PHYSICAL REVIEW B VOLUME 13, NUMBER 3 1 FEBRUARY 1976

Dispersion relation for phonon second-sound waves in superfluid helium

David Benin
Department of Physics, Arizona State University, Tempe, Arizona 85281
(Received 14 August 1975)

The velocity and attenuation of phonon second-sound waves in superfluid ^{4}He at saturated vapor pressure have been calculated, as a function of sound-wave frequency ω, over the entire frequency range at a single temperature 0.25°K. Second sound is obtained as a wavelike normal mode of a model phonon Boltzmann equation containing, in addition to the lifetime $\tau_\parallel$ of a single thermal phonon due to small-angle scattering, a sequence of longer lifetimes characterizing wide-angle scattering of phonons with anomalous dispersion. The calculated second-sound phase velocity shows a dispersion spread out over four orders of magnitude in frequency in the range $\omega\tau_\parallel \lesssim 1$. Moreover, there is a wide frequency range satisfying $\omega\tau_\parallel > 1$ in which a second-sound collective mode still propagates, with the same velocity as a thermal phonon but with an attenuation length much longer than the thermal-phonon mean free path. The existence of a collective mode in the regime $\omega\tau_\parallel > 1$, due to small-angle scattering, supports Maris's proposed explanation of resonancelike dispersion in the first-sound velocity, and also implies that the transition from collective to ballistic propagation in heat-pulse experiments is more complicated than previously supposed.

I. INTRODUCTION

A number of recent experimental and theoretical studies of phonon excitations in superfluid ^{4}He at low pressures have provided much evidence that the phonon spectrum $\omega(q)$ is anomalous at long wavelengths; the phonon frequency increases slightly faster than linearly in the wave vector q. For such an anomalous dispersion, the lifetime $\tau_\parallel$ of a thermal phonon at low temperatures is due to the three-phonon interaction, with all three wave vectors nearly collinear (small-angle scattering).[1] In addition to $\tau_\parallel$, there is a sequence of longer relaxation times characterizing the coupling of phonons over wide angles via many back-to-back small-angle collisions. The existence of long relaxation times characterizing wide-angle scattering, and their importance in the hydrodynamic response of the phonon system, have been examined in an extensive set of computations by Maris.[2-4] His results for the phonon viscosity[2] and for the first-sound velocity and attenuation[3] are in striking agreement with experiment. Maris also finds a number of different collective modes[4] of the phonon system, including a second-sound mode with very spread-out dispersion: the second-sound velocity variation between about $c_0/\sqrt{3}$ and c_0 (c_0 is the phonon velocity) typically extends over about four orders of magnitude in second-sound wave frequency at fixed temperature, or over a factor of 6 or more in temperature at fixed frequency.

Maris's calculations involve extensive numerical treatment of the collision operator in the phonon Boltzmann equation, and are limited to the range of collective mode frequencies $\omega\tau_\parallel \ll 1$. In this work we use a model Boltzmann equation to relate the collective-mode dispersion relation more directly to a spectrum of wide-angle relaxation rates. We also examine the nature of second-sound propagation in the high-frequency regime $\omega\tau_\parallel > 1$, which is of special interest in connection with first-sound propagation. Calculations by Maris[3] and Wehner[5] show an unusual resonancelike dispersion in the first-sound velocity, in agreement with experiment.[6] At for instance $T = 0.25$°K, the resonance is seen in the frequency range $10 \lesssim \omega \lesssim 100$ MHz, which (according to a calculation of $\tau_\parallel$ given below) corresponds to $0.6 \lesssim \omega\tau_\parallel \lesssim 6$. Maris and Wehner both attribute this dispersion to resonant interaction of the first-sound wave with a collective mode of the thermal phonons, which Maris further proposes is the second-sound mode. This explanation would, however, then require that the phonons sustain collective oscillations to frequencies usually thought of as well into the ballistic or single-phonon regime $\omega\tau_\parallel > 1$. The calculations we report here do show this behavior.

