Ising spin system on a Cayley tree: Correlation decomposition and phase transition*

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For N spins, $\sigma_i = \pm 1$, $i \in \{1,2,...,N\}$, interacting via nearest-neighbor ferromagnetic Ising interaction $-J\sigma_i\sigma_j$ on a Cayley tree with branching number B, it is shown that any even-spin correlation function $\langle \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{2K}}\rangle$ decomposes into a product $\langle\sigma_{j_1}\sigma_{j_2}\rangle\cdots\langle\sigma_{j_{2K-1}}\sigma_{j_{2K}}\rangle$ of two-spin correlation functions $\langle\sigma_{j_p}\sigma_{j_{p+1}}\rangle = [\tanh(J/k_BT)]^{d(j_p,j_{p+1})}$, where $d(j_p,j_{p+1})$ is the number of bonds on the unique self-avoiding path connecting σ_{j_p} and $\sigma_{j_{p+1}}$. This generalizes to $B>1$ the known decomposition for an Ising chain (a Cayley tree having $B=1$). The decomposition theorem leads to upper and lower bounds for the zero-field susceptibility, and these bounds become infinite for temperatures $T \leq T_2$ and are finite for $T > T_2$ where $B\tanh^2(J/k_BT_2)=1$. An upper bound is also given for the fourth cumulant of the magnetization. That bound becomes (negatively) infinite for $T<T_4$ where $B^3\tanh^4(J/k_BT_4)=1$. The above exact considerations are consistent with recent results of other authors and provide elementary insight regarding the cumulant divergences and long-range correlation of subsets of surface spins.

I. INTRODUCTION

A model of N spins, $\sigma_i=\pm1$, having nearest-neighbor, ferromagnetic Ising interaction $-J\sigma_i\sigma_j$ on a Cayley tree with branching number B has been a subject of some recent interest for several reasons: (a) The model displays[1–3] a phase transition (divergent susceptibility) without[1–4] a spontaneous magnetization. (b) Previous approximations which neglected surface contributions are invalid here, since careful treatment of the surface is essential for this model. (c) The equilibrium statistical mechanics can be formulated[1–4] in terms of nonlinear recursion relations, the study of which is itself of current importance in mathematical physics and biology.

The purpose of this paper is first to prove for the above model that any even-spin correlation function $\langle\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{2K}}\rangle$ decomposes into a product of two-spin correlation functions $\langle\sigma_{j_p}\sigma_{j_{p+1}}\rangle=[\tanh(J/k_BT)]^{d(j_p,j_{p+1})}$, where $d(j_p,j_{p+1})$ is the number of bonds on the unique self-avoiding path connecting σ_{j_p} and $\sigma_{j_{p+1}}$, and T is the absolute temperature. This generalizes to $B>1$ the known[5] decomposition for an Ising chain (a Cayley tree with $B=1$), and also generalizes to $K>1$ Mukamel's result[6] specialized to zero field.

Having established the decomposition theorem, we use it to obtain upper and lower bounds for the zero-field susceptibility. The bounds are shown to become infinite in the thermodynamic limit for $T\leqq T_2$ and are finite for $T>T_2$, where[1–3] $B\tanh^2(J/k_BT_2)=1$, thus providing a simple proof of the divergence of the zero-field susceptibility for $T\leqq T_2$.

A bound is also given for the fourth cumulant of magnetization, and the bound shows that the cumulant becomes (negatively) infinite for $T<T_4$, where $B^3\tanh^4(J/k_BT_4)=1$. But $T_4>T_2$; therefore the fourth cumulant diverges *before* the susceptibility as T is lowered from infinity.

These divergences of the second and fourth cumulants are consistent with the general asymptotic free-energy analysis by Müller-Hartmann and Zittartz,[3] who found that the 2^l cumulant diverges at T_{2^l} where

$$B\tanh(J/kT_{2^l})=B^{1/2^l},\quad l=1,2,\ldots.$$

The construction of the bounds given here shows explicitly that correlations among a *subset* of surface spins produce the divergences for $l=1,2$.

II. DECOMPOSITION THEOREMS

The main results are obtained here via a previously stated theorem[7] regarding the correlation $\langle\sigma_r\sigma_s\rangle$ between two spins σ_r, σ_s in system A which is linked to another system (call it B) by a single ferromagnetic bond J_{AB}.

