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Effect of isolated inhomogeneities on the galvanomagnetic properties of solids*

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A general expression is obtained for the magnetoresistance tensor of a solid containing a small number of macroscopic inhomogeneities. The result is used to show that a wide variety of inclusions will generate a linear transverse magnetoresistance in a free-electron metal at high fields. Voids or other nonconducting defects are found to produce a linear longitudinal as well as a linear transverse magnetoresistance. The applicability of the theory to the properties of real metals is briefly discussed.

I. INTRODUCTION

It has long been known that the presence of macroscopic sample inhomogeneities may seriously perturb the galvanomagnetic properties of metals and semiconductors. Such defects may be defined as spatial regions where the local conductivity tensor differs from that of the bulk of the sample. A number of inhomogeneities commonly occurring in solids can be, or have been, represented in this fashion—we mention, as examples, contraction voids or regions of localized strain. If these defects have linear dimensions larger than a characteristic mean free path, then they must be treated by the methods of continuum physics. Thus their effects are different from those of impurities and vacancies, which would normally be treated by scattering-theoretic techniques.

The first extensive treatment of such inhomogeneities (in the context of the galvanomagnetic problem) was carried out by Herring,[1] using a perturbative technique valid when spatial fluctuations in the elements of the conductivity tensor are small relative to the conductivity itself. When the fluctuations are not small, an exact treatment is no longer possible, but the galvanomagnetic properties of the system can still be approximately calculated by means of an effective medium or self-consistent-field approach.[2,3] Such methods have been developed by Cohen and Jortner[4] to treat the low-field Hall coefficient of inhomogeneous conductors, and have been generalized so as to allow calculation of the magnetoresistance by Stachowiak[5] and by Stroud.[6] None of these authors (except for Herring in a brief Appendix) has explicitly considered the particularly common limiting case in which the inhomogeneities consist of a few isolated inclusions surrounded by host material. In this paper we develop a method for calculating the influence of such inclusions upon the galvanomagnetic properties of solids. Our results thus supplement and complete those of several earlier workers,[7–9] who have considered the change in low-field Hall coefficient due to these defects. In order to illustrate the utility of the method we shall also carry out several model calculations that illustrate the effect of inhomogeneities on the high-field transport properties of a free-electron metal, and we discuss the possible relation of these calculations to the measured properties of such metals.

II. FORMALISM

We consider a solid of volume V, bounded by surface S, characterized by a resistivity tensor $\overleftrightarrow{\rho}$; and containing a volume fraction f of identical, ellipsoidal inclusions of resistivity tensor $\overleftrightarrow{\rho}'$ (see Fig. 1). Our goal is to calculate the effective resistivity tensor $\overleftrightarrow{\rho}_{\text{eff}}$ of this system. If the mean free path λ characteristic of a current carrier is small compared to a typical linear dimension d of an inclusion, and if d is in turn small compared to a linear dimension, say l, of the sample itself, then $\overleftrightarrow{\rho}_{\text{eff}}$ is uniquely defined and may be determined by a classical calculation. If instead $\lambda \gg d$ (the more familiar situation), then $\overleftrightarrow{\rho}_{\text{eff}}$ is more appropriately determined by a microscopic approach,

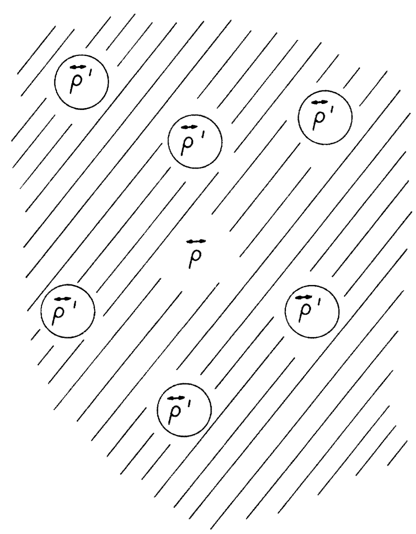

FIG. 1. Schematic of a solid of resistivity tensor $\overleftrightarrow{\rho}$ containing a few inhomogeneities of resistivity tensor $\overleftrightarrow{\rho}$.

