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Self-interacting walks, random spin systems, and the zero-component limit

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Emery's analysis of n-component "spin" systems is extended and simplified for the limits $n = 0, -2, -4, \ldots$. Restrictions on the spin-weighting function implied by an interpretation of the $n \to 0$ limit as a random system are pointed out. Fully explicit relations are also derived for the interpretation of the $n \to 0$ correlation function in terms of self-avoiding and, more generally, *weighted* self-interacting lattice walks.

I. INTRODUCTION

In a recent article, Emery[1] discussed a very general class of n-component classical spin Hamiltonians, and demonstrated various model equivalences as a function of n, in a concise manner, without the use of term-by-term perturbation theory or diagrammatics. In particular, Emery considered[2] the reduced Hamiltonian

$$\mathcal{H}_n(\{\vec{s}_j\}) = \sum_{\alpha=1}^{n} \bar{\mathcal{H}}_I(\{s_j^\alpha\}) - \sum_{i=1}^{N} W\left(\sum_{\alpha=1}^{n} X(s_i^\alpha)\right), \tag{1.1}$$

where the $\vec{s}_i = (s_i^\alpha)$ are n-component classical variables associated with sites $i = 1, 2, \ldots, N$ (which need not be regularly arrayed in a lattice), and $\mathcal{H}_I(\{s_j^\alpha\})$ is a general single-component Hamiltonian; an important example for subsequent developments is the standard pair interaction Hamiltonian

$$\bar{\mathcal{H}}_I(\{s_j\}) = \frac{1}{2}\sum_{i,j} K_{ij} s_i s_j + h \sum_i s_i. \tag{1.2}$$

The function $W(x)$ in (1.1) represents a general interaction which couples the independent Hamiltonians together via the sum of local variables $X_i^\alpha = X(s_i^\alpha)$. For the most part we shall consider the simple variable

$$X(s) = s^2; \tag{1.3}$$

in this case $\exp[-W(|\vec{s}|^2)]$ can be interpreted simply as a spin-length weighting factor for variable length spins, $\vec{s}_i$. However, one may more generally[3] use functions X_i^α which are *combinations* of spins components s_j^α with j neighboring i.

For the partition function $Z_N(n)$ appropriate to (1.1), Emery introduced the integral representation

$$Z_N(n) = \int_{-\infty}^{\infty} d^N z \int_{-i\infty-c}^{i\infty-c} \frac{d^N y}{(2\pi i)^N} \times \exp\left(-\sum_{i=1}^{N} [W(z_i) + y_i z_i] + nNf_I(\{y_j\})\right), \tag{1.4}$$

in which $\int d^N z$ denotes $\int dz_1 \cdots \int dz_N$, etc., and where the reduced single-component free energy $f_I(\{y_j\})$ is defined by

$$\exp[Nf_I(\{y_j\})] = \int_{-\infty}^{\infty} d^N s \exp\left(\bar{\mathcal{H}}_I(\{s_j\}) - \sum_{i=1}^{N} y_i X(s_i)\right), \tag{1.5}$$

so that the $\{y_j\}$ represent a set of local fields. The number of components, n, enters (1.4) *only* as the prefactor of $f_I(\{y_j\})$. This formula thus provides a natural analytic continuation in n.

Emery[1] showed that his representation led formally to a very transparent and general treatment of the $n \to \infty$ limit; this is found to correspond to a class of single-component but constrained systems ("generalized spherical models" or Hartree models)[3-6] with "renormalized" critical exponents.[7] In addition, Emery showed that previous results[8-10] for the special cases $n = -2, -4, -6, \ldots$, could be derived readily. [When $\mathcal{H}_I$ has the pairwise form (1.2) with $K_{ii} = -1$, these values give pure Gaussian behavior provided the spin interaction, or weighting function $W(z)$, vanishes rapidly enough as $z \to 0$ (Refs. 8 and 9); see below also.]