Let system A consist of q spins $\sigma_1, \sigma_2, \ldots, \sigma_q$, and let the Hamiltonian $\mathcal{H}_A$ for system A be of the most general[8] form, possibly containing one-spin, two-spin, ..., q-spin interactions. Let system B contain $N-q$ spins $\sigma_{q+1}, \sigma_{q+2}, \ldots, \sigma_N$, and restrict the Hamiltonian $\mathcal{H}_B$ for system B to even-spin interactions; i.e., $\mathcal{H}_B$ may contain two-spin, four-spin, ... interactions only. The Hamiltonian for the linked system is

$$\mathcal{H}=\mathcal{H}_A+\mathcal{H}_B+\mathcal{H}_{AB}, \tag{1}$$

where

$$\mathcal{H}_{AB}=J_{AB}\sigma_q\sigma_{q+1}. \tag{2}$$

The theorem is

Theorem I: *The canonical ensemble average of any function of the spins in system A only is independent of all interaction parameters contained in* $\mathcal{H}_B$, $\mathcal{H}_{AB}$.

Proof: Introduce new spin variables $t_i=\pm1$ for $i=1,2,\ldots,N$ and write

$$\sigma_j = \begin{cases} t_j & \text{for } j=1,2,\ldots,q, \\ t_q t_j & \text{for } j=q+1,\ q+2,\ldots,N; \end{cases} \quad (3)$$

then the inverse relations are

$$t_j = \begin{cases} \sigma_j & \text{for } j=1,2,\ldots,q, \\ \sigma_q \sigma_j & \text{for } j=q+1,\ q+2,\ldots,N. \end{cases} \quad (4)$$

That transformation leaves $\mathcal{H}_A$ and $\mathcal{H}_B$ unchanged except to replace the σ variables by t variables, whereas $\mathcal{H}_{AB}$ is transformed into $J_{AB}t_{q+1}$. Denote the transformed Hamiltonians by $\overline{\mathcal{H}}_A$ and $\overline{\mathcal{H}}_B$ and let $f(\sigma_1, \ldots, \sigma_q)$ denote an arbitrary function of the spins in system A only. Then the canonical ensemble average

$$\langle f\rangle = Z^{-1} \sum_{\sigma_1,\ldots,\sigma_N} f(\sigma_1,\ldots,\sigma_q) \times \exp[-\beta(\mathcal{H}_A + \mathcal{H}_B + \mathcal{H}_{AB})]$$
$$= \left(\sum_{t_1,\ldots,t_q} f(t_1,\ldots,t_q)\exp(-\beta\overline{\mathcal{H}}_A)\right) \bigg/ \sum_{t_1',\ldots,t_q'} \exp(-\beta\overline{\mathcal{H}}_A), \quad (5)$$

which proves the theorem.

A simple application of the theorem relates to the special case in which system B has pair interactions only:

$$\mathcal{H}_B = -\sum_{q+1\leq k<l\leq N} J_{kl}\sigma_k\sigma_l . \quad (6)$$

The theorem then asserts for $1 \leq r < s \leq q$ that $\langle\sigma_r\sigma_s\rangle$ is independent of J_{AB} and J_{kl} for $q+1 \leq k < l \leq N$. In other words, for the purpose of calculating correlations among spins in system A only, one may "disconnect" system B by putting $J_{AB}=0$.

Remark: Neither the theorem nor the example should lead one to the false conclusion that the spins in system A are statistically independent of the spins in system B. For example, $\langle\sigma_q\sigma_{q+1}\rangle \neq \langle\sigma_q\rangle\langle\sigma_{q+1}\rangle = 0$.

With Theorem I in mind consider the Cayley tree shown in Fig. 1. For definiteness B has been taken equal to 3, and the tree shown has "branched" through $M=4$ generations. The number of spins (vertices) in the zeroth generation is $B^0=1$; in the second generation B new spins are added, etc. The total number of spins on the tree is

$$N = \sum_{0\leq l\leq M} B^l = \frac{B^{M+1}-1}{B-1}, \quad (7)$$

and the Hamiltonian for the system is taken as

$$\mathcal{H} = -J\sum_{\langle i,j\rangle} \sigma_i\sigma_j, \quad J>0, \quad (8)$$

where $\langle i,j\rangle$ refers to nearest-neighbor spins.

