Field-theoretic techniques and critical dynamics. II. Ginzburg-Landau stochastic models with energy conservation

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The critical dynamics of a stochastic Ginzburg-Landau model of an N-component order parameter coupled to a conserved-energy-density field is studied with the help of field-theoretical techniques introduced in previous work. Our results essentially confirm and refine upon those of Halperin, Hohenberg, and Ma. Scaling laws are derived (whenever they hold). A better knowledge of the domain structure of the (N, d) plane and the corresponding critical exponents is obtained, in particular one additional region is shown to be present. Stability criteria lead to a characterization of the leading corrections to dynamical scaling by extra exponents which, except for one of them, are related to known static exponents.

I. INTRODUCTION

The dynamical behavior of Landau-Ginzburg stochastic systems without conservation law in the critical region has now been studied by several methods. The field-theoretical approach has permitted the confirmation of the results of Halperin, Hohenberg, and Ma[1] (HHM), the derivation to all orders in ϵ of dynamical scaling laws, and the carrying to order ϵ^3 the computation of the exponent z.[2] Without entering into the discussion of a purely microscopic approach to systems possessing slow modes coupled to the order parameter, we now proceed and discuss a Ginzburg-Landau stochastic model where a (conserved) energy variable $E(xt)$ and a complex order parameter $\phi(xt)$ are coupled and, as in HHM, satisfy the semiphenomenological generalized Langevin equations

$$\frac{\partial \phi_\alpha(x,t)}{\partial t} = -\Gamma_0 \frac{\delta \mathcal{H}}{\delta \phi_\alpha} + \eta_\alpha(x,t)\,, \tag{1a}$$

$$\frac{\partial E(x,t)}{\partial t} = \Lambda_0 \nabla^2 \frac{\delta \mathcal{H}}{\delta E} + \zeta(x,t)\,. \tag{1b}$$

Here the component index α takes values from 1 to $\frac{1}{2}N$.[3] The functional $\mathcal{H}$, which gives the weight of the equilibrium distribution (proportional to $e^{-\mathcal{H}}$), is given by

$$\mathcal{H}(\phi, \phi^*, E) = \int d^d x \Big((\nabla^2 + r_0)\vec{\phi}(x)\vec{\phi}^*(x) + \frac{g_0}{3!} \times [\vec{\phi}(x)\vec{\phi}^*(x)]^2 + \gamma_0[\vec{\phi}(x)\vec{\phi}^*(x)]E(x) + \tfrac{1}{2}E^2(x) + s_0 E(x)\Big)\,. \tag{2}$$

The random external sources η_α and ζ are governed by Gaussian statistical distributions with zero mean values and mean-square values determined by the fluctuation-dissipation theorem. The physical relevance of such a model is extensively discussed by HHM.

The static correlation functions of the order parameter ϕ are governed by a reduced equilibrium distribution obtained by integrating $e^{-\mathcal{H}}$ over the energy field. The result is again a $(\vec{\phi}^*\vec{\phi})^2$ model where g_0 is replaced by

$$\tilde{g}_0 = g_0 - 3\gamma_0^2\,. \tag{3}$$

Thus the static properties of the order parameter are the usual properties of an isotropic N-vector model.[1]

The dynamics is, of course, dependent upon the coupling γ_0. The treatment of the dynamical properties in the critical region follows the scheme already used in Ref. 2: (a) construction of a Lagrangian which generates correlation functions equal to those deduced from Eqs. (1) and (2), and averaging over the sources; (b) characterization of the primitive divergences and renormalization of the model from which follows a differential renormalization-group (RG) equation satisfied by the dynamical correlation functions; (c) calculation of the coefficients of the RG equation, search for fixed points, and calculation of their domain of stability; and (d) when stable fixed points are identified, scaling is deduced, the critical exponents are computed, and corrections to scaling are characterized.