e.g., via solution of the Boltzmann equation, as noted in the Introduction. If $\lambda < d$ but $d \approx l$, then the results of measurements cannot be described by intensive transport coefficients such as $\overleftrightarrow{\rho}_{\text{eff}}$ but will instead depend on sample shape, lead placement, and similar factors specific to the experimental arrangement. Thus we consider only the regime $\lambda \ll d \ll l$.

To define $\overleftrightarrow{\rho}_{\text{eff}}$ we will find it convenient to impose a boundary condition on the current density $\vec{J}$:

$$\hat{n} \cdot \vec{J}(\vec{x}) = \hat{n} \cdot \vec{J}_0 \tag{2.1}$$

for $\vec{x}$ on S. Here $\hat{n}$ is a unit normal outward from S and $\vec{J}_0$ is a constant vector. $\overleftrightarrow{\rho}_{\text{eff}}$ is then defined by

$$\langle \vec{E} \rangle = \overleftrightarrow{\rho}_{\text{eff}} \langle \vec{J} \rangle , \tag{2.2}$$

where the brackets denote a volume average.

Because of the boundary condition (2.1), the ith Cartesian component of $\langle \vec{J} \rangle$ satisfies

$$\begin{aligned} \langle J_i \rangle &= V^{-1} \int_V J_i \, d^3x = V^{-1} \int_V \vec{\nabla} \cdot (x_i \vec{J}) \, d^3x \\ &= V^{-1} \oint_S x_i \vec{J} \cdot \hat{n} \, d^2x = V^{-1} \oint_S x_i \vec{J}_0 \cdot \hat{n} \, d^2x = (\vec{J}_0)_i , \end{aligned} \tag{2.3}$$

where we have used $\vec{\nabla} \cdot \vec{J} = 0$. Hence $\langle \vec{J} \rangle = \vec{J}_0$ and the calculation of $\overleftrightarrow{\rho}_{\text{eff}}$ reduces to the problem of determining $\langle \vec{E} \rangle$.

We now specialize to the case where the volume fraction f occupied by the inclusion is very small. By this we mean the inclusions are so well separated that the current and field distortions generated by one inclusion do not overlap those of nearby defects. In this regime $\overleftrightarrow{\rho}_{\text{eff}}$ can be found exactly. The problem reduces in essence to that of a single defect of resistivity $\overleftrightarrow{\rho}'$ immersed in a uniform medium of resistivity $\vec{J}$. Far from each defect $\vec{J}(\vec{x}) \to \vec{J}_0$ and $\vec{E}(\vec{x}) \to \overleftrightarrow{\rho} \vec{J}_0$. Inside each defect $\vec{E}$ and $\vec{J}$ are both uniform, provided only that the defect is ellipsoidal, and independent of the symmetry properties of $\overleftrightarrow{\rho}$ and $\overleftrightarrow{\rho}'$ or of the orientation of the principal axes of the ellipsoid relative to those of $\overleftrightarrow{\rho}$ and $\overleftrightarrow{\rho}'$.[10] As is shown in the Appendix, the interior electric field $\vec{E}_{\text{in}}$ is given by

$$\vec{E}_{\text{in}} = [\overleftrightarrow{1} - \overleftrightarrow{\Gamma} \overleftrightarrow{\delta\sigma}]^{-1} \cdot \vec{E}_0, \tag{2.4}$$