Finally, Emery considered the case $n \to 0$ and obtained the striking result that the free-energy-per-spin component

$$f_n = (Nn)^{-1} \ln Z_N(n), \tag{1.6}$$

could then be interpreted directly as the free energy of a single-component system on which were imposed *random fields* y_i [coupled to the $X(s_i)$] with a distribution explicitly related to $W(z)$. (An independent derivation[11] of this result utilizes a rather complex diagram-by-diagram analysis of a Feynman graphical-perturbation theory.) Emery also made some remarks concerning the interpretation, first suggested by de Gennes,[12] of the $n \to 0$ limit, in terms of the excluded volume problem for polymers or random walks.[12-15] However, his treatment here was not very detailed or complete. Furthermore, he assumed that $W(0)$ was zero (or, at least, finite) which excludes the case in which $e^{-W(s^2)}$ vanishes as $s \to 0$; but this case is needed to describe standard fixed-length spins or analogous spin distributions which have zero weight at zero length. We shall refer to spins $\vec{s}$ with weight factors $\exp[-W(|\vec{s}|^2)]$ vanishing as $|\vec{s}|\to 0$, as "stiff" spins. Note in particular that "stiff spins" are not restricted to fixed-length spins such as described by $e^{-W(s^2)}=\delta(s^2-1)$; they also include continuous spins with smooth weight factors like $e^{-W(s^2)} = s^2 e^{-us^4}$, etc.

The relationship of the excluded volume problem to the $n\to 0$ limit for fixed-length spins has been studied by Bowers and McKerrell.[13] By making a diagrammatic analysis of the high-temperature series for pairwise interacting n-vector models, they showed explicitly, that the expansion coefficients $a_l(n)$ of the susceptibility become proportional to c_l, the number of l-step *self-avoiding walks,* when n approaches zero. An alternative diagrammatic derivation of this result was given by Gerber and Fisher,[14] incidental to a derivation of an expansion for the critical temperature $T_c(n, d)$ in inverse powers of the dimensionality d. More recently, in the same connection, Gerber and Fisher[15] considered general spin weights $W(|\vec{s}|^2)$ and showed diagrammatically that the $n\to 0$ limit yields, in general, *self-interacting random walks* with a Boltzmann factor p_r for each r-fold self-intersection. The factors p_r are expressible explicitly in terms of the spin weight function $W(x)$. For stiff spins, it is found that only the totally self-avoiding walks survive.

In this note, we modify and simplify Emery's analysis slightly in order to reconsider the $n\to 0$ limit in more detail. We shall see that the interpretation of the $n=0$ free energy, as deriving from a random system, imposes certain, fairly stringent conditions on the interaction or spin weight function $W(x)$. In particular, stiff spins are excluded. In addition, we show that the method provides a straightforward, transparent, and explicit derivation of the connection with *generally weighted* self-avoiding and self-interacting walks on the appropriate lattice structure. In particular, detailed diagrammatic analyses of the high-temperature or Feynman graphical-perturbation series are not required, nor are path integral representations or continuum limits.

Our analysis of the $n=0$ free energy and the connection with random models is presented in Sec. II. For completeness, the limits $n=-2,-4,\ldots$ are also considered briefly. In Sec. III, we study the pair correlation function and its relation to the random-walk problems.

II. FREE ENERGY IN THE ZERO-COMPONENT LIMIT

To demonstrate most concisely the connection to random systems we suppose, following Emery's lead,[1] that the interaction or weight factor admits an integral representation

$$e^{-W(x)}=\int_C dy\,P(y)e^{-yx}, \tag{2.1}$$

where C is a suitable contour (in general in the complex y plane). Note that this encompasses distributions which contain sums of δ functions, etc. Emery[1] chose C to be the whole imaginary y axis, in which case $P(y)$ is essentially the Laplace transform of $e^{-W(x)}$ [which will always exist for reasonable weight factors; but a shift of the contour to $-c$, as in (1.4), may be needed]. It is significant, however, that if $W(x)$ is suitably restricted, C may be chosen to be (part of) the real axis. If, furthermore, $P(y)$ is to be non-negative, as needed for a probabilistic interpretation, $W(x)$ must satisfy further conditions; in particular, $W(x)$ cannot then increase faster than x as $x\to\infty$.