II. SUMMARY OF RESULTS

For clarity we postpone all technical details and calculations to the subsequent sections and we first summarize the results of this work. We limit ourselves to the only nontrivial case (from the point of view of the renormalization group), i.e., a conserved energy field coupled to a nonconserved order parameter [case (C) of HHM]. The correlation functions of the order parameter and of the energy field depend upon the variables $\tilde{u}$, v, and λ, which are the renormalized equivalents of g_0,

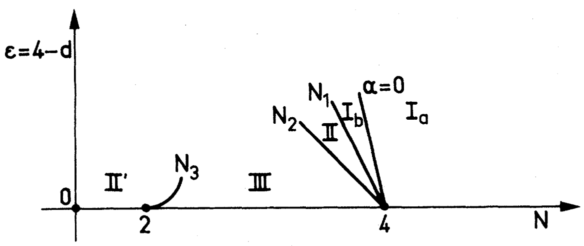

FIG. 1. Topology of the (N, d) plane near $d=4$. The scale is distorted to exhibit the four regions. Regions I_a and I_b are separated by the curve $\alpha=0$. Equations for the lines $N_1(\epsilon)$, $N_2(\epsilon)$ are given in the text by Eqs. (8), (9), and (11).

γ_0^2, and Λ_0/Γ_0. The search for stable fixed points in the space of the parameters $\tilde{u}$, v, and λ leads to several solutions, which determine several regions in the (N, d) plane (Fig. 1). The first two variables $\tilde{u}$ and v are fixed points determined purely by the statics, namely,

(a)

$$\tilde{u}^* = 6\epsilon/(N+8) + O(\epsilon^2) + \cdots, \tag{4}$$

as for the usual Wilson static scaling.[4] It is always stable. The corresponding static critical exponent of the order parameter $\vec{\phi}$ is the usual index η.

(b)

$$v^* = \begin{cases} 0 \quad \text{stable for } \alpha < 0 \text{ (Region } I_a), & \text{(5a)} \\ \dfrac{2\epsilon(4-N)}{N(N+8)} + O(\epsilon^2) \quad \text{stable for } \alpha > 0 \text{ (Regions } I_b, \text{ II, and III)}. & \text{(5b)} \end{cases}$$

The corresponding static critical exponent for the energy field E is η_E,

$$\langle E_k E_{-k}\rangle \underset{\substack{T=T_c \\ k\to 0}}{\sim} k^{\eta_E}, \tag{6}$$

with

$$\eta_E = \begin{cases} 0 \quad \text{for } \alpha < 0, & \text{(7a)} \\ -\alpha/\nu \quad \text{for } \alpha > 0. & \text{(7b)} \end{cases}$$

Finally, three possibilities are left for the fixed point λ^*.

(a) $\lambda^* = +\infty$ (Regions I_a, I_b), which will be shown to be stable for $N > N_1$ with

$$N_1 = 4 - \epsilon(4 + \tfrac{1}{4}c) + O(\epsilon^2), \tag{8}$$

in which we have kept the notation c for the constant

$$c \equiv 6 \ln\tfrac{4}{3} - 1. \tag{9}$$

(b) λ^* finite (Region II), whose stability domain consists of two pieces,

$$N < N_3 = 2 + C\epsilon|\ln\epsilon| \tag{10a}$$

and

$$N_2 < N < N_1, \tag{10b}$$

$$N_2 = 4 - \epsilon(4 + \tfrac{1}{2}c). \tag{11}$$

It is beyond the scope of the ϵ expansion to assert whether the two domains constituting the region II are connected or not.

(c) $\lambda^* = 0$ (Region III), which governs the behavior in the complement of the previous regions. Depending upon the relevant values of v^* and λ^* the N-d plane is split into four regions with different critical dynamics.

