

Renormalization-group treatment of the critical dynamics of superfluid helium, the isotropic antiferromagnet, and the easy-plane ferromagnet

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Phenomenological models with "planar-spin" and "antiferromagnetic" dynamics are introduced, and their critical behavior is analyzed using renormalization-group methods. Dynamic scaling is shown to hold for these models to all orders in $\epsilon = 4 - d$, and the dynamic exponents are expressed entirely in terms of static exponents, in agreement with earlier phenomenological and mode-coupling theories. The magnitudes of the diverging transport and kinetic coefficients are expressed purely in terms of static properties and of universal constants which are calculated to second order in ϵ . Matching conditions are proved between the characteristic frequencies above and below T_c , and the corresponding universal amplitude ratios are calculated to second order in ϵ . The principal applications are to liquid helium and the Heisenberg antiferromagnet RbMnF_3 , where the experimental exponents and amplitudes both agree reasonably well with theory. In the case of liquid helium ("asymmetric" case with $\alpha < 0$) the asymptotic critical behavior is somewhat masked by correction terms, due to the slow approach of the specific heat to its finite value at T_λ . These correction terms are analyzed in detail, and a proposal is made for extracting the true asymptotic behavior from the data. The effects of other conserved fields such as the mass density and momentum in helium, and the energy density in the magnet, are considered, and shown to be irrelevant for the critical behavior of the order parameter.

I. INTRODUCTION

One of the principal achievements of the phenomenological scaling approach to critical phenomena¹⁻⁵ was the prediction,⁴ and subsequent experimental verification,⁶ of a divergent thermal conductivity in liquid helium upon approaching the λ point. The critical exponent for this divergence was determined in terms of the exponents for the superfluid density ρ_s and the specific heat C_p . Moreover, the amplitude of the divergence and the detailed temperature dependence were found to be consistent with a "matching condition"⁴⁻⁶ between thermal diffusion and second sound. Subsequently, more accurate measurements⁷ at different pressures along the λ line have cast some doubt on the precise validity of the matching condition, and of the scaling theory which lies at the basis of this condition, but these are at the very least excellent approximations to the true critical behavior. In addition, calculations based on "mode-mode coupling" theories^{8,9} have also yielded results consistent with dynamic scaling in liquid helium.

In the present paper the renormalization-group method^{10,11} is applied to a classical two-field planar-spin model¹² whose dynamic critical behavior is expected to be identical to that of super-

fluid helium. The scaling picture, and the matching condition which follows from it, are found to hold exactly for the model in the limit as the temperature tends to T_c . The rate of approach to this asymptotic limit, however, is found to be anomalously slow in liquid helium, owing to the near-logarithmic singularity¹³ in the specific heat. Thus, as explained in detail below, there are perceptible corrections to the ideal critical behavior in the experimental range of measurement, as was in fact observed.⁷ The planar-spin model for liquid helium can also be applied to the critical dynamics of an anisotropic (easy-plane) ferromagnet.^{12,14} A related model, which represents the critical dynamics of the isotropic Heisenberg antiferromagnet^{5,14,15} is also discussed in this paper.

All the above models were introduced in an earlier Letter,¹² where some features of the critical behavior were obtained. The models involve two coupled fields—a vector order parameter Ψ which is not conserved, and a conserved density m . There is, however, a difference with case C of the time-dependent Ginzburg-Landau model discussed by Halperin, Hohenberg, and Ma¹¹ (HHM), in that the conserved quantity $M = \int d^d x m(x)$ is now an infinitesimal generator for rotations of the order parameter Ψ , in a sense

which will be made more precise below. This property leads to the existence of propagating hydrodynamic modes in the ordered phase,¹⁴ whereas model *C* contains purely relaxational modes, both above and below T_c .¹¹

In the easy-plane ferromagnet the field Ψ is a two-component vector (or complex scalar) corresponding to the magnetization in the plane, while M is a real scalar corresponding to the magnetization in a direction perpendicular to the plane. In the case where the field H_z conjugate to M is zero, we have $\langle M \rangle = 0$, which we shall refer to as the "symmetric planar-spin model," denoted model *E*. For finite field H_z , the symmetry between M and $-M$ is lost, and the system will be referred to as model *F*, or the "asymmetric planar-spin model." Liquid helium belongs to this latter class, the order parameter Ψ being the expectation value of the complex Bose field, and the conserved field M a linear combination of the energy and particle number.

In the isotropic antiferromagnet,^{12, 15} or model *G*, $\vec{\Psi}$ is a three-component vector representing the staggered magnetization, while $\vec{M}$ is also a

three-component vector, representing the total magnetization. Besides the planar-spin and antiferromagnetic models of the present paper, and the time-dependent Ginzburg-Landau models,¹¹ a number of other dynamical models have been treated by renormalization-group methods.^{12, 15-19} These are summarized in Table I.

For all the two-field models we consider here (models *E-G*), the property of dynamic scaling^{4, 5} may be proved to all orders in the parameter $\epsilon = 4-d$, where d is the dimensionality. Moreover, the dynamic exponents are expressible purely in terms of static exponents, in complete accord with earlier phenomenological and mode-mode coupling theories.^{4, 5, 8, 20, 21} For the asymmetric planar-spin model, and in particular for liquid helium, we find different results for the asymptotic critical behavior, in the case where the specific heat is divergent ($\alpha > 0$), and in the case where it is finite ($\alpha < 0$), as was indeed predicted by the earlier theories.^{4, 5, 8} In the present work we consider in addition corrections to the leading singularities, which reflect the slow approach of the specific heat to its asymptotic limit, and are

TABLE I. Summary of dynamical models.

System	Designation	n	Nonconserved fields	Conserved fields	Nonvanishing Poisson brackets	Reference
Time-dependent Ginzburg-Landau models	<i>A</i>	n	ψ	none	none	11, 18
	<i>B</i>	n	none	ψ	none	
	<i>C</i>	n	ψ	$\bar{\epsilon}$	none	
	<i>D</i>	n	none	$\psi, \bar{\epsilon}$	none	
Two-field planar-spin models	<i>E</i>	2	ψ	m	$\{\psi, m\}$	12
	<i>F</i>	2	ψ	m	$\{\psi, m\}$	This work
Three-field planar-spin models	<i>E'</i>	2	ψ	$m, \bar{\epsilon}$	$\{\psi, m\}$	This work
	<i>F'</i>	2	ψ	$m, \bar{\epsilon}$	$\{\psi, m\}$	
Two-field isotropic antiferromagnet	<i>G</i>	3	$\vec{\psi}$	$\vec{m}$	$\{\vec{\psi}, \vec{m}\}$ $\{\vec{m}, \vec{m}\}$	12, 15 This work
	<i>G'</i>	3	$\vec{\psi}$	$\vec{m}, \bar{\epsilon}$	$\{\vec{\psi}, \vec{m}\}$ $\{\vec{m}, \vec{m}\}$	This work
Fluid	<i>H</i>	1	none	$\psi, \vec{j}$	$\{\psi, \vec{j}\}$	12, 19
One-field isotropic ferromagnet	<i>J</i>	3	none	$\vec{\psi}$	$\{\vec{\psi}, \vec{\psi}\}$	16
	<i>J'</i>	3	none	$\vec{\psi}, \bar{\epsilon}$	$\{\vec{\psi}, \vec{\psi}\}$	
Two-field isotropic ferromagnet	<i>J'</i>	3	none	$\vec{\psi}, \bar{\epsilon}$	$\{\vec{\psi}, \vec{\psi}\}$	...
Multicomponent Bose system	<i>K</i>	$n = 2m$	$\psi_\alpha \alpha = 1, \dots, m$	$\rho_\alpha, \rho_{\alpha\beta}, \mu_{\alpha\beta}$		See Ref. 17

important in the experimental range of temperatures whenever $|\alpha|$ is small (e.g., <0.1). In all cases we predict divergent transport coefficients and a universal "matching condition" between the characteristic frequencies above and below T_c , analogous to the one first proposed by Ferrel *et al.*⁴

Associated with the universal matching condition there is a universal amplitude ratio, which is analogous to a critical exponent in that it is only a property of the specific fixed point reached, and not dependent on the details of the starting Hamiltonian. Unlike the dynamic exponents, however, the amplitude ratios are not expressible purely in terms of static properties; rather, they may be calculated in an ϵ expansion²² for the different fixed points. Such a calculation of amplitude ratios has also recently been carried out by Gunton and Kawasaki⁹ from a mode-mode theory. Their results agree with ours in lowest order for model *E*, but disagree for model *F*.

Comparison of the calculated ratios with measured values⁷ in liquid helium is complicated by the presence of the aforementioned correction terms, which are estimated to lead to an effect of the order of (20–30)%. We have nevertheless carried out such a comparison in liquid helium (model *F*) and RbMnF₃ (model *G*), for which experimental data are available,^{7,23,24} and find rather good agreement with the second-order ϵ expansion. The ϵ expansion appears to converge rather poorly, however, and the agreement may be fortuitous. A more important test of our predictions would result from a careful analysis of the experiments themselves, to see if they are consistent with universality.

A. Outline

In Sec. II the planar-spin models are introduced, and their hydrodynamic properties briefly discussed. Section III is devoted to an analysis of the dynamic renormalization group for both the symmetric and asymmetric cases. In the symmetric case (model *E*) the dynamic exponent is found to be $z = 2 - \frac{1}{2}\epsilon = \frac{1}{2}d/2$, while in the asymmetric case (model *F*), the dynamic exponent has the value $z = \frac{1}{2}d + \bar{\alpha}/2\nu$, where $\bar{\alpha} \equiv \max(\alpha, 0)$. These exponent relations are then shown to be correct to all orders in ϵ , by analyzing the higher-order corrections to the recursion relations. Moreover, the dynamic scaling property and the frequency matching condition are also proved to all orders in ϵ , with the corresponding amplitude ratio expressed in terms of (universal) fixed-point parameters. In Sec. IV we discuss the relation of our two-field models to various magnetic Hamiltonians and to liquid helium, where

additional quantities are conserved, such as energy and momentum. As mentioned above, liquid helium corresponds to model *F*, and has a specific heat exponent which is nearly zero, but almost certainly negative,¹³ so that singular corrections to the leading singularities are never negligible in the experimental range. An analysis of the important correction terms arising from the recursion relations leads to a rather complicated temperature dependence for the diverging thermal conductivity above T_λ , and accounts semiquantitatively for the observed lack of universality.⁷ Model *G*, for the antiferromagnet, is discussed in Sec. V. The results are compared with neutron scattering experiments^{23,24} on RbMnF₃, where the characteristic frequencies for both the staggered magnetization (Ψ) and the total magnetization (M) can be measured above and below T_c . Section VI concludes with a detailed summary of the results of the present paper. Finally, a number of calculations are presented in the Appendixes: Appendix A discusses details of the hydrodynamics of models *E* and *F*, while Appendix B describes the diagrammatic perturbation theory used to calculate the dynamic response functions for the planar-spin models. In Appendix C the recursions are solved in the asymmetric model for $\alpha < 0$, and the principal corrections to scaling are found for this case. Appendix D contains the second-order calculation of the various experimentally observable amplitude ratios, for model *E*, while Appendix E discusses the effects of energy conservation on the various models. Appendix F describes the modifications in the diagrammatic expansion for the case of the isotropic antiferromagnet (model *G*).

II. PLANAR-SPIN MODELS

Let us consider the following phenomenological model¹²:

$$\frac{\partial \psi}{\partial t} = -2 \frac{\Gamma_0}{c_0} \frac{\delta F_0}{\delta \psi^*} - ig_0 \psi \frac{\delta F_0}{\delta m} + \theta, \quad (2.1a)$$

$$\frac{\partial m}{\partial t} = \lambda_0 \nabla^2 \frac{\delta F_0}{\delta m} + 2g_0 \text{Im} \left(\psi^* \frac{\delta F_0}{\delta \psi^*} \right) + \zeta, \quad (2.1b)$$

$$F_0 = \int d^d x \left(\frac{1}{2} c_0 \tilde{r}_0 |\psi|^2 + \frac{1}{2} c_0 |\nabla \psi|^2 + c_0^2 \tilde{u}_0 |\psi|^4 + \frac{1}{2} \chi_0^{-1} m^2 + c_0 \gamma_0 m |\psi|^2 - h_0 m \right), \quad (2.1c)$$

$$\langle \theta(x, t) \theta^*(x', t') \rangle = 4c_0^{-1} \text{Re} \Gamma_0 \delta(x - x') \delta(t - t'), \quad (2.1d)$$

$$\langle \zeta(x, t) \zeta(x', t') \rangle = -2\lambda_0 \nabla^2 \delta(x - x') \delta(t - t'), \quad (2.1e)$$

$$\langle \theta \theta \rangle = \langle \theta^* \theta^* \rangle = \langle \theta \zeta \rangle = \langle \theta^* \zeta \rangle = 0. \quad (2.1f)$$

Here $\psi(x, t)$ and $m(x, t)$ are space- and time-dependent scalar fields with Fourier components $k \leq \Lambda$, ψ being complex and m real, while θ and ζ are Gaussian noise sources. The parameters $\tilde{r}_0$, $\tilde{u}_0$, χ_0 , γ_0 , h_0 , c_0 , g_0 , and λ_0 are real constants, while Γ_0 may be complex. The real part of Γ_0 , as well as λ_0 , χ_0 , c_0 and $u_0 \equiv \tilde{u}_0 - \frac{1}{2}\gamma_0^2\chi_0$ are assumed to be greater than zero. The constant c_0 determines the overall scale of ψ and is often chosen to be unity. In Eq. (2.1) and throughout this paper, a derivative with respect to ψ or ψ^* is to be interpreted as

$$\frac{\delta F_0}{\delta \psi^*} = \frac{1}{2} \left(\frac{\delta F_0}{\delta \text{Re} \psi} + i \frac{\delta F_0}{\delta \text{Im} \psi} \right). \quad (2.2)$$

In the general case where all constants are non-zero, we shall refer to the system described by (2.1) as model F or the asymmetric planar-spin model. In the special case where Γ_0 is real, and $h_0 = \gamma_0 = 0$, the system will be denoted as model E , or the symmetric planar-spin model. The equations of motion are then invariant under the transformation $m \rightarrow -m$ and $\psi \rightarrow \psi^*$. For an anisotropic Heisenberg ferromagnet with an easy plane of magnetization (the x - y plane) this transformation corresponds to a rotation of all spins by 180° about the x axis, which will leave the equations of motion invariant if there is no external field H_z (see Sec. IVA).

Another special case of (2.1) occurs if we set $g_0 = 0$, and take Γ_0 real; we then obtain case C of HHM,¹¹ for $n=2$ [in Ref. 11 the field m was denoted $\tilde{\epsilon}$, χ_0 was C_0 , and h_0 was $\delta\beta_0$.] In principle, one may add various higher-order terms to F_0 , such as terms proportional to $|\psi|^2 m^2$, $|\nabla m|^2$, m^3 , m^4 , $|\psi|^6$, etc. Such terms turn out to be "irrelevant" in the renormalization-group sense¹⁰ for both statics and dynamics at long wavelengths, at least near $d=4$. It is also possible to consider more complicated equations of motion (which need not be Markoffian) as discussed in Sec. IV and Appendix E.

We shall be interested in the response of our system to time-dependent applied fields coupling to ψ and m . These fields are included in the equations of motion by adding to F_0 a perturbation F'_0 given by

$$F'_0 = - \int d^4 x \{ h_m(x, t) m(x, t) + \text{Re} [h_\psi(x, t) \psi^*(x, t)] \}. \quad (2.3)$$

We may define the wave vector and frequency-dependent linear response functions $\chi_\psi(k, \omega)$ and $\chi_m(k, \omega)$ in the usual way, assuming that the system starts out at $t = -\infty$ in a thermal equilibrium state (defined below). These response functions

are related by the fluctuation-dissipation theorem to time-dependent correlation functions for the system in the absence of applied fields.

Two properties of the equations of motion (2.1) deserve special emphasis. First, let us remark that m is a conserved quantity in the absence of a field h_ψ . When $h_\psi = 0$, we have

$$\text{Im} \left(\psi^* \frac{\delta F}{\delta \psi^*} \right) = \nabla \cdot \text{Im}(\psi^* \nabla \psi). \quad (2.4)$$

Furthermore, the noise source ζ vanishes in the limit $k \rightarrow 0$, according to (2.1e), so that ζ may be written as the divergence of a current. Thus Eq. (2.1b) may be written

$$\frac{\partial m}{\partial t} = -\nabla \cdot j^m, \quad (2.5)$$

and therefore M , the space-integral of m , is independent of time. Secondly, we note that the equations of motion are unchanged if $\psi(x, t)$ is multiplied by any constant phase factor, $e^{i\varphi}$.

As discussed in Appendix A, in the absence of time-dependent applied fields there exists a time independent (equilibrium) state in which the probability density $P[\psi, m]$ is the thermal equilibrium distribution

$$P_{\text{eq}}[\psi, m] = Z^{-1} e^{-F_0[\psi, m]}, \quad (2.6)$$

$$Z \equiv \int e^{-F_0[\psi, m]} d[\psi] d[m]. \quad (2.7)$$

Since the field m enters F_0 only quadratically, the Gaussian integral in (2.7) can be performed to eliminate m from the equilibrium distribution (2.6). The remaining equilibrium probability distribution for ψ is then precisely the same as in the usual Ginzburg-Landau-Wilson¹⁰ model (for $n=2$) with shifted coefficients

$$r_0 = \tilde{r}_0 - \gamma_0 \chi_0 h_0, \quad (2.8)$$

$$u_0 = \tilde{u}_0 - \frac{1}{2} \gamma_0^2 \chi_0. \quad (2.9)$$

(See Ref. 11 for a more complete discussion of this point.) It follows that for appropriate choices of the parameters r_0 and u_0 , the system will be at a critical point, characterized by the usual exponents and instantaneous correlations for a system with $n=2$. For r_0 greater than its critical value the system will be in a "normal" or "paramagnetic" state ($T > T_c$), while for r_0 less than the critical value ($T < T_c$) the system will be in a "superfluid" phase characterized by long-range order for ψ .

In the ordered phase, when the system has a continuous broken symmetry, it is convenient to choose a thermal equilibrium state in which the order parameter has a definite phase φ . If beginning at time t_0 we subject the system to a *uni-*

form time-dependent field $h_m(t)$ coupled to m , then the sole effect will be to rotate the phase φ by the amount

$$\delta\varphi(t) = g_0 \int_{t_0}^t h_m(t') dt'. \quad (2.10)$$

Equation (2.10) expresses the fact, already mentioned in the Introduction, that $M = \int d^d x m(x)$ is an infinitesimal generator for rotations of the order parameter Ψ . Alternatively, we may say that there exists a canonical, or "Poisson bracket" relation between M and Ψ , since the equivalent equation in classical Hamiltonian mechanics, would be expressed by the Poisson bracket

$$\{\Psi, M\} = ig_0 \Psi, \quad (2.11)$$

where g_0 is a real nonzero constant.

Equation (2.11) has several important consequences, which are discussed in Appendix A. First, we can derive an equation of motion for the phase $\varphi(t)$ which is exact for the model, and is just the "Josephson equation"²⁵ or "Landau acceleration equation"²⁶ familiar in superfluid hydrodynamics. Secondly, an exact Ward identity can be obtained [Eq. (A9)], linking certain linear and nonlinear response functions. Finally, the coupling between $\varphi(t)$ and m leads to the existence of spin waves in the ordered phase, with a linear dispersion relation

$$\omega_-(k) = c_s k + O(k^2), \quad (2.12)$$

$$c_s^2 = g_0^2 \rho_s / \chi, \quad (2.13)$$

where χ is the susceptibility for the field m , and ρ_s is a stiffness constant, analogous to the superfluid density in helium. These quantities are defined more precisely in Appendix A, where the derivation of Eq. (2.12) is outlined.

The spin waves dominate the spectrum of the correlations of both ψ and m in the long-wavelength limit, below T_c . In the disordered phase ($T > T_c$), the only hydrodynamic mode is a (spin) diffusion mode

$$\omega_m(k) = (\lambda/\chi) k^2, \quad (2.14)$$

which dominates the correlation function $\chi_m(k, \omega)$ for the field m . The correlation function $\chi_\psi(k, \omega)$, on the other hand, has no hydrodynamic mode for $T > T_c$, since ψ is not conserved. We may nonetheless define a characteristic frequency for ψ and write it as

$$\omega_\psi(k) = \Gamma(k)/\chi_\psi(k), \quad (2.15)$$

where

$$\Gamma^{-1}(k) \equiv \left. \frac{\partial \chi_\psi^{-1}(k, \omega)}{\partial (-i\omega)} \right|_{\omega=0}. \quad (2.16)$$

The derivation of the above results, which is discussed in Appendix A, proceeds as for the easy-plane ferromagnet considered in Ref. 14. We remark that for the asymmetric model, $\Gamma(k)$ is in general complex, although $\text{Re} \Gamma$ is always positive. It is convenient to use the symbol $\Gamma(T)$ for the value of Γ at $k=0$. We shall see that $\Gamma(T)$ diverges as $T \rightarrow T_c^+$ but not as fast as χ_ψ . Consequently $\omega_\psi(k=0)$ tends to zero at T_c which is known as critical slowing down.^{5,11}

The thermodynamic derivatives χ , ρ_s , and χ_ψ , and the kinetic coefficients λ and Γ depend on the detailed form of the function F_0 . [Eq. (2.1c)]. In Appendix B we show how these may be expressed formally as power series in the coupling constants u_0 , g_0 , and γ_0 . For $u_0 = \gamma_0 = g_0 = 0$ we have, of course, $\chi_\psi(k) = c_0^{-1}(r_0 + k^2)^{-1}$, $\chi = \chi_0$, $\lambda = \lambda_0$, and $\Gamma = \Gamma_0 c_0^{-1}$.

