Nonlinear spin dynamics in superfluid ^{3}He†

R. Combescot
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853
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A previous approach to spin dynamics in superfluid ^{3}He is generalized to the nonlinear regime, where the direction of the order parameter has large deviations from its equilibrium position. The results are used to study the hydrodynamic regime, to derive an equation for the nonlinear regime in the ringing experiments, and to consider the nonlinear effects arising in spin-echo experiments for small tipping angles. The results for the NMR linewidths (in the linear regime) are compared with a very recent work of Leggett and Takagi. Finally, the behavior of these linewidths is investigated at low temperature, where the hydrodynamic assumption $\omega\tau \ll 1$ is no longer valid.

I. INTRODUCTION

We have recently[1] set up a formalism enabling us to study spin dynamics in superfluid ^{3}He, in the regime of frequencies ω small compared to the gap $\Delta(T)$ and for wavelengths long compared to the coherence length. This theory is an extension of the work of Betbeder-Matibet and Nozières[2] to the case of triplet pairing and is essentially a generalization to the superfluid state of the Landau kinetic equation for the normal state. This generalization is obtained by treating the superfluid state within the BCS weak-coupling theory. Therefore, strong-coupling effects are ignored and can only be accounted by a suitable renormalization of the gap with respect to the BCS value. This theory has been used to study spin waves in the collisionless[1] and in the hydrodynamic regime.[3] For the NMR[4] it reduces to Leggett's[5,6] theory in the hydrodynamic regime and to the results of Maki and Ebisawa[7,8] in the collisionless regime if we assume $\omega \ll \Delta$ in their formulas; it has been used to study[4] the NMR linewidths in the hydrodynamic regime.

However, this theory has been developed so far only in the linear regime, where the deviations of the order parameter from its equilibrium positions are small. On the other hand, there is a growing interest in the nonlinear regime, which is more complex than the linear one, but can allow a better check of the present belief on the nature of ^{3}He superfluid phases. Very interesting experiments have already been performed, which are exploring this regime: ringing experiments[9] and NMR line-shift measurements in the A phase.[10]

Our main purpose in this paper is to generalize our approach of spin dynamics to the nonlinear regime. It happens that this generalization can be achieved rather easily because, instead of linearizing around the equilibrium position of the order parameter as has been done in the linear regime, we can linearize around the instantaneous position of the order parameter. In the low-frequency regime $\omega \ll \Delta$, the deviations from the local equilibrium defined by the instantaneous order parameter are still small, and the linearization can be performed. In this way, after the motion of the order parameter is taken into account by a convenient canonical transformation, the rest of the theory is basically unchanged. In the hydrodynamic regime, we rederive Leggett's equations and also write an equation ruling the space variations of the order parameter.[11]

In this paper, we include a magnetic field and the dipole interaction. This allows us to rederive, in the linear case, equations which have been used in Refs. 3 and 4, but have not been derived in Ref. 1 (referred in this paper as I). For the NMR we discuss the relaxation-time approximation introduced in Ref. 4 in the light of a very recent work of Leggett and Takagi.[12] Then, we discuss the linewidths at low temperature. Applying the relaxation-time approximation to the ringing problem, we derive a differential equation ruling the difference of phase between spin up and spin down. We study this equation in the regime of high-frequency ringing. Finally, we discuss some problems arising from nonlinearity in spin-echo experiments for small tipping angle. Some of these results have already been briefly reported.[13]

II. KINETIC EQUATION

Since the main features of the theory have been described in detail in I and the generalization follows the same lines, we will be somewhat more sketchy here. We start from the BCS Hamiltonian, including a magnetic field with an arbitrary space and time dependence (provided that the time scale is large compared to $\hbar/\Delta$ and that the field is varying slowly over the coherence length). The magnetic energy is assumed to be small compared to the gap. Here we do not take into account the dipole interaction, which will be discussed later. We have

$$\mathcal{H}=\frac{1}{2m}\int d^3r[\vec{\nabla}\psi^\dagger_\alpha(\vec{r})][\vec{\nabla}\psi_\alpha(\vec{r})]$$

$$-\frac{1}{2}\int d^3r\,\psi^\dagger_\alpha(\vec{r})\vec{\omega}_L\cdot\vec{\sigma}_{\alpha\beta}\psi_\beta(\vec{r})+H_{\rm int}\ , \qquad (1)$$

where m is the bare ^{3}He mass and $\vec{\omega}_L$ (the Larmor frequency) is related to the magnetic field $\vec{H}(\vec{r},t)$ by

$$\vec{\omega}_L=\gamma\vec{H}(\vec{r},t)\ , \qquad (2)$$

γ being the nuclear magnetic moment of ^{3}He. In Eq. (1), $H_{\rm int}$ is the BCS interaction.

As in I, we perform a (time- and space-dependent) spin rotation in order to follow the motion of the order parameter: With respect to the moving frame, the order parameter will be space and time independent. But we will not assume, as in I, that the deviations of the order parameter from equilibrium are small. This spin rotation in the nonlinear regime has also been used by Maki[14] in studying spin waves in the B phase in the hydrodynamic regime. In this spin rotation, the old field variables $\psi(\vec{r},t)$ are related to the new ones $\varphi(\vec{r},t)$ by

$$\psi_\alpha(\vec{r},t)=(e^{i\vec{\sigma}\cdot\vec{\theta}(\vec{r},t)})_{\alpha\beta}\varphi_\beta(\vec{r},t)\ , \qquad (3)$$

where σ_x, σ_y, and σ_z are the Pauli matrices.

After the transformation, the Hamiltonian becomes[15]

$$\mathcal{H}=\frac{1}{2m}\int d^3r[\vec{\nabla}\varphi^\dagger_\alpha(\vec{r})][\vec{\nabla}\varphi_\alpha(\vec{r})]$$

$$+\int d^3r[i\vec{\nabla}\varphi^\dagger_\alpha(\vec{r})\cdot\vec{A}_{\alpha\beta}\varphi_\beta(\vec{r})+\text{H. c.}]$$

$$+\int d^3r\,\varphi^\dagger_\alpha(\vec{r})(V_{\alpha\beta}-\tfrac{1}{2}\Omega_{\alpha\beta})\varphi_\beta(\vec{r})\ , \qquad (4)$$

where

$$\vec{A}_{\alpha\beta}=-(i/m)\,e^{-i\vec{\sigma}\cdot\vec{\theta}}\vec{\nabla}(e^{i\vec{\sigma}\cdot\vec{\theta}})\ , \qquad (5)$$

$$V_{\alpha\beta}=-ie^{-i\vec{\sigma}\cdot\vec{\theta}}\frac{\partial}{\partial t}(e^{i\vec{\sigma}\cdot\vec{\theta}})\ , \qquad (6)$$

$$\Omega_{\alpha\beta}=e^{-i\vec{\sigma}\cdot\vec{\theta}}\,\vec{\omega}_L\cdot\vec{\sigma}\,e^{i\vec{\sigma}\cdot\vec{\theta}}\ . \qquad (7)$$

If we write ($\vec{A}_{\alpha\beta}$ and $V_{\alpha\beta}$ are easily checked to be traceless in spin space)

$$\vec{A}_{\alpha\beta}=\vec{A}_\lambda\cdot\sigma^\lambda_{\alpha\beta}\ ,\quad V_{\alpha\beta}=V_\lambda\sigma^\lambda{}_{\alpha\beta}\ ,\quad \Omega_{\alpha\beta}=\Omega_\lambda\sigma^\lambda{}_{\alpha\beta}$$
$$(\lambda=x,\ y,\ z)\ , \qquad (8)$$

$\vec{A}_\lambda$, which is a vector both in spin and orbital space, must be considered as a spin superfluid velocity, corresponding to the difference in superfluid velocity between, say, spin up and spin down. Note that an equivalent definition of the spin superfluid velocity is given by Leggett.[16] In the same way $-V_\lambda$ can be interpreted as an effective field due to the motion of the superfluid: It gives a difference in chemical potential between, say, spin up and spin down. Equation (6) is, in fact, similar to the Josephson relation between chemical potential and time derivative of the phase for ordinary superconductors. Equations (5) and (6) are the generalizations to the nonlinear regime of Eq. (4) in I. Finally, there is no physical difference between $\vec{\Omega}$ and $\vec{\omega}_L$: $\vec{\omega}_L$ is essentially the magnetic field in the fixed frame and $\vec{\Omega}$ is the same field expressed now in the moving frame. Equation (7) is equivalent to saying that $\vec{\Omega}$ is deduced from $\vec{\omega}_L$ by the spin rotation.

Until now, we have not considered Fermi-liquid effects. They could be introduced in a systematic way, as done in I. But, as it seems to be a good approximation and is much simpler, we will assume that only the s-wave part $F_0^a=N_0f_0^a$ of the antisymmetric Fermi-liquid parameters is nonzero (for spin dynamics we are not concerned with the symmetric one). In that case, all we have to do to take into account Fermi-liquid effects is to replace the bare mass m by the effective mass m^* and say that there is a molecular field $f_0^a\delta\vec{\rho}$ which must be added to the magnetic field $\vec{\omega}_L$; here $\delta\vec{\rho}$ is the "magnetization" of the system (without multiplying by the magnetic moment[17]).

Now we can see that the new Hamiltonian [Eq. (4)] is formally identical to the one obtained in the linear regime [Eq. (3) in I]. But now $\vec{A}_\lambda$ is defined by Eq. (5) and $\vec{V}$ must be replaced by

$$\vec{X}=\vec{V}-\tfrac{1}{2}\vec{\Omega}+f_0^a\vec{\rho}\ , \qquad (9)$$

with $\vec{V}$ and $\vec{\Omega}$ defined by Eqs. (6) and (7).[2] We can, as in I, write a kinetic equation for the local distribution-function matrix. In I, we have linearized this equation because the system was assumed to have only small deviations from equilibrium. Here this is no longer the case. But if we deal with perturbations of long wavelengths and of frequency small compared to the gap, the forces which drive the system out of equilibrium and which are proportional to $\vec{A}_\lambda$, V_λ, and Ω_λ, are still small since they are proportional to the wave vector or the frequency. Therefore, *in the moving frame*, the system will still have small departure from equilibrium. However, this already implies a low-frequency long-wavelength expansion which was not the case in the linear regime.