where

$$\begin{aligned} &\overleftrightarrow{\delta\sigma} = \overleftrightarrow{\sigma}' - \overleftrightarrow{\sigma}, \quad \overleftrightarrow{\sigma}' = (\overleftrightarrow{\rho}')^{-1}, \quad \overleftrightarrow{\sigma} = (\overleftrightarrow{\rho})^{-1}, \\ &\Gamma_{ij} = \oint_{S'} \frac{\partial G(\vec{x}')}{\partial x_i'} n_j \, d^2x' , \end{aligned} \tag{2.5}$$

and $G(\vec{x})$ is an electrostatic Green's function satisfying

$$\vec{\nabla} \cdot \overleftrightarrow{\sigma} \vec{\nabla} G(\vec{x} - \vec{x}') = -\delta(\vec{x} - \vec{x}'), \tag{2.6}$$

with the boundary condition $G(\vec{x} - \vec{x}') \to 0$ as $|\vec{x} - \vec{x}'| \to \infty$. In Eq. (2.4) $\vec{E}_0 = \overleftrightarrow{\rho} \vec{J}_0$ is the field far from the inhomogeneity and $\overleftrightarrow{1}$ denotes the 3×3 unit matrix, while in Eq. (2.5) S' represents the surface of a defect centered at the origin and n_i is a Cartesian component of a unit normal directed outward from S'.

Given $\vec{E}_{\text{in}}$, ρ_{eff} can immediately be determined. We have

$$\langle \vec{E} \rangle = V^{-1} \int_V \overleftrightarrow{\rho}(\vec{x}) \vec{J}(\vec{x}) \, d^3x ,$$

where $\overleftrightarrow{\rho}(\vec{x})$ is defined by

$$\overleftrightarrow{\rho}(\vec{x}) = \begin{cases} \overleftrightarrow{\rho}', & \vec{x} \text{ within a defect,} \\ \overleftrightarrow{\rho}, & \text{otherwise.} \end{cases}$$

Introducing $\overleftrightarrow{\delta\rho}(\vec{x}) = \overleftrightarrow{\rho}(\vec{x}) - \overleftrightarrow{\rho}$, we obtain

$$\begin{aligned} \langle \vec{E} \rangle &= \overleftrightarrow{\rho} \vec{J}_0 + f(\overleftrightarrow{\rho}' - \overleftrightarrow{\rho}) \overleftrightarrow{\sigma}' \vec{E}_{\text{in}} \\ &= [\overleftrightarrow{\rho} - f \overleftrightarrow{\rho} \, \overleftrightarrow{\delta\sigma} (\overleftrightarrow{1} - \overleftrightarrow{\Gamma} \, \overleftrightarrow{\delta\sigma})^{-1} \overleftrightarrow{\rho}] \vec{J}_0, \end{aligned}$$

upon using (2.4). It follows from the definition (2.2) and from $\langle \vec{J} \rangle = \vec{J}_0$ that

$$\overleftrightarrow{\rho}_{\text{eff}} = \overleftrightarrow{\rho} - f \overleftrightarrow{\rho} \, \overleftrightarrow{\delta\sigma} (\overleftrightarrow{1} - \overleftrightarrow{\Gamma} \, \overleftrightarrow{\delta\sigma})^{-1} \overleftrightarrow{\rho} . \tag{2.7}$$

Using Eq. (2.7), one may calculate the effect of ellipsoidal inclusions on the galvanomagnetic properties of many materials. Note that $\overleftrightarrow{\rho}_{\text{eff}}$ can be found without the necessity of explicitly evaluating the highly anisotropic fields outside the inhomogeneity. The effects of these fields are included indirectly in the tensor $\overleftrightarrow{\Gamma}$. $\overleftrightarrow{\Gamma}$ depends on the shape of the inclusion and on $\overleftrightarrow{\sigma} = (\overleftrightarrow{\rho})^{-1}$, but can normally be evaluated without difficulty. Thus we expect Eq. (2.7) to be useful in the analysis of measurements on various materials containing macroscopic imperfection.