III. RENORMALIZATION GROUP

A. Lowest-order recursion relations

As mentioned in Sec. II the equilibrium properties of our model are the same as those of the Ginzburg-Landau-Wilson model,¹⁰ [with parameters r_0 and u_0 given by Eqs. (2.8) and (2.9)] and also of model C of HHM. We may thus discuss the static critical behavior by the usual renormalization-group transformation, involving integration over momentum shells, and a scale transformation

$$x \rightarrow x' = b^{-1} x, \quad (3.1a)$$

$$\psi \rightarrow \psi' = b^a \psi, \quad (3.1b)$$

$$m \rightarrow m' = b^c m \quad (3.1c)$$

(the constant c was denoted a_E in Ref. 11). In order to discuss the dynamic critical behavior, we also perform a scale transformation on the frequencies¹¹

$$\omega \rightarrow \omega' = b^z \omega, \quad (3.1d)$$

but we integrate intermediate frequencies from $-\infty$ to ∞ (i.e., we do not integrate over "frequency shells"). The frequency-dependent susceptibilities can be expressed via a diagrammatic expansion of the type introduced by Martin, Siggia, and Rose²⁷ for treating classical dynamical systems. This formalism is slightly more complicated than the usual formalism of quantum-mechanical many-body theory, in that it introduces fictitious fields $\hat{\psi}$ and $\hat{m}$, which must also be rescaled by appropriate powers of b (see Appendix B).

In general, as soon as the renormalization group is applied, the equations of motion become extremely complicated—the new equations must be represented by interaction vertices of all orders

in the fields ψ and m , each one being a function of the wave vectors and frequencies of the attached propagators. Thus, at intermediate stages of the renormalization group, the equations of motion are non-Markoffian, with interactions that are nonlocal in space as well. In the limit $\epsilon \rightarrow 0$, however, the higher-order vertices turn out to be negligible, and the lowest-order vertices may be taken to be constants. To first order in ϵ , the renormalized equations of motion are then of the same form as the starting equations (2.1), but with altered parameters. The recursion relations thus obtained are the basis for our study of the critical dynamics.

The static parameters u_i , r_i , γ_i and χ_i satisfy the same recursion relations as in model C, namely, Eqs. (4.5)–(4.8) of HHM.^{11,28} (We also add an equation for c_i which was not explicitly considered in Ref. 11.) Recursion relations, valid to lowest order in ϵ , may also be obtained for the dynamic parameters Γ_i , λ_i , and g_i (see Appendix B). It is convenient to introduce the quantities

$$v_i \equiv K_d \gamma_i^2 \chi_i \Lambda^{d-4}, \quad (3.2)$$

$$f_i \equiv K_d g_i^2 \Lambda^{d-4} / \lambda_i \operatorname{Re} \Gamma_i, \quad (3.3)$$

and

$$w_i \equiv \Gamma_i \chi_i / \lambda_i \equiv w'_i + i w''_i, \quad (3.4)$$

which are independent of the overall scale of m .

In (3.2) and (3.3) $K_d = 2^{1-d} \pi^{-d/2} [\Gamma(\frac{1}{2}d)]^{-1}$ is the phase-space volume element in d dimensions. The quantities v_i and f_i turn out to be of order ϵ , while w_i is of order unity, at the fixed point.

The recursion relations may now be written

$$r_{i+1} = b^{d-2a} \{ r_i + 8K_d u_i [\Lambda^2(1 - b^{-2}) - 2r_i \ln b] \}, \quad (3.5)$$

$$c_{i+1} = b^{d-2-2a} c_i [1 - O(u_i^2)], \quad (3.6)$$

$$u_{i+1} = b^{d-4a} (u_i - 40u_i^2 K_d \ln b), \quad (3.7)$$

$$\chi_{i+1}^{-1} = b^{d-2c} \chi_i^{-1} (1 - 4v_i \ln b), \quad (3.8)$$

$$\gamma_{i+1} = b^{d-2a-c} \gamma_i (1 - 16u_i K_d \ln b - 4v_i \ln b), \quad (3.9)$$

$$g_{i+1} = b^{z-d+c} g_i, \quad (3.10)$$

$$\Gamma_{i+1} = b^{z+2a-d} \Gamma_i \left[1 - \ln b \left(\frac{4v_i w_i}{1+w_i} - \frac{f_i w'_i}{w_i(1+w_i)} + \frac{4i(f_i v_i w'_i)^{1/2}}{1+w_i} \right) \right], \quad (3.11)$$

$$\lambda_{i+1} = b^{z-d-2+2c} \lambda_i (1 + \frac{1}{2} f_i \ln b). \quad (3.12)$$

[The quantity $K_d = (8\pi^2)^{-1}$ was denoted B in Ref. 11.] It is important to note that in general Γ_i and

$w_i \equiv w'_i + i w''_i$ are complex quantities, whereas the other coefficients are all real.

Using Eqs. (3.2)–(3.12) we find the recursion relations for v_i , f_i , and w_i :

$$v_{i+1} = b^{d-4a} (1 - 32u_i K_d \ln b - 4v_i \ln b), \quad (3.13)$$

$$f_{i+1} = b^{2-2a} f_i \left\{ 1 + \ln b \operatorname{Re} \left[\frac{w_i}{w'_i} \left(\frac{4v_i w_i}{1+w_i} - \frac{f_i w'_i}{w_i(1+w_i)} + \frac{4i(f_i v_i w'_i)^{1/2}}{1+w_i} - \frac{1}{2} f_i \right) \right] \right\}, \quad (3.14)$$

$$w_{i+1} = b^{2+2a-d} w_i \left\{ 1 - \ln b \left[\frac{4v_i w_i}{1+w_i} - \frac{f_i w'_i}{w_i(1+w_i)} + \frac{4i(f_i v_i w'_i)^{1/2}}{1+w_i} + \frac{1}{2} f_i - 4v_i \right] \right\}. \quad (3.15)$$

Note that these equations are independent of the choice of c and z . The exponent a may be determined in the usual manner,¹¹ in terms of the static exponent η , by the condition that c_i remain constant [cf. Eq. (3.6)]:

$$a = \frac{1}{2}(d - 2 + \eta), \quad (3.16)$$

where $\eta = 0$ to order ϵ .

1. Model E ($\gamma_0 = 0$)

In the symmetric case we have $\gamma_i = v_i = 0$, and the recursion relations simplify considerably, since all quantities are real. Let us note that according to Eq. (3.8) χ_i will remain unrenormalized if we choose

$$c = \frac{1}{2}d, \quad (3.17)$$

and to first order in ϵ Eq. (3.16) implies $a = \frac{1}{2}(d - 2)$. When $v_i = 0$ Eqs. (3.14) and (3.15) are easily seen to reach a stable fixed point with

$$f_\infty = \epsilon, \quad (3.18)$$

$$w_\infty = 1. \quad (3.19)$$

[We use the notation f_∞ , w_∞ , etc., for the fixed point values of f_i , w_i , etc., rather than f^* , w^* , in order to avoid confusion with complex conjugation.] Then Eq. (3.11) will lead to a finite nonzero fixed point value Γ_∞ if and only if

$$z - 2 + f_\infty(1 + w_\infty)^{-1} = 0, \quad (3.20)$$

i.e.,

$$z = 2 - \frac{1}{2}\epsilon = \frac{1}{2}d. \quad (3.21)$$

In order to compute the critical exponent of the physical coefficient $\Gamma(T)$ we must relate it to the rescaled quantity Γ_i . Here we shall present an approximate evaluation of $\Gamma(T)$, in the spirit of Wilson and Kogut's calculation¹⁰ for the static case, reserving a more detailed discussion for

Sec. III C and Appendix C. Since at each stage of the renormalization procedure Γ_l is rescaled by $b^{z+2a-d} = b^{-\epsilon/2}$, the physical coefficient is given by

$$\Gamma(T) = b^{\epsilon l/2} \Gamma_l(1 + \sigma_l), \quad (3.22)$$

where σ_l represents corrections arising from intermediate momenta less than Λb^{-l} . The parameter Γ_l approaches its fixed point Γ_∞ and stays there until

$$b^l = \Lambda/\kappa, \quad (3.23)$$

where κ is the inverse correlation range. Furthermore, the correction σ_l is expected to be a finite constant when l satisfies (3.23), whence

$$\Gamma(T) \sim \kappa^{-\epsilon/2} \Gamma_\infty. \quad (3.24)$$

Since λ_∞ is also a finite constant, we may similarly show, using Eqs. (3.12) and (3.17), that

$$\lambda(T) \sim \lambda_\infty \kappa^{-\epsilon/2}, \quad (3.25)$$

which is the dynamic scaling prediction^{4,5} for the symmetric model, in lowest order in ϵ .

2. Model F ($\gamma_0 \neq 0$)

In the asymmetric case, the lowest-order recursion relations cannot be solved analytically, but they may be iterated quite simply on a computer. We first determine the exponent c from the requirement that the rescaled static susceptibility χ_l reach a finite fixed point [see Eqs. (4.11) and (4.13) of HHM¹¹], namely,

$$c = \frac{1}{2}(d - \bar{\alpha}/\nu), \quad (3.26)$$

where

$$\bar{\alpha} = \max(\alpha, 0). \quad (3.27)$$

For the case ($n=2$) under consideration here, we have $\bar{\alpha}/\nu = \frac{1}{5}\epsilon + O(\epsilon^2)$ and $u_0 = \epsilon/40K_4$. Furthermore, Eqs. (3.15) and (3.16) imply that

$$\lim_{l \rightarrow \infty} v_l = v_\infty = \bar{\alpha}/4\nu = \frac{1}{20}\epsilon + O(\epsilon^2) \quad (3.28)$$

and

$$f_\infty = \frac{6}{5}\epsilon. \quad (3.29)$$

Inserting these values into Eqs. (3.14) and (3.15), we obtain by a numerical calculation for $\epsilon \rightarrow 0$

$$w_\infty = 0.732 + 0.480i, \quad (3.30)$$

where the sign of $\text{Im} w_\infty$ is determined by the assumption that $\gamma_0 > 0$. The symmetric fixed point with $v_\infty = 0$, $f_\infty = \epsilon$, $w_\infty = 1$, is an unstable fixed point of the asymmetric model, for $\alpha > 0$.

For the asymmetric case, the requirement that $g_\infty = g_0$ implies, by Eqs. (3.10) and (3.26), that

$$z = d - c = \frac{1}{2}(d + \bar{\alpha}/\nu), \quad (3.31)$$

and that λ_∞ and Γ_∞ are finite nonzero constants. As in model *E* we may again find the critical exponents to lowest order in ϵ from Eqs. (3.11) and (3.12)

$$\Gamma(T) \sim \kappa^{-(z-2)} \Gamma_\infty = \kappa^{-(1/2)(\epsilon - \bar{\alpha}/\nu)} \Gamma_\infty, \quad (3.32)$$

$$\lambda(T) \sim \kappa^{-(z-2 + \bar{\alpha}/\nu)} \lambda_\infty = \kappa^{-(1/2)(\epsilon + \bar{\alpha}/\nu)} \lambda_\infty, \quad (3.33)$$

in agreement with dynamic scaling.^{4,5}

A mode-coupling analysis of systems which are similar to models *E* and *F* has recently been carried out to lowest order in ϵ by Gunton and Kawasaki.⁹ These authors found the same exponent values as we do in both cases. Their critical amplitudes, agree with ours in model *E* [Eqs. (3.18) and (3.19)], but disagree in model *F* where they find (in our notation) $f_\infty = \frac{6}{5}\epsilon$, but $w_\infty = 2$ rather than Eq. (3.30). The reason for the discrepancy is the omission by Gunton and Kawasaki, of the terms involving v_l in Eq. (3.11). It should be noted in particular that their Langevin equations do not yield the correct equilibrium distribution for the asymmetric case.

B. Analysis to all orders in ϵ

We now wish to show that the exponent results of Sec. III A are in fact valid to all orders in ϵ . Since for finite ϵ the renormalized equations of motion are no longer of the simple form (2.1), one must first redefine the nine "slowly varying" parameters r_l , u_l , χ_l , γ_l , c_l , λ_l , g_l , $\text{Re} \Gamma_l$, and $\text{Im} \Gamma_l$ in terms of the limiting behavior of the appropriate frequency- and wave-vector-dependent vertices and propagators, when wave vectors and frequencies tend to zero. For small ϵ , it is still possible formally to write down recursion relations for these nine parameters because the coefficients corresponding to all the other degrees of freedom of the equations of motion may be expressed in terms of the first nine. Recursion relations in terms of these slowly varying parameters suffice for the study of the fixed point, and of the slow transients which decay with exponents of order ϵ . [See the discussion of the static renormalization group given in Sec. 5 of Wilson and Kogut¹⁰.]

For the symmetric model, the recursion relations for f_l and w_l take the form

$$f_{l+1} = b^\epsilon f_l \{1 - \ln b [f_l/(1+w_l) + \frac{1}{2}f_l] + \mathfrak{F}_1(f_l, w_l, u_l, \epsilon)\}, \quad (3.34)$$

$$w_{l+1} = w_l \{1 + \ln b [f_l/(1+w_l) + \frac{1}{2}f_l] + W_1(f_l, w_l, u_l, \epsilon)\}, \quad (3.35)$$

where $\mathfrak{F}_1$ and W_1 contain no terms of lower order than ϵ^2 , $u_l f_l$, u_l^2 , or f_l^2 . Since¹⁰ $u_\infty \propto \epsilon$ and $f_\infty \propto \epsilon$,

the additional terms in (3.34) and (3.35) are of higher order in ϵ , and the fixed point values are seen to be

$$f_\infty = \epsilon + O(\epsilon^2), \quad (3.36)$$

$$w_\infty = 1 + O(\epsilon). \quad (3.37)$$

In contrast to (3.14) and (3.15), the recursion relation (3.10) for g_l , is believed to be exact, to all orders in ϵ . This is closely related to the requirement that the Poisson bracket relationship (2.11) and its consequences, [the Josephson equation (A14) below T_c and the Ward identity (A9) above T_c], are exact relations, which must be preserved by the renormalization group (see Appendix B). It follows that g_l goes to a finite fixed-point value, with

$$g_\infty = g_0, \quad (3.38)$$

if and only if we choose

$$z = d - c. \quad (3.39)$$

Moreover, for $v_l = 0$, χ_l is also unrenormalized ($\chi_l = \chi_0 = \chi_\infty$), if we choose c according to Eq. (3.17). Thus

$$z = \frac{1}{2}d, \quad (3.40)$$

as predicted by dynamic scaling.^{4,5,21} Since f_∞ , w_∞ , g_∞ , and χ_∞ are all finite, Eqs. (3.11) and (3.12) imply that λ_l and Γ_l will also reach finite nonzero fixed-point values.

In a similar manner, we find that for the asymmetric case, the renormalization group approaches a fixed point with λ_∞ , Γ_∞ , and χ_∞ finite, and $g_\infty = g_0$, provided we choose c according to (3.26), and

$$z = \frac{1}{2}(d + \tilde{\alpha}/\nu). \quad (3.41)$$

Furthermore the fixed-point values of f_l and w_l are given to lowest order in ϵ by (3.29) and (3.30), with corrections of order ϵ^2 and ϵ , respectively. The fixed-point Hamiltonian is modified by higher-order terms in ϵ , but not the exponents.

As mentioned in the Introduction, the value of α is believed¹³ to be slightly negative, at $d=3$. In this case the parameter v_l approaches zero as $l \rightarrow \infty$, and the stable fixed point of the asymmetric model (F) is the same as that of the symmetric model (E) [this is similar to the situation in case C of HHM¹¹ for $n=2$]. Thus, strictly speaking, if the fixed-point values of f_l and w_l are to be obtained by analytic continuation from $4-\epsilon$ dimensions, one should use the symmetric fixed-point values (3.36) and (3.37), rather than (3.29) and (3.30).

C. Dynamic scaling laws

The response function $\chi_\psi(k, \omega)$ and $\chi_m(k, \omega)$ may be written in the (exact) form

$$\chi_\psi^{-1}(k, \omega) = c_l b^{(2d-d)l} [-i\omega_l + r_l + k_l^2 + \sigma_l(k_l, \omega_l)], \quad (3.42)$$

$$\chi_m^{-1}(k, \omega) = b^{(2c-d)l} [i\omega_l \Gamma_l / \lambda_l k_l^2 + \chi_l^{-1} + \pi_l(k_l) + (i\omega_l / k_l^2) \pi'_l(k_l, \omega_l)], \quad (3.43)$$

where

$$\omega_l \equiv \omega b^{zl} / \Gamma_l, \quad (3.44)$$

$$k_l \equiv k b^{l^1}, \quad (3.45)$$

and σ_l , π_l , and π'_l may be described as arising from "self-energy corrections," in which the wave vectors of the intermediate propagators are integrated from the cutoff Λ all the way down to zero. These self-energy corrections involve one or more interaction vertices and are formally of order ϵ or higher, relative to the leading terms; at the critical point, however, the self-energies contain divergences if k_l and w_l approach zero.

If the renormalization group leads to a well-behaved fixed point as $l \rightarrow \infty$, then the functions σ_l , π_l , and π'_l become independent of l for large l . Furthermore, when $T = T_c$, if we choose l such that $\max(k_l, \omega_l)$ is some fixed number of order unity, then the self-energy corrections should be finite, and of order ϵ or smaller. If T is slightly above T_c , the renormalization group at first approaches the fixed point, but r_l eventually becomes large and positive. If we choose l such that

$$r_l \approx 1 \quad (3.46)$$

(which occurs for $b^{l^1} \approx \Lambda/\kappa$), then the self-energy corrections are all finite, (and of order ϵ or smaller) even when k_l and ω_l are small. It follows that if the renormalization group for the equations of motion approaches a well-behaved (finite) fixed point at T_c , then χ_m and χ_ψ will have the asymptotic forms

$$\chi_\psi(k, \omega) = c_\infty^{-1} (\Lambda/\kappa)^{2-\eta} K_\psi(\omega/\zeta\kappa^z, k/\kappa), \quad (3.47)$$

$$\chi_m(k, \omega) = \chi K_m(\omega/\zeta\kappa^z, k/\kappa), \quad (3.48)$$

where

$$\zeta \equiv \text{Re} \Gamma_\infty \Lambda^{2-z} = g_0 (K_d)^{1/2} \Lambda^{-z+d/2} \chi_\infty^{-1/2} (w_\infty/f_\infty)^{1/2}, \quad (3.49)$$

while the static susceptibility χ is given by

$$\chi = \chi_0 = \chi_\infty \quad \text{symmetric case} \quad (3.50)$$

$$\chi = \chi_\infty (\Lambda/\kappa)^{\tilde{\alpha}/\nu} [1 + O(\tilde{\alpha}/\nu)] \quad \text{asymmetric case}. \quad (3.51)$$

The asymptotic forms (3.47–48) will be valid in the limit $k \rightarrow 0$, $\kappa \rightarrow 0$, $\omega \rightarrow 0$, for arbitrary ratios of the three variables. Furthermore the functions K_ψ and K_m are universal functions, which depend on the dimensionality and the fixed point reached, but not on details of the original equations of motion. To lowest order in ϵ , we may write

$$K_\psi^{-1}(x, y) = -ixw'_\infty/w_\infty + 1 + y^2, \quad (3.52)$$

$$K_m^{-1}(x, y) = -ixw'_\infty/y^2 + 1, \quad (3.53)$$

where corrections to the above are of order ϵ (these corrections can become large, however, if either x or y are very large; e.g., $K_\psi \sim \epsilon x \ln \kappa$). It follows that for T just above T_c we have

$$\omega_m(k) \equiv \lambda k^2 / \chi = a_1 (\zeta / w'_\infty) \kappa^{z-2} k^2, \quad (3.54)$$

$$\lambda = a_1 (K_d)^{1/2} g_0 (w'_\infty f_\infty)^{-1/2} \kappa^{-\epsilon/2} \chi^{1/2} \equiv R_\lambda g_0 \kappa^{-\epsilon/2} \chi^{1/2}, \quad (3.55)$$

$$\begin{aligned} \omega_\psi(k) &\equiv \Gamma / \chi_\psi = a_2 (\omega_\infty / w'_\infty) \zeta \kappa^z \\ &= g_0 a_2 (K_d)^{1/2} w_\infty (w'_\infty f_\infty)^{1/2} \kappa^{d/2} \chi^{-1/2} \\ &\equiv R_\Gamma g_0 \kappa^{d/2} \chi^{-1/2}, \end{aligned} \quad (3.56)$$

where a_1 and a_2 are universal constants of the form $1 + O(\epsilon)$. Equations (3.54)–(3.56) are the (extended⁵) dynamic scaling result,^{4,5} which has been shown to hold to all orders in ϵ . An important additional result for the critical amplitudes of Γ and λ follows from Eqs. (3.55) and (3.56): these amplitudes are expressed entirely in terms of static quantities and universal constants, apart from the factor g_0 which determines the frequency scale, and is known from the starting Eqs. (2.1) (see Secs. IV and V).

D. Matching conditions

The matching conditions between characteristic frequencies above and below T_c implied by dynamic scaling, may also be proved for the present models. Indeed, since by hydrodynamics the spin-wave frequency (2.12) is determined entirely by g_0 and static quantities, it is clear that certain frequency ratios will be universal, i.e., independent of the parameters of the starting equations.

Let us consider the ratio

$$R_m(k) = \omega_m(k) / \omega_-(k), \quad (3.57)$$

where ω_m is given in Eq. (2.14), and ω_- is the spin-wave frequency (2.12), evaluated at temperatures above and below T_c , respectively, by the same (small) value of $|T - T_c|$. We wish to show that the quantity $R_m \equiv R_m(k = \kappa_-)$, for instance, is universal, where κ_- is the inverse correlation range below T_c .