In Sec. II we present a model phonon Boltzmann equation containing a sequence of wide-angle relaxation rates which can be explicitly solved for the collective-mode dispersion relation. In Sec. III the phase velocity and attenuation length of second sound are evaluated as functions of frequency at $T = 0.25$°K, for a set of relaxation rates which follow from Maris's proposed form for the phonon anomalous dispersion in superfluid helium. At low frequency the second-sound velocity approaches $\langle v_q \rangle/\sqrt{3}$, where $\langle v_q \rangle$ is a thermal average of the phonon group velocity and is weakly temperature dependent, due to thermal averaging of the weak q dependence of v_q if the phonon dispersion is anomalous. This temperature dependence of the hydrodynamic second-sound velocity has been derived previously by Saslow[7] and Brooks and Donnelly.[8] With increasing frequency, we find a second-sound velocity increase from $\langle v_q\rangle/\sqrt{3}$ to $\langle v_q\rangle$ spread over four orders of magnitude in frequency below $\omega\tau_{\parallel}\approx 1$, in agreement with Maris's results. We also find second-sound propagation continuing up to $\omega\tau_{\parallel}\simeq 3000$, with velocity essentially equal to the thermal-phonon velocity $\langle v_q\rangle$ and with an attenuation length longer than the phonon mean free path $\langle v_q\rangle\tau_{\parallel}$. These results can be qualitatively understood by distinguishing between small- and wide-angle phonon scattering (Sec. IV).

II. MODEL PHONON BOLTZMANN EQUATION

We shall describe a collective disturbance of the phonons in superfluid helium in terms of a nonequilibrium phonon distribution

$$\begin{aligned} N(\vec{q},\vec{r},t) &= N^0(\vec{q}) + [N^0(N^0+1)]^{1/2}\psi(\vec{q},\vec{r},t) \\ &= N^0(\vec{q}) + \psi(\vec{q},\vec{r},t)/2\sinh(\tfrac{1}{2}\beta\hbar\omega_q), \\ \psi(\vec{q},\vec{r},t) &= \psi(\vec{q})\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}, \end{aligned} \tag{1}$$

where ω_q is the phonon frequency, $\beta=(k_BT)^{-1}$, $N^0(\vec{q})$ is the equilibrium Bose distribution, and ω and $\vec{k}$ are the frequency and wave vector of the wavelike disturbance. $\psi(\vec{q})$ is taken to obey the phonon Boltzmann equation, with superfluid motion neglected and with the linearized three-phonon collision integral[2,9] $\tilde{C}(\psi)$

$$(i\omega - ikv_q\cos\theta)\psi(\vec{q}) = \tilde{C}(\psi); \tag{2}$$

θ is the angle between $\vec{k}$ and $\vec{q}$, and $v_q = d\omega_q/dq$.

As discussed in the Appendix, it is possible to write $\tilde{C}(\psi)$ formally as a sum of projections of $\psi(\vec{q})$ onto eigenfunctions of the completely continuous part of $\tilde{C}$. We shall retain only those collective modes which arise from projections onto a limited set of slowly relaxing eigenfunctions, whose angular parts in $\vec{q}$ space are spherical harmonics $Y_l^m(\hat{\Omega})$; $\hat{\Omega}$ denotes the angular variables of the wave vector $\vec{q}$. The projection onto a spherical harmonic Y_l^m relaxes at a rate $1/\tau_l$, a rate characterizing coupling of phonons over an angle $\theta\sim\pi/l$ between a maximum of Y_l^m (excess phonons) and an adjacent minimum (too few phonons). Energy and momentum conservation in phonon collisions imply $1/\tau_0 = 1/\tau_1 = 0$. We denote by $1/\tau_{\parallel}$ the inverse lifetime of a thermal phonon due to three-phonon scattering. For small anomalous dispersion, this is small-angle scattering.[1] In contrast, $1/\tau_l$ for small l describes wide-angle scattering resulting from many successive small-angle scatterings, and the sequence $1/\tau_l$ we shall take later to converge to $1/\tau_{\parallel}$ as $l\to\infty$. As outlined in the Appendix, a model Boltzmann equation including these features is

$$\begin{aligned} &(i\omega - ik\langle v_q\rangle\cos\theta)\psi(\vec{q}) \\ &\quad = -\frac{1}{\tau_{\parallel}}\left[\psi(\vec{q}) - \sum_{lm}\left(1-\frac{\tau_{\parallel}}{\tau_l}\right)\phi_{lm}(\vec{q})\left(\frac{1}{\tau_{\parallel}}\phi_{lm},\psi\right)\right], \end{aligned} \tag{3}$$

where

$$\phi_{lm}(\vec{q}) = R(q)Y_l^m(\hat{\Omega}),$$

$$R(q) = [(15/16\pi^4)(\beta\hbar c_0)^5\tau_{\parallel}]^{1/2} q/\sinh(\tfrac{1}{2}\beta\hbar c_0 q),$$