Several other aspects of the result (2.7) deserve mention. First, the equation is essentially analogous to typical microscopic calculations for the contribution of impurities to the resistivity of a solid. Just as in that problem, the presence of the factor f in the result is a feature of the low-concentration or noninteracting limit; the existence of interactions between defects would lead to a more complicated concentration dependence. Note also that (2.7), while it depends on the shape of the defects, does not depend on their size so long as they are small compared to the sample size, but only on the total volume fraction they occupy. Finally, if more than one kind of defect is present in the sample, then their contributions to the total resistivity tensor are simply additive in the low-concentration limit (this may be viewed as a macroscopic "Mathiessen's rule").

III. MODEL CALCULATIONS

In order to illustrate the utility of the formalism just developed, we now use Eq. (2.7) to study the effects of spherical inhomogeneities on the magnetoresistance tensor of a free-electron metal.

For this case, taking the magnetic field to be in the z direction, we have[11]

$$\sigma_{xx}=\sigma_{yy}=\sigma_0/[1+(\omega_c\tau)^2],$$
$$\sigma_{xy}=-\sigma_{yx}=\sigma_0\omega_c\tau/[1+(\omega_c\tau)^2], \quad (3.1)$$
$$\sigma_{zz}=\sigma_0.$$

The other components of $\overleftrightarrow{\sigma}$ vanish. In (3.1) $\omega_c=eH/mc$ is the free-electron cyclotron frequency and τ is a relaxation time characteristic of the metal. Equation (2.6) can be solved for G via a scale transformation and the integral (2.5) for $\overleftrightarrow{\Gamma}$ can be evaluated. $\overleftrightarrow{\Gamma}$ proves to be diagonal with components

$$\Gamma_{zz}=-(1-\sqrt{1-\epsilon}\,\sin^{-1}\sqrt{\epsilon}/\sqrt{\epsilon})/(\sigma_{zz}\epsilon),$$
$$\Gamma_{xx}=\Gamma_{yy}=-\tfrac{1}{2}(\Gamma_{zz}+\sin^{-1}\sqrt{\epsilon}/\sqrt{\epsilon\sigma_{xx}\sigma_{zz}}), \quad (3.2)$$

with $\epsilon=1-\sigma_{xx}/\sigma_{zz}$.

We have used Eqs. (3.1) and (3.2) to determine $\overleftrightarrow{\rho}_{eff}$ for a free-electron metal containing a volume fraction f of spherical voids ($\overleftrightarrow{\sigma}'=0$). The components of $\overleftrightarrow{\rho}_{eff}$ which are of greatest experimental concern are the transverse magnetoresistance $\Delta\rho_{xx}$, the longitudinal magnetoresistance $\Delta\rho_{zz}$, and the Hall coefficient R_H. These are defined by

$$\Delta\rho_{xx}(H)=[\rho_{xx}^{eff}(H)-\rho_{xx}^{eff}(0)]/\rho_{xx}^{eff}(0),$$
$$\Delta\rho_{zz}(H)=[\rho_{zz}^{eff}(H)-\rho_{zz}^{eff}(0)]/\rho_{zz}^{eff}(0), \quad (3.3)$$
$$R_H=[\rho_{xy}^{eff}(H)-\rho_{xy}^{eff}(-H)]/(2H).$$

The results for these three coefficients are plotted in Fig. 2 for $f=0.01$. In the high-field regime, both $\Delta\rho_{xx}$ and $\Delta\rho_{zz}$ vary linearly with field. The asymptotic slopes may be calculated from Eqs. (2.7), (3.1), and (3.2), with the results