Region I_a: $v^* = 0$, $\lambda^* = \infty$. Its boundary is $\alpha = 0$, i.e., $N = 4 - 4\epsilon + O(\epsilon^2)$. Since $v^* = 0$, the energy field decouples in the critical regime and we are led to the same value of z as in Ref. 2,

$$z = 2 + \epsilon^2 \frac{N+2}{2(N+8)^2}\left(6\ln\frac{4}{3} - 1\right) + \epsilon^3 \frac{N+2}{2(N+8)^2} \times \left[\left(6\ln\frac{4}{3} - 1\right)\frac{6(3N+14)}{N+8} - 2.296\right]. \tag{12}$$

However, the structure of corrections to scaling is sensitive to the coupling to the energy field; for instance, the relaxation rate $\omega(k)$ of the order parameter behaves as

$$\omega(k) \sim k^z\{1 + A_1 k^{z-2-\alpha/\nu} + A_2 k^{\omega}\} \tag{13}$$

in which the A_i's are nonuniversal constants. The same correcting powers would be present everywhere, in particular in the energy relaxation rate $\lambda_E(k)$.

Region I_b: $v^* \neq 0$, $\lambda^* = \infty$. It is bounded by the two lines ($\alpha = 0$, $N = N_1$). The value of the critical exponent z coincides with the decoupled value of region I_a although the fixed point is different. However this difference reflects itself in the leading corrections to scaling which take the form

$$\omega(k) \sim k^z\{1 + A_1' k^{\alpha/\nu} + A_2' k^{z-2-\alpha/\nu} + A_3' k^{\omega}\}. \tag{14}$$

Region II: $v^* \neq 0$, $\lambda^* \neq 0$. In this region we find the result which coincides with the conventional scaling theory

$$z = 2 + \alpha/\nu. \tag{15}$$

Corrections to scaling now have the form of the multiplication of the main power by a new polynomial

$$\omega(k) \simeq k^z(1 + A_1'' k^{\alpha/\nu} + A_2'' k^{\omega_\Gamma} + A_3'' k^{\omega}) , \qquad (16)$$

in which ω_Γ is a new dynamical correction exponent. Its ϵ expansion is not the same in the two subdomains, which constitute the region II.

(a) For $N < N_3 = 2 + C\epsilon\,|\ln\epsilon|$, we have

$$\omega_\Gamma = \tfrac{1}{2}\epsilon(2-N)[(4-N)/(N+8)] + O(\epsilon^2) . \qquad (17)$$

(b) For $N_1 < N < N_2$, we have

$$\omega_\Gamma = \frac{\epsilon}{12}\,\frac{(N_2 - N)(N - N_1)}{4 - N - 4\epsilon} + O(\epsilon^3) . \qquad (18)$$

Finally the relaxation rate of the energy mode is governed by the same exponent z [Eqs. (12), (14), and (15)].

Region III: $v^* \neq 0$, $\lambda^* = 0$. In that region, the relevant contributions to the critical dynamics cannot be obtained by letting λ go to zero in the corresponding contributions of region II, because of a nonuniformity in the two limits $\lambda \to 0$ and $\epsilon \to 0$. Up to second order the result is a power law modified by square of logarithms

$$\omega(k) \simeq k^z \ln^2 k, \qquad z = 2 + \epsilon\,\frac{2(4-N)}{N(N+8)} + O(\epsilon^2), \quad 2 < N < 4, \qquad (19)$$

and a breakdown of scaling without guarantee though that it persists to higher orders.

Finally, the role of temperature (neighborhood of T_c) is discussed, and the expected scaling result follows.