Since ψ is a vector order parameter ($n=2$), the static correlation function $\langle \psi(\vec{x}) \psi(0) \rangle$ does not decay exponentially below T_c , and the correlation

range is defined differently⁵ above and below T_c . A consistent definition, which generalizes Josephson's relation²⁹ to arbitrary d , is

$$\kappa_- = \rho_s^{(d-2)^{-1}}, \quad (3.58)$$

where ρ_s , given in Eq. (A16), is related to the k -dependent transverse static response below T_c by¹⁴

$$\chi_\psi(k) = |\psi|^2 / \rho_s k^2, \quad k \rightarrow 0. \quad (3.59)$$

[In the present model we consider the “free energies” F_0 and Ω [Eqs. (2.1c) and (A16)] to be dimensionless, so ρ_s has the dimensions of Λ^{d-2} .] Now according to Eqs. (2.12–13), (3.49) and (3.54) we have

$$R_m^2 = \left[\frac{\omega_m^+(\kappa_-)}{\omega_-(\kappa_-)} \right]^2 = \frac{\lambda^2 \kappa_-^2 \chi_-}{\chi_+^2 g_0^2 \rho_s} = R_\lambda^2 \left(\frac{\chi_-}{\chi_+} \right) \left(\frac{\kappa_-}{\kappa_+} \right)^{4-d}, \quad (3.60)$$

$$R_\lambda^2 \equiv K_d a_1^2 / f_\infty w'_\infty, \quad (3.61)$$

where χ_+ and χ_- are the static susceptibilities of the field m above and below T_c , respectively, κ_+ and κ_- are the corresponding correlation lengths, and R_λ is defined by (3.55). Furthermore, the static ratios κ_+/κ_- and χ_+/χ_- are universal constants, so that R_m is itself predicted to be a universal constant. Using the values from (3.18) and (3.19) and (3.29) and (3.30) we see that for the symmetric case, in the limit $\epsilon \rightarrow 0$,

$$R_\lambda^2 = K_4 \epsilon^{-1} [1 + O(\epsilon)] = 0.0127 \epsilon^{-1} [1 + O(\epsilon)], \quad (3.62)$$

while for the asymmetric case

$$R_\lambda^2 = 0.0144 \epsilon^{-1} [1 + O(\epsilon)]. \quad (3.63)$$

The second-order calculation for the symmetric model, described in Appendix D yields

$$R_\lambda = (K_4 \epsilon^{-1})^{1/2} [1 + 1.335 \epsilon + O(\epsilon^2)], \quad (3.64)$$

which suggests that the expansion may be rather ill behaved. We may note, however, that a significant part of ϵ dependence of R_λ comes from the dimensionality dependence of the phase-space factor, $K_d^{1/2}$, present in (3.61). It may be more advantageous, then, to expand R_λ in the form of Eq. (D29)

$$R_\lambda = (K_d \epsilon^{-1})^{1/2} [1 + 0.597 \epsilon + O(\epsilon^2)]. \quad (3.65)$$

The $O(\epsilon)$ term is then smaller than in (3.64), and one may hope for a more accurate extrapolation to $d=3$.

In the symmetric case (model *E* or model *F* with $\alpha < 0$) χ is continuous at T_c so that $\chi_+ = \chi_-$. For the asymmetric fixed point (model *F* with $\alpha > 0$), the quantity χ_+/χ_- is equal to the specific-heat ratio, evaluated in the ϵ expansion by Brézin *et al.*,³⁰

$$\chi_+/\chi_- = 2^\alpha \left(\frac{1}{4} n \right) (1 + \epsilon) + O(\epsilon^2). \quad (3.66)$$

Similarly, κ_+/κ_- may also be obtained in an ϵ expansion. We shall rather use values of the static ratios obtained³¹ from experiments and high-temperature series in three dimensions, and thus obtain an “experimental” estimate of R_λ from Eq. (3.60) which we can compare with the theoretical values (see Secs. IV and V).

For magnetic systems, one can also measure the characteristic frequency $\omega_\psi(q)$ of the order parameter above T_c . We can then define other universal ratios [see Eqs. (2.12), (2.13), and (3.56)],

$$R_\psi^2 = \left[\frac{\omega_\psi(q=0)}{\omega_\psi(q=\kappa_-)} \right]^2 = \left(\frac{\Gamma}{\chi_\psi} \right)^2 \left(\frac{\chi_-}{g_0^2 \rho_s \kappa_-^2} \right) = R_\Gamma^2 \left(\frac{\kappa_+}{\kappa_-} \right)^d \left(\frac{\chi_-}{\chi_+} \right), \quad (3.67)$$

$$R_\Gamma^2 \equiv a_2^2 K_d w_\infty / f_\infty, \quad (3.68)$$

and

$$R_{\Gamma/\lambda} \equiv \left[\frac{\omega_\psi(q=0)}{\omega_m(q=\kappa_+)} \right] = \frac{R_\psi}{R_m} \left(\frac{\kappa_-}{\kappa_+} \right)^2 = \frac{R_\Gamma}{R_\lambda} = w_\infty (a_2/a_1). \quad (3.69)$$

The last ratio depends only on experimental data above T_c . We shall compare these quantities to measurements on RbMnF₃ in Sec. V.

It should be noted that beyond the lowest order in ϵ the precise values of R_ψ , $R_{\Gamma/\lambda}$ depend on the somewhat arbitrary definition of ω_ψ that we have adopted. In particular the value of a_2 , in (3.56), depends on this definition, although the values of w_∞ and f_∞ do not. The characteristic frequencies and universal ratios defined above are summarized in Table II for later reference.

TABLE II. Some characteristic frequencies and universal dynamic critical ratios used in the text. In the definitions of R_m and R_ψ , the absolute value $|T - T_c|$ must be the same in numerator and denominator.

Characteristic frequencies		
$\omega_m(k) \equiv \lambda \chi^{-1} k^2$	spin-diffusion rate,	$T > T_c$
$\omega_\psi(k) \equiv \Gamma / \chi_\psi$	order-parameter relaxation,	$T > T_c$
$\omega_-(k) \equiv c_s k$	spin-wave frequency,	$T < T_c$
Universal ratios		
$R_\lambda = \lambda / (g_0 \kappa_+^{-\epsilon/2} \chi^{1/2})$		
$R_\Gamma = \Gamma / (g_0 \kappa_+^{d/2} \chi^{-1/2} \chi_\psi)$		
$R_m = \omega_m(\kappa_-) / \omega_-(\kappa_-)$		
$R_\psi = \omega_\psi(0) / \omega_-(\kappa_-)$		
$R_{\Gamma/\lambda} = \omega_\psi(0) / \omega_m(\kappa_+) = R_\Gamma / R_\lambda$		

IV. APPLICATIONS TO SPECIFIC SYSTEMS

A. Hamiltonian models of magnets

Let us consider an anisotropic Heisenberg magnet with Hamiltonian

$$\mathcal{H} = - \sum_{ij} [J_\perp(i-j)(S_x^i S_x^j + S_y^i S_y^j) + J_\parallel(i-j)S_z^i S_z^j] + H_z \sum_i S_z^i, \quad (4.1)$$

where i and j denote points on a d -dimensional lattice and $\vec{S}^i$ is a (three-component) spin operator at point i . The dynamics of this system follows from the usual quantum-mechanical equation of motion

$$-i\hbar \frac{\partial \vec{S}^i}{\partial t} = [\mathcal{H}, \vec{S}^i], \quad (4.2)$$

or the equivalent Poisson bracket relation in the classical limit. If we choose the interaction constants $J_\perp$ and $J_\parallel$ in such a way that the system orders ferromagnetically in the x - y plane, we obtain a model whose hydrodynamic properties were discussed in detail in Ref. 14. The properties of (4.1) are quite similar to those of models *E* and *F* of Sec. II, except for the existence of an additional conserved energy field in the system (4.1). As shown in Ref. 14, this conservation law leads to the existence of a low-lying thermal diffusion mode, which is absent from models *E* and *F*. Thus a more accurate phenomenological model for the study of the dynamic critical behavior of the anisotropic Heisenberg magnet (4.1) would contain three coupled fields—a complex order parameter ψ , representing $S_x - iS_y$, a magnetization m , representing S_z , and an energy density $\tilde{\epsilon}$. Such three-field planar-spin models (which we denote as models *E'* and *F'* for the symmetric and asymmetric cases, respectively) are studied briefly in Appendix E. Above T_c there are two diffusive modes, of which only one has a diverging transport coefficient as $T \rightarrow T_c$. The second mode will thus be slower than the characteristic frequency for the order parameter, which retains an exponent $z = \frac{1}{2}d < 2$. If the specific-heat exponent α is negative then the energy decouples from the other fields in the limit $T \rightarrow T_c$, and the fixed points and leading singularities are the same for models with and without energy conservation (the corrections to scaling have different behavior, however; see Appendix C). If the exponent α is positive, the recursion relation analysis of Appendix E still predicts no change in the exponent z owing to energy conservation, at least in the symmetric model (*E'*). In particular, there will still be a diverging transport coefficient propor-

tional to $\kappa^{-\epsilon/2}$. The critical ratios, such as R_m and R_ψ , on the other hand, are affected by energy conservation, even in lowest order in ϵ (see Appendix E). Moreover, the situation here is very similar to that of model C studied in Ref. 11, where the fixed point reached in the presence of energy conservation is singular for $2 \leq n \leq 4$ and $\epsilon \rightarrow 0$, and it is not clear how well behaved the results will be when corrections of higher order in ϵ are included. It seems probable that a characteristic frequency for the order parameter can still be defined, which obeys the usual dynamic scaling prediction $z = \frac{1}{2}d$, but there may be parts of the scaling function K_ψ of Eq. (3.47) which have another characteristic frequency.

B. Liquid helium

1. Applicability of model F

The usefulness of a pseudospin model to describe superfluid helium was first pointed out by Matsubara and Matsuda.³² Since their work, a number of authors have applied such models to calculations of static and dynamic properties of helium.^{5,14,33-35} In particular, it was shown in Ref. 14 that the easy-plane ferromagnet had hydrodynamic modes which correspond closely to those of superfluid helium.

The order parameter of helium is the expectation value of the boson field $\langle\psi\rangle$, which corresponds to the magnetization in the easy plane of the magnetic system. There are, in addition, three conserved fields,^{4,36} the mass density $\bar{\rho}$, the energy density $\bar{\epsilon}$, and the momentum density $\bar{g}$. One linear combination of $\bar{\rho}$ and $\bar{\epsilon}$ is the "entropy"³⁶ $\bar{q}$, and it is identified with the field m . This field couples with v_s to make second sound below T_λ , and relaxes with the thermal diffusion mode above T_λ . The other linear combination of $\bar{\rho}$ and $\bar{\epsilon}$, which we can call $\tilde{\rho}$, couples with the longitudinal part of $\bar{g}$ to make first sound, both above and below T_λ . Since the first-sound velocity is finite at T_λ this mode remains at high frequency, and is not expected to have any influence on the critical behavior of the other modes.^{4,5,8} Thus the two-field model, Eqs. (2.1), should be a proper starting point for understanding the critical dynamics of pure bulk helium, if attention is confined to phenomena occurring in the frequency range $\omega < c_1\kappa$ (where c_1 is the first-sound velocity). The two-field model is, moreover, the simplest model which incorporates the important conservation laws and hydrodynamic modes of a superfluid.

It is worth pointing out at this stage that if the helium is immersed in a powder,^{34,35} then it is precisely analogous to the three-field model F' , since momentum is no longer conserved. The

field m is then identified with the linear combination $\tilde{\rho}$, which couples with v_s to make *fourth* sound below T_λ , while the other linear combination of $\bar{\epsilon}$ and $\bar{\rho}$ relaxes with the thermal diffusion frequency. The situation is then precisely analogous to the easy-plane ferromagnet discussed above, in which both the energy and S_z are conserved. The uncertainties discussed there, which arise from the existence of a low-frequency thermal diffusion mode in addition to the spin wave, are also expected to exist here.

2. Dynamic exponents and amplitude ratios

In order to apply the results of Sec. II to experiments on helium, it is necessary to specify the dimensional form of the various quantities which enter the theory, and to spell out the detailed correspondence with measurable quantities. Let us suppose that F_0 [Eq. (2.1c)] and Ω [Eq. (A16)] are dimensionless, and that $\Omega = (k_B T)^{-1} \tilde{F}$, where $\tilde{F}$ is the total free energy of the helium system. The usual superfluid density $\bar{\rho}_s$, in units of mass per unit volume, is related to $\tilde{F}$ by

$$\tilde{F} = \tilde{F}(v_s = 0) + \frac{1}{2} \int d^d x \tilde{\rho}_s v_s^2, \quad (4.3)$$

where³⁶ $v_s = (\hbar/m_H) \nabla \varphi$, and m_H is the helium mass, so that according to (A16), (3.58), and (4.3) we have

$$\bar{\rho}_s = (m_H^2 k_B T / \hbar^2) \rho_s = (m_H^2 k_B T / \hbar^2) \kappa_-^{-2}. \quad (4.4)$$

The second-sound velocity of helium is^{26,36}

$$c_2^2 = \bar{\rho}_s k_B^2 T \sigma^2 \tilde{\rho} / C_p \bar{\rho}_n m_H^2, \quad (4.5)$$

where $\sigma = S/R$ is the (dimensionless) entropy per particle, C_p is the constant-pressure specific heat per unit volume, and $\bar{\rho}_n = \bar{\rho} - \bar{\rho}_s$ is the normal fluid density. We shall choose the units of the field m in such a way that $k_B^{-1} C_p$ is identified with χ [Eq. (A7)]. Comparing Eqs. (4.5) and (2.13), and making the approximation (valid near T_λ) that $\rho \approx \rho_n$, we then have

$$g_0 = k_B T \sigma / \hbar. \quad (4.6)$$

As mentioned earlier, m is a particular linear combination of $\tilde{\rho}$ and $\bar{\epsilon}$, which must be chosen so as to yield the second-sound velocity (4.5). From the analysis of Ref. 36 it may be seen that m is proportional to the operator q of Eq. (4.4) of Ref. 36, i.e., $M = \int d^d x m(x)$ is given by

$$M = (k_B T)^{-1} [E - (\tilde{\mu} + k_B T \sigma) N]. \quad (4.7)$$

Here N is the particle number and E the energy, with densities $\tilde{n} = \tilde{\rho} / m_H$ and $\bar{\epsilon}$, respectively, and $\tilde{\mu}$ is the chemical potential per particle. We may

now verify the Josephson equation (A14)

$$\frac{d\varphi}{dt} = g_0 \frac{\partial \Omega}{\partial M} = g_0 (k_B T)^{-1} \frac{\partial \tilde{F}}{\partial N} \frac{\partial N}{\partial M} = -\tilde{\mu}/\hbar, \quad (4.8)$$

which is equivalent to the Landau acceleration equation²⁶

$$\frac{dv_s}{dt} = -\frac{1}{m_H} \nabla \tilde{\mu}. \quad (4.9)$$

[In Eq. (4.8) we have used the relations $\partial M/\partial N = -k_B T \sigma$ and $\partial \tilde{F}/\partial N = \tilde{\mu}$.] It is now easy to verify that χ , as defined in Eq. (A17), is equal to $k_B^{-1} C_p$, and that Eq. (A18) for the current $\tilde{j}^m$ agrees with the usual "entropy current."³⁶ This completes the proof of the equivalence of the dynamics of model F to that of liquid helium near T_λ [when the factor $\tilde{\rho}/\rho_n$ in (4.5) may be set equal to unity].

The observable low-frequency modes in liquid helium are thermal diffusion

$$\omega_T = \lambda k^2 / C_p, \quad (4.10)$$

which we identify with ω_m [Eq. (2.14)] above T_λ , and second sound

$$\omega_-(q) = c_2 k, \quad (4.11)$$

with $c_2 = c_s$ given by (2.13) or (4.5).

According to Eq. (3.55) we have as $T \rightarrow T_\lambda$

$$\lambda = R_\lambda g_0 \kappa_+^{-1/2} (k_B^{-1} C_p)^{1/2} \propto t^{-(1/2)(\nu+\tilde{\alpha})}, \quad (4.12)$$

where R_λ is the universal constant of Eq. (3.61), and

$$t \equiv (T - T_\lambda) / T_\lambda \quad (4.13)$$

is the reduced temperature (which should not be confused with the time!). Since κ_+ is not directly measurable in liquid helium, we shall first express the critical amplitude of λ in terms of the behavior below T_λ , using the matching condition (3.60) and the relation (4.4) between κ_- and $\tilde{\rho}_s$. In the present units the matching condition reads

$$R_m = \lim_{t \rightarrow 0} \frac{\lambda \kappa_-}{C_p c_2} = \lim_{t \rightarrow 0} \frac{\hbar^2 \lambda(t) \tilde{\rho}_s(-t)}{m_H^2 k_B T_\lambda C_p(t) c_2(-t)}. \quad (4.14)$$

All quantities on the right-hand side of (4.14) are measurable in liquid helium, whereas the left-hand side is universal in the limit $T \rightarrow T_\lambda$, and can therefore be calculated using models E or F , e.g., in an ϵ expansion. The temperature and pressure dependence of R_m in liquid helium has been determined near the λ -line by Ahlers,⁷ and universality was only found to hold approximately. Moreover, the exponent prediction in Eq. (4.12) was also not precisely observed in the experiments. Presumably these discrepancies are due to deviations from the asymptotic critical temperature dependence. As we discuss below, the approach to the

asymptotic form is expected to be rather slow, when $|\alpha|$ is small. Even in the absence of precise temperature independence of the measured R_m , however, it is interesting to compare experimental values of R_m with the theoretical estimates made in Sec. III. Using experimental data on thermal conductivity, specific heat, second-sound velocity, and ρ_s , one finds³⁷ that R_m varies by roughly 40% near the λ line from saturated vapor pressure to 22 atm, in the range $10^{-5} < |T - T_\lambda| / T_\lambda < 10^{-3}$. Typical experimental values are of order $R_m^{\text{expt}} = 0.20 \sim 0.28$, if data at all pressures are included, while the value at $t = 10^{-4}$ and saturated vapor pressure (SVP) is³⁷

$$R_m^{\text{expt}} = 0.23. \quad (4.15)$$

In order to compare this value with the calculation of Sec. III, we shall use three-dimensional estimates of the static ratios κ_+/κ_- and C_p^+/C_p^- , to form an "experimental" value of R_λ using Eq. (3.60), and compare this quantity with theory, rather than R_m . In this way we do not make use of the ϵ expansion below T_λ , and we separate the static and dynamic calculations. The inverse correlation range κ_- may be obtained from the measured $\tilde{\rho}_s$, using Eq. (4.4), while κ_+ is found³¹ from the singular part of the specific heat C_p^+ , and high-temperature-series estimates of the universal ratio $C_p^+ \kappa_+^{-3}$. At SVP and for $t = 10^{-4}$ we find³¹

$$\kappa_- = 0.3 |t|^{0.67} \text{ \AA}^{-1} = 6.26 \times 10^{-4} \text{ \AA}^{-1}, \quad (4.16)$$

$$\kappa_+ = 0.7 |t|^{0.67} \text{ \AA}^{-1} = 1.46 \times 10^{-3} \text{ \AA}^{-1}. \quad (4.17)$$

Since experimentally there is good evidence¹³ that $\alpha < 0$ for helium, we choose the value $C_p^+/C_p^- = 1$, appropriate to the symmetric fixed point. Inserting these values and Eq. (4.15) into Eq. (3.60) we found at SVP and $t = 10^{-4}$

$$\bar{R}_\lambda^{\text{expt}} \equiv R_m^{\text{expt}} (\kappa_+/\kappa_-)^{1/2} = 0.36. \quad (4.18)$$

(The bar over $\bar{R}_\lambda^{\text{expt}}$ is meant to indicate that this experimental value was obtained using data both above and below T_λ .) This value is to be compared to the ϵ -expansion estimates of R_λ (3.62)–(3.65). The lowest-order estimate obtained by setting $\epsilon = 1$ in (3.62) and (3.63) and ignoring the $O(\epsilon)$ correction is $R_\lambda = (K_4 \epsilon^{-1})^{1/2} = 0.113$, for the symmetric case, and $R_\lambda = 0.120$ in the asymmetric case. These estimates, which are too small by a factor of 3, may be improved by use of K_3 rather than K_4 in (3.62) to yield $R_\lambda = 0.226$ for the symmetric case. In second order, we obtain from (3.64), $R_\lambda = 0.26$, whereas setting $d = 3$ ($\epsilon = 1$) in (3.65), we find

$$R_\lambda^{\text{theor}} = 0.359. \quad (4.19)$$

The precise agreement between (4.18) and (4.19)

is surely fortuitous, since the experimental values themselves are not universal, and the theoretical values may be subject to large corrections. Nevertheless, the comparison suggests that our method of extrapolation may be quite reasonable.

It is possible to obtain an experimental value using data only above T_λ , by using Eqs. (4.12) and (4.17). Inserting values³⁷ of $\sigma = S/R$, C_p , λ , and κ_+ , appropriate to $t = 10^{-4}$ and SVP into Eqs. (4.6) and (4.12), we find

$$R_\lambda^{\text{expt}} = 0.29, \quad (4.20)$$

which is once again in reasonably good agreement with the theoretical estimate (4.19). The two different experimental values (4.18) and (4.20) do not agree precisely with each other, however, and in order to make a more reliable comparison with theory, the correction terms must be analyzed carefully. An estimate of R_λ based on mode-coupling theory was carried out some time ago by Krueger and Huber,³⁸ who found a value smaller than experiment by a factor of 7.

Dynamic scaling also makes predictions concerning the damping of second sound in liquid helium below T_λ , and it would be interesting to extend the present theory to that case. The experimental situation on second sound damping is still unclear, however, since dynamic scaling predictions were confirmed at low frequency,³⁹ but appear to be in conflict with Brillouin scattering experiments⁴⁰ at high frequencies. We do not know of any explanation for the latter results.