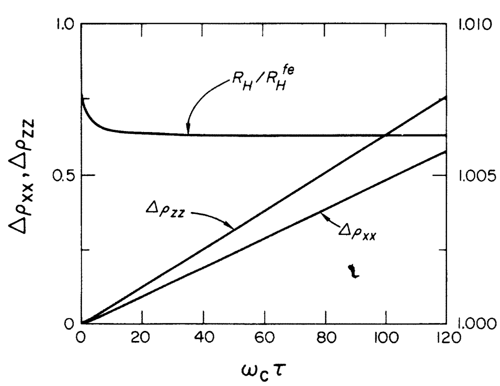

FIG. 2. Transverse magnetoresistance $\Delta\rho_{xx}$, longitudinal magnetoresistance $\Delta\rho_{zz}$, and Hall coefficient R_H for a free-electron metal containing 1% by volume nonoverlapping spherical voids. $\omega_c\tau$ is a dimensionless measure of magnetic field strength, and R_H^{fe} is the field-independent Hall coefficient of the pure free-electron metal. $\Delta\rho_{xx}$ and $\Delta\rho_{zz}$ both vary quadratically with H at small $\omega_c\tau$, although this behavior is masked by the plots.

$$d[\Delta\rho_{xx}(\omega_c\tau)]/d(\omega_c\tau)=\alpha f,$$
$$d[\Delta\rho_{zz}(\omega_c\tau)]/d(\omega_c\tau)=\alpha' f, \quad (3.4a)$$

with

$$\alpha=\tfrac{1}{4}\pi/[1+(\tfrac{1}{4}\pi)^2]=0.49,$$
$$\alpha'=2/\pi=0.64. \quad (3.4b)$$

These results are in agreement with those obtained by Sampsell and Garland[12] via a direct integration of the power dissipated in the vicinity of the voids. The Hall constant, on the other hand, is very nearly field independent and approximately equal to its free-electron value R_H^{fe}. In the low-field limit ($\omega_c\tau\ll 1$), we find

$$R_H/R_H^{fe}=1+\tfrac{3}{4}f, \quad (3.5)$$

which is in agreement to first order in f with the low-field calculations of Juretschke *et al.*[7] and of Cohen and Jortner.[4]

If the defects consist of regions of infinite conductivity — that is, if the components of the defect conductivity tensor $\overleftrightarrow{\sigma}'$ are so large that $\overleftrightarrow{\delta\sigma}(\overleftrightarrow{1}-\overleftrightarrow{\Gamma}\overleftrightarrow{\delta\sigma})^{-1}\approx-\overleftrightarrow{\Gamma}^{-1}$ — then one finds that $\Delta\rho_{xx}$ continues to increase linearly with field at high magnetic fields, but $\Delta\rho_{zz}$ saturates. The asymptotic slope of $\Delta\rho_{xx}$ is given by Eq. (3.4a), with $\alpha=4/\pi=1.27$. It is rather amusing that inhomogeneities of high conductivity lead to a larger magnetoresistance than do zero-conductivity defects.

It can be shown from Eq. (3.2) that any spherical defect will produce a linear transverse magnetoresistance provided either (a) $\delta\sigma_{xy}\approx(\omega_c\tau)^{-1}$ in the high-field regime or (b) $\delta\sigma_{xx}$ and $\delta\sigma_{yy}$ are field independent at large fields. A linear longitudinal magnetoresistance will be generated, however, only by imperfections of strictly zero conductivity. The reason for this is that only nonconducting defects can force a current moving parallel to the H field to flow locally in the direction of much lower conductivity perpendicular to $\vec{H}$. Spherical defects of scalar conductivity $\sigma'\ll\sigma$ can be shown, from (2.7), (3.1), and (3.2), to generate at high fields ($\omega_c\tau\gg 1$) a longitudinal magnetoresistance of the form

$$\Delta\rho_{zz}(H)\cong f/[\sigma'/\sigma_0+\tfrac{1}{2}\pi(\omega_c\tau)^{-1}]. \quad (3.6)$$

Thus in the case $\Delta\rho_{zz}$ is linear in H for $\omega_c\tau\ll\sigma_0/\sigma'$, but departures from linearity and eventual saturation will occur at stronger fields.