with normalization

$$\left(\frac{1}{\tau_{\parallel}}\phi_{lm},\phi_{l'm'}\right) \equiv \frac{1}{\tau_{\parallel}}\int d\vec{q}\,\phi^*_{lm}(\vec{q})\phi_{l'm'}(\vec{q}) = \delta_{ll'}\delta_{mm'};$$

and

$$\begin{aligned} \langle v_q\rangle &= \frac{1}{\tau_{\parallel}}\int_0^\infty dq\, q^2|R(q)|^2 v_q, \\ c_0 &= \lim_{T\to 0\text{ K}}\langle v_q\rangle = \lim_{q\to 0}\frac{d\omega_q}{dq}. \end{aligned} \tag{4}$$

A model Boltzmann equation of the same form as (3) but including superfluid motion has been used by Wehner[5] to discuss first sound in superfluid helium.

Equation (3) may also be written as

$$\begin{aligned} &(z-\cos\theta)\psi(\vec{q}) \\ &\quad = (ik\langle v_q\rangle\tau_{\parallel})^{-1}\sum_{lm}\left(1-\frac{\tau_{\parallel}}{\tau_l}\right)\phi_{lm}(\vec{q})\left(\frac{1}{\tau_{\parallel}}\phi_{lm},\psi\right), \end{aligned} \tag{5a}$$

$$z = (i\omega\tau_{\parallel}+1)/ik\langle v_q\rangle\tau_{\parallel}, \tag{6}$$

or for z not on the real axis between -1 and $+1$:

$$\begin{aligned} \psi(\vec{q}) &= (ik\langle v_q\rangle\tau_{\parallel})^{-1}\sum_{lm}\left(1-\frac{\tau_{\parallel}}{\tau_l}\right) \\ &\quad\times\frac{\phi_{lm}(\vec{q})}{z-\cos\theta}\left(\frac{1}{\tau_{\parallel}}\phi_{lm},\psi\right). \end{aligned} \tag{5b}$$

The model equation (5a) or (5b) is identical in form to model equations occurring in rarefield gas dynamics and neutron transport theory.[10,11] One can distinguish two problems: forced vibrations (ω real and given) or free vibrations ($\vec{k}$ real and given). Either problem has two classes of solutions, collective modes and ballistic phonon modes. Collective modes arise in the following way: For z not on the real axis between -1 and $+1$, the equation has solutions only at isolated points $z_i(\omega)$ or $z_i(k)$. For each such solution z_i, Eq. (6) then provides a dispersion relation relating

ω and $\vec{k}$. For each given ω (or $\vec{k}$), there is also a continuum of ballistic phonon modes, corresponding to solutions for every z *on* the real axis between -1 and $+1$, $z=\cos\theta_0$ say. The solution $\psi(\vec{q})$ of (5a) then contains an additive term $\lambda\delta(\cos\theta - \cos\theta_0)$, and (5a) determines the value of λ, rather than determining a dispersion relation[10]; the only relation between ω and k is [from (6)]

$$\omega = k\langle v_q\rangle \cos\theta_0 + i/\tau_{\parallel}\,.$$

Thus given any temporal (or spatial) variation in the phonon density, such a disturbance may always be partially carried by a single beam of phonons moving with group velocity $\langle v_q\rangle$ in an arbitrary direction θ_0. All such beam solutions have the same lifetime $\tau_{\parallel}$ and attenuation length $\langle v_q\rangle\tau_{\parallel}$. These are "ballistic thermal phonons," and the approximations inherent in the model Boltzmann equation (3) are thus to replace the wave-vector-dependent single phonon scattering rate $\Gamma(q)$ and group velocity v_q by their thermal averages $1/\tau_{\parallel}$ and $\langle v_q\rangle$. Thus the only real way that anomalous phonon dispersion is retained in (3) is in its influence on the phonon scattering—the values of the $1/\tau_l$.