Now consider any two spins on the tree. Again,

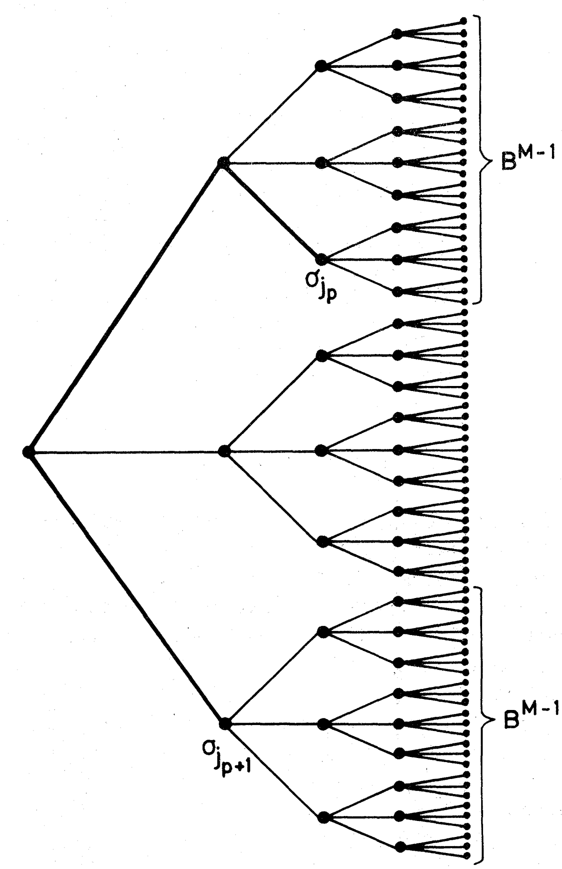

FIG. 1. M (=4)-generation Cayley tree with branching number B (=3).

for definiteness, two spins σ_{j_p} and $\sigma_{j_{p+1}}$ have been labelled in Fig. 1, and the unique self-avoiding path[9] connecting them is shown with a broadened line. The theorem then enables us to partition the tree into two systems A and B. The B system contains all spins attached to the path connecting σ_{j_p} and $\sigma_{j_{p+1}}$ by a particular bond. For example, the upper cluster of 13 spins would constitute a possible choice for the B system. The A system contains all other spins on the tree and clearly includes σ_{j_p}, $\sigma_{j_{p+1}}$, and all spins on the path connecting σ_{j_p} and $\sigma_{j_{p+1}}$. Theorem I says that in calculating $\langle\sigma_{j_p}\sigma_{j_{p+1}}\rangle$ we may prune the tree by disconnecting the B system. The resulting tree may then be similarly partitioned; Theorem I may be applied again and the pruning process continued until one is left with a one-dimensional chain of spins with σ_{j_p} and $\sigma_{j_{p+1}}$ at opposite ends of the chain. For such a chain it is well known[5] and easily verified that

$$\langle\sigma_{j_p}\sigma_{j_{p+1}}\rangle = [\tanh(J/k_BT)]^{d(j_p,j_{p+1})}, \quad (9)$$

where $d(j_p, j_{p+1})$ is the number of bonds on the path connecting σ_{j_p} and $\sigma_{j_{p+1}}$. The above argument obtains for *any* spins on the tree, and Eq. (9) gives Mukamel's result[6] specialized to zero field.

It is worth a bit more effort to generalize Eq. (9) to any even-spin correlation function $\langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{2K}} \rangle$. Odd-spin correlation functions are zero owing to the symmetry of the Hamiltonian. Since the generalization of Eq. (9) is not required for bounding the susceptibility, we simply state the generalization as a theorem here and defer the proof to Appendix A.

Theorem II: *For any subset of spins* σ_{i_1}, $\sigma_{i_2}, \ldots, \sigma_{i_{2K}}$, $K=1, 2, \ldots$, *on a Cayley tree the canonical ensemble average*

$$\langle \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_{2K}} \rangle = \langle \sigma_{j_1}\sigma_{j_2}\rangle\langle \sigma_{j_3}\sigma_{j_4}\rangle \cdots \langle \sigma_{j_{2K-1}}\sigma_{j_{2K}}\rangle, \tag{10}$$

where $\langle \sigma_{j_p} \sigma_{j_{p+1}} \rangle$ *is given by* Eq. (9), *and the Hamiltonian is given by* Eq. (8).

Remark: The proof given in Appendix A shows that although the decomposition indicated in Eq. (10) is not unique, any of the possible decompositions leads to the same value for $\langle \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_{2K}} \rangle$. Since a decomposition of the form of Eq. (10) is well known[5] for an Ising chain (a Cayley tree having $B=1$), Theorem II provides a generalization to $B>1$.