III. LAGRANGIAN VERSION AND RENORMALIZATION OF THE THEORY

The construction of a Lagrangian equivalent to the Langevin-Ginzburg-Landau equations (1) and (2) has been discussed in detail in Ref. 5. The result is a Lagrangian $\mathcal{L}$ given by,

$$-\mathcal{L}[\phi, \phi, E] = \sum_{k,\omega}\left(\frac{-i\omega}{\Gamma_0} + r_0 + k^2\right)\vec{\phi}_{k\omega}\vec{\phi}^*_{k\omega} + \left(\frac{-i\omega}{\Lambda_0 k^2} + 1\right) E_{k\omega}E_{-k-\omega} + s_0 E_{0,0}$$
$$+ \sum_{\omega_i}\int d^d x\,\gamma_0\,E_{\omega_2-\omega_1}(x)\,\vec{\phi}_{\omega_1}(x)\,\vec{\phi}^*_{\omega_2}(x) + \frac{g_0}{3!}[\vec{\phi}_{\omega_1}(x)\cdot\vec{\phi}^*_{\omega_2}(x)][\vec{\phi}_{\omega_3}(x)\cdot\vec{\phi}^*_{\omega_1+\omega_3-\omega_2}(x)] , \qquad (20)$$

in which the[6] ω's are the discrete imaginary Bose frequencies $2i\pi p$, and the wave vectors k are bounded by a cutoff Λ.

The Green's functions generated by averaging over the weight $e^{\mathcal{L}}$ are shown to be order-by-order identical to the linear response generated by the stochastic Eqs. (1) and (2) provided one takes a "classical limit"[2,5] and the continuation from discrete to continuous values of the external frequency. The rules to evaluate the diagrams follow those of Refs. 2 and 5. According to the general scheme, the problem now is to study the large Λ limit of the theory (which characterizes the critical region). Thus we have to study the primitive divergences of the theory and find the counterterms which remove them. We begin with a discussion of the static properties.

A. Renormalization of the static correlation functions

The model is then equivalent to the Wilson theory with the Hamiltonian given by Eq. (2), whose renormalization is well known.[8] At T_c the Hamiltonian with its counter terms reads in terms of the renormalized fields [multiplicatively related to the bare ones of Eq. (2)],

$$\mathcal{H}[\phi, \phi^*, \epsilon] = \int d^d x\, Z_\phi \nabla\vec{\phi}(x)\,\nabla\vec{\phi}^*(x) + Z_E\,E^2(x) + \mu^\epsilon(u/3!)\,Z_u(\vec{\phi}(x)\,\vec{\phi}^*(x))^2$$
$$+ (\mu^\epsilon v)^{1/2} Z_v(\vec{\phi}(x)\,\vec{\phi}^*(x))\,E(x) + Z_\phi r_{0c}\vec{\phi}(x)\,\vec{\phi}^*(x) + Z^{1/2} s_{0c} E(x), \quad (21)$$

in which μ is an arbitrary parameter used to fix the normalizations of the field and the renormalized couplings. The parameter s_{0c} is determined by the requirement that the expectation value of the renormalized field $\langle E\rangle$ vanishes. The renormalization constant Z may then be computed in powers of $\tilde{u}$ and v ($\tilde{u} = u - 3v$) and the (inverse) correlation functions of ϕ and E are related to the bare ones by,

$$\Gamma_{\phi\phi}(k;\, u, v, \mu) = Z_\phi \Gamma^B_{\phi\phi}(k;\, g_0, \gamma_0, \Lambda), \qquad (22a)$$

$$\Gamma_{EE}(k;\, u, v, \mu) = Z_E \Gamma^B_{EE}(k;\, g_0, \gamma_0, \Lambda) , \qquad (22b)$$

$$g_0 = \mu^\epsilon(uZ_u/Z_\phi^2); \quad \gamma_0^2 = \mu^\epsilon(vZ_v^2/Z_\phi^2 Z_E) . \qquad (23)$$