Let us define the Fourier transform of the 4×4 matrix distribution function $n_{k-q/2,\,k+q/2}$ defined in I:

$$n_k(\vec{r},t)=\sum_q e^{i\vec{q}\cdot\vec{r}}n_{k-q/2,\,k+q/2}\ . \qquad (10)$$

For example, the part describing the space-time dependence of the quasiparticle distribution $n_{e,k}(\vec{r},t)$ is related to the Wigner distribution function,

$$[n_{e,k}(\vec{r},\vec{t})]_{\alpha\beta}=\int d^3R\, e^{i\vec{k}\cdot\vec{R}} \times\langle\varphi^{\dagger}_{\alpha}(\vec{r}+\tfrac{1}{2}\vec{R})\varphi_{\beta}(\vec{r}-\tfrac{1}{2}\vec{R})\rangle =\sum_q e^{i\vec{q}\cdot\vec{r}}\langle a^{\dagger}_{k-q/2,\alpha}a_{k+q/2,\beta}\rangle\ . \quad (11)$$

The departure $\delta n_k(\vec{r}, t)$ of this distribution from equilibrium obeys the following kinetic equation:

$$\frac{\partial}{\partial t}(\delta n_k)=-\frac{\vec{k}}{2m^*}\left\{\left[\vec{\nabla}_r\delta n_k\ ,\ \frac{\partial}{\partial\xi_k}(\epsilon^0_k)\right]_+ -\left[\vec{\nabla}_r\delta\epsilon_k\ ,\ \frac{\partial}{\partial\xi_k}(n^0_k)\right]_+\right\}+\frac{1}{i}\{[\delta n_k\ ,\ \epsilon^0_k] +[n^0_k\ ,\ \delta\epsilon_k]+[\delta n_k\ ,\ \delta\epsilon_k]\}\ , \quad (12)$$

where

$$\epsilon^0_k=\begin{pmatrix}\xi_k\delta_{\alpha\beta} & \Delta^{+}_{k\,\alpha\beta}\\ \Delta_{k\,\alpha\beta} & -\xi_k\delta_{\alpha\beta}\end{pmatrix} \quad (13)$$

and

$$n^0_k=\tfrac{1}{2}+\epsilon^0_k\varphi_k/E_k\ ,\quad \varphi_k=-\tfrac{1}{2}\tanh\tfrac{1}{2}\beta E_k\ . \quad (14)$$

Finally

$$\delta\epsilon_k=\begin{pmatrix}M_{k,\beta\alpha} & 0\\ 0 & -M_{-k,\alpha\beta}\end{pmatrix}\ , \quad (15)$$

where we have

$$M_{k,\alpha\beta}(\vec{r},t)=\vec{k}\vec{A}_{\alpha\beta}+V_{\alpha\beta}-\tfrac{1}{2}\Omega_{\alpha\beta}+F^a_0\delta\rho_{\alpha\beta}\ . \quad (16)$$

Equation (11) is essentially identical to Eq. (14) of I, except for the term $[\delta n_k,\ \delta\epsilon_k]$, which can no longer be neglected.

Performing, in the same way as in I, a Bogoliubov-Valatin transformation, we obtain the following kinetic equation[18]:

$$\frac{\partial}{\partial t}\delta\vec{\nu}_k(r,t)+\left(\vec{\nabla}_k E_k\frac{\partial}{\partial\vec{r}}\right)\delta\vec{\mu}_k(r,t) +2\delta\vec{\mu}_k(r,t)\times\delta\vec{E}_k(r,t)=\vec{I}(\delta\vec{\nu}_k)\ , \quad (17)$$

where we have added a collision term $\vec{I}(\delta\vec{\nu}_k)$ on the right-hand side. Here $(\delta\nu_k)_{\alpha\beta}$ is the 2×2 matrix distribution function for *Bogoliubov* quasiparticles and we have written $(\delta\nu_k)_{\alpha\beta}=\delta\vec{\nu}_k\cdot\vec{\sigma}^{T}$; $\delta\vec{\mu}_k$ is the departure of $\delta\vec{\nu}_k$ from local equilibrium

$$\delta\vec{\mu}_k=\delta\vec{\nu}_k-\varphi'_k\delta\vec{E}_k\ , \quad (18)$$

where $\varphi'_k=\partial\varphi_k/\partial E_k$ is, in fact, the energy derivative of the Fermi distribution; $\delta\vec{E}_k(\vec{r}, t)$ is the change in the local energy of the quasiparticles due to the perturbation, the magnetic field, and the Fermi-liquid effects. We see, from the third term of Eq. (17), that $\delta\vec{E}_k(r, t)$ acts as an effective field on the quasiparticles: In the normal state, for example, $\delta\vec{E}_k$ would be essentially the magnetic field plus the molecular field. Here, $\delta\vec{E}_k$ is given by

$$\delta\vec{E}_k=k_i\vec{A}_i+\frac{\xi_k}{E_k}\vec{X}-\left(1-\frac{\xi_k}{E_k}\right)\frac{\vec{d}_k}{|\Delta_k|^2}[\vec{d}^{*}_k(k_i\vec{A}_i-\vec{X})]\ , \quad (19)$$

which is identical to the result for the linear regime. In Eq. (19), the arrow describes a vector in spin space and, in $\vec{A}_i$, the index i ($i=x, y, z$) refers to orbital space; in Eq. (8), we have used the reverse notation. As usual, $\vec{d}_k$ is defined from the order-parameter matrix Δ_k by $\Delta_k=i(\vec{\sigma}\cdot\vec{d}_k)\sigma_y$. Finally, let us note that the kinetic equation could be derived in a more direct way by a semiclassical expansion analogous to the one used by Silin[19] in the normal state.

As in I, we will have to require the spin conservation law to be satisfied in order to determine $\vec{\theta}$: The kinetic equation (17) describes the evolution of the quasiparticles (the "normal fluid") with respect to the superfluid, but we need an equation to tell how the superfluid itself is evolving. This equation is precisely provided by the conservation law. (Actually, we know that the spin conservation law is actually broken by the dipole interaction, as we will see later.) In order to use this "conservation law," we need the kinematic expression of the magnetization $\vec{\rho}$ and the spin current $\vec{j}_i$ in terms of the quasiparticle distribution and the superfluid.

These expressions are obtained from the definition of the magnetization and the spin current,

$$\vec{\rho}(\vec{r})=\psi^{\dagger}_{\alpha}(\vec{r})\vec{\sigma}_{\alpha\beta}\psi_{\beta}(\vec{r})\ ,$$

$$\vec{j}_i(\vec{r})=\frac{1}{2im^*}\left[\psi^{\dagger}_{\alpha}(\vec{r})\vec{\sigma}_{\alpha\beta}\left(\frac{\partial}{\partial x_i}\psi_{\beta}(\vec{r})\right) -\left(\frac{\partial}{\partial x_i}\psi^{\dagger}_{\alpha}(\vec{r})\right)\vec{\sigma}_{\alpha\beta}\psi_{\beta}(\vec{r})\right]\ , \quad (20)$$

and performing first the spin rotation and then the Bogoliubov-Valatin transformation. The spin rotation has, on one hand, a trivial effect: Since $\vec{\rho}$ and $\vec{j}_i$ are vectors and we change the frame, they will be changed (by vector $\delta\vec{\rho}$, we mean a set of three components and not the mathematical object, which is not changed by the rotation). This is exactly the same as going from $\vec{\omega}_L$ to $\vec{\Omega}$. We get rid of that by expressing $\vec{\rho}$ and $\vec{j}_i$ in the moving reference frame, that is, taking their components along the moving axis. In that way, the magnetization is formally not affected by the rotation,

$$\vec{\rho}(\vec{r})=\varphi^{\dagger}_{\alpha}(\vec{r})\vec{\sigma}_{\alpha\beta}\varphi_{\beta}(\vec{r})\ , \quad (21)$$

but, on the other hand, we obtain an additive gauge term in the spin current,

$$\vec{j}_i(\vec{r}) = \frac{1}{2im^*}\left[\varphi^{\dagger}_{\alpha}(r)\vec{\sigma}_{\alpha\beta}\left(\frac{\partial}{\partial x_i}\varphi_{\beta}(\vec{r})\right) - \left(\frac{\partial}{\partial x_i}\varphi^{\dagger}_{\alpha}(\vec{r})\right)\vec{\sigma}_{\alpha\beta}\varphi_{\beta}(\vec{r})\right] + \rho\vec{A}_i(r) \ . \quad (22)$$

This is formally identical to what has been obtained in I, and after the Bogoliubov transformation we obtain as in I

$$\vec{\rho}(\vec{r}) = \sum_k \frac{\xi_k}{E_k}\,\delta\vec{\nu}_k(\vec{r}) + \left(1 - \frac{\xi_k}{E_k}\right)\frac{\vec{d}_k[\vec{d}^*_k\,\delta\vec{\nu}_k(\vec{r})]}{|\Delta_k|^2} + \sum_k \frac{\varphi_k}{E_k^3}\,\vec{d}_k\times[\vec{X}(r)\times\vec{d}^*_k] \ , \quad (23)$$

$$\vec{j}_i(\vec{r}) = \frac{1}{m^*}\sum_k k_i\left[\delta\vec{\nu}_k(\vec{r}) - \left(1 - \frac{\xi_k}{E_k}\right)\frac{\vec{d}_k[\vec{d}^*\delta\vec{\nu}_k(\vec{r})]}{|\Delta_k|^2}\right] + \sum_k \frac{\varphi_k}{E_k^3}\,\frac{k_i k_j}{m^*}\,\vec{d}_k[\vec{A}_j(\vec{r})\,\vec{d}^*_k] + \rho A_i(\vec{r}) \ . \quad (24)$$

In each of these expressions, the first term can be considered as the contribution of the normal fluid and the second term the contribution of the superfluid. Now we have to consider the conservation law itself and we need to include the dipole interaction.

III. CONSERVATION LAW AND DIPOLE INTERACTION

In principle, in order to include the dipole interaction, we should add it to the original Hamiltonian and carry on the process of deriving the kinetic equation. Fortunately, it is not necessary to do so because of the weakness of the dipole interaction. We know[5,6] that only the off-diagonal part in the dipole energy is important. The characteristic frequency which is introduced in the kinetic equation is of the order of the average dipole interaction between two ^{3}He atoms, $E_d \sim \gamma^2 k_F^3 (\Delta/E_F)^2$; this corresponds to frequencies of order 10^{-1} to 1 Hz, far lower than the frequencies actually considered. Therefore, the dipole interaction is negligible in the kinetic equation.

