\frac{\partial\Sigma}{\partial E_{BC}}\bigg|_{\Gamma^\nu_D} = t^\Gamma_{ji} \tag{6.5}$$

is a symmetric portion of the stress tensor. The equations of motion of the internal coordinates,[2]

$$m^\nu \ddot{y}^{T\nu} = q^\nu \mathcal{E}_i - \rho^0 x_{i,A}\frac{\partial\Sigma}{\partial\Gamma^\nu_A}\bigg|_{E_{CD}}, \tag{6.6}$$

can be solved for the stored energy derivative,

$$\rho\frac{\partial\Sigma}{\partial\Gamma^\nu_B}\bigg|_{E_{CD}} = J^{-1}q^\nu\mathcal{E}_k X_{B,k} - \rho^\nu \ddot{y}^{T\nu}_k X_{B,k}. \tag{6.7}$$

Elimination of this derivative from Eq. (6.4) now yields

$$t^y_{ij} = t^\Gamma_{ij} + P_i\mathcal{E}_j - \sum_\nu{}' \rho^\nu y^{T\nu}_i \ddot{y}^{T\nu}_j , \tag{6.8}$$

with the use of Eq. (2.7). This may be substituted into the expression for t^L_{ij}, Eq. (3.9), with the result

$$t^L_{ij} = t^\Gamma_{ij} + (\mathcal{E}_i P_j + P_i\mathcal{E}_j) + m_{ij} - \rho\dot{x}_i\dot{x}_j - \sum_\nu{}' \rho^\nu y^{T\nu}_i\ddot{y}^{T\nu}_j . \tag{6.9}$$

An examination of this expression for t^L_{ij} shows that all parts of it except the last term are symmetric upon interchange of the tensor indices. The last term is asymmetric causing the total stress tensor t^L_{ij} to be *asymmetric*. It is apparent that the asymmetry results from the presence of the $N-1$ vector internal coordinates. Because of the presence of the second (material) time derivative in that term, this term will be very small for frequencies well below the resonances of the internal coordinates. In this low-frequency regime the total stress tensor becomes symmetric.

Antisymmetric total stress tensors have been shown before to arise from the presence of internal structure, namely Cosserat continua possessing directors and couple stresses.[6] Since our antisymmetric contribution vanishes in the low-frequency limit, it does not have as its origin a couple stress since the latter would remain finite at zero frequency. Couple stresses would appear in our theory if we went to the next order of derivatives, i.e., if we included the effects of wave-vector dispersion.

In summary, we have shown by the arguments of Sec. III that t^L_{ij} is the total stress tensor representing momentum flow across surfaces fixed in the laboratory or spatial frame. We have now shown that this tensor is asymmetric. We demonstrated in the preceding paper[2] that the closed system of a dielectric crystal and the electromagnetic fields, which is represented by this total stress tensor, conserves angular momentum. Finally, we pointed out that these two conditions can apply to the same system because the internal coordinates, which are needed to account for the various modes of internal motion of the crystal, can carry angular momentum. This is a *general conclusion*, applying to all dielectric crystals.

VII. APPLICATION TO ANGULAR MOMENTUM FLOW

It is apparent that the line of argument of Secs. III and IV can be applied to other conserved quantities to obtain correct identification of the flow of that quantity and jump conditions that the flow satisfies. If we begin from the angular momentum conservation equation found from the equations of motion as in the preceding paper[2] or from the conservation law involving the canonical quantities found directly from the Lagrangian, these arguments lead us to identify the total angular momentum density $\vec{\omega}$ in the spatial frame as

$$\vec{\omega} \equiv \vec{x} \times \left(\rho \frac{d\vec{x}}{dt} + \epsilon_0 \vec{E} \times \vec{B}\right) + \vec{l}, \tag{7.1}$$

where $\vec{l}$ is the angular momentum density resident in the internal motions,

$$\vec{l} \equiv {\sum_\nu}' \rho^\nu \left(\vec{y}^{T\nu} \times \frac{d\vec{y}^{T\nu}}{dt}\right), \tag{7.2}$$

and to identify the flow of angular momentum k^L_{ij} as

$$k^L_{ij} \equiv l_i \dot{x}_j - \epsilon_{ikl} x_k t^L_{lj}. \tag{7.3}$$

These quantites satisfy the spatial frame angular momentum conservation equation

$$\frac{\partial \omega_i}{\partial t} + k^L_{ij,j} = 0. \tag{7.4}$$

From this equation is obtained the jump condition across a surface fixed in the spatial coordinate system,

$$[k^L_{ij}] n_j = 0. \tag{7.5}$$

For a surface fixed in the material coordinate system the jump condition by the arguments of Sec. IV becomes

$$[k^B_{ij}] n_j = 0, \tag{7.6}$$

where

$$k^B_{ij} \equiv k^L_{ij} - \omega_i \mathring{x}_j$$
$$= - \epsilon_{ikl} x_k t^L_{lj} - \left[\vec{x} \times \left(\rho \frac{d\vec{x}}{dt} + \epsilon_0 \vec{E} \times \vec{B}\right)\right]_i \dot{x}_j. \tag{7.7}$$

VIII. APPLICATION TO ENERGY FLOW

If the arguments of Secs. III and IV are applied to the energy-conservation equation, we are led to identify the energy density in the spatial frame of the dielectric crystal plus electromagnetic field as

$$W \equiv \tfrac{1}{2} \rho \frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt} + \frac{1}{2} {\sum_\nu}' \rho^\nu \frac{d\vec{y}^{T\nu}}{dt} \cdot \frac{d\vec{y}^{T\nu}}{dt} + \tfrac{1}{2} \epsilon_0 \vec{E} \cdot \vec{E}$$
$$+ (1/2\mu_0) \vec{B} \cdot \vec{B} + \rho \Sigma \tag{8.1}$$

and to identify the flow of energy as

$$S^L_j \equiv \left(\frac{1}{2} \frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt} + \frac{1}{2} {\sum_\nu}' \rho^\nu \frac{d\vec{y}^{T\nu}}{dt} \cdot \frac{d\vec{y}^{T\nu}}{dt} + \rho \Sigma\right) \mathring{x}_j$$
$$- (t^y_{ij} + \mathcal{E}_i P_j) \mathring{x}_i + (1/\mu_0)(\vec{E} \times \vec{B})_j. \tag{8.2}$$

These quantities satisfy the spatial frame energy conservation equation

$$\frac{\partial W}{\partial t} + S^L_{j,j} = 0. \tag{8.3}$$

From this equation is obtained the jump condition on the energy flow across a surface fixed in the spatial coordinate system,

$$[S^L_j] n_j = 0. \tag{8.4}$$

By the reasoning of Sec. IV the jump condition on the energy flow across a surface fixed in the material coordinate system is

$$[S^B_j] n_j = 0, \tag{8.5}$$

where

$$S^B_j \equiv S^L_j - W \mathring{x}_j$$
$$= - (t^y_{ij} + \mathcal{E}_i P_j) \dot{x}_i + (1/\mu_0)(\vec{E} \times \vec{B})_j$$
$$- [\tfrac{1}{2} \epsilon_0 \vec{E} \cdot \vec{E} + (1/2\mu_0) \vec{B} \cdot \vec{B}] \dot{x}_j. \tag{8.6}$$

The flow of energy $\vec{S}^L$ of Eq. (8.2) (or the analogous quantity $\vec{S}^B$ relevant to a moving body surface) includes both mechanical and electromagnetic energy. The division between the two types is indistinct, however, because terms such as $-\mathcal{E}_i P_j \dot{x}_i$ contain both electromagnetic fields and mechanical variables.

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