We now apply these theorems, particularly Eq. (9), to obtain upper and lower bounds for the susceptibility, and will subsequently use the more general result, Eq. (10), to bound the fourth cumulant of the magnetization.

III. SUSCEPTIBILITY LOWER BOUND AND PHASE TRANSITION

Equation (9) leads directly to upper and lower bounds for the zero-field isothermal susceptibility χ_M [with N given by Eq. (7)]

$$\beta^{-1}\chi_M = N^{-1} \sum_{1 \leq i \leq j \leq N} \langle \sigma_i \sigma_j \rangle, \tag{11}$$

$$\beta^{-1}\chi_M = 1 + 2N^{-1} \sum_{1 \leq i < j \leq N} \langle \sigma_i \sigma_j \rangle, \tag{12}$$

$$\beta^{-1}\chi_M = 1 + 2N^{-1} \sum_{1 \leq i < j \leq N} [\tanh(\beta J)]^{d(i,j)}, \tag{13}$$

where the sum is over all pairs of vertices and $d(i,j)$ is the number of bonds in the unique, self-avoiding path between vertex i and vertex j.

Refer to Fig. 1 and consider the upper and lower clusters of *surface* spins, each cluster containing B^{M-1} spins. The number of bonds in the path from any spin in the upper cluster to any spin in the lower cluster is $2M$. According to Eq. (9), the correlation $\langle \sigma_i \sigma_j \rangle$ between a spin σ_i in, say, the upper cluster and a spin σ_j in the lower cluster is

$$\langle \sigma_i \sigma_j \rangle = u^{2M}, \tag{14}$$

where $u = \tanh \beta J$. Taking the $(B^{M-1})^2$ pairs formed from a spin in the upper cluster and a spin in the lower cluster, one has only a partial sum of terms in the susceptibility and hence a lower bound

$$\begin{aligned}\beta^{-1}\chi_M &\geq [(B-1)/(B^{M+1}-1)]\, B^{2M-2} u^{2M} \\ &\geq (B-1)\, B^{-3} (u^2 B)^M \\ &\to \infty \quad \text{for } u^2 B > 1,\ M \to \infty. \end{aligned} \tag{15}$$

Thus

$$\chi \equiv \lim_{M\to\infty} \chi_M = \infty \tag{16}$$

for

$$B \tanh^2(J/k_B T) > B \tanh^2(J/k_B T_2) \equiv 1. \tag{17}$$

This is intended as a proof of the zero-field susceptibility divergence for $T < T_2$. The proof is based on a simple theorem and avoids an analysis of nonlinear coupled recursion relations.[1–4] The construction of the bound shows how a subset of surface spins produces the susceptibility divergence.

Before obtaining an upper bound on the susceptibility for $T > T_2$, it is interesting to qualitatively consider the connection between long-range correlations and the established divergence of χ. Notice that there are no *infinite*-range correlations for any $T>0$ since $0 < \tanh(J/k_B T) < 1$, and from Eq. (9),

$$m^* \equiv \left(\lim_{d(i,j)\to\infty} \lim_{N\to\infty} \langle \sigma_i \sigma_j \rangle \right)^{1/2} = 0,$$

consistent with the absence[1–4] of spontaneous magnetization. Nevertheless, one can study the long-range correlations. For that purpose we imbed the Cayley tree in a regular two-dimensional square lattice so that the distance b between neighboring vertices in any column (labelled by l in Fig. 1) is the same as the distance between neighboring columns. Such an imbedding obviously leaves many lattice sites unoccupied. Consider two surface spins σ_p, σ_m (in column $l=M$), one at the top and one at the bottom. The distance r between those spins is $(B^M - 1)\, b$; whereas the correlation $\langle \sigma_p \sigma_m \rangle$ by Eq. (9) is

$$\begin{aligned}\langle \sigma_p \sigma_m \rangle &= u^{2M} \\ &= r^{-\alpha},\end{aligned}$$

where

$$\alpha \equiv -2M \ln u / \ln[(B^M - 1)\, b].$$

If we take $b=1$ as our length scale and demand that

$$r^{1-\alpha} \to 0 \quad \text{for } r \to \infty$$

for integral convergence in two-dimensions, then

$$1 - \alpha < 0,$$

which, for large M, implies that

$$-2M\ln u/(M\ln B)>1,$$

or, equivalently,

$$u^2B<1.$$

Conversely, the condition $u^2B>1$, which by Eq. (15) gives a divergent χ, also implies long-range correlation within the framework of this qualitative argument.