One technique of solution of (3), discussed by Case and Hazeltine,[11] amounts to taking inner products of both (5a) and (5b) to construct the moments

$$\rho_l(k,\omega) \equiv \frac{4\pi}{2l+1}\sum_m Y_l^{m}(\hat{k})\left(\frac{1}{\tau_{\parallel}}\,\phi_{lm},\psi\right) = \frac{1}{\tau_{\parallel}}\int d\vec{q}\,R(q)P_l(\cos\theta)\psi(\vec{q})\,.$$

$\rho_0(k,\omega)$ is proportional to the Fourier transform of the energy density fluctuation. (5a) then gives the recursion relation[11]

$$(l+1)\rho_{l+1}+l\rho_{l-1} - (2l+1)[z-(1-\tau_{\parallel}/\tau_l)/ik\langle v_q\rangle\tau_{\parallel}]\rho_l = 0\,, \quad (7a)$$

while (5b) gives the coupled set of equations

$$\rho_{l'} = (ik\langle v_q\rangle\tau_{\parallel})^{-1} \times\sum_l (2l+1)(1-\tau_{\parallel}/\tau_l)P_{l<}(z)Q_{l>}(z)\rho_l\,. \quad (7b)$$

[To obtain (7b) one may evaluate an angular integral

$$\int d\hat{\Omega}\,P_{l'}(\cos\theta)Y_l^{m}(\hat{\Omega})/(z-\cos\theta)$$

by using the rotation theorem for the spherical harmonic, to obtain the integral

$$\frac{1}{2}\int_{-1}^{1} d(\cos\theta)P_{l'}(\cos\theta)P_l(\cos\theta)/(z-\cos\theta) = P_{l<}(z)Q_{l>}(z)\,,$$

where $l_<$ ($l_>$) is the smaller (larger) of l, l', and P_l and Q_l are Legendre functions of the first and second kinds.] Case and Hazeltine write (7a) as

$$\rho_l(k,\omega) = h_l(k,\omega)\rho_0(k,\omega)\,, \quad (8)$$

$$(l+1)h_{l+1}+lh_{l-1} - (2l+1)[z-(1-\tau_{\parallel}/\tau_l)/ik\langle v_q\rangle\tau_{\parallel}]h_l = 0\,, \quad (9)$$

$$h_{-1}\equiv 0,\quad h_0\equiv 1\,.$$

An approximate dispersion relation valid for $\omega\tau_{\parallel}\ll 1$ results from recursing (9) for $l=1,2,\ldots, L+1$ and assuming the moments truncate: $h_{L+1}(k,\omega)=0$ is then an Lth approximation to the dispersion relation for collective modes. Maris's discussion[4] of collective modes in superfluid ^{4}He involves an equivalent approximation. The Case-Hazeltine method, valid for all frequencies, is to substitute (8) into the right-hand side of (7b) for $l'=0$, and rearrange to obtain

$$\rho_0\Lambda(k,\omega)=0,$$

$$\Lambda(k,\omega) = 1-(ik\langle v_q\rangle\tau_{\parallel})^{-1} \times\sum_l (2l+1)(1-\tau_{\parallel}/\tau_l)Q_l(z)h_l(k,\omega)\,. \quad (10)$$

The zeros of the dispersion function $\Lambda(k,\omega)$ in the z-plane cut from -1 to $+1$ thus give the dispersion relations for collective phonon modes. By combining (9) with recursion relations for $Q_l(z)$, Case and Hazeltine show that (10) may also be written

$$\Lambda(k,\omega) = \lim_{L\to\infty}\Lambda_L(k,\omega)\,,$$
$$\Lambda_L(k,\omega) = (L+1)[h_{L+1}(k,\omega)Q_L(z) - h_L(k,\omega)Q_{L+1}(z)]\,. \quad (11)$$

III. COMPUTATION OF SECOND-SOUND DISPERSION RELATION

The effect of anomalous phonon dispersion on the phonon scattering is contained in the wide-angle relaxation rate spectrum $1/\tau_l$, but at present an explicit connection between the phonon frequency ω_q and $1/\tau_l$ has been given only for small l,[9] so we shall resort to Maris's numerical computation of $1/\tau_l$. We examine the phonon dispersion relation

$$\omega_q = c_0\left(q+\gamma q^3\frac{1-(q/q_A)^2}{1+(q/q_B)^2}\right)\,, \quad (12)$$

with $\gamma=1.11$ Å^2, $q_A=0.5418$ Å^{-1}, $q_B=0.3322$ Å^{-1}.