3. Correction terms

Since the exponent α is expected to be negative in liquid helium,¹³ the appropriate fixed point of model F at $d=3$ is the symmetric one, as in model E , and the dynamic exponents are those of Eqs. (3.21) and (3.25). Nevertheless, there is a difference between models E and F for $|\alpha|$ small, since the asymmetric model will possess significant correction terms which reflect the slow approach of the specific heat to its (finite) value at T_c . We have analyzed the recursion relations (3.5)–(3.15) for model F in the case $\alpha < 0$, keeping the parameter v_i finite, i.e., not equal to its fixed-point value $v_\infty = 0$. In this way we show [see Appendix C] that Eq. (4.12) has the correction terms

$$\lambda(t) = R_\lambda g_0 \kappa^{-\epsilon/2} [k_B^{-1} C_p(t)]^{1/2} \varphi(\alpha_e) \times (1 + D_0 t^{\alpha_0} + D_1 t^{\alpha_1} + D_2 t^{\alpha_2} + D_3 t^{\alpha_3}), \quad (4.21)$$

where $\alpha_e(t)$ is the (nonuniversal) "effective exponent" of the specific heat

$$\alpha_e(t) \equiv - \frac{d \ln C_p(t)}{d \ln t}, \quad (4.22)$$

which is positive and approaches zero slowly as $t \rightarrow 0$ for $\alpha < 0$ ($\alpha_e \sim t^{|\alpha|}$). In Eq. (4.21) $\varphi(\alpha_e)$ is a universal function of its argument, with an asymptotic expansion of the form

$$\varphi(x) = 1 + \varphi_1 x + \varphi_2 x^2 + \dots \quad (4.23)$$

The coefficient φ_1 has been evaluated to lowest order, and is equal to

$$\varphi_1 = (5/18\epsilon\nu)[1 + O(\epsilon)]. \quad (4.24)$$

A function similar to $\varphi(\alpha_e)$ occurs for the specific heat itself. As demonstrated in Eqs. (C36)–(C38)

$$C_p(t)/C_p(-t) = (1 + \bar{P}\alpha_e)^{-1}, \quad t > 0, \quad (4.25)$$

where $\bar{P} \equiv \alpha^{-1}[(A'/A) - 1] \equiv PA'/A$ is a universal constant,⁴⁰ whose ϵ expansion can be obtained from the results of Ref. 30,

$$\bar{P} = 2(n+8)/n\epsilon = (10/\epsilon)[1 + O(\epsilon)]. \quad (4.26)$$

Experimentally, $\bar{P}$ can be obtained rather accurately, and is found to be equal to^{13,41}

$$\bar{P} \approx 4 \quad (4.27)$$

in liquid helium.

The last factor in (4.21) comes from the usual corrections to scaling,⁴² with universal exponents x_i and nonuniversal coefficients D_i . The exponent $x_0 \approx 0.5\epsilon$ arises from static corrections,⁴² and is the same as in the specific heat,¹³ and the coefficient D_0 has the same pressure dependence as the corresponding nonuniversal static coefficient. The exponents x_1 , x_2 , and x_3 , on the other hand, are the three independent dynamic correction exponents, which are of order ϵ and are calculated to lowest order in Appendix C. The value of α_e is approximately 0.15 in the experimental range of temperatures,³⁷ and it will be interesting to see whether the data on the thermal conductivity⁷ can be explained quantitatively by (4.21), with a simple ansatz for $\varphi(\alpha_e)$. Equation (4.21), which also holds for $\alpha \geq 0$, has the same limiting form as the original dynamic scaling prediction of Ferrell *et al.* [Eq. (5.33) of Ref. 4]. However, (4.21) contains in addition the correction terms in the last two factors, which are expected to make a measurable difference in the experimental range.

4. Hamiltonian model for helium

Various authors⁴³ have attempted to study the critical dynamics of helium by applying the renormalization group directly to a Hamiltonian model of helium involving only a single complex field ψ . The simplest example of such a model can be obtained from our Equation (2.1) by treating Γ_0 as pure imaginary (no dissipation) and dropping

all couplings to the field m :

$$\frac{\partial \psi}{\partial t} = -2i\Gamma_0'' \frac{\delta \mathcal{H}}{\delta \psi^*} \quad (4.28)$$

$$\mathcal{H}[\psi] = \int d^d x \left(\frac{1}{2} r_0 |\psi|^2 + \frac{1}{2} |\nabla \psi|^2 + u_0 |\psi|^4 \right), \quad (4.29)$$

with an appropriate cutoff at short wavelengths. If the renormalization group is properly applied to this model, however, one immediately encounters a difficulty on the first iteration. The renormalized four-point vertex $u_4(k_i, \omega_i)$ is a singular function of the wave vectors and frequencies, when the net wave vector and frequency transfers go to zero. These singularities are due to the absence of damping in the bare propagators, and are reflections of the conservation laws in the system. Equations (4.28-29) conserve the total "particle number,"

$$N = \int |\psi|^2 d^d x, \quad (4.30)$$

as well as the total energy $E = \mathcal{H}[\psi]$. Furthermore, as long as the cutoff is introduced in such a way that the Hamiltonian $\mathcal{H}$ is translationally invariant, the total momentum will also be conserved.

We expect that after many iterations of the renormalization group, the vertex u_4 will contain a number of simple poles near zero frequency transfer (when the momentum transfer is small) while the renormalized propagator for ψ will be damped and well behaved. A careful analysis of the singularities in u_4 would then involve keeping track of the positions and residues of these poles. This procedure would presumably be equivalent to introducing propagators and coupling constants for appropriate auxiliary conserved fields, as has been done phenomenologically in the present paper.

Alternatively, one might attempt to remove the singularities in the vertex $u_4(k_i, \omega_i)$ by simply introducing an imaginary part in the frequencies of the *initial* propagator for ψ , while keeping a structureless bare coupling constant u_0 . This procedure violates the conservation laws, however, and one is led by further iterations to the fixed point for a time-dependent Ginzburg-Landau model with no conserved quantity⁴⁴ (model A of HHM¹¹), rather than to a fixed point appropriate to helium.

V. ISOTROPIC ANTIFERROMAGNET

A. Two-field model

As mentioned in the Introduction, there is a simple variant of the two-field planar-spin model which describes an isotropic antiferromagnet.^{12,15} In this model, which we shall denote model G, $\vec{\psi}$ and $\vec{m}$ are (three-component) vector fields repre-

sented the staggered and total magnetizations, respectively. The equations of motion may be taken as

$$\frac{\partial \vec{\psi}}{\partial t} = -\Gamma_0 \frac{\delta F_0}{\delta \vec{\psi}} + g_0 \vec{\psi} \times \frac{\delta F_0}{\delta \vec{m}} + \vec{\theta}, \quad (5.1a)$$

$$\frac{\partial \vec{m}}{\partial t} = \lambda_0 \nabla^2 \frac{\delta F_0}{\delta \vec{m}} + g_0 \vec{\psi} \times \frac{\delta F_0}{\delta \vec{\psi}} + g_0 \vec{m} \times \frac{\delta F_0}{\delta \vec{m}} + \vec{\xi}, \quad (5.1b)$$

where

$$F_0[\vec{\psi}, \vec{m}] = \int d^d x \left[\frac{1}{2} r_0 |\vec{\psi}|^2 + \frac{1}{2} |\nabla \vec{\psi}|^2 + \tilde{u}_0 |\vec{\psi}|^4 + \frac{1}{2} \chi_0^{-1} |\vec{m}|^2 + s_0 (\vec{\psi} \cdot \vec{m})^2 \right]. \quad (5.1c)$$

In Eq. (5.1c) the symbol $|\vec{\psi}|^2$ denotes the scalar product $(\vec{\psi} \cdot \vec{\psi})$. The noise terms $\vec{\theta}$ and $\vec{\xi}$ have correlations analogous to Eqs. (2.1d)-(2.1f). The parameter s_0 is irrelevant for $T > T_c$ and can be taken to be zero; the term $g_0 \vec{m} \times \delta F_0 / \delta \vec{m}$ in Eq. (5.1b) vanishes identically when $s_0 = 0$, and in any case it does not contribute to the correlation functions at T_c , to first order in ϵ , at least. This term should be included in general, however, to preserve the Poisson bracket relations of the antiferromagnet. For simplicity we have normalized $\vec{\psi}$ in such a way that the coefficient of $|\nabla \vec{\psi}|^2$ remains unity.

The recursion relations of model G are quite analogous to those of model E, with modifications arising in certain coefficients, owing to the different number of components ($n=3$). The static recursion relations are the same as in model C of HHM¹¹ (for $n=3$), and the dynamic parameters satisfy

$$\Gamma_{i+1} = b^{z-2} \Gamma_i [1 + 2 \ln b f_i (1 + w_i)^{-1}], \quad (5.2)$$

$$\lambda_{i+1} = b^{z-2} \lambda_i (1 + \frac{1}{2} \ln b f_i), \quad (5.3)$$

$$g_{i+1} = b^{z-d/2} g_i. \quad (5.4)$$

It is then easy to see, from the recursion relations for f_i and w_i that the fixed point has the "symmetric" exponent

$$z = \frac{1}{2} d, \quad (5.5)$$

and the parameter values

$$f_\infty = \epsilon + O(\epsilon^2), \quad (5.6)$$

$$w_\infty = 3 + O(\epsilon). \quad (5.7)$$

The characteristic frequency for the order parameter then goes to zero as

$$\omega_\psi(k=0) = \Gamma / \chi_\psi \propto \kappa^{d/2}, \quad (5.8)$$

while the spin diffusion mode goes as

$$\omega_m(k) = \lambda k^2 / \chi, \quad (5.9)$$

with

$$\lambda \propto \kappa^{-\epsilon/2}, \quad (5.10)$$

and $\chi \rightarrow \text{const}$, in agreement with results of dynamic scaling⁵ and mode coupling theories.^{8,9,21} The ratios R_λ , R_Γ , and $R_{\Gamma/\lambda}$; defined in Table II, are given by

$$R_\lambda = K_d^{1/2} (f_\infty w_\infty)^{-1/2} = (24\pi^2 \epsilon)^{-1/2}, \quad (5.11)$$

$$R_\Gamma = K_d^{1/2} (w_\infty / f_\infty)^{1/2} = (3/8\pi^2 \epsilon)^{1/2}, \quad (5.12)$$

$$R_{\Gamma/\lambda} = w_\infty = 3, \quad (5.13)$$

to lowest order in ϵ . The second-order calculation of Appendix F yields in this case

$$R_\lambda = K_d^{1/2} (3\epsilon)^{-1/2} [1 + 0.271\epsilon + O(\epsilon^2)], \quad (5.14)$$

$$R_\Gamma = K_d^{1/2} (3/\epsilon)^{1/2} [1 - 0.605\epsilon + O(\epsilon^2)], \quad (5.15)$$

and

$$R_{\Gamma/\lambda} = 3[1 - 0.876\epsilon + O(\epsilon^2)]. \quad (5.16)$$

A calculation of the amplitude ratio $R_{\Gamma/\lambda}$ was performed earlier by Joukoff-Piette and Résibois,²⁰ using mode-coupling theory. Their result, $R_{\Gamma/\lambda} = \beta_1/\alpha_1 = 2.5$ is rather close to our first-order result in Eq. (5.13). A similar calculation by Huber and Krueger²⁰ obtained $R_{\Gamma/\lambda} \approx 3.3$. In a recent paper Freedman and Mazenko¹⁵ have treated model G to lowest order in ϵ , using renormalization-group techniques. Their results for f_0 and w_0 agree with ours in lowest order. In addition, these authors have calculated the scaling function K_ψ [Eq. (3.47)] to the next order in ϵ .

B. Hamiltonian antiferromagnetic models

Let us consider a spin system with Hamiltonian

$$\mathcal{H} = - \sum_{ij} J_{ij} (S_x^i S_x^j + S_y^i S_y^j + S_z^i S_z^j), \quad (5.17)$$

and J_{ij} so chosen that the system orders antiferromagnetically. It was shown in Sec. 6 of Ref. 14 that the hydrodynamic behavior of this system corresponds to that of a three-field model with a vector order parameter $\vec{\psi}$ (the staggered magnetization) a conserved vector field $\vec{m}$ (the total magnetization), and a conserved energy density $\tilde{\epsilon}$. Such a model, which we can denote model G' , is analogous to model E' , considered in Appendix E. A similar analysis may be carried out for G' to show that the isotropic antiferromagnet (5.17) will have the critical exponents of model G, and also the same amplitude ratios, for $\alpha < 0$ (as seems to be the case⁴⁵ for $d=3$, $n=3$).

As pointed out in Ref. 11, a real magnet has phonon degrees of freedom in addition to the spin degrees of freedom in (5.17), and it is difficult to

say whether a model with or without conservation of energy is a better representation of a real material. In any case it is only the correction terms and not the leading singularities which are affected by energy conservation, for $\alpha < 0$.

Finally, let us mention that in the presence of a magnetic field the Heisenberg antiferromagnet corresponds to model F (or F'), and in particular the static properties are those of an XY system ($n=2$).^{14,46}

C. Application to RbMnF₃

The dynamic critical behavior of the Heisenberg antiferromagnet RbMnF₃ has been studied in detail using inelastic neutron scattering.^{23,24} Although the accuracy of the measurements is considerably less than in the case of bulk measurements on fluids, the ability to study the complete fluctuation spectrum of both the staggered²³ and total²⁴ magnetization leads to an excellent semi-quantitative understanding of the isotropic antiferromagnet. As has already been stressed,^{23,24,34} the exponent predictions of dynamic scaling theory are reasonably well borne out by the measurements. There is, however, an apparent difference^{23,31} between exponents above and below T_N , which is not understood at present. This difference leads to a weak t dependence of the critical ratios κ_+/ κ_- , R_m , and R_ψ (see below). In addition, one could hope for improvements in the accuracy of the measurements and closeness of approach to the critical point, but it is not clear how these improvements are best achieved.

It should be pointed out that although the frequency spectrum of $\chi_\psi(k, \omega)$ given in Eqs. (3.47) and (3.52) is Lorentzian, this property is not expected to hold in higher order in ϵ , and in particular for $d=3$. A non-Lorentzian spectrum was indeed found recently by Freedman and Mazenko¹⁵ in first order in ϵ for $k > \kappa$. Moreover the neutron experiments²³ on RbMnF₃ gave evidence for large deviations from simple Lorentzian behavior at T_c , since they were fit to a three-peak structure. Even above T_c , for $k \ll \kappa$, there is no reason to believe that $\chi_\psi(k, \omega)$ will be exactly Lorentzian, although the data of Ref. 23 were fit to such a form, with a characteristic frequency

$$\hbar\omega_\psi(k=0) = 12.8\kappa_+^{1.46} \text{ meV}, \quad (5.18)$$

$$\kappa_+ = 0.476 t^{0.71} \text{ \AA}^{-1}. \quad (5.19)$$

The spin diffusion mode, on the other hand, is hydrodynamic, and does have a Lorentzian form at long wavelengths, with a frequency measured²⁴ to be

$$\hbar\omega_m(k) = 1.09\kappa_+^{-0.5} k^2 \text{ meV}. \quad (5.20)$$

[In Eqs. (5.18) and (5.20), κ_+ and k are expressed in $\text{\AA}^{-1}$, and we have used⁴⁷ $a = 4.24 \text{\AA}^{-1}$ for the lattice parameter at T_N .] Below T_N , the spin-wave frequency has been measured as^{23,47}

$$\hbar\omega_-(k) = 25.2 t^{0.27} k \text{ meV}, \quad (5.21)$$

with k again in $\text{\AA}^{-1}$. We shall discuss elsewhere³¹ our estimates of the static ratio κ_+/κ_- obtained from the neutron scattering experiments, which yield

$$\kappa_+/\kappa_- = 2.97 t^{0.17}. \quad (5.22)$$

This result has the unexplained temperature dependence alluded to earlier. We may now obtain experimental values for the dynamic ratios in Eqs. (3.60), (3.68), and (3.69), noting that for the antiferromagnetic model (5.1), χ is finite at T_c , and $\chi_+/\chi_- = 1$. Using Eqs. (5.18)–(5.22) for $d=3$, we find

$$R_\psi^{\text{expt}} = \frac{\omega_\psi(0)}{\omega_-(\kappa_-)} = \left(\frac{\omega_\psi(0)}{\omega_-(\kappa_+)} \right) \left(\frac{\kappa_+}{\kappa_-} \right) = 1.07 t^{0.22}, \quad (5.23)$$

$$R_{\Gamma/\lambda}^{\text{expt}} = \frac{\omega_\psi(0)}{\omega_m(\kappa_+)} = 1.16 t^{-0.03}, \quad (5.24)$$

$$R_m^{\text{expt}} = \frac{\omega_m(\kappa_-)}{\omega_-(\kappa_-)} = 0.10 t^{-0.09}. \quad (5.25)$$

The reduced ratios of (3.61) and (3.68) are

$$\bar{R}_\Gamma^{\text{expt}} = R_\psi^{\text{expt}} (\kappa_-/\kappa_+)^{d/2} = 0.21 t^{-0.02} = 0.23 - 0.24 \quad (5.26)$$

(using $d=3$ in this experimental value) and

$$\bar{R}_\lambda^{\text{expt}} = \bar{R}_\Gamma^{\text{expt}} / R_{\Gamma/\lambda}^{\text{expt}} = 0.18 t^{0.01} = 0.18 - 0.17, \quad (5.27)$$

where the notation $\bar{R}^{\text{expt}}$ again indicates that data both above and below T_N were used. In these equations we have evaluated the ratios for the temperature range $t = 10^{-1} \sim 10^{-2}$ where the experiments^{23,24} were carried out. The quoted experimental amplitudes vary with the fitting function and it is necessary to include the unexplained temperature dependence when comparing with theory.

We may also obtain an experimental value for R_λ using data purely above T_N , from the measured spin-diffusion frequency (5.20) and Eq. (3.55). The quantity g_0 which enters Eq. (3.55) is

$$g_0 = k_B T g \mu_B / \hbar, \quad (5.28)$$

if $\vec{\psi}$ and $\vec{m}$ are measured in emu/cm³, and

$$\chi = k_B T \tilde{\chi}, \quad (5.29)$$

where $\tilde{\chi}$ is the usual susceptibility per cm³ in electromagnetic units. Inserting the value⁴⁸

$$\tilde{\chi} = 3.9 \times 10^{-4}, \quad (5.30)$$

into Eq. (3.55) and using (5.19) we find

$$R_\lambda^{\text{expt}} = 0.17, \quad (5.31)$$

which agrees with (5.27) to the extent that the experimental data are consistent with strong scaling. The mode-coupling calculation by Huber and Krueger²⁰ obtained a value of R_Γ which was rather close to experiment, but a value of R_λ which was too small by a factor of 2.7, so that $R_{\Gamma/\lambda}$ was too large by a similar factor.

Since RbMnF₃ corresponds to a symmetric model, it does not contain slowly varying corrections arising from the small value of α , as in helium. On the other hand, the experiments did not go very close to T_c and the departures from scaling in (5.26) and (5.27) (involving only data above T_c) can be accounted for by ordinary static and dynamic corrections to scaling, with the exponents x_i of Eq. (4.21). The relatively strong temperature dependence in the static ratio κ_+/κ_- of Eq. (5.22), which influences R_ψ and R_m [Eqs. (5.23) and (5.25)] and involves data below T_c , seems less likely to be merely a correction term, and further investigation of this point would be worthwhile. For the purpose of estimating R_λ , however, the temperature dependence of κ_+/κ_- seems to cancel, since the two estimates (5.27) and (5.31) are in agreement.

The theoretical values of R_λ , $R_{\Gamma/\lambda}$, and R_Γ can be estimated from the lowest-order expressions (5.11)–(5.13) by setting $\epsilon = 1$,

$$R_\lambda = 0.065, \quad (5.32)$$

$$R_\Gamma = 0.19, \quad (5.33)$$

$$R_{\Gamma/\lambda} = 3. \quad (5.34)$$

The second-order expressions (F18) and (F19) of Appendix F may be extrapolated using $K_3 = (2\pi^2)^{-1}$ to yield

$$R_\lambda = 0.165, \quad (5.35)$$

$$R_\Gamma = 0.155, \quad (5.36)$$

$$R_{\Gamma/\lambda} = 0.93. \quad (5.37)$$

Although the second-order estimates involve large corrections to the first-order values, the rough agreement of these estimates with the experimental values, (5.24)–(5.27) and (5.31), gives us some confidence in our extrapolation procedures.

VI. CONCLUSION

Let us conclude by summarizing the principal results of the present work.

(i) Phenomenological models with planar-spin and antiferromagnetic dynamics are introduced, and their critical behavior analyzed using renormali-

zation-group methods. These models are applicable to easy-plane ferromagnets, isotropic antiferromagnets, and superfluid helium. Dynamic scaling^{4,5} is shown to hold to all orders in $\epsilon = 4 - d$, and the dynamic exponents are expressed entirely in terms of static exponents (Sec. III). For symmetric models (magnetic systems in zero applied field), the characteristic frequency of the order parameter is given by

$$\omega_\psi(k=0) \propto \kappa^{d/2}, \quad (6.1)$$

where κ is the inverse correlation range. For asymmetric models (liquid helium and magnetic models in an applied field), the corresponding relation is

$$\omega_\psi(k=0) \propto \kappa^{(d+\tilde{\alpha})/2}, \quad (6.2)$$

where $\tilde{\alpha} \equiv \max(\alpha, 0)$, and the transport coefficient (thermal conductivity) diverges as

$$\lambda \propto \kappa^{(4-d+\tilde{\alpha})/2}, \quad (6.3)$$

for $2 < d < 4$. For $d=3$ we obtain $\lambda \propto t^{-\nu/2} C_p^{1/2}$, in agreement with the prediction of Ferrell *et al.*⁴ for liquid helium.

(ii) Matching conditions are proved between characteristic frequencies above and below T_c , and the corresponding amplitude ratios are shown to be universal, i.e., dependent only on the dimensionality and the fixed point reached. For liquid helium the universal ratio is

$$R_m = \lim_{t \rightarrow 0^+} \frac{\lambda(t) \hbar^2 \tilde{\rho}_s(-t)}{m_H^2 \tilde{\rho} k_B T_\lambda C_p(t) c_2(-t)}, \quad (6.4)$$

where t is the reduced temperature, $\tilde{\rho}_s$ the superfluid density, m_H the helium mass, $\tilde{\rho}$ the mass density, $C_p(t)$ the specific heat above T_λ , and $c_2(-t)$ the second-sound velocity. Inserting experimental values on the right-hand side of Eq. (6.4) we obtain a number which agrees reasonably well with the second-order ϵ -expansion estimate. The amplitude of the divergence of λ may be related to experimental data purely above T_c , by the universal amplitude

$$R_\lambda \equiv \lim_{t \rightarrow 0^+} \lambda(t) \kappa^{1/2}(t) / g_0 C_p^{1/2}(t), \quad (6.5)$$

where the inverse correlation range $\kappa(t)$ is obtained³¹ from the singular part of the specific heat, and the constant g_0 is determined from the entropy at T_λ . The experimental value of R_λ at $t=10^{-4}$ and SVP is in good agreement with the second-order ϵ -expansion for this quantity. Equation (6.5) represents the transport coefficient λ purely in terms of equilibrium properties and a calculated coefficient.