IV. SUSCEPTIBILITY UPPER BOUND

An alternative form for the zero-field susceptibility expression, Eq. (13), is obtained by writing $u=\tanh\beta J$ and considering the number C_n of two-vertex connecting paths of bond length n. Then

$$\beta^{-1}\chi_M=1+2N^{-1}\sum_{1\leq n\leq 2M} C_n u^n. \tag{18}$$

An explicit expression for C_n has been given by graph theorists[10] and is contained in Appendix B, which also includes a derivation of the following bound:

$$C_n\leq GB^{M+n/2}, \quad 0\leq n\leq 2M. \tag{19}$$

The quantity G depends only on B and is positive and finite for $B<1$.

Substitution of Eq. (19) into Eq. (18) gives

$$\beta^{-1}\chi_M\leq 1+\frac{2(B-1)}{B^{M+1}-1}\;\frac{(u\sqrt{B})^{2M+1}-u\sqrt{B}}{(u\sqrt{B})-1}\,GB^M, \tag{20}$$

where the right-hand side

$$\begin{aligned}&\to\infty \quad \text{for } u^2B\geq 1,\quad M\to\infty,\\ &<\infty \quad \text{for } u^2B<1.\end{aligned} \tag{21}$$

Thus, Eqs. (15) and (20) establish that χ is bounded for $T>T_2$ and unbounded for $T\leq T_2$, with T_2 given by Eq. (17). This is consistent with the results given by Matsuda,[2] von Heimburg and Thomas,[1] and Müller-Hartmann and Zittartz.[3]

V. FOURTH CUMULANT OF MAGNETIZATION: UPPER BOUND

The zero-field isothermal susceptibility expressed by Eq. (11) is essentially the second derivative of the free energy per spin with respect to the magnetic field In zero field the odd derivatives are zero, so that the next nonzero derivative is the fourth derivative D_4

$$\beta^{-3}D_4=N^{-1}\sum_{i_1,\ldots,i_4} C_4(i_1,\ i_2,\ i_3,\ i_4), \tag{22}$$

where

$$\begin{aligned}C_4(i_1,\ i_2,\ i_3,\ i_4)=&\langle\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}\rangle-\langle\sigma_{i_1}\sigma_{i_2}\rangle\\ &\times\langle\sigma_{i_3}\sigma_{i_4}\rangle-\langle\sigma_{i_1}\sigma_{i_3}\rangle\langle\sigma_{i_2}\sigma_{i_4}\rangle\\ &-\langle\sigma_{i_2}\sigma_{i_3}\rangle\langle\sigma_{i_1}\sigma_{i_4}\rangle.\end{aligned} \tag{23}$$

The decomposition theorem, Eq. (10), shows that $\langle\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}\rangle$ will always be cancelled by one of the three terms involving pair correlations; thus, $C_4(i_1,\ i_2,\ i_3,\ i_4)\leq 0$ as required by a more general inequality given by Griffiths, Hurst, and Sherman[11]

To obtain a lower bound for D_4, consider any pair σ_1', σ_2' of spins in the upper surface cluster. There are $\frac{1}{2}B^{M-1}(B^{M-1}-1)$ such pairs. Similarly there are $\frac{1}{2}B^{M-1}(B^{M-1}-1)$ pairs of spins $\sigma_{1''}, \sigma_{2''}$ on the lower surface cluster. The cumulant associated with the four spins is

$$\begin{aligned}C_4(1',2',1'',2'')=&\langle\sigma_{1'}\sigma_{2'}\sigma_{1''}\sigma_{2''}\rangle\\ &-\langle\sigma_{1'}\sigma_{2'}\rangle\langle\sigma_{1''}\sigma_{2''}\rangle\\ &-\langle\sigma_{1'}\sigma_{1''}\rangle\langle\sigma_{2'}\sigma_{2''}\rangle\\ &-\langle\sigma_{1'}\sigma_{2''}\rangle\langle\sigma_{1''}\sigma_{2'}\rangle.\end{aligned} \tag{24}$$