The independence with respect to μ of the bare theory yields the differential RG equations

$$\left(\mu\frac{\partial}{\partial\mu} + W_1(\tilde{u}, v)\frac{\partial}{\partial\tilde{u}} + W_2(\tilde{u}, v)\frac{\partial}{\partial v}\right)\Gamma_{\phi\phi} = \eta_\phi(\tilde{u}, v)\Gamma_{\phi\phi}(k;\, \tilde{u}, v, \mu),$$
$$\left(\mu\frac{\partial}{\partial\mu} + W_1(\tilde{u}, v)\frac{\partial}{\partial\tilde{u}} + W_2(\tilde{u}, v)\frac{\partial}{\partial v}\right)\Gamma_{EE} = \eta_E(\tilde{u}, v)\,\Gamma_{EE}(k;\, \tilde{u}, v, \mu), \qquad (24)$$

in which

$$W_1 = \mu \frac{\partial}{\partial \mu}\bigg|_0 \tilde{u}, \tag{25a}$$

$$W_2 = \mu \frac{\partial}{\partial \mu}\bigg|_0 v, \tag{25b}$$

$$\eta_\phi(\tilde{u}, v) = \mu \frac{\partial}{\partial \mu}\bigg|_0 \ln Z_\phi ,$$

$$\eta_E(\tilde{u}, v) = \mu \frac{\partial}{\partial \mu}\bigg|_0 \ln Z_E , \tag{26}$$

where $\mu(\partial/\partial\mu)|_0$ stands, in general, for a derivative taken with fixed values of the bare parameters $(g_0, \gamma_0, \Lambda, \ldots)$.

Simplifications follow from the fact that the energy field for the statics is essentially like $\vec{\phi}\vec{\phi}^*$, apart from irreducibility and renormalization questions treated in Appendix A. The resulting properties are (a)

$$W_1(\tilde{u}, v) = W(\tilde{u}), \tag{27}$$

in which $W(\tilde{u})$ is the function calculated for the statics of Hamiltonian (2) with $\gamma_0 = 0$,[8]

$$W(\tilde{u}) = -\epsilon\tilde{u} + \tfrac{1}{6}(N+8)\tilde{u}^2(1+\tfrac{1}{2}\epsilon) - \tfrac{1}{12}(3N+14)\tilde{u}^3 + \cdots \tag{28}$$

(b)

$$W_2(\tilde{u}, v) = -v(\eta_E(\tilde{u}, v) + 2\hat{\eta}(\tilde{u}) + \epsilon), \tag{29}$$

in which $\hat{\eta}(\tilde{u})$ is the static contribution associated with $\vec{\phi}\vec{\phi}^*$ for the same Hamiltonian

$$\hat{\eta}(\tilde{u}) = -\tfrac{1}{6}(N+2)\tilde{u}(1+\tfrac{1}{2}\epsilon) + \tfrac{5}{72}(N+2)\tilde{u}^2 + \cdots. \tag{30}$$

The fixed-point value of $\hat{\eta}$ is $\hat{\eta}^* = 1/\nu - 2$. The E-field exponent η_E is shown in Appendix A to take the form

$$\eta_E(\tilde{u}, v) = -vB(\tilde{u}), \tag{31}$$

where the function $B(\tilde{u})$ already introduced in Ref. 8 is related to the $\langle\phi^2, \phi^2\rangle$ correlation function

$$B(\tilde{u}) = \tfrac{1}{2}N(1+\tfrac{1}{2}\epsilon) + O(\tilde{u}^2, \epsilon^2). \tag{32}$$

Finally, the ϕ-field exponent η_ϕ, whose fixed point value is the usual η, is

$$\eta_\phi(\tilde{u}, v) \equiv \eta(\tilde{u}) = \tfrac{1}{72}(N+2)\tilde{u}^2(1+\tfrac{5}{4}\epsilon - \tfrac{1}{12}(N+8)\tilde{u}) + \cdots. \tag{33}$$