On the other hand, we know that it is not negligible in the conservation law. As we will see, this conservation law is typically of the form $\partial\vec{\rho}/\partial t \sim D$, where D is a dipole term which has variations of order of E_d when the order parameter $\vec{d}_k$ is rotating. On the other hand, when $\vec{d}_k$ is rotating with characteristic frequency ω, it produces an effective field of order ω from Eq. (6). The corresponding magnetization $\vec{\rho}$ is obtained, in order of magnitude, from the susceptibility of the system. But, because of the Pauli principle, this susceptibility has a strongly reducing factor $1/E_F$. Finally, $\vec{\rho} \sim \omega/E_F$ and $\partial\vec{\rho}/\partial t \sim D$ gives $\omega \sim (E_D E_F)^{1/2}$ (all the energies are evaluated in frequency units); this characteristic frequency $(E_D E_F)^{1/2}$ is precisely of the order of the longitudinal NMR frequency and naturally not negligible. Let us note that it is somewhat paradoxical that the dipole interaction can be neglected in the kinetic equation and not in the conservation law, since they both derive from a single primitive kinetic equation[1] before the low-frequency expansion. This is due to strong cancellations which occur in the derivation of the conservation law and leads to the factor $1/E_F$ mentioned above. In the hydrodynamic regime, neglecting the dipole interaction in the kinetic equation is equivalent to neglecting it in the second Leggett equation [Eq. (4.13) of Ref. 6]. But we have to keep it in the conservation law, in the same way as Leggett does in his first equation [Eq. (4.12) of Ref. 6].

To derive the conservation law, we write

$$\frac{\partial\vec{\rho}}{\partial t} = \frac{1}{i}[\delta\vec{\rho},\ H] \ , \quad (25)$$

where the Hamiltonian H includes the dipole interaction, and calculate the commutator by using the Hartree-Fock approximation. Except for the dipole interaction, the interaction conserves the spin and does not give rise to any contribution. The kinetic-energy term gives a $\partial\vec{j}_i/\partial x_i$ term as usual. After the spin rotation, the dipole interaction leads to two terms. One is the same as if the system would be in global equilibrium, the order parameter having its local value. The other is due to the disequilibrium of the quasiparticle distribution with respect to this equilibrium. But we have seen that this departure from equilibrium is small: The second term is of order ω/T_c compared to the first one, and we can neglect it. Actually, this first term is exactly the one derived by Leggett in his first equation. Finally, our conservation law is essentially the first Leggett equation, except that we have to take into account the diffusion of the magnetization, which gives the $\partial\vec{j}_i/\partial x_i$ term. We obtain

$$\frac{\partial\vec{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\vec{j}_i = \vec{\rho}\times\vec{\omega}_L(\vec{r}) - \frac{\pi\gamma^2}{4}\sum_{k,k'}\frac{\varphi_k\varphi_{k'}}{E_k E_{k'}} \times[(\vec{Q}\times\vec{d}_k)(\vec{Q}\cdot\vec{d}_{k'}) + (k \rightleftharpoons k')] \ , \quad (26)$$

where $\vec{Q} = (\vec{k} - \vec{k}')/|\vec{k} - \vec{k}'|$ and the summations over k and k' also mean summations over spin up and spin down.

Equation (26) is written in the fixed reference frame. It may be more convenient to write it in the moving frame. In that case, $\vec{\omega}_L(r)$ must be replaced by $\vec{\Omega}_L(r)$. But we must also take care in using Eqs. (23) and (24) so that they give the components of $\vec{\rho}$ and $\vec{j}_i$ along moving axes. Therefore, if we want to calculate $\partial\vec{\rho}/\partial t$ and $\partial\vec{j}_i/\partial x_i$, we can take the derivative of Eqs. (23) and (24), but we must add terms which are well known in classical mechanics[20] and account for the motion of the axes. In our case, these terms are $2\vec{\rho}\times\vec{V}$ for $\partial\vec{\rho}/\partial t$ and

$2m^*\vec{j}_i\times\vec{A}_i$ for $\partial\vec{j}_i/\partial x_i$. Finally, we can rewrite the conservation law as

$$\frac{\partial\vec{\rho}}{\partial t}+\frac{\partial\vec{j}_i}{\partial x_i}+2\vec{\rho}\times\vec{X}+2m^*\vec{j}_i\times\vec{A}_i + \frac{\pi\gamma^2}{4}\sum_{k,k'}\frac{\varphi_k\varphi_{k'}}{E_kE_{k'}}\left[(\vec{Q}\times\vec{d}_k)(\vec{Q}\cdot\vec{d}_{k'})+(k\rightleftharpoons k')\right]=0\ , \tag{27}$$

where $\vec{\rho}$ and $\vec{j}_i$ are now given by Eqs. (23) and (24). Together with the kinetic equation (17), it gives a closed set of equations allowing us to treat nonlinear spin dynamics for all frequencies small compared to the gap. As long as the kinetic equation (17) can be solved for $\delta\vec{\nu}_k$, this set can be reduced to a single partial differential equation for $\vec{\theta}$.

Let us add an important remark in the case of the axial (ABM) state which is believed to represent the A phase. In that case, the direction of $\vec{d}_k$ is independent of $\vec{k}$. We choose this direction to be along the y axis in the moving frame. We also note that the somewhat unpleasant formulas (23) and (24) mean simply (we drop the k index for clarity)

$$\rho_y=\sum\delta\nu_y\ ,\quad \rho_{x,z}=\sum\frac{\xi}{E}\,\delta\nu_{x,z}+X_{x,z}\sum\frac{\varphi}{E^3}\,|\Delta|^2\ , \tag{28}$$

$$(j_i)_y=\sum\frac{k_i}{m^*}\,\frac{\xi}{E}\left[\delta\nu_y-\varphi'\,\frac{\xi}{E}\,k_j(A_j)_y\right]\ , \tag{29}$$

$$(j_i)_{x,z}=\sum\frac{k_i}{m^*}\,\delta\nu_{x,z}+\rho(A_i)_{x,z}\ .$$

In the same way, Eq. (19) reduces to

$$\delta E_y=(\xi/E)k_j(A_j)_y+X_y\ ,\quad \delta E_{x,z}=k_j(A_j)_{x,z}+(\xi/E)X_{x,z}\ . \tag{30}$$

The y component of the kinetic equation (17) can be rewritten as

$$\frac{\partial}{\partial t}\,\delta\nu_y+\frac{k_i}{m^*}\,\frac{\xi}{E}\,\frac{\partial}{\partial x_i}\,(\delta\nu_y-\varphi'\,\delta E_y) +2(\delta\nu_z\,\delta E_x-\delta\nu_x\,\delta E_z)=I_y(\delta\vec{\nu})\ . \tag{31}$$

Because the collisions conserve the total magnetization, we expect $\sum_k I_y(\delta\vec{\nu}_k)=0$. Summing Eq. (31) over $\vec{k}$ and using Eqs. (28)–(30), we obtain

$$\frac{\partial}{\partial t}\,\rho_y+\frac{\partial}{\partial x_i}\,(j_i)_y+2(\rho_zX_x-\rho_xX_z) +2m^*[(j_i)_z(A_i)_x-(j_i)_x(A_i)_z]=0\ . \tag{32}$$

But if we take the y component of Eq. (27), we obtain exactly the same equation because the dipole term does not contribute for this component. Therefore, one of the three components of the conservation law Eq. (27) is automatically satisfied. This result is associated with the fact that spin rotations along $\vec{d}_k$ do not physically change the order parameter. Accordingly, spin rotations with $\vec{\theta}$ parallel to $\vec{d}$ have no physical meaning and can be chosen arbitrarily, which explains why the equation which would determine them is disappearing.

We note also that we can repeat the preceding argument in the normal state and prove that the conservation law is identically satisfied by the kinetic equation. This is again expected since the spin rotation $\vec{\theta}$ has no longer any physical meaning in the normal state.

Finally, let us show how the equations used in Refs. 3 and 4 for the linear regime result from the present equations. In these papers, a static magnetic field was considered. Since we are in the linear regime, the contributions to $\delta\vec{\nu}_k$, $\delta\vec{E}_k$, $\vec{\rho}$ due to the fluctuations (let us call them $\delta\vec{\nu}'_k$, $\delta\vec{E}'_k$, $\vec{\rho}'$) are small compared to those ($\delta\vec{\nu}^0_k$, $\delta\vec{E}^0_k$, $\vec{\rho}^0$) due to the magnetic field,

$$\delta\vec{\nu}_k=\delta\vec{\nu}^0_k+\delta\vec{\nu}'_k\ ,\quad \delta\vec{E}_k=\delta\vec{E}^0_k+\delta\vec{E}'_k\ ,\quad \vec{\rho}=\vec{\rho}^0+\vec{\rho}'\ . \tag{33}$$

More precisely, $\delta\vec{\nu}^0_k$, $\delta\vec{E}^0_k$, and $\vec{\rho}^0$ correspond to the magnetic field contribution to zero order in the perturbation, the first order being included in $\delta\vec{\nu}'_k$, $\delta\vec{E}'_k$, $\vec{\rho}'$. For example,

$$\vec{X}=\vec{V}-\tfrac{1}{2}\vec{\Omega}+f_0^a\vec{\rho} \simeq\frac{\partial\vec{\theta}}{\partial t}-\tfrac{1}{2}(\vec{\Omega}-\vec{\omega}_L)+f_0^a\vec{\rho}'-\tfrac{1}{2}\vec{\omega}_L+f_0^a\vec{\rho}^0\ , \tag{34}$$

which gives [$\vec{\Omega}$ is obtained from Eq. (7) by expanding for small $\vec{\theta}$; see Eq. (54)]

$$\vec{X}=\vec{X}^0+\vec{X}'\ ,\quad \vec{X}^0=-\tfrac{1}{2}\vec{\omega}_L+f_0^a\vec{\rho}^0\ , \qquad \vec{X}'\simeq\frac{\partial\vec{\theta}}{\partial t}+\vec{\omega}_L\times\vec{\theta}+f_0^a\vec{\rho}'\ , \tag{35}$$

and

$$\delta\vec{E}^0_k=\frac{\xi}{E}\,\vec{X}^0+\left(1-\frac{\xi}{E}\right)\frac{\vec{d}_k}{|\Delta|^2}\,(\vec{d}_k^*\vec{X}^0)\ ,\quad \delta\vec{\nu}^0_k=\varphi'\delta\vec{E}^0_k\ . \tag{36}$$

Finally, to first order in the perturbation, $\delta\vec{E}_k$ can be replaced by $\delta\vec{E}^0_k$ in the third term of Eq. (17) and also we note that

$$\delta\vec{\mu}_k=\delta\vec{\nu}_k-\varphi'\delta\vec{E}_k=\delta\vec{\nu}'_k-\varphi'\,\delta\vec{E}'_k=\delta\vec{\mu}'_k\ . \tag{37}$$

In this way, we have rederived the equations used in Refs. 3 and 4. All the quantities which are primed here ($\delta\vec{\mu}'_k$, $\delta\vec{E}'_k$, $\vec{X}'$) correspond to the same quantities unprimed in these papers ($\delta\vec{\nu}_k$, $\delta\vec{E}_k$, $\vec{X}$); $\vec{\rho}'$ was called $\delta\vec{\rho}$ in the moving frame and $\delta\vec{\rho}$ in the fixed frame. The conservation law was written in the fixed frame.