Quite generally, if there is a power series expansion

$$e^{-W(x)}=e_0-e_1x+e_2x^2+\cdots+(-)^j e_j x^j+O(x^{j+\delta}) \tag{2.2}$$

for some j with $\delta>0$, one will have, by differentiating (2.1), the relations

$$e^{-W(0)}=e_0=\int_C P(y)\,dy, \tag{2.3}$$

$$e_l=\frac{1}{l!}\int_C y^l P(y)\,dy \quad (l\le j). \tag{2.4}$$

We may also note, for future reference, that

$$\int_0^\infty x^m e^{-W(x)}dx=m!\int_C dy\,P(y)/y^{m+1}, \tag{2.5}$$

provided the moments exist (as they normally should).

On introducing the representation (2.1) into the definition

$$Z_N(n)=\int d^N\vec{s}\,\exp[\mathcal{H}_n(\{\vec{s}_j\})], \tag{2.6}$$

for the total n-component partition function, one finds that the integrals over the separate sets of components $\{s_j^\alpha\}$ factorize and one obtains

$$Z_N(n)=\int_C d^N y \prod_{i=1}^N P(y_i)\exp[nNf_I(\{y_j\})], \tag{2.7}$$

where the *single-component free energy* $f_I(\{y_j\})$ was defined in (1.5) and is, in particular, *independent* of n. This, of course, corresponds to Emery's expression (1.4). We want to study f_n, the reduced free-energy-per-spin component, defined in (1.6) in the limit $n\to 0$, understood via the analytic continuation provided by (2.7). By L'Hospital's rule, we have

$$f_0=\lim_{n\to 0}\frac{\ln Z_N(n)}{Nn}=\lim_{n\to 0}\frac{\partial \ln Z_N(n)}{N\partial n}, \tag{2.8}$$

which yields immediately

$$f_0=\int_C d^N y \prod_{i=1}^{N}[P(y_i)/e_0] f_I(\{y_j\}), \tag{2.9}$$

where we have used the relation (2.3) to replace $\int P(y_i)dy_i$ by e_0. (We may note that if the thermodynamic limit, $N\to\infty$, is to be taken before the $n\to 0$ limit this step requires more justification.)

The interpretation of f_0 as the free energy of a system with fields y_i coupling to $X(s_i)$ varying randomly from site to site, follows from (2.9) *provided the contour C may be restricted to the real axis.* (This point was not made by Emery.[1]) Evidently, the probability that the field y_i lies in the (real) interval (a,b) is given by

$$\mathcal{P}(a\leq y_i<b)=e_0^{-1}\int_a^b P(y)\,dy. \tag{2.10}$$

Clearly, for a probabilistic interpretation, one must have $P(y)\geq 0$ and $e_0>0$. In the simplest case in which the fields take just two values c_1 and c_2 with probabilities p_1 and p_2, an appropriate interaction factor is given by

$$e^{-[W(x)-W(0)]}=p_1e^{-c_1x}+p_2e^{-c_2x}, \tag{2.11}$$

where, in fact, $W(0)$ may be arbitrary.

It is clear, however, that the interpretation as a random system fails if e_0 vanishes because, in particular, the integral $\int_{\mathrm{Re}} P(y)\,dy$ cannot vanish if $P(y)$ is never negative. Furthermore, we see that, at best, the representation (2.9) for f_0 becomes indeterminate when e_0 is zero.

In the important case where $\overline{\mathcal{H}}_I(\{s\})$ has the pairwise form (1.2) and one puts $X_i=(s_i)^2$ and sets $h=0$, one finds[1] simply

$$f_I\{y_j\}=\tfrac{1}{2}\ln\pi-\tfrac{1}{2}N^{-1}\ln|\hat{y}-\tfrac{1}{2}\hat{K}|, \tag{2.12}$$

where $|\cdots|$ denotes the determinant of the $N\times N$ matrix formed from

$$\hat{y}=[y_i\delta_{ij}],\quad \hat{K}=[K_{ij}],\; K_{ij}=K_{ji},\; K_{ii}=-1. \tag{2.13}$$

Then one may interpret f_n as the free energy of a system of pairwise interacting spins $\vec{s}_i$ with weights $W(|\vec{s}|^2)$. In the limit $n\to 0$ one obtains a Gaussian system of scalar spins with random mean-square spin lengths. However, if the original spins are stiff, the $n\to 0$ limit *cannot* be interpreted in terms of a random system.