This is Maris's dispersion relation D, which he shows is consistent with measured neutron scattering, phonon viscosity, and ultrasonic attenuation, at saturated vapor pressure.[2,3] For $T=0.25\,^\circ\mathrm{K}$ and $l \leqslant 15$ we use Maris's numerical computations of $1/\tau_l$ for this dispersion relation (Fig. 2 of Ref. 4). For $l>15$ we use

$$1/\tau_l = \frac{1}{\tau_{\parallel}}\,(1-e^{-(\sigma l)^2/2}), \quad \sigma = 0.0269\,, \tag{13}$$

$$\frac{1}{\tau_{\parallel}} = 32.9\,\frac{\hbar}{\rho}\,(u+1)^2 (k_B T/\hbar c_0)^5\,, \tag{14}$$

with Grüneisen constant u, liquid density ρ, and velocity c_0 as in Refs. 2–4. The thermal phonon lifetime $\tau_{\parallel}$ due to small-angle scattering has been calculated as in Ref. 5 and 9. The form for $1/\tau_l$ is as suggested for $l \gg 1$ by Wehner,[5] who observed in the somewhat unrealistic case $\omega_q = c_0(q+\gamma q^3)$ that σ may be interpreted as a mean scattering angle for small-angle phonon collisions. It seems plausible that a similar interpretation holds for the phonon dispersion (12), in which case σ is calculable from (12) somehow. Here we simply observe that the form (13) roughly approximates Maris's calculated $1/\tau_l$ for $l \gtrsim 8$, and $\sigma = 0.0269$ results from fitting to Maris's calculation for $l=15$ (see Fig. 1).

For this relaxation rate spectrum, the second-sound dispersion relation was determined as follows. We consider the problem of forced vibrations, so that ω is real and k complex in (5) and (6). For fixed ω, each of the Λ_L defined by (11) and (9) is analytic in the z-plane cut from -1 to $+1$, and has isolated zeros. As ω is varied, each zero moves along a continuous path in the z-plane; each such path corresponds by Eq. (6) to a dispersion relation $k=k(\omega)$ for a different collective mode. The second-sound modes are the two paths on which the phase velocity $\omega/\mathrm{Re}k \to \pm\langle v_q\rangle/\sqrt{3}$ as $\omega \to 0$. As L is increased, old paths are deformed, and new paths corresponding to more highly damped collective modes appear. For $L=5$, points on the path of second-sound zeros were computed by iterative numerical solution of $\Lambda_5=0$. For each frequency considered, the $L=5$ zero was then used as a starting approximation in iterative numerical solution of $\Lambda_6=0$, and so on upwards in L until convergence of the phase velocity and damping to better than 1% was achieved. We found this procedure free of instabilities and able to follow the second sound mode zero unambiguously.

Figure 2 shows the second-sound phase velocity $\omega/\mathrm{Re}k$ and inverse attenuation length $\alpha = \mathrm{Im}k$ determined in this way for dispersion relation (12) at $0.25\,^\circ\mathrm{K}$. At a maximum frequency $\omega\tau_{\parallel} \cong 2749$ the

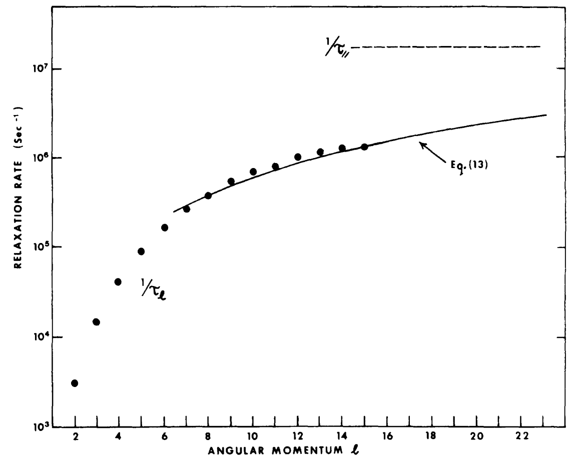

FIG. 1. Relaxation rate spectrum at 0.25 °K. Points denote Maris's calculations (Ref. 4). See text, Sec. III.

second-sound zeros in the z plane pass onto the branch cut and the collective mode breaks up into a beam of ballistic phonons. In Fig. 2, this maximum frequency is very nearly (but not exactly) the frequency at which the attenuation length $1/\alpha$ has decreased to equal the phonon mean free path $\langle v_q\rangle\tau_{\parallel}$. It is possible to give a formal continuation of the dispersion relation to higher frequencies by following the zero across the cut onto a second Riemann sheet of the dispersion function, and this has been done in Fig. 2.