IV. HYDRODYNAMIC REGIME

In the hydrodynamic regime, by definition, all the frequencies involved are small, compared to the inverse lifetime of the quasiparticles due to collisions: $\omega\tau\ll 1$. Therefore, the quasiparticle distribution has always enough time to relax to-

ward local equilibrium. This local equilibrium is the Fermi distribution corresponding to the local energy of the quasiparticles. Since this energy is changed by an amount $\delta\vec{E}_k$ with respect to global equilibrium, the local quasiparticle distribution is different by an amount[21]

$$\delta\vec{\nu}_k = f'\delta\vec{E}_k = \varphi'\delta\vec{E}_k \tag{38}$$

from the global equilibrium (f' is the derivative of the Fermi distribution). In other words, $\delta\vec{\mu}_k = 0$ is the solution of the kinetic equation (17) in the hydrodynamic regime. Substituting Eq. (38) into Eqs. (23) and (24), and taking Eq. (19) into account, we obtain exactly as in the linear regime

$$\vec{\rho} = \overleftrightarrow{\kappa}(-\vec{X}) , \tag{39}$$

$$\vec{j}_i = (\overleftrightarrow{\rho}_s)_{ij}\vec{A}_j . \tag{40}$$

Here $\overleftrightarrow{\kappa}$ is the static susceptibility tensor without Fermi-liquid corrections,

$$\kappa_{\alpha\beta} = N_0 \int \frac{d\Omega_k}{4\pi} d\xi \left[\delta_{\alpha\beta} - (1+\varphi')(\hat{d}_k)_\alpha(\hat{d}_k)_\beta\right] , \tag{41}$$

and is related to the susceptibility $\overleftrightarrow{\chi}$ with Fermi-liquid corrections by $\overleftrightarrow{\chi}^{-1} = \overleftrightarrow{\kappa}^{-1} + f_0^a$. Equation (39) merely states that, in the hydrodynamic regime $\omega\tau \ll 1$, the magnetization $\vec{\rho}$ corresponding to the effective field $(-\vec{X})$ is given by the static susceptibility. Equation (39) can also be rewritten as

$$\vec{\rho} = \overleftrightarrow{\chi}\left(\tfrac{1}{2}\vec{\Omega} - \vec{V}\right) . \tag{42}$$

In the same way, the spin superfluid density $(\overleftrightarrow{\rho}_s)_{ij}$ gives the spin current $\vec{j}_i$ corresponding to the spin superfluid velocity $\vec{A}_i$, in the quasistatic regime $\omega\tau \ll 1$. This relation is quite similar to the relation between particle current and ordinary superfluid velocity, but here we have no normal density because there is no normal spin velocity.[3] Explicitly[3] we have

$$(\rho_s)_{\alpha\beta ij} = \frac{N_0}{m^*}\int \frac{d\Omega}{4\pi} d\xi\, k_i k_j \frac{|\Delta|^2}{E^2} \times\left(\varphi' - \frac{\varphi}{E}\right)\left[\delta_{\alpha\beta} - (\hat{d}_k)_\alpha(\hat{d}_k)_\beta\right] . \tag{43}$$

Equations (40) and (42), together with the conservation law Eq. (27), describe completely the spin dynamics in the hydrodynamic region. Equation (42) is, actually, identical to the second Leggett equation. To see this, we note that for a vector $\vec{a}$, we have the following relation between its components $\vec{a}_m$ in the moving frame and its components $\vec{a}_f$ in the fixed frame:

$$\vec{a}_f\vec{\sigma} = e^{+i\vec{\sigma}\cdot\vec{\delta}}\vec{a}_m\vec{\sigma}e^{-i\vec{\sigma}\cdot\vec{\delta}} , \tag{44}$$

as in Eq. (7). Applying it to $\vec{d}_k$, and taking the time derivative of Eq. (44), we obtain

$$\frac{\partial}{\partial t}\vec{d}_k = 2\vec{d}_k\times\vec{V} \tag{45}$$

(where $\vec{V}$ is now expressed in the fixed frame). Equation (45), together with Eq. (42) which gives

$$\vec{V} = \tfrac{1}{2}\vec{\Omega} - (\overleftrightarrow{\chi}^{-1})\vec{\rho} \tag{46}$$

is equivalent to the second Leggett equation. (The factors 2 are only a matter of definition: The true magnetization of the system is $2\vec{\rho}$.)

Let us note that, except for the factor 2 and the sign, $\vec{V}$ is simply what is called in classical mechanics the instantaneous velocity of the moving frame; any vector $\vec{a}$ fixed in the moving frame satisfies

$$\frac{\partial}{\partial t}\vec{a} = 2\vec{a}\times\vec{V} . \tag{47}$$

Equation (47) can be considered as an alternative definition of $\vec{V}$, instead of Eq. (6).

In the same way, the spatial derivatives of $\vec{d}_k$ obey

$$\frac{\partial}{\partial x_i}\vec{d}_k = 2\vec{d}_k\times\vec{A}_i , \tag{48}$$

which again can be considered as an alternative definition for the superfluid velocity $\vec{A}_i$ (it is in this way that it defined by Leggett[11,16]). Using Eq. (40), we can write Eq. (48) formally as

$$\frac{\partial}{\partial x_i}\vec{d}_k = 2\vec{d}_k\times[(\overleftrightarrow{\rho}_s)^{-1}]_{ij}\vec{j}_j , \tag{49}$$

which plays the same role for the space derivative of the order parameter as the second Leggett equation for the time derivative.

In the uniform regime (no space dependence), we can make the following remark. If we neglect the dipole interaction and have no magnetic field, the problem is exactly identical to the top problem of classical mechanics. The magnetization corresponds to the angular momentum and is conserved as it is for the angular momentum. As noted above, $\vec{V}$ (in fact $-2\vec{V}$) is the instantaneous velocity of the top. Finally, the magnetization is related to the top velocity by Eq. (42), which shows that $\overleftrightarrow{\chi}$ corresponds to the inertia tensor. The solutions of this problem are well known.[20] Actually, they are very simple in the case of the axial [Anderson-Brinkman-Morel (ABM)] state and the isotropic [Balian-Werthamer (BW)] state. In the present case, Eq. (27) is simply

$$\frac{\partial\vec{\rho}}{\partial t} + 2\vec{\rho}\times\vec{X} = 0 . \tag{50}$$

In the isotropic state, the susceptibility $\overleftrightarrow{\kappa}$ is isotropic and $\vec{\rho}$ and $\vec{X}$ are parallel. From Eq. (50), $\vec{\rho}$ is constant in the moving frame. But since it is also constant in the fixed frame, this means merely that the moving frame (that is $\vec{d}_k$) is rotating around $\vec{\rho}$ with an angular velocity $\vec{\rho}/\chi$, according

to Eq. (42). In the axial state, we will see that $\vec{\rho}$ is always perpendicular to $\vec{d}$, so that $\vec{\rho}$ and $\vec{X}$ are also parallel, according to Eq. (39) (the susceptibility $\vec{\kappa}$ is isotropic for directions perpendicular to $\vec{d}$). Again, $\vec{d}$ is rotating around $\vec{\rho}$. Now, if we have a static magnetic field, we can eliminate it by performing a Larmor transformation: In a frame rotating with angular velocity $(-\vec{\omega}_L)$, we have again a free-top problem, so that the total motion is the superposition of the rotation around $\vec{\rho}$ and of the precession of $\vec{\rho}$ around the magnetic field.

Coming back to the space-dependent case, we can show that $\vec{\rho}$ is perpendicular to $\vec{d}$ in the A phase, which generalizes a known result for the space-independent case. This results directly from the component of Eq. (27) parallel to $\vec{d}$. In this direction, the dipole term is zero and from Eqs. (40) and (43) the spin current is also zero. Finally, from Eqs. (41) and (43), $\vec{\kappa}$ and $(\vec{\rho}_s)_{ij}$ are isotropic in the plane perpendicular to $\vec{d}$, and $(\vec{\rho}_s)_{ij}=(\vec{\rho}_s)_{ji}$, so that $\vec{\rho}\times\vec{X}$ and $\vec{j}_i\times\vec{A}_i$ have zero contribution in this direction, according to Eqs. (39) and (40). Thus, we have

$$\frac{\partial}{\partial t}(\vec{\rho}\cdot\vec{d})=0 \ . \tag{51}$$

Equation (51) tells that, in the hydrodynamic regime, $\vec{\rho}\cdot\vec{d}$ is fixed at some value, but it does not tell which one. To find it, we must consider the corrections due to the relaxation processes. From Eq. (51), we see that there is no driving force on $\vec{\rho}\cdot\vec{d}$, which means that $\vec{\rho}\cdot\vec{d}$ will relax to some equilibrium value. For this value, there will be no relaxation at all. Accordingly, this value corresponds necessarily to the local equilibrium value $\vec{\rho}\cdot\vec{d}=0$, so that after some time corresponding to the relaxation, $\vec{\rho}$ will be perpendicular to $\vec{d}$.