But

$$\langle\sigma_{1'}\sigma_{2'}\sigma_{1''}\sigma_{2''}\rangle=\langle\sigma_{1'}\sigma_{2'}\rangle\langle\sigma_{1''}\sigma_{2''}\rangle \tag{25}$$

and

$$\langle\sigma_{1'}\sigma_{1''}\rangle=\langle\sigma_{2'}\sigma_{2''}\rangle=\langle\sigma_{1'}\sigma_{2''}\rangle=\langle\sigma_{1''}\sigma_{2'}\rangle=u^{2M}, \tag{26}$$

where $u=\tanh\beta J$, therefore

$$C_4(1',2',1'',2'')=-2u^{4M} \tag{27}$$

and

$$\beta^{-3}D_4\left(-\frac{B-1}{B^{M+1}-1}\right)\left(\frac{B^{M-1}(B^{M-1}-1)}{2}\right)^2 2u^{4M}\to-\infty \tag{28}$$

for

$$B^3u^4>1,\quad M\to\infty. \tag{29}$$

Define T_4 by

$$k_B T_4/J\equiv[\tanh^{-1}(B^{-3/4})]^{-1}, \tag{30}$$

and recall from Eq. (17) that

$$k_B T_2/J\equiv[\tanh^{-1}(B^{-1/2})]^{-1}. \tag{31}$$

The upper bound shows that D_4 diverges (to $-\infty$) as $M\to\infty$ for $T<T_4$. But $T_2<T_4$; therefore, as the temperature is lowered from infinity, D_4 diverges *before* the susceptibility diverges.

The detailed asymptotic free-energy analysis of Müller-Hartmann and Zittartz[3] led them to describe this as a "continuous phase transition," since there is a line of transitions for T in the interval $(0, T_\infty)$.

ACKNOWLEDGMENTS

I am pleased to thank Professor R. J. Elliott and Dr. J. Groeneveld for beneficial conversations, and I am greatly indebted to L. Matthews and H. Jenkins for contributing the graph-counting results contained in Appendix B.

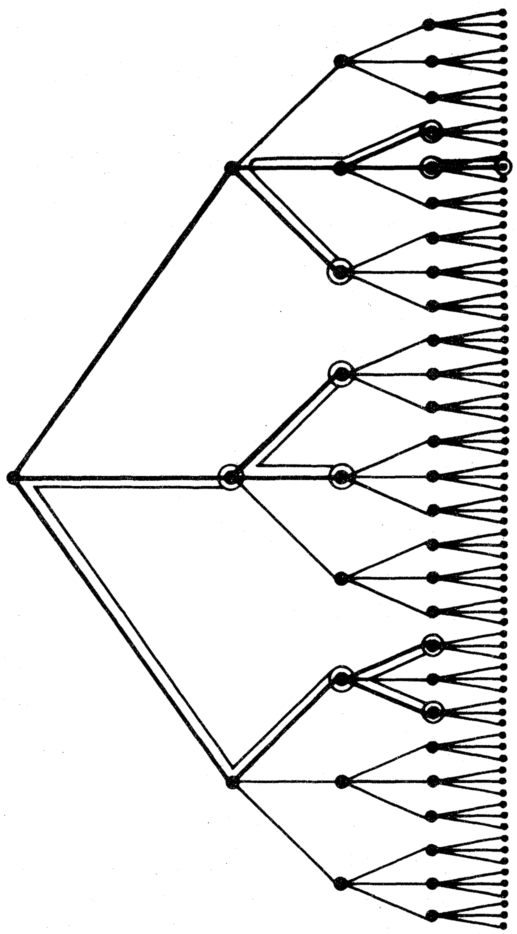

FIG. 2. Ten-spin correlation decomposition on a Cayley tree.

APPENDIX A

To establish Theorem II, consider any subset R consisting of r spins on the tree. Take, e.g., the spins on the circled vertices in Fig. 2. Let r be an even positive integer. Form a subtree by connecting all pairs of spins in R by both paths as indicated, e.g., by the broadened lines in Fig. 2. The subtree is itself a Cayley tree and consists of a set Q of q spins where $R \subset Q$.

On the subtree, partition the set R into *disjoint* pairs by drawing nonoverlapping (edge-disjoint) bond paths between spins as indicated. Such a partitioning is always possible. Denote two spins in a connected pair by σ_{j_p}, $\sigma_{j_{p+1}}$ so that

$$R = (\sigma_{j_1}, \sigma_{j_2}; \sigma_{j_3}, \sigma_{j_4}; \ldots ; \sigma_{j_{2K-1}}\sigma_{2K}),$$

where $2K = r$.