To learn the effects of imperfection shape on $\overleftrightarrow{\rho}_{eff}$, we have also computed that tensor for a free-electron metal containing very long cylindrical voids parallel to the y axis. Equation (2.7) continues to be applicable in this case since a cylinder is a limiting case of a highly elongated ellipsoid. $\overleftrightarrow{\Gamma}$ is again found to be diagonal, with components

$$\Gamma_{zz}=(\sqrt{1-\epsilon}-1)/(\sigma_{zz}\epsilon),$$

$$\Gamma_{xx} = (\Gamma_{zz} + 1/\sigma_{zz})/(1-\epsilon),$$
$$\Gamma_{yy} = 0, \tag{3.7}$$

with $\epsilon = 1 - \sigma_{xx}/\sigma_{zz}$. Combining (3.1) and (3.7) with (2.7) leads to expressions for $\Delta\rho_{xx}$ and $\Delta\rho_{zz}$. In the high-field regime both are found to vary linearly with H. The slopes are given by Eq. (3.4a), with $\alpha = \alpha' = 1$. Thus the existence of a linear magnetoresistance is insensitive to the shape of the inclusion.

IV. DISCUSSION

The original motivation for this work was to examine the hypothesis[13] that the anomalous high-field transport properties of potassium and other nearly-free-electron metals could be accounted for by the presence of inhomogeneities. We now turn to a discussion of this hypothesis in the model calculations just described.

The salient experimental features in potassium are as follows: Both the transverse and the longitudinal magnetoresistance are observed to increase linearly with magnetic field in the regime $\omega_c\tau \gg 1$. The dimensionless slopes ("Kohler slopes") $d(\Delta\rho_{xx})/d(\omega_c\tau)$ and $d(\Delta\rho_{zz})/d(\omega_c\tau)$ vary widely from sample to sample but generally fall in the range 10^{-2}–10^{-4}. The ratio of slopes $[d(\Delta\rho_{zz})/d(\omega_c\tau)]/[d(\Delta\rho_{xx})/d(\omega_c\tau)]$ is reported by Lass[14] to be relatively sample independent and to average about 1.3. The Kohler slopes are not strongly correlated with measures of impurity content such as the residual resistivity ratio (i.e., the ratio of the room-temperature zero-field resistivity to the resistivity at 0 °K).[15] Nor do they depend on temperature. The Hall coefficient differs by no more than a few percent from its free-electron value, and appears to saturate at high fields.[16,17]

The explanation of these data on the basis of inhomogeneities would thus require the presence of (0.01–1)% voids or other nonconducting (or very poorly conducting) imperfections. There are, however, two major objections to consider. First, no explicit quantitative experiment has been carried out to our knowledge which has demonstrated the presence of such defects in a carefully prepared single-crystal specimen of potassium. Clearly such a measurement would be necessary before the defect theory could be accepted. (Conversely, it would be very desirable if nonconducting defects could be introduced into potassium or another free-electron metal in controlled quantites in order to verify the effects calculated in Sec. III.) The second objection concerns the size of the defects. For the theory of Sec. II to be strictly applicable, the defects must be large compared to a mean free path. In a high-purity sample of potassium at cryogenic temperatures the mean free path may exceed 0.1 mm. Inhomogeneities greatly exceeding this size would appear unlikely. It could be argued that in strong magnetic field the appropriate "mean free path" is actually the cyclotron radius, which is smaller than λ by a factor of $(\omega_c\tau)^{-1}$, but even defects of this size would extend for thousands of angstroms in a good potassium sample. It may be that the effects predicted in Sec. III for large defects would occur even for defects small compared to λ and large compared to an interatomic separation, but the present paper cannot shed additional light on this possibility. Thus the applicability of the present formalism to potassium must remain an open question. There would appear to be little doubt, however, that at least some of the unexplained linear magnetoresistances reported in free-electron metals[18] can be attributed to inhomogeneities.