Since there exist transformations (of the Kac-Baker-Hubbard type) which, for *fixed* n, replace stiff spins by "soft" spins, it might be surmised that a suitable spin transformation could remove these difficulties of the zero-component limit. This question is treated in Appendix A where it is shown that the problem as $n\to 0$ is merely transformed but *not* removed.

Before proceeding to discuss the correlation function for this model, where a definite $e_0\equiv 0$ limit does exist, we may observe how straightforwardly the Gaussian results at $n=-2l$ with $l=1,2,\ldots$, follow from the representation (2.7). In these cases, the factor $\exp(nNf_I)$ becomes simply $|\hat{y}-\frac{1}{2}\hat{K}|^l$. On expansion of the determinant this yields a polynomial in the y_i of degree l which must be multiplied by $\prod_i P(y_i)$ and integrated. But now we see directly from (2.3) that *if the coefficients $e_1,e_2,\ldots,e_l$ vanish identically* in (2.2) [as implied by $W(x)=O(x)^{l+\delta}$)], the integrals over all nonzero powers of the y_i also vanish. Consequently, we may set $y_j\equiv 0$ in (2.6) and so find

$$\begin{aligned} Z_N(-2l)&=e_0^N\exp[-2lNf_I(\{0\})],\\ &=e_0^N(2\pi)^{-lN}|-\hat{K}|^l, \end{aligned} \tag{2.14}$$

which, apart from the factor of e_0, is the pure Gaussian result (meaningful only when all eigenvalues of $-\hat{K}$ are positive). Note, as observed by Knops,[10] that for stiff spins the partition function actually vanishes identically in the $n=-2l$ limit. This can be understood heuristically because when n is negative, the integrand of the spherically symmetric spin-length integral, $\int_0^\infty e^{-W(s^2)}s^{n-1}ds$, is concentrated around $s=0$.

It is natural to ask what the restrictions on the weighting function $W(x)$, necessary for a probabilistic interpretation, might imply about the behavior of random spin models? We have no definite answer to this question. Note, however, that in addition to the exclusion of stiff spins, the positivity of $P(x)$ implies that $W(x)$ cannot increase more rapidly than linearly with x as $x\to\infty$; when $X_i=(s_i)^2$, this means the spin weighting cuts off no faster than a Gaussian for large s_i. If one subscribes to a sufficiently broad interpretation of the universality hypothesis for critical behavior (as indicated, say, by the formal $\epsilon=4-d$ expansion of renormalization-group theory for nonstiff spins with $|\vec{s}|^4$ weighting) one would, nevertheless, probably feel that no special limitations are placed on the random-system behavior. Conversely, if one is disturbed by such problems as the apparent breakdown of hyperscaling in the three-dimensional Ising model (i.e., $d\nu\neq 2-\alpha$), or the difference between the critical exponents of real alloys and real fluids, one may feel constrained to pause for further thought about the possibilities of critical behavior in random systems which is not simply continuous with that of better understood, $n>0$ systems.

III. ZERO-COMPONENT CORRELATIONS AND SELF-INTERACTING WALKS

The two-spin correlation function in the zero-field pairwise interaction models follows directly from

$$\Gamma_{ij}(n;\hat{K})=\frac{1}{n}\sum_{\alpha=1}^{n}\langle s_i^\alpha s_j^\alpha\rangle = N\frac{\partial f_n}{\partial K_{ij}}, \tag{3.1}$$

with the understanding $K_{ij}\equiv K_{ji}$. We will consider the limit $n\to 0$ under the condition $e_0\neq 0$, although we will show that, with suitable normalization, one may let e_0 vanish. A diagrammatic analysis of the zero-field high-temperature expansion in powers of K_{ij} without the restriction $e_0=0$ has been given and shown generally to yield self-interacting walks.[15] (See also Refs. 13 and 14 where only fixed-length spins are considered.) In the present approach we obtain from (2.9) and (2.12) the result

$$\Gamma_{ij}(0;\hat{K})=\int_C d^N y\prod_{i=1}^{N}[P(y_i)/e_0]\Gamma_{ij}^G(\hat{K};\{y_k\}), \tag{3.2}$$

where the correlation functions in the Gaussian model are given by

$$\Gamma_{ij}^G(\hat{K};\{y_k\})=\tfrac{1}{2}[(\hat{y}-\tfrac{1}{2}\hat{K})^{-1}]_{ji}, \tag{3.3}$$

which arises since the derivative of a determinant with respect to a particular element is the corresponding cofactor. The form (3.2) clearly displays the $n=0$ correlations as those of a randomly weighted Gaussian model.