According to Theorem I, the canonical ensemble average of any function of the spins in Q may be expressed as an equivalent average with respect to the Hamiltonian $\mathcal{H}_Q$ of the subtree alone. Thus

$$\langle \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{2K}}\rangle = \langle \sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_p}\sigma_{j_{p+1}}\cdots\sigma_{j_{2K-1}}\sigma_{j_{2K}}\rangle_{\mathcal{H}_Q}. \quad \text{(A1)}$$

Now the path connecting the pair σ_{j_p}, $\sigma_{j_{p+1}}$ contains $d(j_p, j_{p+1})$ bonds and therefore $d(j_p, j_{p+1}) - 1$ internal vertices (in addition to the terminal vertices where σ_{j_p} and $\sigma_{j_{p+1}}$ are located). Label the internal vertices consecutively as they appear along the path from σ_{j_p} to $\sigma_{j_{p+1}}$ so that if $d(j_p, j_{p+1}) - 1 = 3$, say, then the internal vertices would be labelled $j_{p,1}$, $j_{p,2}$, and $j_{p,3}$;

$$\sigma_{j_p}\sigma_{j_{p+1}} = (\sigma_{j_p}\sigma_{j_{p,1}})(\sigma_{j_{p,1}}\sigma_{j_{p,2}})(\sigma_{j_{p,2}}\sigma_{j_{p,3}})(\sigma_{j_{p,3}}\sigma_{j_{p+1}}), \quad \text{(A2)}$$

which is associated with the product

$$\exp(\beta J\sigma_{j_p}\sigma_{j_{p,1}})\cdots\exp(\beta J\sigma_{j_{p,3}}\sigma_{j_{p+1}}) \quad \text{(A3)}$$

in the ensemble average where $\beta J \equiv J/k_B T$. Each term in Eq. (A2) provides a contribution $\tanh\beta J$; consequently, the product of L disjoint pairs gives

$$\begin{aligned}\langle \sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_p}\sigma_{j_{p+1}}\cdots\sigma_{j_{2L-1}}\sigma_{j_{2L}}\rangle_{\mathcal{H}_Q} &= \langle\sigma_{j_1}\sigma_{j_2}\rangle_{\mathcal{H}_Q}\cdots\langle\sigma_{j_p}\sigma_{j_{p+1}}\rangle_{\mathcal{H}_Q}\cdots\langle\sigma_{j_{2L-1}}\sigma_{j_{2L}}\rangle_{\mathcal{H}_Q}\\ &= \langle\sigma_{j_1}\sigma_{j_2}\rangle\cdots\langle\sigma_{j_p}\sigma_{j_{p+1}}\rangle\cdots\langle\sigma_{j_{2L-1}}\sigma_{j_{2L}}\rangle, \end{aligned} \quad \text{(A4)}$$

where $\langle\sigma_{j_p}\sigma_{j_{p+1}}\rangle$ is given by Eq. (9).

The above proof shows that the subtree, in effect, decomposes into a collection of independent Ising chains with spins in R at the ends of the chains. Although the decomposition is not unique, it is seen that any decomposition gives the same contribution, i.e , $\tanh\beta J$ from each member of a fixed set of bonds. The product, in spite of different factorizations, will therefore be invariant.

A proof equivalent to the above can be obtained by introducing new spin variables to replace each product of nearest-neighboring spins. That transformation was used by Eggarter[3] to calculate the partition function and has also been used, e.g., by Dobson[12] for one-dimensional systems.

APPENDIX B

In this appendix we describe a derivation kindly provide by Matthews and Jenkins[10] of the number C_n of two-vertex connecting paths of bond length $n > 0$ in an M-generation Cayley tree (see Fig. 1).

A path is of the form shown in Fig. 3, where v is the vertex nearest the $l = 0$ vertex, n is the bond length of the path, and r is length of the shorter branch at v. Thus, $n \geqq 2r$, and such a path exists for any v in the $0, 1, 2, \ldots, (M - n + r)$th generation. Such vertices will be referred to as "feasible" vertices, of which there are

$$\sum_{0 \leq j \leq M-n+r} B^j = (B^{M-n+r+1} - 1)/(B - 1) \quad \text{(B1)}$$

for $M - n + r \geqq 0$, $r \geqq 0$.

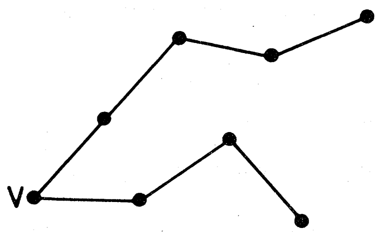

FIG. 3. Typical two-vertex connecting path of bond length n (=7) with shorter branch of length r (=3).