We consider finally the possibility that the present results will be altered by interactions between defects. In strong magnetic fields, a defect of diameter d will generate in the surrounding medium a current distortion which propagates a distance of order $d(\omega_c\tau)$ parallel to $\vec{H}$.[12] At fields such that $f(\omega_c\tau) > 1$, these distortions have a substantial probability of overlapping neighboring defects; interaction effects will then start to influence $\overleftrightarrow{\rho}_{\text{eff}}$ by producing departures from a strictly linear magnetoresistance dependence. Approximate calculations of these interaction effects will be reported in a subsequent publication.

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APPENDIX: DERIVATION OF EQ. (2.4)

We consider an isolated ellipsoidal inhomogeneity of conductivity tensor $\overleftrightarrow{\sigma}'$ embedded in an infinite uniform conductor of conductivity tensor $\overleftrightarrow{\sigma}$. Far from the inhomogeneity the electric field $\vec{E}(\vec{x})$ approaches a constant $\vec{E}_0$ and the current density $\vec{J}(\vec{x})$ approaches $\overleftrightarrow{\sigma}\vec{E}_0$. The electrostatic equations are $\vec{\nabla}\cdot\vec{J} = 0$, $\vec{\nabla}\times\vec{E} = 0$; these combined with the boundary condition on $\vec{E}$ imply that the electrostatic potential $\Phi(\vec{x})$ satisfies

$$\nabla\cdot\overleftrightarrow{\sigma}(\vec{x})\vec{\nabla}\Phi(\vec{x}) = 0,$$
$$\Phi(\vec{x}) \to -\vec{E}_0\cdot\vec{x} \quad \text{as } \vec{x}\to\infty, \tag{A1}$$

where $\overleftrightarrow{\sigma}(\vec{x})$ is defined by

$$\overleftrightarrow{\sigma}(\vec{x}) = \overleftrightarrow{\sigma}' \quad \text{for } \vec{x} \text{ inside the inclusion,}$$
$$= \overleftrightarrow{\sigma} \quad \text{otherwise.} \tag{A2}$$

With the introduction of the free-space Green's function defined by (2.6), Eq. (A1) may be rewritten as an integral equation:

$$\Phi(\vec{x}) = -\vec{E}_0 \cdot \vec{x} + \int G(\vec{x}-\vec{x}\,')\,\vec{\nabla}' \cdot [\overleftrightarrow{\delta\sigma}(\vec{x}\,')\,\vec{\nabla}'\Phi(\vec{x}\,')]\; d^3x', \quad \text{(A3)}$$

where $\overleftrightarrow{\delta\sigma}(\vec{x}\,') = \overleftrightarrow{\sigma}(\vec{x}\,') - \overleftrightarrow{\sigma}$ and the integral runs over all space. Since the bracketed quantity is constant within the inclusion and zero outside, the last term in (A3) can be converted to a surface integral with the help of a parts integration and an application of the divergence theorem. The result is

$$\Phi(\vec{x}) = -\vec{E}_0 \circ \vec{x} + \oint_{S'} G(\vec{x}-\vec{x}\,')(\overleftrightarrow{\sigma}' - \overleftrightarrow{\sigma})\vec{E}_{\text{in}} \cdot \hat{n}\, d^2x', \quad \text{(A4)}$$

where S' is as before the surface of the defect and $\hat{n}$ an outward normal from it. Taking the negative gradient of each side, setting $\vec{x}=0$, and using $G(-\vec{x}\,') = G(\vec{x}\,')$ then yields

$$\vec{E}_{\text{in}} = \vec{E}_0 + \overleftrightarrow{\Gamma}(\overleftrightarrow{\sigma}' - \overleftrightarrow{\sigma}) \circ \vec{E}_{\text{in}}, \quad \text{(A5)}$$

from which (2.4) follows immediately. (Note that this derivation depends strongly on the uniformity of $\vec{E}_{\text{in}}$ within the inhomogeneity.)

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