We will now consider spin waves in the axial state in the absence of a magnetic field. Let us first neglect the dipole interaction. By direct inspection of the equations, we can see that, as in the linear regime, rotations of the moving frame around a direction fixed in space are still solutions. To satisfy $\vec{\rho}\cdot\vec{d}=0$, this direction has to be perpendicular to $\vec{d}$, so that $\vec{d}$ is rotating around this direction and perpendicular to it. From Eq. (27), the angle of rotation θ satisfies

$$\chi\frac{\partial^2\theta}{\partial t^2}-(\rho_s)_{ij}\frac{\partial^2\theta}{\partial x_i\,\partial x_j}=0 \ , \tag{52}$$

where χ and $(\rho_s)_{ij}$ are the susceptibility and the spin superfluid density perpendicular to $\vec{d}$. Therefore, the corresponding modes propagate exactly as in the linear case, with the same (anisotropic) velocity.[3] If we now consider the dipole interaction, we see that rotations around a fixed direction are still solutions. The dipole term merely modifies Eq. (52), which becomes

$$\frac{\partial^2\theta}{\partial t^2}-\frac{(\rho_s)_{ij}}{m^*\chi}\frac{\partial^2\theta}{\partial x_i\,\partial x_j}=-\omega_0^2\sin\theta \ . \tag{53}$$

This equation has been studied by Maki.[22] Let us emphasize, however, that since the equations are, in general, nonlinear, the linear combination of two solutions is, in general, not a solution, which makes the study of particular solutions far less useful than in the linear regime.

Finally, we derive for completeness the equation ruling $\vec{\theta}$ and the spin-wave relation dispersion in the linear regime. In this regime, we have from Eqs. (5)–(7),

$$\vec{A}_i=\frac{1}{m^*}\frac{\partial}{\partial x_i}\vec{\theta} \ , \quad \vec{V}=\frac{\partial}{\partial t}\vec{\theta} \ , \quad \vec{\Omega}=\vec{\omega}_L-2\vec{\omega}_L\times\vec{\theta} \ . \tag{54}$$

In Eq. (27), $\vec{j}_i\times\vec{A}_i$ is of second order in $\vec{\theta}$ and can be neglected. On the other hand, as we have seen, $\vec{\rho}$ and $\vec{X}$ are always parallel in the actual motion so that $\vec{\rho}\times\vec{X}$ disappears. The dipole term in Eq. (27) reduces to $-\overleftrightarrow{\varphi}^{(0)}\vec{\theta}$ in the linear regime, where $\overleftrightarrow{\varphi}^{(0)}$ has been defined by Leggett.[6] With the help of Eqs. (40) and (42), Eq. (27) becomes

$$\frac{\partial^2\vec{\theta}}{\partial t}+\vec{\omega}_L\times\frac{\partial\vec{\theta}}{\partial t}-\frac{(\vec{\rho}_s)_{ij}}{m^*\chi}\frac{\partial^2\vec{\theta}}{\partial x_i\,\partial x_j}+\overleftrightarrow{\Omega}^2\vec{\theta}=0 \ , \tag{55}$$

where χ is the susceptibility perpendicular to $\vec{d}$ for the A phase. The tensor $\overleftrightarrow{\Omega}^2=\overleftrightarrow{\varphi}^{(0)}/\chi$ gives the longitudinal resonance frequencies: In the axial state $\overleftrightarrow{\Omega}$ has a zero eigenvalue parallel to $\vec{d}$, and eigenvalue equal to the longitudinal NMR frequency ω_0 in the plane perpendicular to $\vec{d}$; in the isotropic state where $\vec{d}_k=R(\hat{k})$, the eigenvalue for a direction parallel to the axis of the rotation R is equal to the longitudinal NMR frequency ω_0, and the eigenvalues perpendicular to this axis are zero.

Now we look for the relation dispersion of modes with frequency ω and wave vector $\vec{q}$. In the axial state, we take $\vec{d}$ along the y axis and $\vec{\omega}_L$ along the z axis. The y component of Eq. (55) gives

$$\omega_L\theta_x=i\omega\theta_y \ . \tag{56}$$

Using this result in the x and z components, we find that the equations for these components are decoupled, so that we have one mode corresponding to an oscillation of the magnetization parallel to the field, $\vec{d}$ oscillating perpendicular in the field in the x-y plane. The relation dispersion of this mode is

$$\omega^2=\omega_0^2+(1/m^*\chi)(\rho_s)_{ij}q_iq_j \ , \tag{57}$$

where $(\rho_s)_{ij}$ is the spin superfluid density for directions perpendicular to $\vec{d}$. The other mode corresponds to an oscillation of $\vec{d}$ in the y-z plane, while the magnetization is precessing elliptically around the field, exactly as in the transverse NMR, $\vec{d}$ and $\vec{\rho}$ always being perpendicular. The cor-

responding dispersion relation is

$$\omega^2 = \omega_L^2 + \omega_0^2 + (1/m^*\chi)(\rho_s)_{ij} q_i q_j \ . \tag{58}$$

In the isotropic state, we have to solve

$$\omega^2 \vec{\theta} + i\omega \vec{\omega}_L \times \vec{\theta} - \vec{\Omega}^2 \vec{\theta} - (1/m^*\chi)\, q_i\, q_j (\vec{\rho}_s)_{ij}\, \vec{\theta} = 0 \ . \tag{59}$$

Equation (59) is not simple in general since the eigenvectors of $q_i q_j (\vec{\rho}_s)_{ij}$ are parallel and perpendicular to $R(\vec{q})$, those of $\vec{\Omega}$ are parallel and perpendicular to the axis of R, and those of the operator $\vec{\omega}_L \times$ are parallel and perpendicular to ω_L. Nevertheless, solving Eq. (59) amounts to solving a the third-degree equation, which can always be done. Naturally, in some particular cases, one has simple solutions.[3]

V. RELAXATION EFFECTS

We will now consider some effects coming from the fact that the quasiparticle distribution does not relax in an infinitely short time toward the hydrodynamic equilibrium defined by Eq. (38). The quasiparticle distribution is relaxing toward the equilibrium defined by the superfluid, but at the same time the superfluid is moving so that the system is never in complete equilibrium. Therefore, we will have dissipation processes caused by this disequilibrium which give rise to NMR linewidth, spin diffusion, etc.

To be able to treat these problems, we have to handle the term $\vec{I}(\delta\vec{\nu}_k)$ in the kinetic equation which arises essentially from collisions between quasiparticles. This collision term is generally very complicated and solutions of the kinetic equation can only be found in some limiting cases. However, when it can be used, this approach is naturally extremely fruitful. This kind of treatment has been used recently to study the viscosity and the thermal conductivity of superfluid ^{3}He near T_c and at low temperature.[23] An alternative method is to make a relaxation-time approximation on the collision term. This leads naturally to much simpler calculations, and if the approximation satisfies the conservation laws and more generally the physical features of the collision term, we can expect at least a semiquantitative agreement with experiment. This is the solution that we have taken so far. Since the collision term is zero at local equilibrium when Eq. (38) is satisfied, we use the following approximation:

$$\vec{I}(\delta\vec{\nu}_k) = -(\delta\vec{\nu}_k - \varphi' \delta\vec{E}_k)/\tau(T) \ . \tag{60}$$

Naturally, the main shortcoming of this approximation is that we do not know the temperature dependence of the relaxation time $\tau(T)$ and we need the exact collision term to have a hint about it. Alternatively, we can use one experiment to determine it or compare different experiments to eliminate $\tau(T)$. We must take care also that $\tau(T)$ can depend on the process that we consider. It should depend also, in general, on the spin polarization that we consider and $1/\tau(T)$ could be replaced by a tensor. Finally, let us stress that, since the conservation law Eq. (27) is explicitly required, the relaxation-time approximation Eq. (60) satisfied it automatically. However, we will see that this does not completely eliminate problems associated with spin conservation.

We will now consider the problem of NMR linewidths and first rederive rapidly the results of Ref. 4. The actual experiments are done in the linear regime and satisfy the hydrodynamic condition $\omega\tau \ll 1$. Also, there is naturally no space dependence. The kinetic equation (17) becomes very simple since $\delta\vec{\mu}_k \sim O(\omega\tau)$: The third term in the left-hand side of Eq. (17) can be neglected and we are left with

$$\frac{\partial}{\partial t}\, \delta\vec{\nu}_k = -(\delta\vec{\nu}_k - \varphi'\, \delta\vec{E}_k)/\tau(T) \ . \tag{61}$$

If we consider a solution of frequency ω, Eq. (61) has the solution

$$\delta\vec{\nu}_k = [1 + i\omega\tau(T)]\, \varphi'\, \delta\vec{E}_k \ . \tag{62}$$

Let us first consider the axial state. From Eqs. (28) and (30), the projections of $\vec{\rho}$ and $\vec{X}$ perpendicular to $\vec{d}$ will be parallel, so that we have again $\rho_y = 0$, from the component of Eq. (27) parallel to the y axis. From Eq. (28) we have nothing to determine X_y, since it is not entering explicitly the expression for ρ_y. This naturally is linked to the indetermination noted at the end of Sec. II. So we can set for convenience $X_y = 0$ and, as for Eq. (55), we obtain from Eq. (27),

$$\frac{\partial \rho_{x,z}}{\partial t} - \chi\omega_0^2\, \theta_{x,z} = 0 \ . \tag{63}$$

Let us take the magnetic field along the z axis. From $X_y = 0$ and the definition of $\vec{X}$, Eq. (56) still holds. From Eqs. (28) and (62),

$$\begin{aligned} &\rho_{x,z} = -N_0 [1 + i\omega\tau f(T)]\, X_{x,z} \ , \\ &f(T) = \int \frac{d\Omega}{4\pi}\, d\xi (-\varphi') \left(\frac{\xi}{E}\right)^2 \ , \end{aligned} \tag{64}$$

where N_0 is the density of states: $N_0 = m^* k_F/\pi^2$. From Eqs. (9), (54), and (56),

$$\begin{aligned} &X_x = -i\omega\theta_x + i(\omega_L^2/\omega)\theta_x + f_0^a \rho_x \ , \\ &X_z = -i\omega\theta_z - \tfrac{1}{2}\omega_L \theta_z + f_0^a \rho_z \ . \end{aligned} \tag{65}$$