Before analyzing (3.2) further, we introduce precisely the concept of a *weighted self-interacting walk*[15] on a lattice or general graph. Consider first, for simplicity, random walks in which at each step the walker moves always to a nearest-neighbor site. Let $q_l(i,j;k_2,k_3,\ldots)$ be the number of such walks which leave site i on the first step and arrive at site j on the lth step having visited k_2 distinct sites precisely twice, k_3 distinct sites precisely three times, and so on. If a *self-intersection of order* r (i.e., a site visited r times) carries a Boltzmann factor p_r (with $p_1\equiv 1$), the total partition function for l-step walks from i to j is

$$Q_l(i,j)=\sum_{k_2}\sum_{k_3}\cdots q_l(i,j;k_2,k_3,\ldots)p_2^{k_2}p_3^{k_3}\cdots. \tag{3.4}$$

For a *self-avoiding walk,* all the factors p_2, $p_3,\ldots$ must be set to zero. With the convention $q_0(i,j)=\delta_{ij}$, one may then form the grand partition function or generating function[16]

$$Q(\zeta;i,j)=\sum_{l=0}Q_l(i,j)\zeta^l. \tag{3.5}$$

More generally we may consider a random walk on the sites i in which arbitrary length steps can be taken at each point of the walk but where a direct step from site i' to site j' carries a *weight* $z_{i'j'}$. The partition function $Q_l(i,j)$ may then be generalized by introducing the weights which gives

$$Q_0(i,j)=\delta_{ij},\quad Q_1(i,j)=z_{ij}, \tag{3.6}$$

and

$$Q_l(i,j)=\sum_{C(i,j;l)}\prod_{r=2}p_r^{k_r}\prod_{i',j'}z_{i'j'}^{g_{i'j'}}, \tag{3.7}$$

where the sum runs over all l step walks from i to j (with steps of any length) and $g_{i'j'}$ is the number of times the step from i' to j' appears in the walk.[17]

Now the correctly weighted noninteracting, on free walks (with all $p_m=1$) on a graph of N sites can be generated in terms of the matrix $\hat{z}=[z_{ij}]$. Thus, by the rules of matrix multiplication, one has

$$\begin{aligned}[(\hat{z})^l]_{ij}&=\sum_{i'}\sum_{i''}\cdots\sum_{j'}z_{ii'}z_{i'i''}\cdots z_{j'j}\\&=\sum_{C(i,j;l)}\prod_{i',j'}z_{i'j'}^{g_{i'j'}}=Q_l^0(i,j),\end{aligned} \tag{3.8}$$

where the superscript zero denotes free walks. The corresponding free grand partition function, for small enough ζ, is then seen to be

$$\begin{aligned}Q^0(\zeta;i,j)&=[\hat{1}+\zeta\hat{z}+\zeta^2\hat{z}^2+\cdots]_{ij}\\&=[(\hat{1}-\zeta\hat{z})^{-1}]_{ij}.\end{aligned} \tag{3.9}$$

This result clearly bears a close resemblance to the expression (3.3) for the Gaussian-model correlation function; indeed, for uniform fields, $y_i\equiv y_0$, the two expressions can easily be brought into one-one correspondence. However, since we are interested in self-interacting rather than free walks, we will, instead, replace $\frac{1}{2}$ by $\frac{1}{2}\zeta$ and expand the Gaussian expression as

$$\begin{aligned}(\hat{y}-\tfrac{1}{2}\zeta\hat{K})^{-1}&=\hat{y}^{-1}+\zeta\hat{y}^{-1}(\tfrac{1}{2}\hat{y}^{-1}\hat{K})\\&\quad+\zeta^2\hat{y}^{-1}(\tfrac{1}{2}\hat{y}^{-1}\hat{K})^2+\cdots,\end{aligned} \tag{3.10}$$

and then substitute in (3.3) and in (3.2). On recalling that $\hat{y}=(y_i\delta_{ij})$, we see that $\hat{y}^{-1}=[\delta_{ij}/y_i]$ and $\frac{1}{2}\hat{y}^{-1}\hat{K}=[\frac{1}{2}K_{ij}/y_i]$. Thus, consider the coefficient of ζ^0 in $\Gamma_{ij}(0)$; by (3.2) and (2.3) this reduces to

$$\Gamma_{ij}(0;\hat{K}\equiv 0)=\tfrac{1}{2}\delta_{ij}\int_C dy_i\,P(y_i)/y_i e_0\equiv\overline{m}_2^0\delta_{ij}. \tag{3.11}$$