B. Renormalization of the dynamical correlation functions

Returning to the bare Lagrangian (20), we apply the rules,[7] which give the perturbation expansion once all the internal summations over discrete frequencies are performed. The counting of the superficial degree of divergence of the diagrams is the same as for the statics, the external frequency ω being merely an external parameter. The degree of divergence of the purely dynamical correlation function (obtained by subtracting its static $\omega = 0$ part) is decreased by two since ω has the dimension of the square of a wave vector. Therefore the only primitive divergence of the dynamical part is a logarithmic divergence (for $d = 4$) of the two-point function of the order parameter and one only needs a single (multiplicative) renormalization. The Lagrangian leading to a finite renormalized perturbation theory writes, at $T = T_c$, as in Ref. 2,

$$-\mathcal{L}[\phi, \phi^*, E] = \sum_{k,\omega}\left(-\frac{i\omega Z_\phi}{\Gamma_0 Z_\Gamma} + k^2\right)\vec{\phi}_{k\omega}\vec{\phi}^*_{k\omega} + \sum_{k,\omega}\left(-\frac{i\omega Z_\phi}{\lambda\Gamma_0 Z_\Gamma} + 1\right)E_{k\omega}E_{-k,-\omega}$$
$$+ \sum_{\omega_i}\int d^dx\,(\mu^\epsilon v)^{1/2}E_{\omega_2-\omega_1}(x)(\vec{\phi}_{\omega_1}(x)\cdot\vec{\phi}^*_{\omega_2}(x)) + \mu^\epsilon u(\vec{\phi}_{\omega_1}(x)\cdot\vec{\phi}^*_{\omega_2}(x))(\vec{\phi}_{\omega_3}(x)\cdot\vec{\phi}^*_{\omega_1+\omega_3-\omega_2}(x)) - \delta\mathcal{L}, \tag{34}$$

$$-\delta\mathcal{L}[\phi, \phi^*, E] = \sum_{k,\omega}\left(-\frac{i\omega Z_\phi}{\Gamma_0 Z_\Gamma}(Z_\Gamma - 1) + (Z_\phi - 1)k^2 + r_{0c}Z_\phi\right)\vec{\phi}_{k\omega}\vec{\phi}^*_{k\omega} + \sum_{k,\omega}(Z_E - 1)E_{k\omega}E_{-k,-\omega}$$
$$+ \sum_{\omega_i}\int d^dx\,(\mu^\epsilon v)^{1/2}(Z_v - 1)E_{\omega_2-\omega_1}(x)\vec{\phi}_{\omega_1}(x)\cdot\vec{\phi}^*_{\omega_2}(x)$$
$$+ \mu^\epsilon u(Z_u - 1)(\vec{\phi}_{\omega_1}(x)\cdot\vec{\phi}^*_{\omega_2}(x))(\vec{\phi}_{\omega_3}(x)\cdot\vec{\phi}^*_{\omega_1+\omega_3-\omega_2}(x)), \tag{35}$$

where we have exhibited the counter terms $\delta\mathcal{L}$ and used the renormalized parameter

$$\lambda = \Lambda_0 Z_\phi/\Gamma_0 Z_\Gamma Z_E . \tag{36}$$

The renormalizability expresses the fact that the dynamical correlation functions remain finite when the cutoff Λ is sent to infinity (in $d < 4$) provided they are written in terms of u, v, λ, and the renormalized external frequency

$$\zeta = \omega Z_\phi/\Gamma_0 Z_\Gamma . \tag{37}$$

The function Z_Γ is then defined by an extra normalization condition, which is conveniently taken as

$$\frac{\partial}{\partial(-i\zeta)}\Gamma_{\phi\phi}(-i\zeta, k=0;\ u,\ v,\ \lambda;\ \mu)\bigg|_{-i\zeta=\mu^2} = 1. \tag{38}$$

With this choice Z_Γ is real and $\Gamma_{\phi\phi}$ is a real function of $-i\zeta$, related to the bare correlation function by

$$\Gamma^B_{\phi\phi}(-i\omega/\Gamma_0,\ k;\ g_0,\ \gamma_0,\ \Lambda_0/\Gamma_0;\ \Lambda)$$

$$=Z_\phi^{-1}\Gamma_{\phi\phi}(-i\zeta;\; k;\; u,\; v,\; \lambda;\; \mu)\,. \tag{39}$$