The method of upward recursion of the sequence Λ_L shows directly how sensitive the dispersion relation is to the entire spectrum of relaxation rates $1/\tau_l$. In fact the phase velocity is at all frequencies given to better than $\frac{1}{2}$% of its "final" value by stopping at $L=15$, which in effect retains only the rates $1/\tau_0$ through $1/\tau_{15}$ and lumps all higher rates together at the limit point $1/\tau_{\parallel}$. The same $L=15$ approximation gives the attenuation α to at worst about 3% of its final value, for $\omega\tau_{\parallel} \leqslant 1$. However, in the high-frequency region $\omega\tau_{\parallel} \gg 1$ the attenuation is very sensitive to the details of the spectrum for large l, and convergence to 1% at the highest frequencies requires $L \simeq 120$. Since Eq. (13) for $1/\tau_l$ for large l is not much better than a conjecture, our results for the attenuation in the region $\omega\tau_{\parallel} \gg 1$ should then be considered qualitative.

IV. DISCUSSION

The second-sound dispersion relation in Fig. 2 has the following features:

(a) The low-frequency hydrodynamic regime, in which the second-sound velocity is $\langle v_q\rangle/\sqrt{3}$, is defined by $\omega\tau_2 \ll 1$; τ_2 is the viscosity lifetime.[2,9]

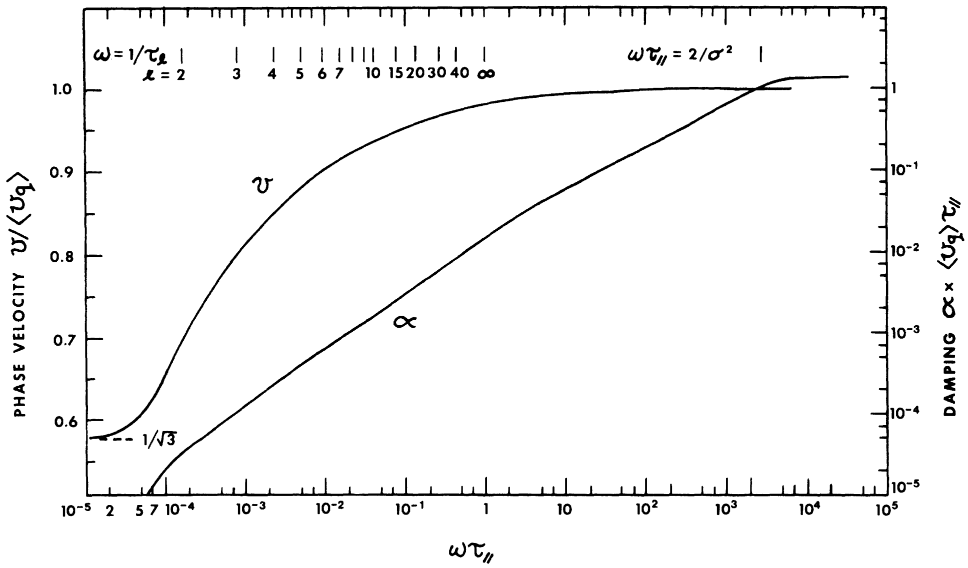

FIG. 2. Velocity and damping of second sound at 0.25°K. The phase velocity v and inverse attenuation length α are in units of thermal phonon group velocity and thermal phonon inverse free path.

The hydrodynamic velocity is weakly temperature dependent, due to thermal averaging of the anomalous, weak q dependence of the group velocity $v_q = d\omega_q/dq$. At 0.25°K, evaluation of (4) with (12) gives $\langle v_q \rangle = 1.015c_0$.

(b) As ω increases through a wide dispersive region $1/\tau_2 \lesssim \omega \lesssim 1/\tau_\parallel$, the slower wide-angle scattering processes become unable to couple phonons quickly enough, and the surplus phonons carrying the disturbance from equilibrium gradually narrow down to $\vec{q}$'s in a small cone along the second-sound wave vector $\vec{k}$. With this increasing collinearity of the surplus phonons, the velocity of the disturbance increases toward the phonon velocity $\langle v_q \rangle$. At 0.25°K, the dispersive region extends over about four orders of magnitude in frequency.

(c) For $\omega\tau_\parallel \sim 1$ collective second-sound propagates with velocity within a few percent of the ballistic thermal phonon velocity $\langle v_q \rangle$, but unlike ballistic phonons the damping is still weak enough to propagate second sound over macroscopic distances. Equation (14) gives $\tau_\parallel = 5.79\times 10^{-8}$ sec at 0.25°K, corresponding to a ballistic phonon mean-free-path 0.014 mm. In contrast, for $\omega\tau_\parallel = 1$, $\omega = 17$ MHz, the second-sound attenuation length from Fig. 2 is considerably longer: 0.94 mm.