Now, on using (2.5), the coefficient of δ_{ij} may easily be recognized as the mean-square noninteracting spin length

$$\overline{m}_2(n)=\frac{\langle|\vec{s}|^2\rangle_{\hat{K}=0}}{n}=\frac{\int_0^\infty s^2e^{-W(s^2)}s^{n-1}ds}{n\int_0^\infty e^{-W(s^2)}s^{n-1}ds}, \tag{3.12}$$

continued analytically to $n=0$ by using (2.2). [See Ref. 15, but note that $\overline{m}_2(0)$ diverges if $e_0=0$.] To obtain a scale free result, we should clearly normalize Γ_{ij} by dividing by m_2^0.

Consider then the coefficient of ζ^1 in $\Gamma_{ij}(0)/m_2^0$, namely,

$$\tfrac{1}{4}K_{ji}\int_C dy_j \int_C dy_i\, P(y_j)P(y_i)/y_j y_i e_0^2 m_2^0 = m_2^0 K_{ji}. \tag{3.13}$$

This, in turn, indicates that K_{ji} should also be normalized by dividing by m_2^0. Both this and the previous normalization are quite standard in deriving such high-temperature expansions.[15]

Comparison of (3.11) and (3.13) with (3.6) now indicate the identities

$$Q(1;i,j) = \Gamma_{ij}(0)/m_2^0, \tag{3.14}$$

$$K_{ij} = z_{ji}/m_2^0. \tag{3.15}$$

To justify the first identity, however, we must consider the general coefficient of ζ^l in $\Gamma_{ij}(0)/m_2^0$. The powers of the matrix $\frac{1}{2}\hat{y}^{-1}m_2^0\hat{K}$ will clearly generate the correctly weighted walks and will also introduce factors y_i^r if site i is visited r times by the corresponding walk. As in the derivation of (3.11) and (3.13), sites which are not visited at all introduce a factor $p_0 = 1$. Since each of the r incoming steps carries a factor m_2^0 by virtue of (3.15), we see that the remaining Boltzmann factors for an r-fold self-intersection are

$$\begin{aligned} p_r &= \int_C dy_i\, P(y_i)/(2y_i m_2^0)^r e_0, \\ &= \frac{I(2r)}{(r-1)!\,I(2)}\left(\frac{e_0}{2I(2)}\right)^{r-1}, \end{aligned} \tag{3.16}$$

where we have used (2.5), (3.11), and introduced the moments

$$I(l) = \int_0^\infty e^{-W(s^2)} s^{l-1}\, ds, \tag{3.17}$$

which are well defined for $l > 0$. This agrees precisely with the result of Gerber and Fisher.[15]

We may check from (3.16) that p_1 is unity as required. Furthermore, we note that e_0 now appears only in the numerator defining p_r. Evidently, the limit $e_0 = 0$ yields $p_2 = p_3 = \cdots = 0$, so that all self-intersections are forbidden. Thus, in the $n\to 0$ limit for *stiff spins,* the normalized correlation function describes only *self-avoiding walks.*[13,14] (Note that all other details of the spin-weighting factor enter only into the normalization.)

Finally, for all spin weights $W(|\vec{s}|^2)$, we may conclude generally

$$\begin{aligned} &\lim_{n\to 0}\Gamma_{ij}(n;\hat{K}=\hat{z}^T/\overline{m}_2(n))/\overline{m}_2(n) \\ &\qquad = Q(1;i,j;\{z_{ij}\},\{p_r\}), \end{aligned} \tag{3.18}$$

where $Q(ij)$ is the grand partition function [at $\zeta = 1$; see (3.5) and (3.7)] for walks with step weights z_{ij} and self-intersection Boltzmann factors p_r given by (3.16); the superscript T denotes a matrix transpose, while from (3.12) and (3.17) we have $\overline{m}_2(n) = I(n+2)/nI(n)$. This completes our explicit identification of the $n\to 0$, zero-field correlation function for pairwise spin Hamiltonians with weight $W(|\vec{s}|^2)$, with weighted self-avoiding and self-interacting walks. The result agrees with that found in Ref. 15 through a diagram-by-diagram analysis of the high-temperature series. Self-avoiding walks arise from any stiff-spin weighting function (for which $e^{-W(0)} = 0$), while more general spin weights yield interacting walks.