The z component corresponds to the longitudinal resonance and the x component to the transverse resonance. From Eqs. (63)–(65) we have the frequencies of these resonances. The real parts naturally give the well-known results[5] that the longitudinal resonance frequency is ω_0 and the transverse one $(\omega_0^2 + \omega_L^2)^{1/2}$. We obtain the full

linewidth by taking twice the imaginary part of the frequency. For the longitudinal and the transverse linewidths, we have, respectively (with $F_0^a = N_0 f_0^a$),

$$\Delta\omega_L = \omega_0^2(T)\,\tau(T)\,\frac{f(T)}{1+F_0^a} \quad , \tag{66}$$

$$\Delta\omega_T = \frac{\omega_0^4(T)}{\omega_0^2(T)+\omega_L^2}\,\tau(T)\,\frac{f(T)}{1+F_0^a} \quad . \tag{67}$$

In the isotropic state, the transverse resonance is not affected by the dipole interaction and we are only concerned with the longitudinal resonance where we can ignore the magnetic field. The derivation follows exactly the same lines as for the axial state. The only modification is that, from Eq. (23), we now have

$$\rho_z = -\tfrac{1}{3}N_0\{[2+Y(T)]+i\omega\tau[2f(T)+Y(T)]\}X_z \, , \tag{68}$$

where $Y(T)$ is the Yoshida function; we recall that the susceptibility of the isotropic state is

$$\chi(T) = N_0\,\frac{\kappa(T)}{1+F_0^a\kappa(T)} \quad , \quad \kappa(T) = \tfrac{1}{3}[2+Y(T)] \, , \tag{69}$$

and we obtain for the longitudinal linewidth in the isotropic state

$$\Delta\Omega_L = \Omega_L^2(T)\,\tau(T)\,\frac{1-\frac{2}{3}[1-f(T)]/\kappa(T)}{1+F_0^a\kappa(T)} \quad , \tag{70}$$

where $\Omega_L(T)$ is the longitudinal resonance frequency. For a discussion of these results, we refer to Ref. 4.

Leggett and Takagi[12] have very recently worked out a phenomenological approach to relaxation effects. We now want to compare our results with theirs and give some comments induced by their work. First, we believe that the physics which is behind the two theories is the same: The relaxation mechanism considered is in both cases the relaxation of the quasiparticles toward the local equilibrium defined by the superfluid through collisions between quasiparticles. Then we remark that both theories actually agree for the NMR linewidths provided that we make the following identification between the relaxation time τ_{CE} introduced in Ref. 4 [that is the one appearing in Eqs. (66), (67), and (70)] and the relaxation time τ_{LT} of Leggett and Takagi. In the ABM or axial state,

$$\tau_{CE}^{ABM}(T) = \frac{\tau_{LT}^{ABM}(T)}{1-f(T)} \quad , \tag{71}$$

and in the BW or isotropic state,

$$\tau_{CE}^{BW}(T) = \tfrac{3}{2}\kappa(T)\,\frac{\tau_{LT}^{BW}(T)}{1-f(T)} \quad . \tag{72}$$

Since the relaxation times are not introduced in the same way in the two theories, it is not astonishing that they are not the same. Equations (71) and (72) are not mere coincidence: We can derive them by looking in our language to the process for which τ_{LT} is defined. Following Leggett and Takagi, let us assume that there is no dipole interaction: The total magnetization is conserved. The quasiparticles relax toward the local equilibrium, but the superfluid is also moving in such a way that the total magnetization is conserved. The relaxation time τ_{LT} is the relaxation time corresponding, for example, to the relaxation of the superfluid toward its equilibrium. To derive it, let us consider the simple case of the longitudinal resonance. The kinetic equation (61) is

$$\frac{\partial}{\partial t}(\delta\nu_z) = -\,\frac{\delta\nu_z - \varphi'\,\delta E_z}{\tau_{CE}} \quad . \tag{73}$$

Let us also forget Fermi-liquid effects for simplicity. In that case, Eqs. (28) and (30) give

$$\delta E_z = \frac{\xi}{E}\,V_z \, , \quad \rho_z = \sum\frac{\xi}{E}\,\delta\nu_z + V_z\sum\frac{\varphi}{E^3}\,|\Delta|^2 \, , \tag{74}$$

where, by integrating by parts over ξ,

$$\sum\frac{\varphi}{E^3}\,|\Delta|^2 = N_0\int\frac{d\Omega}{4\pi}\,d\xi\,\frac{\varphi}{E^3}\,|\Delta|^2 = -\,N_0[1-f(T)] \, . \tag{75}$$

Taking the derivative of Eq. (74) and using Eq. (73), we have

$$\frac{\partial}{\partial t}\{\rho_z + N_0[1-f(T)]\,V_z\}$$

$$= -\,\frac{1}{\tau_{CE}}\{\rho_z + N_0[1-f(T)]\,V_z + N_0 f(T) V_z\} \, , \tag{76}$$

or, since ρ_z is constant,

$$N_0\,\frac{\partial V_z}{\partial t} = -\,\frac{\rho_z + N_0 V_z}{\tau_{CE}[1-f(T)]} \quad . \tag{77}$$

We see that the relaxation time for the difference V_z between the chemical potential of spin up and spin down is $\tau_{CE}[1-f(T)]$, which proves Eq. (71). In exactly the same way, we can derive Eq. (72). More generally, in the nonlinear regime, from Eq. (8) of Leggett and Takagi, we can identify their $\vec{\eta}$ by

$$\vec{\eta} = -\,[1-f(T)]\,[N_0\kappa(T)\,\vec{X} + \vec{\rho}\,] \, , \tag{78}$$

where $\kappa(T)$ must be understood as equal to 1 in the axial state.

If the kinetic equation has the simple form Eq. (61), we obtain in the axial state from Eq. (28),

$$\frac{\partial}{\partial t}\{\vec{\rho} + N_0[1-f(T)]\,\vec{X}\} = -\,\frac{1}{\tau_{CE}}\,(\vec{\rho} + N_0\,\vec{X}) \tag{79}$$

for directions perpendicular to $\vec{d}$, and in the isotropic state from Eq. (23),

$$\frac{\partial}{\partial t}\{\vec{\rho}+\tfrac{2}{3}N_0[1-f(T)]\vec{X}\}=-\frac{1}{\tau_{CE}}[\vec{\rho}+N_0\kappa(T)\vec{X}]\ . \tag{80}$$

On the other hand, if $\vec{\rho}$ and $\vec{X}$ are parallel, Eq. (27) gives

$$\frac{\partial}{\partial t}\vec{\rho}=\vec{R}_D\ , \tag{81}$$

where $-\vec{R}_D$ is the dipole term in Eq. (27). Combining Eq. (79) or (80) with Eq. (81), we obtain Eq. (7) of Leggett and Takagi, which can be rewritten in the moving frame as

$$\frac{\partial\vec{\eta}}{\partial t}=(1-\lambda)\vec{R}_D-\frac{\vec{\eta}}{\tau_{LT}}\ , \tag{82}$$

assuming again $\vec{\rho}$ and $\vec{X}$ parallel. We see that if the kinetic equation has the form Eq. (61), one can write directly a differential equation for X [Eqs. (79) or (80)] which allows us to avoid the kinetic equation. But if the kinetic equation does not reduce to Eq. (61), it is not likely to be so. The kinetic equation reduces to Eq. (61) in two cases: In the regime $\omega\tau\ll 1$, as we have seen, and also in the longitudinal resonance, which is a one-dimensional problem which again makes the third term on the left hand side of Eq. (17) disappear. In these two cases and if $\vec{\rho}$ and $\vec{X}$ are parallel (which is automatically satisfied in the longitudinal resonance), both theories agree. But otherwise, they will probably disagree.

Let us come back to Eqs. (71) and (72). Since $f(T)$ is going rapidly to zero when the temperature decreases, and $\kappa(T)$ is going to $\frac{2}{3}$, τ_{CE} and τ_{LT} are identical far from T_c. On the other hand, for $T\to T_c$, the ratio τ_{CE}/τ_{LT} diverges since $f(T_c)=1$. The interesting question is to know what is the behavior of the relaxation time near T_c. We will see, by looking at the processes which cause the relaxation, that τ_{CE} must diverge at T_c. This question has also been discussed by Ambegaokar.[24] As we have seen, the relaxation is due to collisions between quasiparticles. When two normal quasiparticles collide, their total spin is conserved. On the other hand, we are interested for the kinetic equation in the collisions between Bogoliubov quasiparticles, not normal quasiparticles. But, from Eq. (28), for example, we can see that the spin carried by a Bogoliubov quasiparticle is ξ/E for directions perpendicular to $\vec{d}$: It depends on the energy of the quasiparticle. Since this energy is redistributed during the collision, the total spin of the Bogoliubov quasiparticles is not conserved during this process (since the total spin of the system is conserved during the collision, the excess spin is transferred to the superfluid, as noted by Leggett and Takagi[12]). For this reason, we can write a simple relaxation-time approximation Eq. (60) for the Bogoliubov quasiparticles, which does not have to conserve the spin explicitly. The fact that the total spin is conserved during the collisions appears only in the conservation law. However, when we approach T_c, Bogoliubov quasiparticles are more and more like normal quasiparticles and Eq. (60) should have a more and more spin-conserving structure. Now, in the normal state the limit of the relaxation process considered in the NMR is the relaxation of a difference in chemical potential between spin up and spin down. But this is an equilibrium situation since, because of the spin conservation, this difference in chemical potential cannot change. Therefore, the collision term is *identically* zero in that case, which implies that the corresponding relaxation time is infinite. This shows that τ_{CE} must go to infinity for $T\to T_c$. Let us emphasize, however, that in the normal state, whatever the value of τ_{CE}, the limit of our formalism automatically conserves the spin, since we require Eq. (26) to be satisfied and it reduces to the spin conservation law in the normal state. But if we want to know what the value of τ_{CE} is, we must look in detail at the collision term.