The ratio R_m for the antiferromagnet involves spin diffusion above T_N and spin waves below. Moreover, the characteristic frequency (6.1) for the order parameter (staggered magnetization) has also been measured and the corresponding ratios determined. The experimental values in RbMnF_3 are in rough agreement with the estimates obtained from the ϵ expansion (Sec. V).

(iii) For the asymmetric model, the nature of the dynamic fixed point reached depends on the sign of the exponent α [see Eq. (6.2)]. For $\alpha < 0$, $\tilde{\alpha} = 0$ and the asymmetric model reaches a *symmetric* fixed point, whereas for $\alpha > 0$, there is an asymmetric fixed point with distinct exponents and critical amplitudes. Since liquid helium has $\alpha < 0$ but very close to zero, there will be significant corrections to the asymptotic critical behavior reflecting the slow approach of the specific heat to its finite value at T_c . These corrections, which are analyzed in some detail in Sec. IV and Appendix C, tend to mask the scaling behavior at finite distances from T_λ , and may serve to explain the small observed departures⁷ from dynamic scaling and universality in liquid helium.

(iv) The applicability of our simple phenomenological models to real materials is studied by analyzing more complicated models with additional fields. In the case of the Bose liquid it is argued that these additional fields only affect the dynamics at frequencies of order $c_1 \kappa$, which are much higher than ω_ψ , since the first-sound velocity c_1 remains finite at T_λ . For magnetic systems, energy conservation is shown not to affect the asymptotic critical behavior, since $\alpha < 0$ in three dimensions. In the experimental range, however, energy conservation may lead to significant corrections.

ACKNOWLEDGMENTS

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APPENDIX A: HYDRODYNAMICS OF THE PLANAR-SPIN MODEL

We may derive the hydrodynamics of the model in Eqs. (2.1) without specifying the form of the functional $F_0[\psi, m]$. Thus the derivation given in this appendix is not restricted to the specific example of Eq. (2.1c). Let us write a Fokker-Planck equation for the probability density $P[\psi, m]$ in the form

$$\frac{dP}{dt} = \int d^d x \left\{ \frac{\delta}{\delta m} \left[-\lambda_0 \nabla^2 \left(\frac{\delta P}{\delta m} + P \frac{\delta F_0}{\delta m} \right) - 2 g_0 P \text{Im} \left(\psi^* \frac{\delta F_0}{\delta \psi^*} \right) \right] + 2 \text{Re} \frac{\delta}{\delta \psi} \left[\frac{2 \Gamma_0}{c_0} \left(\frac{\delta P}{\delta \psi^*} + P \frac{\delta F_0}{\delta \psi^*} \right) + i g_0 \psi \frac{\delta F_0}{\delta m} P \right] \right\}. \quad (\text{A1})$$

In the absence of time-dependent applied fields, a time-independent solution of (A1) is the thermal equilibrium distribution

$$P_{\text{eq}}[\psi, m] = Z^{-1} e^{-F_0[\psi, m]}, \quad (\text{A2})$$

$$Z = \int e^{-F_0[\psi, m]} d[\psi] d[m]. \quad (\text{A3})$$

The occurrence of this equilibrium solution depends crucially on the facts that the same constant g_0 appears in (2.1a) and (2.1b), and that the constants λ_0 and Γ_0/c_0 in (2.1d) and (2.1e) are the same as in (2.1a) and (2.1b). Indeed, requiring that (A2) be an equilibrium state puts important restrictions on the possible equations of motion that can occur.

Let us define an entropy function $\mathcal{S}(P)$ and a thermodynamic potential $\Omega(P)$ by

$$\mathcal{S} \equiv \int P \ln P d[\psi] d[m], \quad (\text{A4})$$

$$\Omega \equiv \int P F_0 d[\psi] d[m] - \mathcal{S}. \quad (\text{A5})$$

It is clear that Ω is minimized by taking $P = P_{\text{eq}}$, and that

$$\Omega(P_{\text{eq}}) = -\ln Z. \quad (\text{A6})$$

In the ordered phase, when the system has a broken symmetry, a thermal equilibrium state in which the order parameter has a definite phase φ may be chosen of the following form:

$$P_{\text{eq}, \varphi} = \lim_{h_\psi \rightarrow 0^+} Z^{-1} \exp[-F_0 - \text{Re}(h_\psi e^{-i\varphi} \Psi)], \quad (\text{A7})$$

$$\Psi \equiv \int d^d x \psi(x). \quad (\text{A8})$$

[For a system of finite volume, h_ψ must be taken small but finite in (A7).] The thermodynamic potential per unit volume for (A7) will differ only infinitesimally from that of (A2), in the limit of infinite volume, and it may also be verified that $P_{\text{eq}, \varphi}$ is a time-independent solution of the equations of motion in this limit.

A. M as the generator of rotations of Ψ

Let us consider a situation where the system is prepared in some specified state $\rho_0(\psi, m)$ at time $t = t_0$, and then subjected to a *uniform*, time-dependent field coupling to M [i.e., $h_m(x, t) = h_m(t)$]. If $h_\psi = 0$ for $t > t_0$, then the only term in the equations of motion affected by h_m is the second term on the right-hand side of (2.1a). This term has the sole effect of rotating the order parameter by the uniform phase $\delta\varphi(t) = g_0 \int_{t_0}^t h_m(t') dt'$. [This $\delta\varphi$ is in addition to any time dependence that would have been present in the absence of h_m .] There are several important consequences of this rotation.

(i). In the normal phase (above T_c), let us consider the second-order response function $\chi_{\psi; m, \psi}$ which gives the expectation value of $\psi(x, t)$ in the presence of weak time-varying fields $h_m(x', t')$ and $h_\psi(x'', t'')$. It follows that

$$\begin{aligned} \int \chi_{\psi; m, \psi}(x, t, x', t', x'', t'') d^d x' \\ = \begin{cases} i g_0 \chi_\psi(x - x'', t - t'') & \text{for } t > t' > t'', \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (\text{A9})$$

Here χ_ψ is the ordinary linear response function for a time-varying field h_ψ coupling to ψ . Equation (A9), which is a Ward identity, will be useful in the general renormalization-group analysis of Sec. III.

(ii). In the ordered phase, let us consider a probability density P which minimizes Ω , subject to a *constraint* that the total "magnetization"

$$M = \int d^d x [m(x)] \quad (\text{A10})$$

have some specified expectation value $\bar{M}$. Let us suppose, also, that the phase of the order parameter is specified to be φ_0 at time t_0 . Then the density matrix at time t_0 is given by

$$P = [Z(\bar{M})]^{-1} \lim_{h_\psi \rightarrow 0^+} \exp[-F_0 + \mu M - h_\psi (e^{-i\varphi_0} \Psi + e^{i\varphi_0} \Psi^*)], \quad (\text{A11})$$

where μ is a Lagrange multiplier which obeys the relation

$$\mu = \frac{d\Omega}{d\bar{M}}, \quad (\text{A12})$$

and the thermodynamic potential Ω is related to the normalization constant $Z(\bar{M})$ by

$$\ln Z(\bar{M}) = \mu \bar{M} - \Omega. \quad (\text{A13})$$

Note that μ is analogous to the chemical potential of helium. Comparing (A11) with (A7), and involving our previous remarks on the effect of a uniform field coupled to M , we see that the probability density P will not change in time except for a variation of the phase φ given by

$$\frac{d\varphi}{dt} = g_0 \frac{d\Omega}{d\bar{M}}. \quad (\text{A14})$$

Equation (A14) is the Josephson equation²⁵ mentioned in Sec. II.

B. Hydrodynamics

The long-wavelength, low-frequency behavior of our model will be very similar to that of the easy-plane ferromagnet discussed in Ref. 14, except that in the present case there is no field corresponding to the conserved energy density. The

hydrodynamic states in the paramagnetic phase are now described by a single conserved density $m(x, t)$ which is assumed to vary slowly in space and time. In the ordered phase the hydrodynamic states are described by two slowly varying fields $m(x, t)$ and $\varphi(x, t)$. For small long-wavelength fluctuations the thermodynamic potential Ω will have the form

$$\Omega = \Omega_{\text{eq}} + \frac{1}{2} \int d^d x \chi^{-1} m^2, \quad T > T_c \quad (\text{A15})$$

$$\Omega = \Omega_{\text{eq}} + \int d^d x \left(\frac{1}{2} \chi^{-1} m^2 + \frac{1}{2} \rho_s |\nabla \varphi|^2 \right), \quad T < T_c \quad (\text{A16})$$

where ρ_s is a stiffness constant, analogous to the superfluid density in helium, and χ is the susceptibility given by

$$\chi^{-1} = d^2 \Omega(\bar{M}) / d\bar{M}^2. \quad (\text{A17})$$

From the equations of motion (2.1), the thermodynamic potential $\Omega(P)$ may be shown to be a monotonic decreasing function of time (in the absence of any time-dependent perturbations). Using this fact, and Eqs. (A12)–(A17), we may derive the hydrodynamic properties of the present model in the ordered phase, in complete analogy to the derivation in Ref. 14. (Here Ω plays a role similar to the entropy in Ref. 14.) In the limit of long wavelengths, below T_c , one finds a dissipationless flow

$$\vec{j}^m = g_0 \rho_s \vec{\nabla} \varphi, \quad (\text{A18})$$

which leads to a propagating spin-wave mode, undamped in the long-wavelength limit, with frequency given by Eqs. (2.12) and (2.13) of the text. The response functions $\chi_\psi(q, \omega)$ and $\chi_m(q, \omega)$ are both dominated at long wavelengths by the spin-wave modes in the ordered phase. Above T_c we may write

$$\chi_m(q, \omega) = \lambda q^2 / (-i\omega + \lambda q^2 / \chi), \quad (\text{A19})$$

for the correlations of m , but χ_ψ does not have a hydrodynamic mode and is not in general dominated by a simple pole. Nevertheless, one can define a “characteristic frequency” for ψ by its low-frequency behavior, according to Eqs. (2.15) and (2.16).

APPENDIX B: DIAGRAMMATIC EXPANSION AND THE RENORMALIZATION GROUP

A. General formalism

In order to implement the renormalization-group calculations described in Sec. III we need a perturbation expansion for the frequency-dependent correlation functions which follow from the equations of motion (2.1). We shall use the method of

Martin, Siggia, and Rose²⁷ which converts the classical formalism at finite temperatures into a form analogous to zero-temperature quantum field theory, (i.e., with frequency integrals rather than sums). This is accomplished by introducing a fictitious “adjoint” operator for each classical variable, and appropriate commutation relations to generate the dynamics. For the model in Eqs. (2.1), in addition to the three physical fields $\psi(x, t)$, $\psi^*(x, t)$, and $m(x, t)$, we define the adjoints $\hat{\psi}(x, t)$, $\hat{\psi}^*(x, t)$, and $\hat{m}(x, t)$ by the equal-time commutation relations,

$$[\psi(x, t), \hat{\psi}(x', t)] = A_0 \delta^{(d)}(x - x'), \quad (\text{B1a})$$

$$[\psi^*(x, t), \hat{\psi}^*(x', t)] = A_0^* \delta^{(d)}(x - x'), \quad (\text{B1b})$$

$$[m(x, t), \hat{m}(x', t)] = \delta^{(d)}(x - x'). \quad (\text{B1c})$$

All other equal-time commutators between the fields ψ , ψ^* , $\hat{\psi}$, $\hat{\psi}^*$, m , and $\hat{m}$ vanish. The physical fields ψ , ψ^* , and m commute with each other even at different times, as is required for a classical system. The constant A_0 in (B.1) is arbitrary, and may be taken to be unity, as in Ref. 27. We introduce it here for later convenience. We shall define correlation functions

$$C_{AB}(k, \omega) = T_F \langle [A(x, t) B(x', t')]_+ \rangle, \quad (\text{B2})$$

where T_F denotes a Fourier transform, the $+$ is a time-ordering operation, and A and B are chosen among the 6 quantities ψ , ψ^* , $\hat{\psi}$, $\hat{\psi}^*$, m , and $\hat{m}$. Of the 36 functions C_{AB} only nine are non-zero, namely $C_{\psi\psi^*}$, $C_{\psi\hat{\psi}}$, $C_{m\hat{m}}$, $C_{\hat{\psi}\hat{\psi}^*}$, $C_{\hat{\psi}\psi}$, $C_{\hat{\psi}^*\psi^*}$, $C_{\hat{\psi}^*m}$, and $C_{m\hat{\psi}^*}$. The first three are the physical correlation functions, whereas the last six are responses to infinitesimal changes in the thermal noise sources, and have only an indirect relationship to the physical response functions $\chi_\psi(k, \omega)$ and $\chi_m(k, \omega)$. The latter may, however, be determined from the correlation functions $C_{m\hat{m}}$ and $C_{\psi\hat{\psi}^*}$ by means of the fluctuation-dissipation theorem and the Kramers-Kronig relations.

We may note that of the nonzero correlation functions, only four are independent, e.g., $C_{\psi\hat{\psi}^*}$, $C_{\psi\hat{\psi}}$, $C_{m\hat{m}}$, and $C_{m\hat{\psi}^*}$. The remaining nonzero functions are determined from these by the general relations

$$C_{AB}(k, \omega) = C_{BA}(-k, -\omega) = C_{B^*A^*}(k, \omega)^*. \quad (\text{B3})$$

It is convenient to introduce 2×2 matrices G and D for the correlation functions involving the ψ 's and m 's separately, i.e., to define

$$\begin{aligned} G_{11} &= C_{\psi\psi^*}, \quad G_{12} = C_{\psi\hat{\psi}} = [C_{\hat{\psi}^*\psi^*}]^* = G_{21}^*, \\ G_{22} &= C_{\hat{\psi}\hat{\psi}^*} \equiv 0, \quad D_{11} = C_{m\hat{m}}, \quad D_{12} = C_{m\hat{\psi}^*} \\ &= [C_{\hat{m}\psi}]^* = D_{21}^*, \quad D_{22} = C_{\hat{m}\hat{\psi}} \equiv 0. \end{aligned} \quad (\text{B4})$$

These correlation functions may be obtained perturbatively in terms of the bare propagators

$$G_{11}^0(k, \omega) = \frac{4\text{Re}\Gamma_0}{c_0[-i\omega + \Gamma_0(\gamma_0 + k^2)]^2}, \quad (\text{B5a})$$

$$G_{12}^0(k, \omega) = A_0[-i\omega + \Gamma_0(\gamma_0 + k^2)]^{-1}, \quad (\text{B5b})$$

$$D_{11}^0(k, \omega) = \frac{2\lambda_0 k^2}{[-i\omega + \lambda_0 \chi_0^{-1} k^2]^2}, \quad (\text{B5c})$$

$$D_{12}^0(k, \omega) = (-i\omega + \lambda_0 \chi_0^{-1} k^2)^{-1}, \quad (\text{B5d})$$

and the interaction

$$\begin{aligned} \mathcal{H}_I = - \int d^4x [& A_0^{-1} (2\gamma_0 \Gamma_0 \hat{\psi} m \psi + 4\tilde{u}_0 \Gamma_0 c_0 \hat{\psi} \psi |\psi|^2 + i g_0 \chi_0^{-1} \hat{\psi} \psi m \\ & + i g_0 \gamma_0 c_0 \hat{\psi} \psi |\psi|^2) + \text{c.c.} - c_0 \lambda_0 \gamma_0 \hat{m} \nabla^2 |\psi|^2 \\ & - \frac{1}{2} c_0 i g_0 \hat{m} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*)]. \end{aligned} \quad (\text{B6})$$

It was shown in Ref. 27 that the ordinary Feynman rules of quantum field theory may be applied to (B6) to find the dressed propagators. Let us define self-energy matrices

$$\Sigma = (G^0)^{-1} - (G)^{-1}, \quad (\text{B7a})$$

$$\Pi = (D^0)^{-1} - (D)^{-1}, \quad (\text{B7b})$$

and interaction vertices U^0 (linking four G^0 propagators), and V^0 (linking two G_0 propagators and one D^0). Note that the matrix elements Σ_{11} and Π_{11} are zero, and that $\Sigma_{12} = \Sigma_{21}^*$, $\Pi_{12} = \Pi_{21}^*$. We represent the self-energies diagrammatically by drawing a solid line for the matrix G^0 , a wavy line for the matrix D^0 with a four-point vertex for U^0 , and a three-point vertex for V^0 , as in Fig. 1. The non-zero matrix elements of U^0 and V^0 may easily be obtained from the interaction (B6), and are given by

$$U_{\psi\psi\psi\psi}^0 = U_{\psi\psi\psi^*\psi^*}^0 = (4\tilde{u}_0 \Gamma_0 + i\gamma_0 g_0) c_0 / A_0, \quad (\text{B8a})$$

$$U_{\psi\psi^*\psi\psi^*}^0 = U_{\psi\psi^*\psi^*\psi}^0 = (4\tilde{u}_0 \Gamma_0^* - i\gamma_0 g_0) c_0 / A_0^*, \quad (\text{B8b})$$

$$V_{m\psi}^0 = (2\gamma_0 \Gamma_0 + i g_0 \chi_0^{-1}) / A_0, \quad (\text{B9a})$$

$$V_{m\psi^*}^0 = (2\gamma_0 \Gamma_0^* - i g_0 \chi_0^{-1}) / A_0^*, \quad (\text{B9b})$$

$$V_{m\psi\psi}^0 = \lambda_0 \gamma_0 k_1^2 c_0 + \frac{1}{2} i g_0 (k_2^2 - k_3^2) c_0. \quad (\text{B9c})$$

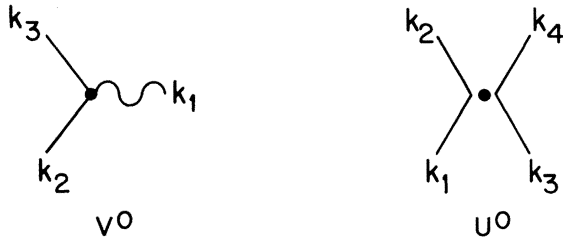


FIG. 1. Bare vertices in perturbation expansion. Straight lines represent the order parameter ψ , wavy line represents the field m .

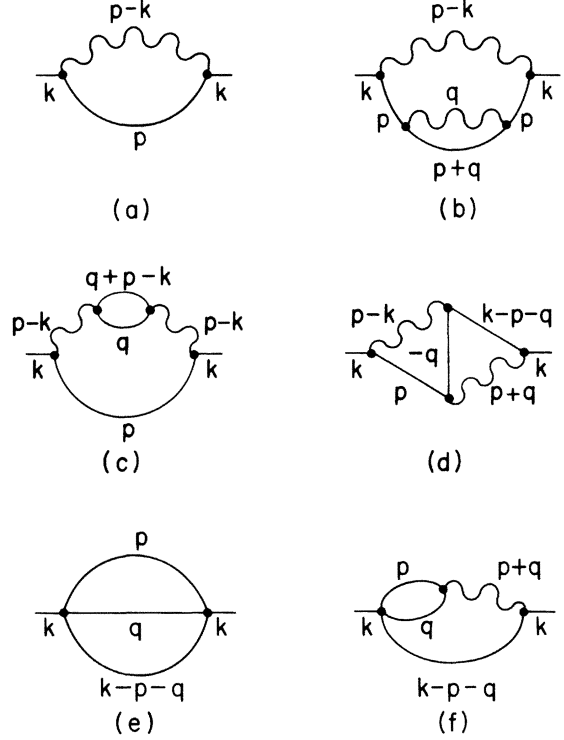


FIG. 2. Some diagrams contributing to the self-energy Σ , for the ψ propagator.

In the last equation, k_1 , k_2 , and k_3 are the wave vectors associated with the fields $\hat{m}$, ψ , and ψ^* , respectively.

Some typical diagram for Σ and Π are given in

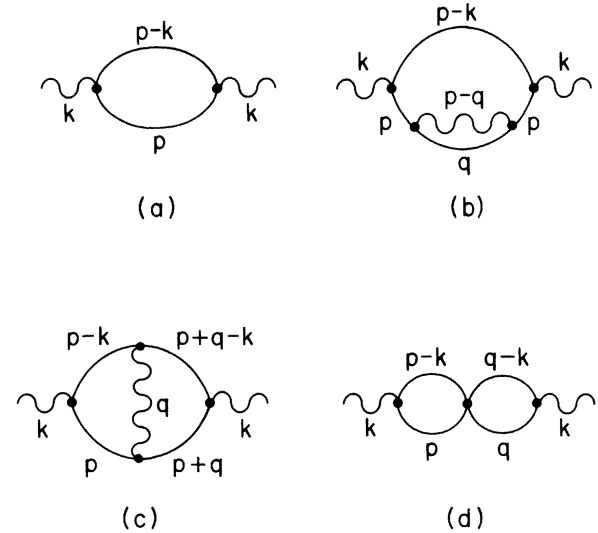


FIG. 3. Some diagrams contributing to the self-energy Π , for the m -propagator.

Figs. 2 and 3. Note that each of these diagrams may represent the sum of several terms, as different elements of the matrices are considered. For example, the contribution of the lowest-order diagram for Σ_{21} , indicated in Fig. 2(a) is actually the sum of two different terms, as illustrated in Fig. 4.

B. Renormalization group

At the l th stage of the renormalization procedure, the equations of the model may be described in terms of renormalized propagators G_l and D_l , together with a set of interaction vertices, each of which are matrices with indices referring to ψ , ψ^* , $\hat{\psi}$, etc. We shall write the propagators in the form

$$G_{12}^l = G_{21}^{l*} = \frac{A_l}{-i\omega + \Gamma_l(r_l + k^2)}, \quad (\text{B10a})$$

$$G_{11}^l = \frac{4\text{Re}\Gamma_l}{c_l | -i\omega + \Gamma_l(r_l + k^2) |^2}, \quad (\text{B10b})$$

$$D_{12}^l = D_{21}^{l*} = \frac{1}{-i\omega + \lambda_l k^2 \chi_l^{-1}}, \quad (\text{B10c})$$

$$D_{11}^l = \frac{2\lambda_l k^2}{| -i\omega + \lambda_l k^2 \chi_l^{-1} |^2}, \quad (\text{B10d})$$

where the constants c_l, r_l , and χ_l^{-1} are obtained from the usual static recursion relations.