For each feasible vertex the number of paths is

$$B^n \quad \text{for } r=0, \tag{B2}$$

$$\tfrac{1}{2}B^r(B-1)\,B^{r-1}=\tfrac{1}{2}B^{2r-1}(B-1) \quad \text{for } n=2r, \tag{B3}$$

$$B^r(B-1)\,B^{n-(r+1)}=(B-1)\,B^{n-1} \quad \text{for } n>2r, \tag{B4}$$

as may be readily verified by considering a path-number matrix with rows labelled $r=0,1,\ldots,$ and columns labelled $n=0,1,\ldots.$

For $M-n+r\geqq 0$ let A_{rn} denote the number of paths of length n with shorter branch r:

$$A_{rn}=\frac{B^{M-n+r+1}-1}{B-1}\begin{cases} B^n, & r=0, \quad \text{(B5)}\\ \tfrac{1}{2}B^{2r-1}(\beta-1), & n=2r>2, \quad \text{(B6)}\\ B^{n-1}(B-1), & 2\leqq 2r\leqq n-1. \quad \text{(B7)}\end{cases}$$

The number C_n which we seek is now obtained from elementary geometric sums:

$$C_n=\begin{cases} \sum_{0\leqq r\leqq n/2}{}' A_{rn}, & n \text{ even}, \quad \text{(B8)}\\ \sum_{0\leqq r\leqq (n-1)/2}{}' A_{rn}, & n \text{ odd}, \quad \text{(B9)}\end{cases}$$

where the prime means that the sums are restricted to $M-n+r\geqq 0$.

To compute the sums consider the following cases:

Case I: n a positive even integer, $M<n\leqq 2M$:

$$C_n=[\tfrac{1}{2}(B^{M-n/2+1}-1)+S_{12}B^{M-n+1}-S_{11}]\,B^{n-1}. \tag{B10}$$

Case II. n a positive odd integer, $M<n\leqq 2M-1$:

$$C_n=(S_{22}B^{M-n+1}-S_{21})\,B^{n-1}. \tag{B11}$$

Case III. n a positive even integer, $n\leqq M$:

$$C_n=[(B^{M-n+1}-1)/(B-1)]\,B^n + \tfrac{1}{2}(B^{M-n/2+1}-1)\,B^{n-1}+(S_{32}\,B^{M-n+1}-S_{31})\,B^{n-1}. \tag{B12}$$

Case IV. n a positive odd integer, $n\leqq M$:

$$C_n=[(B^{M-n+1}-1)/(B-1)]\,B^n + (S_{42}\,B^{M-n+1}-S_{41})\,B^{n-1}. \tag{B13}$$

For Eqs. (B10)–(B13),

$$S_{11}=M-\tfrac{1}{2}n, \tag{B14}$$

$$S_{12}=(B^{n/2}-B^{n-M})/(B-1), \tag{B15}$$

$$S_{21}=M-\tfrac{1}{2}(n-1), \tag{B16}$$

$$S_{22}=(B^{(n+1)/2}-B^{n-M})/(B-1), \tag{B17}$$

$$S_{31}=\tfrac{1}{2}n-1, \tag{B18}$$

$$S_{32}=(B^{n/2}-B)/(B-1), \tag{B19}$$

$$S_{41}=\tfrac{1}{2}(n-1), \tag{B20}$$

$$S_{42}=(B^{(n+1)/2}-B)/(B-1). \tag{B21}$$

Consideration of Matthews and Jenkins's above expressions for C_n leads to the inequality

$$C_n\leqq GB^{M+n/2} \quad \text{for } 0\leqq n\leqq 2M, \tag{B22}$$

where G is a positive quantity depending only on B.

In principle, one can substitute Eqs. (B10)–(B13) into Eq. (18) and compute the indicated summation to arrive at an explicit expression for the susceptibility for a Cayley tree with general B and M. One finds that $u^2B=1$ coincides with the radius of convergence; however, the resulting expressions are somewhat unwieldy and provide little insight beyond what has already been gained via our bounds and the work of von Heimburg and Thomas,[1] Matsuda,[2] and Müller-Hartmann and Zittartz.[3]

A generalization to the Potts and Ashkin-Teller models of ideas in Refs. 1–4 has been given by Y. K. Wang and F. Y. Wu.[13]

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