(d) Collective second sound propagates far into the regime $\omega\tau_\parallel \gg 1$, with weaker damping than ballistic phonons. This result may be qualitatively understood as follows. At high frequencies, only small-angle scattering is important, and the collective disturbance consists of wavefronts of coupled surplus phonons with $\vec{q}$'s in a narrow cone of angular width σ about $\vec{k}$; σ is the mean scattering angle for small-angle scattering. For collective propagation the surplus phonons must ride together on a wavefront for a long enough time $\tau_\parallel$ for them to couple to each other. Owing to the finite angular width of the cone of surplus phonons, those on the edges and those down the middle of the cone have slightly different group velocity components along $\vec{k}$ and traverse different distances along the direction of propagation of the wave in time $\tau_\parallel$. For collective propagation this distance difference must then be small compared to the second sound wavelength, or

$$\langle v_q \rangle \tau_\parallel - \langle v_q \rangle \cos\sigma\, \tau_\parallel \lesssim 1/k', \quad k' = \mathrm{Re}\,k.$$

With $\cos\sigma \cong 1 - \frac{1}{2}\sigma^2$ this gives $k'\langle v_q \rangle \tau_\parallel \lesssim 2/\sigma^2$, or, since to lowest order in σ the second-sound phase velocity $\omega/k' = \langle v_q \rangle$,

$$\omega\tau_\parallel \lesssim 2/\sigma^2 .$$

With $\sigma = 0.0269$ as in (13) above, this gives $\omega\tau_\parallel \lesssim 2800$ as a criterion for the existence of collective second sound, in agreement with the value $\omega\tau_\parallel \cong 2749$ at which the second-sound zero in the z plane disappears into the branch cut, signaling the breakup of the collective mode into ballistic phonons.

We stress again that the dispersion relation of Fig. 2 probes the phonon frequency anomaly through a *spectrum* of relaxation rates $1/\tau_2$, $1/\tau_3, \ldots, 1/\tau_\parallel$. Narayanamurti, Dynes, and Andres[12] in an analysis of their experimental work on heat pulses examine a model dispersion relation with all phonon-phonon scattering lumped into a single rate $1/\tau_{pp}$. That model predicts a second-sound velocity approaching $c_0/\sqrt{3}$ for $\omega\tau_{pp} \lesssim \frac{1}{10}$ and c_0 for $\omega\tau_{pp} \gtrsim 10$, i.e., a dispersion extending over only about two orders of magnitude in frequency, in contrast to the behavior of Fig. 2. The frequency dispersion was not directly measured; rather Narayanamurti *et al.* measured the temperature dependence of the velocity of heat pulses at fixed pulse central frequency, and determine $\tau_{pp} \propto T^{-3}$. This temperature dependence is weaker than that of both τ_2 and $\tau_{\parallel}$,[13] and is we believe an artificiality resulting from use of a single phonon scattering rate. The present calculation suggests a temperature dependence of pulse velocity extending from the temperature where $\omega\tau_2(T) \approx 1$ down to that where $\omega\tau_{\parallel}(T) \approx 1$. The very different magnitudes of $\tau_2(T)$ and $\tau_{\parallel}(T)$ make this temperature range wide, and in order to duplicate this wide interval by ranging a *single* parameter $\omega\tau_{pp}$ from $\sim\frac{1}{10}$ to 10, τ_{pp} needs to vary artificially slowly.

Two problems remain to be solved before a real comparison with heat pulse experiments is possible. First, a theory of the temperature dependence of the wide-angle scattering rates $1/\tau_l$ is lacking; approximations have been given only for small l.[9] Second, in this work a single collective mode of the phonon Boltzmann equation has been examined, while a real heat pulse in superfluid ^{4}He might be expected to couple to all collective modes plus ballistic modes. This aspect has been formulated as a boundary-value problem and discussed by Maris[4]; in the present context it involves study of the full response function $1/\Lambda(k,\omega)$, rather than a single one of its poles as above.