The constants Γ_l, λ_l , and A_l are defined in terms of the partial self-energy matrices $\Sigma^l(k, \omega)$ and $\Pi^l(k, \omega)$ in which only intermediate wave vectors greater than $b^{-1}\Lambda$ are included. Specifically we identify

$$A_l^{-1} \equiv \left(A_0^{-1} - \frac{i\partial \Sigma_{21}^l(0, \omega)}{\partial \omega} \Big|_{\omega=0} \right) b^{(d-a-\hat{a})l}, \quad (\text{B11a})$$

$$\Gamma_l \equiv A_l \left(A_0^{-1} \Gamma_0 - \frac{\partial \Sigma_{21}^l(k, 0)}{\partial k^2} \Big|_{k=0} \right) b^{(a+d-a-\hat{a}-\hat{a})l}, \quad (\text{B11b})$$

$$\lambda_l \equiv \chi_l \left(\lambda_0 \chi_0^{-1} - \lim_{k \rightarrow 0} k^{-2} \Pi_{21}^l(k, 0) \right) b^{(a-2)l}, \quad (\text{B11c})$$

where $a, \hat{a}$, and c are the rescaling exponents for the fields $\psi, \hat{\psi}$, and m , respectively. We have chosen the rescaling exponent for $\hat{m}$ as $\hat{c} = d - c$.

The differences between the actual value of Σ^l and Π^l at finite k , and their forms in the limit $k, \omega \rightarrow 0$, must be treated as two-point vertices, which will be included in the diagrammatic expansion for the renormalization group. Thus, for example, we define a two-point vertex W^l by

$$W_{\hat{\psi}\psi}^l(k, \omega) = [G^l(k, \omega)^{-1}]_{21} - b^{(a+d-a-\hat{a})l} \{ [G^0(k', \omega')^{-1}]_{21} - \Sigma_{21}^l(k', \omega') \}, \quad (\text{B12})$$

$$W_{\hat{\psi}\hat{\psi}}^l(k, \omega) = \frac{-4\text{Re}\Gamma_l}{c_l |A_l|^2} - b^{(a+d-2\hat{a}-2a)l} \{ [G^0(k', \omega')^{-1}]_{22} - \Sigma_{22}^l(k', \omega') \}, \quad (\text{B13})$$

where $k' = kb^{-1}$ and $\omega' = \omega b^{-2l}$. The two-point vertices are small for small ϵ , and may be neglected in deriving recursion relations to lowest order in ϵ (see below).

In a similar manner we may consider the partially renormalized three- and four-point vertices V^l and U^l . In general, these will be functions of their wave vectors and frequencies. To lowest order in ϵ , these functions may be replaced by their values in the limit $k, \omega \rightarrow 0$, and may be written in the same form as (B8) and (B9), with γ_0, Γ_0, g_0 , etc., replaced by γ_l, Γ_l, g_l , etc. The constants γ_l and $\bar{u}_l$ are determined by the static renormalization group.

C. Calculation of lowest-order recursion relations

The lowest-order diagrams for Σ and Π are those of Fig. 2(a) and 3(a), respectively. These diagrams have the value

$$\begin{aligned} \Pi_{21}^{(a)}(k, \omega) = & -g_0^2 \chi_0^{-1} \int \frac{d^d p}{(2\pi)^d} \left(\frac{[p^2 - (p+k)^2]^2}{(r_0 + p^2)[r_0 + (p+k)^2] \{ -i\omega + \Gamma_0[r_0 + (p+k)^2] + \Gamma_0^*(r_0 + p^2) \}} \right) \\ & + 2k^2 \lambda_0 \gamma_0^2 \int \frac{d^d p}{(2\pi)^d} \left[\left(\frac{1}{r_0 + p^2} + \frac{1}{r_0 + (p+k)^2} \right) \left(\frac{\Gamma_0 + \Gamma_0^*}{-i\omega + \Gamma_0[r_0 + (p+k)^2] + \Gamma_0^*(r_0 + p^2)} \right) \right], \end{aligned} \quad (\text{B14})$$

$$\begin{aligned} \Sigma_{21}^{(a)}(k, \omega) = & -\frac{(r_0 + k^2)g_0^2}{A_0 \chi_0} \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{(r_0 + p^2) \{ -i\omega + \Gamma_0(r_0 + p^2) + \lambda_0 \chi_0^{-1}(p-k)^2 \}} \right) \\ & + \frac{4\gamma_0^2 \chi_0 \Gamma_0}{A_0} \int \frac{d^d p}{(2\pi)^d} \left(\frac{\Gamma_0(r_0 + p^2) + \lambda_0 \chi_0^{-1}(p-k)^2}{(r_0 + p^2) \{ -i\omega + \Gamma_0(r_0 + p^2) + \lambda_0 \chi_0^{-1}(p-k)^2 \}} \right) \\ & + \frac{2ig_0 \gamma_0}{A_0} \int \frac{d^d p}{(2\pi)^d} \left(\frac{\Gamma_0(r_0 + p^2) + \lambda_0 \chi_0^{-1}(p-k)^2}{(r_0 + p^2) \{ -i\omega + \Gamma_0(r_0 + p^2) + \lambda_0 \chi_0^{-1}(p-k)^2 \}} \right) \\ & + \frac{2(r_0 + k^2)}{A_0} ig_0 \gamma_0 \Gamma_0 \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{(r_0 + p^2) \{ -i\omega + \Gamma_0(r_0 + p^2) + \lambda_0 \chi_0^{-1}(p-k)^2 \}} \right). \end{aligned} \quad (\text{B15})$$

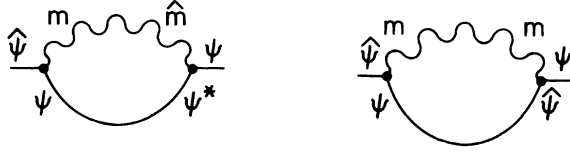


FIG. 4. Two different terms included in diagram 2a, corresponding to two possible choices of the matrix elements of the propagators.

In (B14) and (B15) we have performed the integration over intermediate frequencies, and have summed over the contributing components of the matrices G^0 and D^0 , but we have left explicit the momentum integrals. Recursion relations for λ_{l+1} and Γ_{l+1} are obtained by substituting r_l , g_l , $\gamma_l, \dots$, for r_0 , g_0 , $\gamma_0, \dots$, on the right-hand sides of (B14) and (B15), and restricting the momentum integrals to the shell $\Lambda b^{-1} < p < \Lambda$. The values of the static parameters are determined by the static recursion relations, and the changes in λ_{l+1} and Γ_{l+1} are extracted from the behavior at $\omega \rightarrow 0$ and $k \rightarrow 0$, according to Eqs. (B11). Finally, rescaling is performed and Eqs. (3.11) and (3.12) are obtained. The recursion relation for the quantity A_{l+1} , which was not needed in the text, is given by

$$A_{l+1} = A_l^{(d-a-\hat{a})} \left(1 - \frac{(4\gamma_l^2 \chi_l \Gamma_l + 2ig_l \gamma_l) K_4 \ln b}{\Gamma_l + \lambda_l \chi_l^{-1}} \right). \quad (\text{B16})$$

The coupling constant g_l must be extracted from the asymptotic form of the renormalized three-point vertex V^l . To lowest order in ϵ , we approximate V^l by matrix elements of the same form as (B9) but with renormalized coupling constants g_l , γ_l , etc. The deviation of the full vertex V^l from the simple form of (B9) can be shown to be small, i.e., finite and higher order in ϵ (see below.) Similarly, the four-point vertex U^l may be approximated by the simple form (B8) with renormalized parameters $\tilde{u}_l$, etc.

All the parameters other than g_l , which enter the approximate form of V^l and U^l , are determined either from the static renormalization group, or from the previously discussed dynamic self-energies Σ_{21}^l and Π_{21}^l . Furthermore, one can show that recursion relation for g_l can be written in the form (3.10), to all orders in ϵ . The fact that there is no nontrivial renormalization of g_l is closely related to the exactness of the Josephson relation and the Ward identity. This nonrenormalization of g_l has been checked, to lowest order in ϵ , by explicit computation of the renormalized vertices $V_{m\psi\psi}$ and $V_{m\psi\psi^*}$. The diagrams which enter to lowest order are shown in Fig. 5. The computation is fairly simple in the symmetric case, but



FIG. 5. Lowest-order diagrams for the renormalization of the vertex V .

rather complicated in the asymmetric case where each vertex has both a dissipative and nondissipative part. We shall not give details here.

It may be noted that the coefficient A_l , will be independent of l , for large l , if and only if the nonphysical scaling exponent $\hat{a}$ is chosen appropriately. We see from (3.16) and (B15), that $\hat{a}$ will be equal to 3, in the limit $d \rightarrow 4$. We may also remark that the exponent $\hat{a}$ is in general complex at the asymmetric fixed point.

In principle, we could have introduced a coefficient B_l , analogous to A_l , in the numerator of D_{12}^l . The equations analogous to (B11a) and (B16) would be trivial, however, because Π is proportional to k^2 , and $\partial \Pi_{21}(k, \omega) / \partial \omega|_{\omega=0}$ must vanish in the limit $k \rightarrow 0$. The choice $\hat{c} = d - c$ leads to B_l independent of l ; and we choose $B_l = 1$.

D. Justification to all orders in ϵ

The detailed justification of the above procedures follows closely the discussion of the static case given by Wilson and Kogut.¹⁰ We mention here only a few of the more subtle points.

(i) The self-energies and vertices generated by the renormalization group are claimed to be regular functions of the wave vectors and frequencies, in the limit $k \rightarrow 0$ and $\omega \rightarrow 0$. It is clear that the integrals arising from any diagram in our perturbation theory have integrands which remain finite and regular for any frequency on the real axis as long as all intermediate momenta are restricted to a shell, with $b^{-1}\Lambda < p < \Lambda$. Furthermore the integrals over intermediate frequency variables can be carried out from $-\infty$ to $+\infty$, without any difficulties, because the integrands fall off sufficiently rapidly at infinity and contain no poles or singularities closer to the real axis than the minimum of $2\text{Re}\Gamma_l \Lambda^{2b-2}$ and $(\lambda_l \chi_l^{-1} + \text{Re}\Gamma_l) \Lambda^{2b-2}$.

The integrations over intermediate wave vectors can lead to no divergences, because the regions of integration are finite. Unfortunately, however, the use of sharp cutoffs in k space does lead to nonanalyticities of a weaker sort, e.g., a term in the self-energy proportional to $|k|$ —because the volumes of integration in various diagrams are non-analytic functions of the incoming momenta.⁴⁹

These spurious singular terms lead to no fundamental change in the renormalization group because the singularities introduced at any stage l are cancelled by singularities of the opposite sign in the next stage of the integration. Alternatively, one may employ a renormalization group with “soft” cutoffs and avoid these difficulties.¹⁰ We emphasize that these problems of sharp cutoffs are also a feature of the *static* renormalization group and are not consequences of the extension to dynamics.

(ii) In developing the renormalization group, we must keep explicit track of the slowly-varying parameters, i.e., those parameters which are multiplied by a non-negative power of b (either b^2 or b^0) under the scaling operation¹¹ R_b^s , in the limit of four dimensions. The remaining variables correspond to fast transients and may be ignored (to lowest order in ϵ) or eliminated in favor of the slow variables (at higher orders in ϵ). The slow variables in the present case are the constant part of Σ_{21}^l , the part proportional to k^2 , the constant part of Σ_{22}^l , the parts of Π_{21}^l and Π_{22}^l proportional to k^2 , and the parts of V^l and U^l listed in (B8) and (B9).

(iii) The slowly-varying constants are not all independent—there are various constraints that must be obeyed if detailed balance and the fluctuation-dissipation theorem are to be satisfied. To lowest order in ϵ , for example, Π_{22}^l and Π_{21}^l are related, for small k , by

$$\Pi_{22}^l(k, 0) = -\chi_m(k, 0)\Pi_{21}^l(k, 0). \quad (\text{B17})$$

The relation between Σ_{22}^l and Σ_{21}^l , to lowest order in ϵ , is given by Eq. (B13), with $W_{\psi\psi}^l$ set equal to zero. Also, the coupling constants g_l and γ_l , appearing in different parts of the vertices V^l and U^l must all be the same. Indeed, we see that to lowest order in ϵ , the system of propagators and vertices at stage l is equivalent to a set of Langevin equations of the form (2.1), with renormalized coupling constants. We have already remarked in Sec. II, that in order for the proper equilibrium distribution to be reached, it is necessary for the constants Γ_0 , and λ_0 in (2.1d) and (2.1e) to be the same as in (2.1a) and (2.1b), and similarly for the constant g_0 in (2.1a) to be the same as in (2.1b).

(iv) In higher order in ϵ , the relation between Σ_{22}^l and Σ_{21}^l is not direct. The two-point vertex W^l , defined by (B12) and (B13), is small (of relative order ϵ or higher) but nonzero. In principle, $W^l(k, \omega)$ may be expressed in terms of the slowly-varying coupling constants by inverting the renormalization group equations and the detailed balance condition. (Cf. Wilson and Kogut and also the analysis of Wilson, Ref. 10.) Similar consid-

erations apply to the relation between $V_{\hat{m}\psi\psi}^*$ and $V_{m\psi\hat{\psi}}^*$, etc.

(v) It should be remarked that Π^l and $V_{\hat{m}\psi\psi}^l$ are proportional to k^2 as $k \rightarrow 0$, to all orders in ϵ . Also, matrix elements such as Σ_{11}^l , Π_{11}^l , $V_{m\psi\psi}^*$, $U_{\psi\psi^*\psi\psi}^*$ vanish identically. Matrix elements such as $V_{\hat{m}\psi\hat{\psi}}^l$ or $V_{\hat{m}\psi^*\hat{\psi}}^l$ do not vanish identically, but are small (higher order in ϵ).

(vi) As remarked above, the precise relationship between the physical correlation functions or response functions and the parameters of the off-diagonal Green's functions G_{12} and D_{12} becomes complicated beyond the lowest order in ϵ . We have, for example

$$\chi_\psi(k, \omega) = G_{12}(k, \omega)[A_0^{-1}\Gamma_0 + \Lambda_\psi(k, \omega)], \quad (\text{B18})$$

where Λ_ψ is a vertex correction which has a diagrammatic expansion similar to $\Sigma_{21}(k, \omega)$, except that the first interaction in Λ_ψ must be taken to be $\frac{1}{2}ig_0\hat{m}\psi^*$. We also have

$$\chi_m(k, \omega) = D_{12}(k, \omega)[\lambda_0 k^2 + \Lambda_m(k, \omega)]. \quad (\text{B19})$$

The diagrammatic expansion for Λ_m is the same as for Π_{21} , except that the first vertex must be $-ig(\hat{\psi}\psi + \hat{\psi}^*\psi^*)$. In the symmetric case, model E, where $\gamma_0 = 0$, and $\chi = \chi_0$, it is easy to see that

$$\Lambda_m(k, \omega) = -\chi\Pi_{21}(k, \omega). \quad (\text{B20})$$

Thus, according to (A19), we may express the physical transport coefficient λ as

$$\lambda = \lambda_0 - \lim_{k \rightarrow 0} k^{-2}\Pi_{12}(k, 0)\chi. \quad (\text{B21})$$

A simple relationship between Λ_ψ and Σ_{21} , analogous to (B20) does not exist even in the symmetric case, since there are diagrams such as Fig. 2(e) which contribute to Σ_{21} , but not to Λ . On the basis of our renormalization-group analysis, however, we know that

$$\Gamma(T) = a_3\Gamma', \quad (\text{B22})$$

where

$$\Gamma' \equiv \chi_\psi(0, 0) \frac{r_0\Gamma_0 A_0^{-1} - \Sigma_{21}(0, 0)}{A_0^{-1} - i\partial\Sigma_{21}(0, \omega)/\partial\omega|_{\omega=0}}, \quad (\text{B23})$$

and where a_3 is finite and universal, as $T \rightarrow T_c$, and equal to unity in the limit $\epsilon \rightarrow 0$. We shall see in Appendix D that for model E, $a_3 = 1 + O(\epsilon)$ and that a_3 cancels the denominator of Eq. (B23).

APPENDIX C: CORRECTION TERMS IN THE ASYMMETRIC MODEL

We wish to study the recursion relations (3.8)–(3.15) for model F in the case $\alpha < 0$, i.e., when $v_\infty = 0$. When $|\alpha|$ is small v_l approaches zero very slowly as $l \rightarrow \infty$, and we shall retain the corrections of order v_l in Eqs. (3.14) and (3.15). It is con-

venient to rewrite these recursion relations in differential form,

$$\frac{d\lambda}{dl} = \lambda(-\frac{1}{2}\epsilon + \frac{1}{2}f), \quad (C1)$$

$$\frac{d\chi}{dl} = 4v\chi, \quad (C2)$$

$$\frac{dv}{dl} = v(\alpha/\nu - 4v), \quad (C3)$$

$$\frac{df}{dl} = \frac{f}{w'} \text{Re}[w(\epsilon - \frac{1}{2}f - A)], \quad (C4)$$

$$\frac{dw}{dl} = w(A - \frac{1}{2}f + 4v), \quad (C5)$$

$$A \equiv (1+w)^{-1}[fw'/w - 4vw - 4i(fvw')^{1/2}], \quad (C6)$$

$$w = w' + iw'', \quad (C7)$$

where the quantities λ , f , χ , v , and w are all functions of l . It is clear from Eq. (C3) that $v \rightarrow 0$ when $l \rightarrow \infty$, for $\alpha < 0$. In order to find the large- l behavior of the various quantities, we expand the equations about the symmetric fixed point $w_\infty = 1$, $f_\infty = \epsilon$, $v_\infty = 0$. It is important to note that $w''(l)$ is proportional to $[v(l)]^{1/2}$ [see Eqs. (C5) and (C6)], so that we must go to second order in $w''(l)$. Let us write

$$f(l) \equiv \epsilon[1 + Z(l)], \quad (C8)$$

$$w'(l) \equiv 1 + X(l), \quad (C9)$$

$$w''(l) \equiv Y(l), \quad (C10)$$

$$v(l) \equiv \epsilon k^2(l). \quad (C11)$$

Then to second order in $k(l)$ we find

$$\frac{dY}{dl} = -\frac{3}{4}\epsilon Y - 2\epsilon k, \quad (C12)$$

$$\frac{dX}{dl} = -\frac{1}{4}\epsilon X - \frac{1}{8}\epsilon Y^2 + 2\epsilon k^2 + \epsilon kY, \quad (C13)$$

$$\frac{dZ}{dl} = -\epsilon Z + \frac{1}{4}\epsilon X - \epsilon kY + \frac{1}{8}\epsilon Y^2 + 2\epsilon k^2. \quad (C14)$$

These equations may now be solved for $Y(l)$, $X(l)$, and $Z(l)$ in terms of $k(l)$, assuming $k(l)$ to be slowly varying. The general form of these equations is

$$\frac{dh}{dl} = -ah(l) + s(l), \quad (C15)$$

with $s(l)$ slowly varying. The solution is

$$h(l) = e^{-al} \int_{l_0}^l e^{al'} s(l') dl', \quad (C16)$$

which may be written as

$$h(l) = s(l) \int_{l_0}^l dl' \exp[a(l' - l) + \sigma(l)(l' - l) + \dots], \quad (C17)$$

where $\sigma(l) \equiv d \ln s(l)/dl < 0$, and we assume that $|\sigma| < a$ [i.e., $s(l)$ is slowly varying]. We may then extend the integral in (C17) down to $l_0 \rightarrow -\infty$, and obtain

$$h(l) = s(l)(a + \sigma)^{-1} = a^{-1}s(l) - a^{-2}\dot{s}(l) + \dots, \quad (C18)$$

where the dot in $\dot{s}(l)$ denotes a derivative with respect to l . Equation (C18) can be applied to Eqs. (C12)–(C14), in turn, to obtain

$$Y(l) = -\frac{3}{8}k + (32/9\epsilon)\dot{k}(l) + \dots, \quad (C19)$$

$$X(l) = 8k^2 - \frac{1}{2}Y^2 + 4kY + (4/\epsilon)Y\dot{Y} - (64/\epsilon)k\dot{k} - (16/\epsilon)(\dot{k}Y + k\dot{Y}) + \dots, \quad (C20)$$

$$Z(l) = 4k^2 + (40/9\epsilon)k\dot{k} + \dots, \quad (C21)$$

where the terms left out of Eqs. (C19)–(C21) are of higher order in k , $\dot{k}$, etc. Equations (C8) and (C1) yield

$$f(l) = \epsilon + 4v + (40/18\epsilon)\dot{v} + O(\dot{v}^2, \ddot{v}) \quad (C22)$$

and

$$\frac{d \ln \lambda}{dl} = \frac{d \ln \chi}{dl} \left(\frac{1}{2} + \nu \varphi_1 \frac{d \ln v}{dl} \right), \quad (C23)$$

with

$$\varphi_1 = (5/18\epsilon\nu)[1 + O(\epsilon)]. \quad (C24)$$

If we define

$$\varphi(l) = \text{const} \lambda(l) \chi^{-1/2}(l), \quad (C25)$$

then Eq. (C23) reads

$$\frac{d \ln \varphi}{dl} = \varphi_1 4\nu \frac{dv}{dl}, \quad (C26)$$

where we have used (C2). Equation (C26) may be solved to yield

$$\varphi = \text{const} e^{4\varphi_1 \nu v} = 1 + 4\varphi_1 \nu v + \dots. \quad (C27)$$

In Eq. (C27) we have chosen the undetermined constant of Eq. (C25) such that $\varphi(\infty) = 1$.