The fact that τ_{CE} diverges at T_c does not tell in principle what is the behavior of τ_{LT} at T_c. However, comparison with experiment makes it rather likely that τ_{LT} has a finite limit at T_c, which corresponds to τ_{CE} diverging like $(1-T/T_c)^{-1/2}$. If this is so, one can understand the rather good agreement with experiment obtained in Ref. 4 by tentatively taking τ_{CE} of the order of a typical relaxation time in the normal state. In Eq. (67) for the transverse linewidth, $\tau(T)$ is multiplied by $\omega_0^4(T)$, which behaves like $(1-T/T_c)^2$: The divergence of $\tau(T)$ appears near T_c, in a region where $\Delta\omega_T$ is small anyway because of the overwhelming factor $\omega_0^4(T)$. On the other hand, for the longitudinal linewidth $\Delta\omega_L$ will behave like $(1-T/T_c)^{1/2}$ instead of $(1-T/T_c)$ if τ_{LT} is regular near T_c instead of τ_{CE}. This will be easier to verify experimentally, although the general behavior will not be changed drastically. But we emphasize that the regularity of τ_{LT} at T_c is still a matter of likelihood and only a study of the collision term can settle the problem on a theoretical point of view.[28]

We now turn toward the question of the NMR linewidth at low temperature. If $\tau(T)$ would have an upper bound at low temperature, Eqs. (66), (67), and (70) would settle the problem: $\omega_0(T)$ is expected to be some constant at low temperature and the linewidths would go to zero, since $f(T)$, for example, is going to zero for zero temperature. However, $\tau(T)$ is actually expected to grow to infinity when the temperature is going to zero, so that Eqs. (66), (67), and (70), which are derived in the approximation $\omega\tau\ll 1$, will no longer be

valid. Let us consider for simplicity the longitudinal linewidth: The kinetic equation is given by Eq. (61), and with Eqs. (28), (30), and (27) [which reduce to Eq. (81)], we can easily derive the dynamic susceptibility $\chi_{zz}(\omega)$ parallel to the field. We obtain in the axial state

$$\chi_{zz}(\omega)=\frac{N_0\omega_0^2}{1+F_0^a}\,\frac{1-i\omega\tau[1-f(T)]}{(\omega_0^2-\omega^2)-i\omega\tau\{\omega_0^2[f(T)/(1+F_0^a)]+[1-f(T)](\omega_0^2-\omega^2)\}}\ , \tag{83}$$

which is valid at any frequency. The imaginary part $\chi''_{zz}(\omega)$ is proportional to the measured absorption in the NMR experiment,

$$\chi''_{zz}(\omega)=\frac{N_0\omega_0^4}{(1+F_0^a)^2}\,\frac{\omega\tau f(T)}{(\omega_0^2-\omega^2)^2+\omega^2\tau^2\{\omega_0^2[f(T)/(1+F_0^a)]+[1-f(T)](\omega_0^2-\omega^2)\}^2}\ . \tag{84}$$

The susceptibility $\chi''_{zz}(\omega)$ may be easily checked to satisfy the sum rules.[25] At low temperature, we have $f(T)\ll 1$ (which is obtained far before $\omega_0\tau\sim 1$), and we have

$$\chi''_{zz}(\omega)=\frac{N_0\omega_0}{2(1+F_0^a)}\,\frac{\Gamma}{(\omega-\omega_0-\Delta\omega_0)^2+\Gamma^2}\ , \tag{85}$$

where $\Delta\omega_0$ is a line shift (with respect to the usual resonance frequency ω_0), which should be always very small,

$$\Delta\omega_0=\frac{\omega_0}{2}\,\frac{f(T)}{1+F_0^a}\,\frac{(\omega_0\tau)^2}{1+(\omega_0\tau)^2}\ , \tag{86}$$

and Γ is half the full linewidth,

$$\Delta\omega_L=2\Gamma=\omega_0\,\frac{f(T)}{1+F_0^a}\,\frac{\omega_0\tau}{1+(\omega_0\tau)^2}\ . \tag{87}$$

We see that at low temperature, even if $\omega_0\tau$ is going to infinity, the linewidth should go to zero, because $f(T)$ is going to zero and $\omega_0\tau/(1+\omega_0^2\tau^2)$ is always less than $\frac{1}{2}$. This can be seen also directly from Eq. (83), where $\chi_{zz}(\omega)$ becomes real when $f(T)$ goes to zero, whatever $\omega\tau$ is. At high temperature, when $\omega_0\tau\ll 1$, Eq. (87) reduces naturally to Eq. (66). This behavior of the linewidth at low temperature is very similar to the behavior of sound attenuation in normal ^{3}He. First-sound attenuation is proportional to τ and increases when the temperature decreases. But when $\omega\tau\sim 1$, first sound becomes zero sound with attenuation proportional to $1/\tau$, which decreases when temperature decreases. Here the situation is very similar except that superfluidity brings, in addition, the factor $f(T)$, which goes to zero at zero temperature. We can also say that, at zero temperature, we have a pure superfluid regime: There are no excited quasiparticles, no attenuation, and no linewidth. From Eq. (87), we see that the low-temperature regime appears for $\omega_0\tau\sim 0.5$. On the other hand, a longitudinal resonance frequency of 100 MHz, which corresponds to $\omega_0\sim 6\times 10^5$, has already been obtained. The relaxation time should be at least $\tau_D(T_c/T)^2$, where τ_D is the spin-diffusion relaxation time at T_c. For $T/T_c=0.2$, which has almost been obtained experimentally already, this gives $\omega_0\tau\sim 0.5$. So it should be possible to see experimentally the decrease of the longitudinal linewidth without any problem. For the transverse resonance, we can do a similar calculation. In the same way, the effect of the growth of $\tau(T)$ on the linewidth will saturate for $\omega_R\tau\sim 1$ and the linewidth will start to decrease when the temperature goes below this point. Here the frequency ω_R, coming in $\omega_R\tau$, is the resonance frequency: By going to higher fields, one increases ω_R, which puts the onset of the low-temperature regime at higher temperature. So this onset should be easier to observe in the transverse resonance.

More precisely, we can derive for the transverse resonance the following result which holds at any frequency (small compared to the gap):

$$\omega^2=\omega_L^2+\omega_0^2\,\frac{1-\Lambda[1+F_0^a+(\omega_L^2/\omega^2)]F_0^a/(1+F_0^a)^2}{1-\Lambda\{1-[\omega_L/\omega(1+F_0^a)]^2\}}\ , \tag{88}$$

where Λ is given by

$$\Lambda=\int\frac{d\Omega}{4\pi}\,d\xi(-\varphi')\,\frac{(\xi/E)^2}{1-[\omega_L/\omega(1+F_0^a)]^2(\xi/E)^2+i/\omega\tau}\ . \tag{89}$$

In the hydrodynamic regime $\omega\tau\ll 1$, Λ reduces to $-i\omega\tau f(T)$, so that Eq. (88) agrees with Eq. (67) in that regime. In the collisionless regime, on the other hand, $\omega\tau\gg 1$ and Eq. (88) reduces to Maki and Ebisawa's result.[7,8] In all practical cases, we have $\Lambda\ll 1$ because either $\omega\tau\ll 1$ (temperature not too low and moderate field), or $(-\varphi')\ll 1$. Only in strong fields, near T_c, would $\Lambda\ll 1$ not apply. For $\Lambda\ll 1$, Eq. (88) becomes

$$\omega^2=\omega_L^2+\omega_0^2+\frac{\omega_0^4}{\omega^2}\,\frac{\Lambda}{1+F_0^a}\ , \tag{90}$$

and the corresponding linewidth is

$$\Delta\omega_T=\frac{\omega_0^4}{\omega_R^3}\,\frac{\mathrm{Im}\Lambda(\omega_R)}{1+F_0^a}\ , \tag{91}$$

where $\omega_R^2=\omega_L^2+\omega_0^2$. Equation (91) reduces to Eq. (87) for $\omega_L=0$. As we have said, we see that the

effect of the growth of $\tau(T)$ on Λ saturates for $\omega_R\tau \sim 1$. We note also that, for $\omega_R\tau \gg 1$, Λ behaves like $1/\omega_R\tau$ if $\omega_L \ll \omega_R$, but is independent of τ if $\omega_L \sim \omega_R$.

To derive Eq. (88), we have used the fact that the relaxation time, for the NMR, for spin polarization parallel to $\vec{d}$ should be infinite. This happens for the same reason as in the normal state: From Eq. (28), the superfluid does not carry any magnetization for this polarization, so that the collisions should conserve the spin of the quasiparticles. This implies, as in the normal state, that the corresponding NMR relaxation time should be infinite. This result was of no importance for the study of the NMR linewidth in the regime $\omega\tau \ll 1$, since only the relaxation time for directions perpendicular to $\vec{d}$ was coming in. Note also that this results is only approximate: If we use a relaxation-time approximation, we must take it to be infinite in order to be consistent, but in an improved calculation, some relaxation will come in the direction parallel to $\vec{d}$, modifying the quasiparticle distribution (but conserving the total spin parallel to $\vec{d}$).

Let us come to the nonlinear regime and derive the equation ruling the angle of rotation β of the $\vec{d}$ vector in the longitudinal resonance. We consider only the axial state, but in the isotropic state the argument would follow exactly the same line. Actually, we have already almost derived this equation. We have to use Eq. (76), but V_z must be replaced by $X_z = V_z + f_0^a \rho_z$ since we want to take Fermi-liquid effects into account; also, ρ_z is no longer constant if we have nonzero dipole forces, but from Eq. (27) it satisfies

$$\frac{\partial \rho_z}{\partial t} = \frac{N_0\omega_0^2}{4(1+F_0^a)} \sin(4\theta) \ . \tag{92}$$

The right-hand side is the well-known dipole term in the axial state [from Eq. (45) our angle θ is related to β by $\beta = -2\theta$]. Taking the derivative of Eq. (76) we obtain

$$\tau_1 \dddot{\alpha} + \ddot{\alpha} + \omega_0^2 \tau_2 \dot{\alpha} \cos\alpha + \omega_0^2 \sin\alpha = 0 \ , \tag{93}$$

where $\alpha = 2\beta$ and

$$\begin{aligned} \tau_1 &= \tau_{CE}[1 - f(T)] \ , \\ \tau_2 &= \tau_{CE}\left(1 - \frac{f(T)\,F_0^a}{1+F_0^a}\right) \ . \end{aligned} \tag{94}$$