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APPENDIX: TRANSFORMATION OF SPIN VARIABLES

In Sec. II we have shown that the direct interpretation of the free energy in the $n\to 0$ limit as the free energy of a random system fails when the original spin model has stiff spins, i.e., when e_0 vanishes in (2.3). A particularly important example of stiff spins is, as mentioned, provided by fixed-length spins in which the restriction

$$|\vec{s}|^2 = \sum_{\alpha=1}^{n}(s^\alpha)^2 = \mathcal{S}_n^2$$

is imposed at each site. The failure of the direct interpretation is more general, however.

It is known from the work of Kac,[18] Baker,[19] Hubbard,[20] and others that a system with pairwise interactions of the form (1.2) involving stiff spins, $\vec{s}_i$, can be transformed to a model with "soft" spins, $\vec{\sigma}$, in which the new weight factor $\exp[-\overline{W}(\sigma^2)]$ does *not* vanish at $\sigma = |\vec{\sigma}| = 0$. It might be supposed that the difficulties uncovered in Sec. II would not arise in the transformed description. In fact, as we now indicate, the difficulties are merely transformed so that the conclusions of Sec. II remain unaltered.

The basic transformation of the interaction Boltzmann factor is expressed by[19]

$$\begin{aligned} &D(\hat{K})\exp\left[\frac{1}{2}\sum_{i,j}K_{ij}\vec{s}_i\cdot\vec{s}_j\right] \\ &\quad = \int_{-\infty}^{\infty} d^N\vec{\sigma}\,\exp\left\{-\frac{1}{2}\sum_{i,j}[\hat{K}^{-1}]_{ij}\vec{\sigma}_i\cdot\vec{\sigma}_j + \sum_i \vec{\sigma}_i\cdot\vec{s}_i\right\}, \end{aligned} \tag{A1}$$

where $\vec{\sigma}_i$ is the new n-component spin variable, $[\hat{K}^{-1}]_{ij}$ is the element of the inverse matrix of $\hat{K}=[K_{ij}]$, supposed positive definite and nonsingular, and $D=(2\pi)^{Nn/2}[\mathrm{Det}(\hat{K})]^{n/2}$ is the normalization factor. The zero-field partition function is

$$Z_N(n) = B_{Nn}\int d^N\vec{s}\,\exp\left[\frac{1}{2}\sum_{ij}K_{ij}\vec{s}_i\cdot\vec{s}_j-\sum_j W(|\vec{s}_j|^2)\right], \tag{A2}$$

where we allow for an arbitrary normalization factor B_{Nn} in case this proves important when $n\to 0$. [A natural normalization is to require $Z_N(n)\equiv 1$ for free or noninteracting spins ($K_{ij}\equiv 0$ for $i\neq j$).] On substituting with (A1) we find $Z_N(n)$ expressed in terms of the partition function of a new model of N spins, $\vec{\sigma}_i$, with pairwise interaction matrix $-\hat{K}^{-1}$ and a new spin weight function $\overline{W}_n(|\vec{\sigma}|^2)$ defined by

$$\begin{aligned} e^{-\overline{W}_n(\sigma^2)} &= Y_n(\sigma)\\ &=\langle e^{\vec{s}\cdot\vec{\sigma}-W(|\vec{s}|^2)}\rangle_{\vec{s}},\\ &=b_n\int d\vec{s}\,e^{\vec{s}\cdot\vec{\sigma}-W(|\vec{s}|^2)}\Big/\int d\vec{s}\,e^{-W(|\vec{s}|^2)}. \end{aligned}\tag{A3}$$