In order to relate the function $v(l)$ to observable quantities, we note that the physical susceptibility $\chi(T)$ is expressible in terms of the rescaled quantity $\chi(l)$ by [cf. Eq. (3.8)],

$$\chi(T) = (\kappa/\Lambda)^{2\alpha-d} \chi(l - l_t) = \chi(l = l_t), \quad (C28)$$

$$l_t \equiv l_0 - \ln(\kappa/\Lambda) = l_0 - \nu \ln t; \quad (C29)$$

the second equality in (C28) follows from Eq. (3.26), for $\alpha < 0$. Let us define the "effective exponent"

$$\alpha_e(t) \equiv -\frac{d \ln \chi(T)}{d \ln t} = \nu \frac{d \ln \chi(T)}{dl} \bigg|_{l=l_t}, \quad (C30)$$

which according to Eq. (C2), is given by

$$\alpha_e(t) = \alpha_e(l) = 4\nu v(l). \quad (C31)$$

Equation (C27) can then be rewritten

$$\varphi = 1 + \varphi_1 \alpha_e + O(\alpha_e^2). \quad (C32)$$

The temperature dependence of α_e is found from the differential equation (C3), which can be written as

$$\frac{d\alpha_e}{dl} = \frac{\alpha_e}{\nu} (\alpha - \alpha_e). \quad (C33)$$

The solution is

$$\alpha_e = \frac{-\alpha A \exp[\alpha/\nu(l-l_0)]}{A - \alpha - A \exp[\alpha/\nu(l-l_0)]} = \frac{A t^{-\alpha}}{1 + (A/\alpha)(t^{-\alpha} - 1)}, \quad (C34)$$

with $A \equiv \alpha_e(l=l_0)$; in the second equality in (C34) we used (C29). The quantity $\chi(l)$ can now be found by solving Eq. (C2), and $\chi(T)$ obtained from (C30), as

$$\chi(T)/\chi_0 = (A/\alpha)(t^{-\alpha} - 1) + 1, \quad (C35)$$

where $\chi_0 = \chi(l=l_0) = \chi(t=1)$, is the value “far from T_c .” The quantities χ_0 and A are nonuniversal since they depend on l_0 , or the cut-off Λ .

Note that in liquid helium the “susceptibility” $\chi(T)$ is the specific heat $C_p(T)$, as explained in Sec. IV B. The parameter $\alpha_e(t)$ is simply related to the ratio $C_p^+(t)/C_p^-(-t)$. We may write

$$C_p^+(T)/C_0 = (A/\alpha)(t^{-\alpha} - 1) + 1, \quad t > 0, \quad (C36a)$$

$$C_p^-(T)/C_0 = (A'/\alpha)|t|^{-\alpha} - A/\alpha + 1, \quad t < 0. \quad (C36b)$$

From Eqs. (C36) and (C34) we find

$$C_p^+(t)/C_p^-(t) = [1 + \bar{P}\alpha_e(t)]^{-1}, \quad (C37)$$

where we defined a universal quantity, Eq. (4.26),

$$\bar{P} \equiv \alpha^{-1}(A'/A - 1) \equiv A'P/A. \quad (C38)$$

From the relation between the physical transport coefficient $\lambda(T)$ and the rescaled function $\lambda(l)$, we find, using (C25), (3.3), and (3.4),

$$\lambda(T) = (\kappa/\Lambda)^{-\epsilon/2} \lambda_\infty [\chi(l_t)/\chi_0]^{1/2} \varphi(l_t), \quad (C39)$$

$$\lambda(T) = g_0 (w_\infty f_\infty)^{1/2} \kappa^{-\epsilon/2} [C_p(T)]^{1/2} \varphi(\alpha_e). \quad (C40)$$

In obtaining Eq. (C40), we have neglected the exponentially decaying transients in $f(l)$, arising from the terms independent of $\nu(l)$ [or $k(l)$] in Eqs. (C12)–(C14). These will give rise to powers of t with combinations of the following exponents corresponding, respectively, to the eigenoperators w' , w'' , and $\epsilon^{-1}f - \frac{1}{3}w'$:

$$x_1 = \frac{1}{4}\epsilon\nu + O(\epsilon^2), \quad (C41)$$

$$x_2 = \frac{3}{4}\epsilon\nu + O(\epsilon^2), \quad (C42)$$

$$x_3 = \epsilon\nu + O(\epsilon^2). \quad (C43)$$

Thus the final expression for $\lambda(T)$ is as given in Eq. (4.21) of the text.

The foregoing calculation applied to the case $\alpha < 0$, where the susceptibility $\chi(T)$ goes to a finite limit at T_c , and α_e vanishes when $t \rightarrow 0$ [cf. Eq. (C34)], so that the expansion in (C32) should become applicable as $T \rightarrow T_c$. For $\alpha > 0$, Eqs. (C40) and (C37) can still be shown to be valid, but α_e tends towards α at T_c , and $\varphi(\alpha_e)$ goes to a finite constant, which affects the amplitude of $\lambda(T)$ in Eq. (C40), in the asymptotic limit.

APPENDIX D: SECOND-ORDER CALCULATION FOR MODEL E

In order to find the amplitude ratios R_λ and R_Γ to second order in ϵ we shall use the Feynman-graph expansion method of Wilson,⁵⁰ rather than the recursion relations. This means that we do perturbation theory in the Hamiltonian (B6) with parameters $\tilde{u}_0$, χ_0 , Γ_0 , λ_0 , g_0 , γ_0 , etc., which depend on ϵ , and are fixed in such a way as to eliminate slow transients from the perturbation theory. We shall only consider the symmetric case, where $\gamma_0 = 0$, so that Γ_0 is real, χ is equal to the bare susceptibility χ_0 , and $\tilde{u}_0 = u_0$. The physical coefficients λ and Γ will be obtained from Eqs. (B21)–(B23), so that we need only calculate the off-diagonal self-energies Σ_{21} and Π_{21} .

Since g_0 is not renormalized, we shall use it to fix the overall frequency scale. We also choose $A_0 = c_0 = 1$. Let us introduce the dimensionless quantities

$$f_0 \equiv K_d g_0^2 \Lambda^{-\epsilon} / \lambda_0 \Gamma_0, \quad (D1)$$

$$w_0 \equiv \Gamma_0 \chi_0 / \lambda_0, \quad (D2)$$

where

$$K_d^{-1} \equiv 2^{d-1} \pi^{d/2} \Gamma(\frac{1}{2}d) = 8\pi^2 [1 - \frac{1}{2}\epsilon(\ln 4\pi + 1 - C_E) + O(\epsilon^2)] \\ = 8\pi^2 [1 - 1.477\epsilon + O(\epsilon^2)]. \quad (D3)$$

We shall fix f_0 and w_0 , as functions of ϵ , by the requirement that the scaling relations (3.55) and (3.56) be satisfied order by order.

The diagrams contributing to Σ_{12} up to second order in ϵ are shown in Fig. 2. The three-point vertex is V^0 [Eq. (B9)], and the four-point vertex U^0 [Eq. (B8)]. The propagators will be taken in the form (B5), except that r_0 is replaced by $r = \chi_\psi^{-1}(k=0)$, i.e., one performs a “mass renormalization.”⁵⁰ After considerable rearrangement, the contribution from the diagrams in Fig. 2 can be reduced to the following form:

$$\Sigma_{21}^{(a)}(k=0, \omega) = -r g_0^2 \chi_0^{-1} \int_0^\Lambda \frac{d^4 p}{(2\pi)^4} (r+p^2)^{-1} [-i\omega + \Gamma_0(r+p^2) + \lambda_0 \chi_0^{-1} p^2]^{-1}, \quad (D4)$$

$$\begin{aligned} \Sigma_{21}^{(b)}(0, 0) + \Sigma_{21}^{(c)}(0, 0) &= r (g_0^2 \chi_0^{-1})^2 \int_0^\Lambda \frac{d^4 p}{(2\pi)^4} \int_0^\Lambda \frac{d^4 q}{(2\pi)^4} (r+q^2)^{-1} [\lambda_0 \chi_0^{-1} p^2 + \Gamma_0(r+p^2)]^{-2} \\ &\quad \times \left(\frac{1}{\lambda_0 \chi_0^{-1} [p^2 + (p+q)^2] + \Gamma_0(r+q^2)} + \frac{2}{(r+p^2) \Gamma_0} \frac{(p+q)^2 - q^2}{p^2 + q^2 + (p+q)^2 + 3r} \right), \end{aligned} \quad (D5)$$

$$\begin{aligned} \Sigma_{21}^{(d)}(0, 0) &= r (g_0^2 \chi_0^{-1})^2 \int_0^\Lambda \frac{d^4 p}{(2\pi)^4} \int_0^\Lambda \frac{d^4 q}{(2\pi)^4} [(\lambda_0 \chi_0^{-1} p^2 + \Gamma_0 p^2 + \Gamma_0 r)(\lambda_0 \chi_0^{-1} q^2 + \Gamma_0 q^2 + \Gamma_0 r)]^{-1} \\ &\quad \times \left(\frac{[p^2 - (p+q)^2][q^2 - (p+q)^2]}{(r+p^2)(r+q^2)[r+(p+q)^2] \Gamma_0 [p^2 + q^2 + (p+q)^2 + 3r]} + \frac{1}{[\lambda_0 \chi_0^{-1} (p^2 + q^2) + \Gamma_0 r + \Gamma_0 (p+q)^2][r+(p+q)^2]} \right), \end{aligned} \quad (D6)$$

$$\Sigma_{21}^{(e)}(0, \omega) = 128 u_0^2 \Gamma_0 \int_0^\Lambda \frac{d^4 p}{(2\pi)^4} \int_0^\Lambda \frac{d^4 q}{(2\pi)^4} \frac{1}{(r+p^2)(r+q^2)[r+(p+q)^2]} \frac{\Gamma_0 [3r+p^2+q^2+(p+q)^2]}{-i\omega + \Gamma_0 [3r+p^2+q^2+(p+q)^2]}. \quad (D7)$$

The diagram coming from Fig. 2(f) gives no contribution in the symmetric model for $k \rightarrow 0$.

Similarly, the diagrams for $\Pi_{21}(k, \omega)$ shown in Fig. 3 lead to the integrals

$$\Pi_{21}^{(a)}(k, 0) = -g_0^2 / \chi_0 \Gamma_0 \int_0^\Lambda \frac{d^4 p}{(2\pi)^4} 2(k \cdot p)^2 (p^2 + r)^{-3}, \quad (D8)$$

$$\begin{aligned} \Pi_{21}^{(b)}(k, 0) &= \frac{1}{2} (g_0^2 / \chi_0 \Gamma_0)^2 k^2 \int_0^\Lambda \frac{d^4 p}{(2\pi)^4} \int_0^\Lambda \frac{d^4 q}{(2\pi)^4} \{ p^2 (r+p^2)^{-3} \\ &\quad \times (r+q^2)^{-1} [\Gamma_0 (p^2 + q^2 + 2r) + \lambda_0 \chi_0^{-1} (p+q)^2]^{-1} \}, \end{aligned} \quad (D9)$$

$$\begin{aligned} \Pi_{21}^{(c)}(k, 0) &= \frac{1}{2} (g_0^2 / \chi_0 \Gamma_0)^2 k^2 \int_0^\Lambda \frac{d^4 p}{(2\pi)^4} \int_0^\Lambda \frac{d^4 q}{(2\pi)^4} \{ (p \cdot q) (r+p^2)^{-2} \\ &\quad \times (r+q^2)^{-2} [\Gamma_0 (p^2 + q^2 + 2r) + \lambda_0 \chi_0^{-1} (p+q)^2]^{-1} \}, \end{aligned} \quad (D10)$$

and the diagram in Fig. 3(d) does not contribute. The quantities f_0 and w_0 are known in lowest order from the recursion relation analysis, and in general we write them in the form

$$f_0 = \epsilon (1 + f'_0 + \dots), \quad (D11)$$

$$w_0 = 1 + w'_0 + \dots. \quad (D12)$$

We find, to second order in ϵ , for $r \rightarrow 0$,

$$\begin{aligned} \Sigma_{21}^{(a)}(0, 0) &= -\frac{r \Gamma_0 f_0}{2(1+w_0)} \left\{ w_0 \ln \left(\frac{w_0}{1+w_0} \right) \left[1 + \frac{\epsilon}{2} \ln \left(\frac{\Lambda^2}{r} \right) \right] \right. \\ &\quad \left. + \ln \left(\frac{\Lambda^2}{r} \right) + \frac{\epsilon}{4} \ln^2 \left(\frac{\Lambda^2}{r} \right) \right\}, \end{aligned} \quad (D13)$$

$$\left. \frac{\partial \Sigma_{21}^{(a)}}{\partial i\omega} \right|_{\omega=0} = \frac{f_0 w_0}{2(1+w_0)} \left[1 + (1+w_0) \ln \left(\frac{w_0}{1+w_0} \right) \right] + O(\epsilon^2), \quad (D14)$$

$$\begin{aligned} \Pi_{21}^{(a)}(k, 0) &= \lambda_0 k^2 \chi_0^{-1} f_0 d^{-1} \left[\frac{3}{2} - \ln(\Lambda^2/r) \right. \\ &\quad \left. + \frac{3}{4} \epsilon \ln(\Lambda^2/r) - \frac{1}{4} \epsilon \ln^2(\Lambda^2/r) \right]. \end{aligned} \quad (D15)$$

The other integrals, which are multiplied by $f_0^2 \sim \epsilon^2$, may be evaluated for $d=4$, inserting the lowest-order values for f_0 and w_0 from (D11) and

(D12). A rather lengthy calculation yields, for $r \rightarrow 0$,

$$\begin{aligned} \Sigma_{21}^{(b)} + \Sigma_{21}^{(c)} &= \Gamma_0 r f_0^2 2^{-6} [2 \ln^2(\Lambda^2/r) + \ln(\Lambda^2/r) \\ &\quad \times (20 \ln 2 - 1 - 12 \ln 3)], \end{aligned} \quad (D16)$$

$$\Sigma_{21}^{(d)} = \Gamma_0 r f_0^2 2^{-3} [\ln(\Lambda^2/r) (12 \ln 2 - 6 \ln 3 - 1)], \quad (D17)$$

$$\Sigma_{21}^{(e)}(0, \omega) = \Sigma_{21}^{(e)}(0, 0) + i\omega 96 \ln \frac{4}{3} K_4^2 u_0^2 \ln(\Lambda^2/r). \quad (D18)$$

The term $\Sigma_{21}^{(e)}(0, 0)$ is just equal to Γ_0 times the contribution of the diagram 2(e) to the renormalization of r , and thus drops out of the correction to Γ' . We choose $u_0 = \frac{1}{5} \pi^2 \epsilon$ fixed at its lowest-order value,¹¹ for $n=2$. In addition, we find

$$\begin{aligned} \Pi_{21}^{(b)}(k, 0) &= (\lambda_0 k^2 / \chi_0) f_0^2 2^{-5} [\ln(\Lambda^2/r) (6 \ln 3 - 12 \ln 2 - 1) \\ &\quad + \ln^2(\Lambda^2/r)], \end{aligned} \quad (D19)$$

$$\Pi_{21}^{(c)}(k, 0) = (\lambda_0 k^2 / \chi_0) f_0^2 2^{-3} [\ln(\Lambda^2/r) (6 \ln 2 - 3 \ln 3 - 1)]. \quad (D20)$$

We now insert the above results into Eqs. (B23)

and (B21) for Γ' and λ , with $A_0 \equiv 1$ and $\chi_0 \equiv \chi$, for the symmetric case. To linear order in ϵ we obtain, using Eqs. (D11)–(D20),

$$\begin{aligned} \Gamma'/\Gamma_0 &= (1 - \frac{1}{4}\epsilon + \frac{1}{4}\epsilon \ln 2) [1 + \frac{1}{4}\epsilon \ln(\Lambda^2/r) \\ &\quad + \frac{1}{32}\epsilon^2 \ln^2(\Lambda^2/r) + \epsilon^2 \ln(\Lambda^2/r) \\ &\quad (\frac{1}{4}f'_0 - \frac{1}{8}w'_0 - \frac{15}{8}\ln 2 + \frac{15}{16}\ln 3 + \frac{9}{64} - \frac{3}{50}\ln \frac{4}{3})], \end{aligned} \quad (D21)$$

$$\begin{aligned} \lambda/\lambda_0 &= (1 - \frac{3}{8}\epsilon) [1 + \frac{1}{4}\epsilon \ln(\Lambda^2/r) + \frac{1}{32}\epsilon^2 \ln^2(\Lambda^2/r) \\ &\quad + \epsilon^2 \ln(\Lambda^2/r) (\frac{1}{4}f'_0 + \frac{3}{16}\ln 3 - \frac{3}{8}\ln 2 + \frac{1}{8})]. \end{aligned} \quad (D22)$$

According to dynamic scaling [see Eqs. (3.5) and (3.6)], we must have

$$\lambda \sim \kappa^{-\epsilon/2} \sim \gamma^{-(\epsilon/2)(2-\eta)^{-1}}, \quad (D23)$$

$$\Gamma' \sim \kappa^{-\epsilon/2+\eta} \sim \gamma^{-(\epsilon/2-\eta)(2-\eta)^{-1}}. \quad (D24)$$

We may verify these relations to linear order in ϵ , by comparing the terms proportional to $\epsilon \ln(\Lambda^2/r)$ with the terms of order $\epsilon^2 \ln^2(\Lambda^2/r)$ in (D21) and (D22). This exponentiation condition confirms the validity of the lowest-order expressions for f_0 and w_0 in Eqs. (D11) and (D12). Moreover, from the condition that the scaling relations (D23) and (D24) should also hold in *sec-ond* order in ϵ , we find conditions on the terms proportional to $\epsilon^2 \ln(\Lambda^2/r)$ in (D21) and (D22), which imply

$$f'_0 = \frac{3}{4} \ln \frac{4}{3} - \frac{1}{2} = -0.284, \quad (D25)$$

$$w'_0 = \frac{41}{200} - \frac{162}{25} \ln \frac{4}{3} = -1.659, \quad (D26)$$

where we have used the value (for $n=2$) $\eta = \frac{1}{50}\epsilon^2$.

In order to find the critical amplitude ratios we return to Eq. (D22), which we rewrite as

$$\lambda = \lambda_0 (1 - \frac{3}{8}\epsilon) (\Lambda^2/r)^{\epsilon/4} = R_\lambda g_0 \chi^{1/2} \kappa^{-\epsilon/2}, \quad (D27)$$

where

$$R_\lambda = (1 - \frac{3}{8}\epsilon) (w_0 f_0)^{-1/2} K_d^{1/2}, \quad (D28)$$

and we have used Eqs. (D1), (D2), and the relation⁵¹ $r = \kappa^2 [1 + O(\epsilon^2)]$. Since the correction to w_0 in (D26) is rather large, we shall calculate $w_0^{1/2} = 1 + \frac{1}{2}w'_0\epsilon + \dots$, which is the quantity entering (D28). Inserting also f_0 to second order in ϵ from (D11) and (D25), we find

$$\begin{aligned} R_\lambda &= K_d^{1/2} \epsilon^{-1/2} (1 - \frac{3}{8}\epsilon - \frac{1}{2}w'_0\epsilon - \frac{1}{2}f'_0\epsilon) \\ &= K_d^{1/2} \epsilon^{-1/2} (1 + 0.597\epsilon). \end{aligned} \quad (D29)$$

In a similar manner we find

$$\Gamma' = \Gamma_0 (1 - \frac{1}{4}\epsilon + \frac{1}{4}\epsilon \ln 2) \kappa^{-\epsilon/2+\eta}. \quad (D30)$$

In order to calculate the physical quantity Γ , we must know the vertex correction $\Lambda(k, \omega)$ in

(B18) to first order in ϵ . Only the diagram Fig. 2(a) contributes, and we have, to first order in ϵ ,

$$\Lambda_\psi(k, \omega) = -(\gamma + k^2)^{-1} \Sigma_{21}^{(a)}(k, \omega). \quad (D31)$$

Using (B18), (B23), (A20), and (2.16), we then find, for the symmetric model with $A_0=1$

$$\Gamma = \Gamma' \left(1 - \frac{i \partial \Sigma_{21}(0, \omega)}{\partial \omega} \Big|_{\omega=0} \right). \quad (D32)$$

Thus, for R_ψ of Eq. (3.67) we find

$$R_\psi^2 = \frac{\Gamma^2 \chi}{r^{-2} g_0^2 \kappa^d} = R_\Gamma^2 \left(\frac{\kappa_+}{\kappa_-} \right)^d, \quad (D33)$$

$$\begin{aligned} R_\Gamma &= \Gamma_0 g_0^{-1} \chi_0^{1/2} (1 - \frac{1}{4}\epsilon \ln 2) \\ &= K_d^{1/2} (w_0/f_0)^{1/2} (1 - \frac{1}{4}\epsilon \ln 2). \end{aligned} \quad (D34)$$

Expanding, once again, the quantities $w_0^{1/2}$ and $f_0^{1/2}$ we obtain

$$R_\Gamma = K_d^{1/2} \epsilon^{-1/2} [1 - 0.860\epsilon + O(\epsilon^2)]. \quad (D35)$$

APPENDIX E: EFFECTS OF ENERGY CONSERVATION

Let us consider the following three-field model

$$\frac{\partial \psi}{\partial t} = -2\Gamma_0 \frac{\delta F_0}{\delta \psi^*} - i g_0 \psi \frac{\delta F_0}{\delta m} + \theta, \quad (E1a)$$

$$\frac{\partial m}{\partial t} = \lambda_0 \nabla^2 \frac{\delta F_0}{\delta m} + 2g_0 \text{Im} \left(\psi^* \frac{\delta F_0}{\delta \psi^*} \right) + \zeta, \quad (E1b)$$

$$\frac{\partial \bar{\epsilon}}{\partial t} = \lambda_0^E \nabla^2 \frac{\delta F_0}{\delta \bar{\epsilon}} + \zeta^E, \quad (E1c)$$

$$\begin{aligned} F_0 &= \int d^d x \left(\frac{1}{2} \tilde{r}_0 |\psi|^2 + \frac{1}{2} |\nabla \psi|^2 + \tilde{u}_0 |\psi|^4 \right. \\ &\quad \left. + \frac{1}{2} \chi_0^{-1} m^2 + \frac{1}{2} C_0^{-1} \bar{\epsilon}^2 + \gamma_0^E \bar{\epsilon} |\psi|^2 \right), \end{aligned} \quad (E1d)$$

where m and $\bar{\epsilon}$ are real fields representing the z component of magnetization S_z and energy density, respectively, and ψ is a complex order parameter representing $S_x - iS_y$. The noise sources θ , ζ , and ζ^E have Gaussian correlations, as in (2.1). By the symmetry properties of the system under 180° rotations about the x axis, it may be seen that the $\bar{\epsilon}$ equation cannot possess reversible terms (proportional to g_0), or terms containing $\delta F_0/\delta m$. The model defined by Eq. (B1), which we shall call model E' reduces to model E when $\gamma_0^E=0$, and to model C of HHM¹¹ when $g_0=0$. It corresponds to the planar ferromagnet in the absence of a z magnetic field discussed in Refs. 14 and 34. We can introduce a corresponding asymmetric three-field model (F') by adding to F_0 terms linear in m , as in (2.1c), but we shall not treat this model explicitly here. The recursion relations for model E' may be obtained analogously to those of Sec. III for model E . The static recur-

sion relations are identical to (3.5)–(3.9), with in addition an equation for $v_i^E \equiv K_d(\gamma_i^E)^2 C_i$,

$$v_{i+1}^E = b^\epsilon v_i^E [1 - \ln b(32K_d\mu_i - 4v_i^E)]. \quad (\text{E2})$$