APPENDIX

Previously it has been shown[9] that the linearized three-phonon collision integral $\tilde{C}(\psi)$ in the phonon Boltzmann equation (2) has the formal spectral representation

$$\tilde{C}(\psi) = -\Gamma(q)\left(\psi(\vec{q}) - \sum_{nlm}(1-\lambda_{nl})\phi_{nlm}(\vec{q})(\Gamma\phi_{nlm}, \psi)\right), \tag{A1}$$

where $\Gamma(q)$ is the wave-vector-dependent single phonon small-angle scattering rate, and $\phi_{nlm}(\vec{q})$ and $1-\lambda_{nl}$ are eigenfunctions and eigenvalues of the completely continuous part of the collision integral,[9] with normalization

$$(\Gamma\phi_{nlm}, \phi_{n'l'm}) \equiv \int d\vec{q}\,\Gamma(q)\phi^*_{nlm}(\vec{q})\phi_{n'l'm'}(\vec{q}) = \delta_{nn'}\delta_{ll'}\delta_{mm'}\,. \tag{A2}$$

The angular parts of $\phi_{nlm}(\vec{q})$ are spherical harmonics

$$\phi_{nlm}(\vec{q}) = R_{nl}(q)Y_l^m(\hat{\Omega})\,,$$

and the index n denotes the number of radial nodes.

We seek those collective modes of the phonons consisting of disturbances from equilibrium whose radial distributions of surplus phonons are conserved, or nearly conserved, in phonon-phonon collisions. Such disturbances comprise the set $\phi_{0lm}(\vec{q})$ of eigenfunctions with no radial nodes.[4,9] Thus:

(i) We assume that $\psi(\vec{q})$ has projections onto only the $n=0$ functions ϕ_{0lm}, and restrict the sum over n in (A1) to $n=0$.

(ii) Following earlier work,[9] we write $\lambda_{0l} = \tau_{\parallel}/\tau_l$, where

$$1/\tau_{\parallel} = \frac{\int d\vec{q}\,\Gamma(q)q^2/\sinh^2(\frac{1}{2}\beta\hbar c_0 q)}{\int d\vec{q}\,q^2/\sinh^2(\frac{1}{2}\beta\hbar c_0 q)}$$

is a thermal average of the scattering rate $\Gamma(q)$. The set $1/\tau_l$ comprises the spectrum of wide-angle relaxation rates.[9]

(iii) We assume that $\Gamma(q)$ may be replaced in (A1) by its thermal average $1/\tau_{\parallel}$.

(iv) We take the radial part $R_{0l}(q)$ of $\phi_{0lm}(\vec{q})$ only to zeroth order in the phonon dispersion anomaly[9]; $R_{0l}(q) \cong R(q)$ as given following Eq. (3) above. Under these assumptions, the Boltzmann equation (2) may be reduced to (3) by equating projections of both sides onto the subspace of $n=0$ eigenfunctions.

[1]See, for instance, H. J. Maris and W. E. Massey, Phys. Rev. Lett. 25, 220 (1970); N. A. Lockerbie, A. F. G. Wyatt, and R. A. Sherlock, Solid State Commun. 15, 567 (1974).

[2]H. J. Maris, Phys. Rev. A 8, 1980 (1973).

[3]H. J. Maris, Phys. Rev. A 8, 2629 (1973).

[4]H. J. Maris, Phys. Rev. A 9, 1412 (1974).

[5]R. K. Wehner, Phys. Rev. A 9, 2625 (1974).

[6]W. R. Junker and C. Elbaum, Phys. Rev. Lett. 34, 186 (1975), and references therein.

[7]W. M. Saslow, Phys. Rev. A 5, 1491 (1972).

[8]J. S. Brooks and R. J. Donnelly, Phys. Rev. A 9, 1444 (1974).

[9]D. Benin, Phys. Rev. B 11, 145 (1975).

[10] G. E. Uhlenbeck and G. W. Ford, *Lectures in Statistical Mechanics* (American Mathematical Society, Providence, R. I., 1963), Chap. 5; J. R. Mika, Nucl. Sci. Eng. 11, 415 (1961).

[11] K. M. Case and R. D. Hazeltine, J. Math. Phys. 12, 1970 (1971).

[12] V. Narayanamurti, R. C. Dynes, and K. Andres, Phys. Rev. B 11, 2500 (1975).

[13] $\tau_{\parallel} \propto T^{-5}$, and the more complicated temperature dependence of τ_2 is stronger than T^{-4} throughout the phonon region $T \lesssim 0.6\,°K$; see Refs. 2 and 9.