Here b_n represents an arbitrary normalization factor related to B_{Nn}. We note that for Ising ($n=1$) spins restricted by $s_i=\pm 1$ one finds $\overline{W}_n(\sigma^2)=-\ln\cosh\sigma$. Similarly, for classical Heisenberg ($n=3$) spins of unit length the weight is $\overline{W}_n(\sigma^2)=-\ln[(\sinh\sigma)/\sigma]$. Quite generally $\overline{W}_n(\sigma^2)$ varies as $-\bar{s}|\vec{\sigma}|$ when $\sigma^2\to\infty$ (where $\bar{s}>0$ is fixed) so that in the $\vec{\sigma}$-spin model the weight function does *not* serve to establish a cutoff at large $|\vec{\sigma}|$. On the contrary it is only the negative bilinear interactions which provide convergence as $|\vec{\sigma}|\to\infty$. In effect, the $\vec{\sigma}$-spin model is Gaussian for large $|\vec{\sigma}|$ (although this is not, of course, the situation for small $\vec{\sigma}$).

Now the angular integrations in (A3) can be performed leaving

$$Y_n(\sigma)=\frac{b_n\Gamma(\frac{1}{2}n)\int_0^\infty s^{n-1}\mathcal{K}_n(s\sigma)e^{-W(s^2)}ds}{\int_0^\infty s^{n-1}e^{-W(s^2)}ds}, \tag{A4}$$

where, in terms of the modified Bessel function[21] $I_\nu(z)$, the kernel is

$$\mathcal{K}_n(z)=(\tfrac{1}{2}z)^{1-(n/2)}I_{(n/2)-1}(z)=1/\Gamma(\tfrac{1}{2}n)+\mathcal{R}_n(z). \tag{A5}$$

The second equality isolates the singular part as $n\to 0$; the remainder

$$\mathcal{R}_n(z)=\sum_{k=1}^{\infty}\frac{(\frac{1}{2}z)^{2k}}{k!\,\Gamma(\frac{1}{2}n+k)}, \tag{A6}$$

is absolutely convergent for all $n>-2$, positive for real z, and varying as z^2 for small z. Substitution in (A4) and comparison with (2.1) yields

$$\begin{aligned} Y_n(\sigma)&=\int_C \overline{P}_n(y)e^{-y\sigma^2}dy\\ &=b_n+c_n\int_0^\infty s^{n-1}\mathcal{R}_n(s\sigma)e^{-W(s^2)}ds, \end{aligned}\tag{A7}$$

with

$$c_n=b_n\Gamma(\tfrac{1}{2}n)\Big/\int_0^\infty s^{n-1}e^{-W(s^2)}ds, \tag{A8}$$

where $\overline{P}_n(y)$ should, in the limit $n\to 0$, be directly related to the probability distribution of the random fields y_j [see (2.9)].

Now from (A7) with $\sigma=0$, we have

$$Y_n(0)=\int_C \overline{P}_n(y)dy=b_n, \tag{A9}$$

so that a probabilistic interpretation ($\overline{P}_0>0$) is consistent provided $b_0=\lim_{n\to\infty}b_n>0$. For nonzero σ the integral in (A7) remains well behaved as $n\to 0$. Furthermore, the factor c_n behaves as $2b_n e^{W(0)}$ when $n\to 0$ provided the $\vec{s}$-spins are not stiff. In these circumstances, as before, an interpretation of the zero-component limit as describing a random system is hence permitted. Conversely, for stiff spins, with $e^{-W(0)}=0$, one sees that c_n/b_n diverges to infinity. If b_0 remains nonzero, it follows that $Y_n(\sigma)$ is undefined (i.e., infinite everywhere) for nonzero σ in the limit $n\to 0$. The implications of this behavior can be understood by choosing $b_n\to 0$ such that c_n remains finite as $n\to 0$. In the limit, $\overline{P}_n(y)$ becomes a function with well-defined first, second, ... moments [corresponding to the successive terms in (A6)] but with *vanishing zeroth moment* [since $b_0=0$; see (A9)]. Such a function, $\overline{P}_0(y)$, must have negative parts and hence cannot be a probability distribution. If b_0 is not allowed to vanish, $\overline{P}_0(y)$ simply becomes a function with unbounded oscillations. The additional terms entering in the application of L'Hospital's rule in (2.8), through the n dependence of $\overline{P}_n(y)$, do not alter these conclusions.

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