The dynamic recursion relations can be written in terms of f_i and w_i of Eqs. (3.3) and (3.4) and

$$\mu_i \equiv \lambda_i^E / \Gamma_i C_i \quad (\text{E3})$$

(note that Γ_i is real since the model is symmetric, i.e., even in m). These equations are

$$f_{i+1} = b^\epsilon f_i \left[1 + \ln b \left(\frac{4v_i^E}{1+\mu_i} - \frac{f_i}{1+w_i} - \frac{1}{2} f_i \right) \right], \quad (\text{E4})$$

$$w_{i+1} = w_i \left[1 + \ln b \left(\frac{f_i}{1+w_i} - \frac{1}{2} f_i - \frac{4v_i^E}{1+\mu_i} \right) \right], \quad (\text{E5})$$

$$\mu_{i+1} = \mu_i \left[1 - \ln b \left(4v_i^E + \frac{f_i}{1+w_i} - \frac{4v_i^E}{1+\mu_i} \right) \right]. \quad (\text{E6})$$

Repeating the analysis of model C , or model F (Sec. III), for Eq. (E2), we find

$$4v_\infty^E = \tilde{\alpha}/\nu = \frac{1}{5}\epsilon + O(\epsilon^2), \quad (\text{E7})$$

where $\tilde{\alpha} = \max(\alpha, 0)$. Thus for $\alpha < 0$, $v_\infty^E = 0$ and the energy becomes uncoupled from ψ and m near T_c . The exponents and amplitude ratios are the same in models E' and E , though of course the correction terms will differ. For $\alpha > 0$, we may combine Eqs. (E4) and (E5) to find

$$f_\infty = \epsilon = 2f_\infty/(1+w_\infty) - 8v_\infty^E/(1+\mu_\infty), \quad (\text{E8})$$

irrespective of the values of w_∞ and μ_∞ . Then Eq. (E6) may be rewritten, using (E7), as

$$\mu_{i+1} = \mu_i [1 - \ln b(\tilde{\alpha}/\nu + \frac{1}{2}\epsilon)], \quad (\text{E9})$$

which only has solutions

$$\mu_\infty = 0, \quad (\text{E10})$$

$$\mu_\infty = \infty, \quad (\text{E11})$$

of which (E10) is the stable one. It then follows from Eq. (E8) that

$$w_\infty = \frac{3}{7} + O(\epsilon). \quad (\text{E12})$$

Since both f_∞ and w_∞ are finite nonzero constants, we may repeat the analysis of Sec. III to find

$$z = \frac{1}{2}d, \quad (\text{E13})$$

$$\lambda \propto \kappa^{-\epsilon/2}, \quad (\text{E14})$$

$$\Gamma \propto \kappa^{-\epsilon/2}, \quad (\text{E15})$$

whereas the thermal conductivity λ^E is unrenormalized, and the thermal diffusivity has a scaling exponent¹¹

$$z_E = 2 + \tilde{\alpha}/\nu. \quad (\text{E16})$$

As mentioned in Sec. IV, we anticipate that higher-order corrections to the recursion relations may yield some of the same difficulties as were encountered^{11,52} in model C for $2 < n < 4$, owing to the singular nature of the fixed point reached [see Eq. (E10)]. Nevertheless, dynamic scaling may hold for a suitable definition of the characteristic frequencies.

It is instructive to see what would have happened if the field $\tilde{\epsilon}$ had been introduced as a nonconserved quantity. The operator $(\lambda_0^E \nabla^2)$ in Eq. (E1c) would then be replaced by a constant Γ_0^E , and the noise source ζ^E modified accordingly. The recursion relation for Γ_i^E would then read

$$1/\Gamma_{i+1}^E = b^{-2}(1/\Gamma_i^E)[1 + O(\epsilon)].$$

It follows that $(1/\Gamma_i^E)$ is a “fast transient,” which rapidly tends to zero with increasing l . The field $\tilde{\epsilon}$ then responds instantaneously to fluctuations of ψ and m , and may be eliminated in favor of an instantaneous interaction between the latter fields. The critical behavior is thus simply that of the two-field model E . Alternatively, one could have eliminated the nonconserved field $\tilde{\epsilon}$ from the equations of motion at the beginning, at the price of introducing interactions which are nonlocal in time. Since the retardation is of order C_0/Γ_0^E , which is short compared to the order parameter relaxation times near T_c , one would not expect the $\tilde{\epsilon}$ to affect the critical properties. Indeed, frequency-dependent interactions at intermediate stages of the renormalization group were already encountered in earlier discussions (Sec. III C) and were argued to be irrelevant.

More generally, it appears that any number of nonconserved fields may be added to the models listed in Table I, without any effect on the critical dynamics, provided that the overall symmetry and Poisson bracket relations for the order parameter and the various conserved fields are not altered.

APPENDIX F: PERTURBATION THEORY AND SECOND-ORDER CALCULATION FOR THE ANTIFERROMAGNET

The diagrammatic formalism for model G is very similar to that for model E . In the antiferromagnet the physical fields are the six real variables ψ_α and m_α , $\alpha = 1, 2, 3$. We introduce adjoint fields $\hat{\psi}_\alpha$ and $\hat{m}_\alpha$, and define correlation functions in the same way as before. We may again define 2×2 matrices G and D , which are related to the nonzero correlation functions by

$$G_{11} = C_{\psi_\alpha \psi_\alpha}, \quad (\text{F1a})$$

$$G_{12} = C_{\psi_\alpha \hat{\psi}_\alpha} = [C_{\hat{\psi}_\alpha \psi_\alpha}]^* = G_{21}^*, \quad (\text{F1b})$$

$$D_{11} = C_{m_\alpha m_\alpha}, \quad (\text{F1c})$$

$$D_{12} = C_{m_\alpha \hat{m}_\alpha} = [C_{\hat{m}_\alpha m_\alpha}]^* = D_{21}^*. \quad (\text{F1d})$$

The bare propagators G_{12}^0 , D_{12}^0 , and D_{11}^0 are the same as in (B5), but (B5a) must be replaced by

$$G_{11}^0(k, \omega) = 2\Gamma_0 / (c_0 | -i\omega + \Gamma_0(r_0 + k^2)|^2). \quad (\text{F2})$$

Equation (B6) is now replaced by

$$\begin{aligned} \mathcal{H}_I = - \int d^d x \left(g_0 \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} (A_0^{-1} \chi_0^{-1} m_\alpha \psi_\beta \hat{\psi}_\gamma + c_0 \hat{m}_\alpha \psi_\beta \nabla^2 \psi_\alpha) \right. \\ \left. + 4\Gamma_0 u_0 A_0^{-1} \sum_{\alpha\beta} \hat{\psi}_\alpha \psi_\alpha \psi_\beta^2 \right), \end{aligned} \quad (\text{F3})$$

where $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric unit tensor. [See Eq. (5.7).]

The calculation of diagrams is identical to that for model *E*, except for simple numerical factors arising from the sums over components in (F3) and the factor of $\frac{1}{2}$ difference between (F2) and (B5a). For any given value of the parameters u_0 , g_0 , Γ_0 , etc., the contributions of the various diagrams in Figs. 2 and 3 may be written as

$$\Sigma_{21}^{(i)}(\text{model } G) = s^i \Sigma_{21}^{(i)}(\text{model } E), \quad (\text{F4})$$

$$\Pi_{21}^{(i)}(\text{model } G) = p^i \Pi_{21}^{(i)}(\text{model } E), \quad (\text{F5})$$

where

$$\begin{aligned} s^b = 4, \quad s^c = s^a = s^d = p^b = 2, \\ s^e = \frac{5}{4}, \quad p^a = p^c = 1. \end{aligned} \quad (\text{F6})$$

Thus $\Sigma_{21}^{(a)}$ is now given by Eqs. (D13) and (D14) multiplied by 2, whereas $\Pi_{21}^{(a)}$ is precisely given by (D15). The lowest-order value of f_0 is the same as for model *E*, but the lowest-order value of w_0 is now 3. For the diagrams other than $\Pi_{21}^{(a)}$ and $\Sigma_{21}^{(a)}$ it suffices to use these lowest-order values in the integrals, and we find, for $d=4$ and $r \rightarrow 0$, taking into account (F6):

$$\begin{aligned} \Sigma_{21}^{(b)} + \Sigma_{21}^{(c)} = \Gamma_0 r f_0^2 2^{-7} [4 \ln(\Lambda^2/r) + \ln(\Lambda^2/r) \\ \times (3 + 42 \ln 7 + 6 \ln 3 - 132 \ln 2)], \end{aligned} \quad (\text{F7})$$

$$\Sigma_{21}^{(d)} = \Gamma_0 r f_0^2 2^{-5} \ln(\Lambda^2/r) (40 \ln 2 - 7 \ln 7 - 9 \ln 3 - 2), \quad (\text{F8})$$

$$\Sigma_{21}^{(e)}(0, \omega) = \Sigma_{21}^{(e)}(0, 0) + i \omega u_0^2 K_4^2 (120) \ln \frac{4}{3} \ln(\Lambda^2/r), \quad (\text{F9})$$

$$\begin{aligned} \Pi_{21}^{(b)}(k, 0) = (\lambda_0 k^2 / \chi_0) f_0^2 2^{-5} [\ln^2(\Lambda^2/r) + \ln(\Lambda^2/r) \\ \times (30 \ln 15 - 120 \ln 2 - 1)], \end{aligned} \quad (\text{F10})$$

$$\begin{aligned} \Pi_{21}^{(c)}(k, 0) = (\lambda_0 k^2 / \chi_0) f_0^2 2^{-3} \ln(\Lambda^2/r) \\ (60 \ln 2 - 15 \ln 15 - 1). \end{aligned} \quad (\text{F11})$$

The constant u_0 must now be chosen as $\frac{2}{11} \pi^2 \epsilon$. Let us write

$$w_0 = 3 + w'_0 \epsilon + O(\epsilon^2), \quad (\text{F12})$$

$$f_0 = \epsilon + f'_0 \epsilon^2 + O(\epsilon^3). \quad (\text{F13})$$

We then find

$$\begin{aligned} \Gamma'/\Gamma_0 = (1 - \frac{3}{4} \epsilon - \frac{9}{4} \epsilon \ln \frac{3}{4}) \{ 1 + \frac{1}{4} \epsilon \ln(\Lambda^2/r) + \frac{1}{32} \epsilon^2 \ln^2(\Lambda^2/r) \\ + \frac{1}{32} \epsilon^2 \ln(\Lambda^2/r) [8f'_0 - 2w'_0 - 19 \ln 2 \\ + \frac{27}{2} \ln 3 + \frac{5}{4} - \frac{240}{121} \ln \frac{4}{3}] \}, \end{aligned} \quad (\text{F14})$$

$$\begin{aligned} \lambda/\lambda_0 = (1 - \frac{3}{8} \epsilon) [1 + \frac{1}{4} \epsilon \ln(\Lambda^2/r) + \frac{1}{32} \epsilon^2 \ln^2(\Lambda^2/r) \\ + \epsilon^2 \ln(\Lambda^2/r) (\frac{1}{4} f'_0 - \frac{15}{4} \ln 2 + \frac{15}{16} \ln 15 + \frac{1}{8})]. \end{aligned} \quad (\text{F15})$$

Applying the scaling laws (D23) and (D24), we obtain

$$f'_0 = -0.258, \quad (\text{F16})$$

$$w'_0 = -3.102. \quad (\text{F17})$$

The amplitude ratios defined by Eqs. (3.55) and (3.56) are

$$\begin{aligned} R_\chi = K_d (3\epsilon)^{-1/2} (1 - \frac{3}{8} \epsilon - \frac{1}{6} w'_0 \epsilon - \frac{1}{2} f'_0 \epsilon) \\ = K_d (3\epsilon)^{-1/2} (1 + 0.271 \epsilon), \end{aligned} \quad (\text{F18})$$

$$\begin{aligned} R_\Gamma = K_d^{1/2} (3/\epsilon)^{1/2} (1 - \frac{3}{4} \epsilon - \frac{9}{4} \epsilon \ln \frac{3}{4} + \frac{1}{6} w'_0 \epsilon - \frac{1}{2} f'_0 \epsilon) \\ = K_d (3/\epsilon)^{1/2} (1 - 0.605 \epsilon). \end{aligned} \quad (\text{F19})$$

We have used a relation analogous to (D30)–(D32),

$$\Gamma = \Gamma_0 (1 + \frac{3}{4} \epsilon \ln \frac{3}{4}) \kappa^{-\epsilon/2 + \eta}. \quad (\text{F20})$$

An equation equivalent to (F20) was recently found by Freedman and Mazenko.¹⁵ As a result of a comparison of our Eq. (F20) with their work, we uncovered an error in an earlier version of the present paper, which had led to incorrect values for several of our universal ratios, at the linear order in ϵ .

Note that since Γ_0 is not measurable, Eq. (F20) cannot be compared directly to experiment. It is only once one has determined w'_0 and f'_0 from a second-order calculation that one can make contact with experimental amplitudes via (F19).

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- ¹B. Widom, *J. Chem. Phys.* **43**, 3892 (1965); **43**, 3898 (1965).
 - ²L. P. Kadanoff, *Physics* (N.Y.) **2**, 263 (1966).
 - ³A. Z. Patashinskii and V. L. Pokrovskii, *Zh. Eksp. Teor. Fiz.* **50**, 439 (1966) [*Soviet Phys.-JETP* **23**, 292 (1966)].
 - ⁴R. A. Ferrell, N. Menyhard, H. Schmidt, F. Schwabl, and P. Szépfalusy, *Phys. Rev. Lett.* **18**, 891 (1967); *Ann. Phys. (N.Y.)* **47**, 565 (1968).
 - ⁵B. I. Halperin and P. C. Hohenberg, *Phys. Rev.* **177**, 952 (1969).
 - ⁶G. Ahlers, *Phys. Rev. Lett.* **21**, 1159 (1968).
 - ⁷G. Ahlers, in *Proceedings of the Twelfth International Conference on Low Temperature Physics*, edited by E. Kanda (Keigaku, Tokyo, 1971), p. 21; and private communication.
 - ⁸L. P. Kadanoff and J. Swift, *Ann. Phys. (N.Y.)* **50**, 312 (1968); K. Kawasaki, *ibid.* **61**, 1 (1970).
 - ⁹J. D. Gunton and K. Kawasaki, *J. Phys. A* **8**, L9 (1975), and (unpublished).
 - ¹⁰K. G. Wilson and J. Kogut, *Phys. Rep.* **12**, 75 (1974); K. G. Wilson, *Rev. Mod. Phys.* **47**, 773 (1975).
 - ¹¹B. I. Halperin, P. C. Hohenberg and S. Ma, *Phys. Rev. B* **10**, 139 (1974).
 - ¹²B. I. Halperin, P. C. Hohenberg and E. D. Siggia, *Phys. Rev. Lett.* **32**, 1289 (1974).
 - ¹³G. Ahlers, *Phys. Rev. A* **8**, 530 (1973); K. H. Mueller, F. Pobell, and G. Ahlers, *Phys. Rev. Lett.* **34**, 513 (1975).
 - ¹⁴B. I. Halperin and P. C. Hohenberg, *Phys. Rev.* **188**, 898 (1969).
 - ¹⁵R. Freedman and G. F. Mazenko, *Phys. Rev. Lett.* **34**, 1575 (1975), and (unpublished).
 - ¹⁶S. Ma and G. F. Mazenko, *Phys. Rev. Lett.* **33**, 1384 (1974); and *Phys. Rev. B* **11**, 4077 (1975).
 - ¹⁷B. I. Halperin, *Phys. Rev. B* **11**, 178 (1975); B. I. Halperin and E. D. Siggia (unpublished).
 - ¹⁸M. Suzuki and G. Igarashi, *Prog. Theor. Phys.* **49**, 1070 (1973); M. Suzuki, *ibid.* **50**, 1767 (1973); **51**, 1254 (1974); Y. Kuramoto, *ibid.* **52**, 711 (1974); H. Yahata, *ibid.* **52**, 871 (1974); M. Suzuki and F. Tanaka, *ibid.* **52**, 722 (1974); K. K. Murata, (unpublished).
 - ¹⁹E. D. Siggia, B. I. Halperin, and P. C. Hohenberg *Phys. Rev. B* (to be published).
 - ²⁰P. Résibois and M. D. Leener, *Phys. Rev.* **178**, 806 (1969); P. Résibois and C. Piette, *Phys. Rev. Lett.* **24**, 514 (1970); C. Joukoff-Piette and P. Résibois, *Phys. Lett. A* **42**, 531 (1973); D. L. Huber and D. A. Krueger, *Phys. Rev. Lett.* **24**, 111 (1970).
 - ²¹P. C. Hohenberg, M. De Leener and P. Résibois, *Physica* **65**, 505 (1973).
 - ²²K. G. Wilson and M. E. Fisher, *Phys. Rev. Lett.* **28**, 240 (1972).
 - ²³A. Tucciarone, H. Y. Lau, L. M. Corliss, A. Delapalme, and J. M. Hastings, *Phys. Rev. B* **4**, 3206 (1971).
 - ²⁴A. Tucciarone, J. M. Hastings and L. M. Corliss, *Phys. Rev. B* **8**, 1103 (1973).
 - ²⁵P. W. Anderson, *Rev. Mod. Phys.* **38**, 298 (1966).
 - ²⁶L. D. Landau and E. M. Lifshitz, *Fluid Dynamics* (Addison-Wesley, Reading, Mass., 1959), Chap. 16.
 - ²⁷P. C. Martin, E. D. Siggia, and H. A. Rose, *Phys. Rev. A* **8**, 423 (1973).
 - ²⁸Equation (4.8) of Ref. 11 contains a number of errors. The correct equation should read

$$\gamma_{l+1} = b^{d-2a-aE} [\gamma_l - 4(n+2)u_l \gamma_l B \ln b - 2n\gamma_l^2 C_l B \ln b].$$
 - ²⁹B. D. Josephson, *Phys. Lett.* **21**, 608 (1966).
 - ³⁰E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, *Phys. Lett. A* **47**, 285 (1974).
 - ³¹P. C. Hohenberg, A. Aharony, B. I. Halperin, and E. D. Siggia, *Phys. Rev. B* (to be published). Note that the definition of $\kappa_- = (\xi_-)^{-1}$ in Eqs. (3.58) and (4.4) differs by a factor of 4π in three dimensions, from that of Eq. (5.63) of Ref. 5, and Eq. (5.4) of Ref. 4. See also, M. Ferer, *Phys. Rev. Lett.* **33**, 21 (1974); and A. Aharony, *Phys. Rev. B* **9**, 2107 (1974).
 - ³²T. Matsubara and H. Matsuda, *Prog. Theor. Phys.* **16**, 416 (1956); **16**, 569 (1956); **17**, 19 (1957).
 - ³³M. E. Fisher, *Rep. Prog. Phys.* **30**, 615 (1967).
 - ³⁴P. C. Hohenberg, in *Critical Phenomena*, edited by M. S. Green (Academic, New York, 1971), p. 285.
 - ³⁵P. C. Hohenberg, in *Physics of Quantum Fluids*, edited by R. Kubo and F. Takano (Syokabo, Tokyo, 1971), p. 77.
 - ³⁶P. C. Hohenberg and P. C. Martin, *Ann. Phys. (N.Y.)* **34**, 291 (1965).
 - ³⁷G. Ahlers (private communication).
 - ³⁸D. A. Krueger and D. L. Huber, *Phys. Lett.* **33**, 149 (1970).
 - ³⁹J. A. Tyson, *Phys. Rev. Lett.* **21**, 1235 (1968).
 - ⁴⁰G. Winterling, F. S. Holmes, and T. J. Greytak, *Phys. Rev. Lett.* **30**, 427 (1973).
 - ⁴¹M. Barmatz, P. C. Hohenberg, and A. Kornblit, *Phys. Rev. B* **12**, 1947 (1975).
 - ⁴²F. Wegner, *Phys. Rev. B* **5**, 6529 (1972).
 - ⁴³M. Suzuki and G. Igarashi, *Phys. Lett. A* **47**, 361 (1974); Y. Yamashita and T. Tsuneto, *Prog. Theor. Phys.* **51**, 949 (1974); M. Suzuki, *ibid.* **53**, 97 (1975).
 - ⁴⁴C. De Dominicis, E. Brézin, and J. Zinn-Justin, *Phys. Rev. B* (to be published); E. Abrahams and T. Tsuneto, *Phys. Rev. B* **11**, 4498 (1975); F. Tanaka *Prog. Theor. Phys.* **54**, 289 (1975).
 - ⁴⁵A. Kornblit and G. Ahlers, *Phys. Rev. B* **8**, 5163 (1973).
 - ⁴⁶D. R. Nelson, J. M. Kosterlitz, and M. E. Fischer, *Phys. Rev. Lett.* **33**, 813 (1975).
 - ⁴⁷J. M. Hastings (private communications).
 - ⁴⁸W. J. Ince, MIT Lincoln Lab Technical Report (1965) (unpublished).
 - ⁴⁹K. G. Wilson, comment made at Conference on the Renormalization Group-Quantum Field Theory, Temple University, 1973, edited by J. D. Gunton and M. S. Green (unpublished), p. 30.
 - ⁵⁰K. G. Wilson, *Phys. Rev. Lett.* **28**, 548 (1972).
 - ⁵¹See Eq. (3.22) of M. E. Fisher and A. Aharony, *Phys. Rev. B* **10**, 2818 (1974).
 - ⁵²B. I. Halperin, P. C. Hohenberg, and S. Ma, *Phys. Rev. B* (to be published).