This equation has also been derived independently by Ambegaokar[24] and Leggett and Takagi.[12] Its implication for the ringing experiment[9] will be discussed elsewhere. Here, because of its implication for spin-echo experiments, we want only to consider the regime where the initial velocity $\dot{\alpha} = \lambda_0$ is high compared to the resonance frequency ω_0. In this regime (which corresponds to a ringing experiment with a large change in magnetic field[9]), the solution of Eq. (93) is of the form

$$\alpha(t) = t\lambda(t) + \mu(t)\sin[t\lambda(t) + \varphi(t)] \ , \tag{95}$$

where $\lambda(t)$, $\mu(t)$, and $\varphi(t)$ are functions varying on a time scale $T \sim (\lambda_0/\omega_0)^2(1/\Delta\omega_L)$, that is, very slow in comparison to $\alpha(t)$. Moreover, $\lambda(t)$ is large compared to ω_0 and $\mu(t)$ is small compared to 1. We can verify that the solution, Eq. (95), is correct by considering first a period of time of order $1/\lambda_0$, where $\lambda(t)$, $\mu(t)$, and $\varphi(t)$ are essentially constant. Putting Eq. (95) into Eq. (93), we obtain

$$\begin{aligned} \mu(t) &= \left(\frac{\omega_0}{\lambda(t)}\right)^2 \left(\frac{1+[\tau_2\lambda(t)]^2}{1+[\tau_1\lambda(t)]^2}\right)^{1/2} \ , \\ \sin\varphi(t) &= \frac{\lambda(t)(\tau_2 - \tau_1)}{\{1+[\tau_1\lambda(t)]^2\}^{1/2}\{1+[\tau_2\lambda(t)]^2\}^{1/2}} \ , \end{aligned} \tag{96}$$

where we see that $\mu(t)$ is indeed small compared to 1 if $\omega_0 \ll \lambda(t)$. Then multiplying Eq. (93) by $\dot{\alpha}$ and averaging over a period $2\pi/\lambda$ we get for $\lambda(t)$,

$$\begin{aligned} \frac{d\lambda^2}{dt} &= -\mu\lambda^2(\omega_0^2\tau_2\cos\varphi - \mu\lambda^2\tau_1) \\ &= -\omega_0^4 \frac{\tau_2 - \tau_1}{1+[\tau_1\lambda(t)]^2} \ , \end{aligned} \tag{97}$$

which can easily be integrated. This equation shows that $\lambda(t)$ is evolving on a time scale

$$\begin{aligned} T &= \left(\frac{\lambda_0}{\omega_0}\right)^2 \frac{1+(\lambda_0\tau_1)^2}{\Delta\omega_L} \ , \\ \Delta\omega_L &= \omega_0^2(\tau_2 - \tau_1) \ . \end{aligned} \tag{98}$$

For $t \ll T$, we have

$$\lambda(t) = \lambda_0(1 - t/2T) \ , \tag{99}$$

which shows that $\lambda(t)$ is decreasing linearly at the beginning. But the solution, Eq. (93), is not restricted to $t \ll T$, but to $\lambda(t) \gg \omega_0$, and in general, one must integrate Eq. (97) and we will have deviations from the linearity. We note that the time T can easily be made fairly large: With $\omega_0 \sim 6\times10^5$, switching off a field of 150 G produces $\lambda_0 \sim 10\omega_0$, which gives T of the order of $\frac{1}{10}$ sec. Now we have from Eq. (92),

$$\begin{aligned} \rho_z = \rho_0 &+ \frac{N_0\omega_0}{1+F_0^a}\frac{\omega_0}{\lambda(t)} \times \sin[t\lambda(t)] \\ &- \frac{N_0}{1+F_0^a}[\lambda_0 - \lambda(t)] \ . \end{aligned} \tag{100}$$

We see that ρ_z has a small-amplitude oscillation of frequency $\lambda(t)$ around its average value and that this average value is varying very slowly, on a time scale T. A measurement of T itself would be of some interest since it would provide an experimental value for τ_1, and in this way all the parameters coming in Eq. (93) would be known

experimentally through the measurement of ω_0, $\Delta\omega_L$, and T. To do that, one should have $\lambda_0 \tau_1 \gtrsim 1$, which is available by a 180° pulse in a 1 kG field near T_c. The conclusion from Eq. (100) is that the effect of the dipole interaction on ρ_z is fairly small and can be ignored in first approximation if $\lambda_0 \gg \omega_0$. This is somewhat obvious since the condition $\omega_0 \ll \lambda_0$ allows us to consider the dipole interaction as a perturbation and to expect its effect to be small. What is less obvious is that the relaxation time T could be very long compared to $1/\Delta\omega_L$. One can understand this result by saying that the rotation of $\vec{d}$ is so fast that the system feels only the average of the dipole force, which is zero. This averaging is actually not complete, and the corrections bring a finite but large relaxation time T. At small frequencies, $\lambda_0 \lesssim \omega_0$, there is no averaging and the relaxation time would be of the order of $1/\Delta\omega_L$.

Finally, we discuss some problems arising in spin-echo experiments. It has recently been suggested that spin-echo experiments for small tipping angles would be of interest.[26] However, in the standard spin-echo technique, one would apply a small-angle pulse, then a series of 180° pulses. One of these 180° pulses would bring the magnetization almost opposite to the magnetic field; the next one would bring it back almost in the direction of the magnetic field. In Ref. 26, it was assumed that the system had only a small departure from equilibrium. But naturally a magnetization almost opposite to the magnetic field does not satisfy this condition. Here we want to investigate what is the diffusion coefficient (measured in the spin-echo experiment) in this goemetry (magnetization almost opposite to the magnetic field).

As has been seen in Ref. 26, the dipole interaction can perturb to a large extent the measurement of the diffusion coefficient, unless rather high magnetic fields (say, greater than 500 G) are used. Under these conditions, the dipole interaction can be neglected. As we have just seen, the relaxation time T for the magnetization is very large and the component of the magnetization parallel to the field can be considered as constant for the free-precession period t_0 between two pulses, which is typically of order of 1 msec. We also note that t_0 is very long compared to the relaxation time τ so that the system can be considered to be in hydrodynamic equilibrium during the free-precession periods. Therefore, in our situation in the axial state, for example, the vector $\vec{d}$ will essentially rotate in the plane perpendicular to the magnetic field, with an angular velocity $2\omega_L$.

Now let us consider the effect of a small magnetic field gradient parallel to the direction of the field. It brings spatial inhomogeneity which creates spin current, resulting in a diffusion of the magnetization. The main difference with the linear regime (magnetization almost in the direction of the field) is that we have now an important magnetization current with spin polarization parallel to the field (naturally, all the spin currents correspond to transport of magnetization parallel to the field). Indeed, we have $V_z = \partial\theta_z/\partial t = \omega_L$, which gives $A_{zz} = (t/m^*)(\partial\omega_L/\partial z)$. Through Eq. (29) we obtain the spin current, and the conservation law Eq. (27) expressed in the fixed frame gives

$$\frac{\partial \rho_z}{\partial t} = -\frac{t}{m^*}(\rho_s)_{zz}\frac{\partial^2 \omega_L}{\partial z^2} \quad , \tag{101}$$

where $(\rho_s)_{zz}$ is the zz component of the spin superfluid density. If the gradient is homogeneous, we have $\partial^2\omega_L/\partial z^2 = 0$ and the magnetization along θ_z will not decay because of this spin current. On the other hand, if the gradient is not homogeneous, $\partial^2\omega_L/\partial z^2 \sim \omega_L/L^2$, where L is the characteristic length of the inhomogeneity. From Eq. (101) we see that magnetization is decaying over a characteristic time $T_1 \sim L/C_s$, where C_s is the spin-wave velocity.[3] Clearly the process for this decay is magnetization transport through spin-wave propagation at the velocity C_s, and not ordinary spin diffusion. Since C_s grows like $(1 - T/T_c)^{1/2}$ near T_c and tends to saturate at lower temperature, the behavior of T_1 as a function of temperature agrees qualitatively with the one observed recently by Corruccini and Osheroff.[27] (Actually, a wide range of tipping angles was used in their experiments.) Also in their experiment, L is of the order of 1 cm (the magnetization would be carried out of the NMR coil region). Since C_s should saturate around 10 m/sec, this gives a saturation value of 10^{-3} sec for T_1, which is also in qualitative agreement with their results.

If we have $\partial^2\omega_L/\partial z^2 = 0$ or, more generally, an experimental situation where the decay of ρ_z due to the spin current is negligible, we can calculate the relaxation of the transverse magnetization for a tipping angle around 180°. Here we will not go into the details of the calculation, which will be reported elsewhere, but simply give the result in the A phase: The decay rate of the transverse magnetization is the same for a tipping angle around 180° as for a small tipping angle. This means that there should be no problem in using the standard spin-echo techniques with small tipping angles for measuring the effective diffusion coefficient: The result of Ref. 26 is unaffected by the 180° pulse. On the other hand, the way in which the phase of the magnetization is evolving is affected by the 180° pulse. In other words, the coefficient D_1 of Ref. 26 is affected by the 180° pulse, but not D_2, which is the one which is related to the decay rate and the only one usually measured.

Finally, we note that the relaxation time used in this spin-diffusion calculation should have no singularity near T_c and should go to the usual spin-diffusion relaxation time when the system is going to be normal.

VI. CONCLUSION

In this paper we have generalized to the nonlinear regime a method for treating spin dynamics, which had been only used so far in the linear regime. This generalization is rather easy and rests on the fact that, even in the nonlinear regime, one can write a semiclassical kinetic equation for the quasiparticle distribution in the low-frequency long-wavelength regime. Therefore, this formalism provides a general method for spin dynamics in superfluid ^{3}He. In the hydrodynamic regime, one rederives from this formalism the Leggett's equation and one obtains a third equation ruling the space dependence of the order parameter. We have also used this formalism to derive a third-order equation ruling the phase in the ringing experiments and have studied this equation for high frequencies, with the conclusion that the relaxation of the magnetization is very slow in this regime. We have also sketched some results which may be obtained for spin-echo experiments with small tipping angles in the A phase: The decay rate of the transverse magnetization is not affected by the use of 180° pulses.

Coming back to the linear regime, we have discussed the NMR linewidths in view of the recent work of Leggett and Takagi.[12] We have shown that the results of both theories agree, the apparent discrepancy coming only from the fact that the relaxation times are not defined in the same way in both theories. We have discussed the temperature behavior of these relaxation times. Finally, we have investigated the low-temperature behavior of the NMR linewidths with the general conclusion that it goes to zero when the temperature is going to zero. The onset of this behavior should not be difficult to observe experimentally.

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