The associated RG equation is thus

$$\left(\mu\frac{\partial}{\partial\mu}+W_1\frac{\partial}{\partial\tilde u}+W_2\frac{\partial}{\partial v}+W_3\frac{\partial}{\partial\lambda}-\eta_\phi-\eta_\Gamma\zeta\frac{\partial}{\partial\zeta}\right)\times\Gamma_{\phi\phi}(-i\zeta;\; k;\; u, v, \lambda;\; \mu)=0, \tag{40}$$

where W_1 and W_2 are given by Eqs. (27)–(29) and

$$W_3(\tilde u, v, \lambda)=\mu\frac{d}{d\mu}\bigg|_0\lambda, \tag{41}$$

$$\eta_\Gamma(\tilde u, v, \lambda)=\mu\frac{d}{d\mu}\bigg|_0\ln\frac{Z_\Gamma}{Z_\phi}, \tag{42}$$

yielding with the definition (36),

$$W_3(\tilde u, v, \lambda)=-\lambda[\eta_\Gamma(\tilde u, v, \lambda)+\eta_E(\tilde u, v)]\,. \tag{43}$$

The dynamic RG(39) equation will be completely determined by the new function $\eta_\Gamma(\tilde u, v, \lambda)$, that is by the knowledge of Z_Γ given by the normalization condition (38). The result of the computation detailed in Appendix C is

$$\begin{aligned}\eta_\Gamma(\tilde u, v, \lambda)=&\frac{v}{1+\lambda}\left(1-\frac{\epsilon}{2}+\frac{\epsilon}{2}\ln(1+\lambda)\right)+\tilde u^2\,\frac{N+2}{72}\left(c-\tfrac{5}{4}\epsilon+\frac{N+8}{12}\tilde u\right)\\&+\tilde u v\frac{N+2}{6(1+\lambda)}\,(1+3\ln\tfrac{4}{3}-\ln(1+\lambda))+\frac{v^2N}{4(1+\lambda)^2}\,(1-\lambda+3\ln\tfrac{4}{3}-\ln(1+\lambda))\\&+\frac{v^2}{2(1+\lambda)^3}\left(1-\lambda-\lambda^2+3(1+\lambda)\ln\tfrac{4}{3}-(2+\lambda)\ln(2\lambda)-(\lambda+2)\ln\left(1+\frac{\lambda}{2}\right)+2(\lambda+2)\ln(1+\lambda)\right)\;.\end{aligned} \tag{44}$$

In Eq. (44) we have kept the terms of order $\tilde u^3$ and not v^3, $v^2\tilde u$, etc.,..., because they will play a role in the vicinity of $N=4$, for which we will explore values of v of order $\tilde u^2$.

C. Consequences of the dynamical RG equation

The dynamical behavior, the role of the fixed points, their stability, and the corrections to scaling are best displayed by direct integration of the partial differential equation (39). The method of characteristics exhibits the behavior under dilation of the solutions. Indeed from Eq. (39) we obtain under an arbitrary rescaling $\mu\to\rho\mu$,

$$\Gamma_{\phi\phi}(-i\zeta;\; k;\; u,\; v,\; \lambda;\; \mu)=\exp\left(-\int_1^\rho\frac{d\rho'}{\rho'}\,\eta_\phi(u(\rho'))\right)\times\Gamma_{\phi\phi}(-i\zeta(\rho);\; k;\; u(\rho),\; v(\rho),\; \lambda(\rho);\; \rho\mu), \tag{45}$$

in which

$$\rho\frac{d\tilde u}{d\rho}=W_1(\tilde u(\rho))\qquad \tilde u(1)=\tilde u, \tag{46a}$$

$$\rho\frac{dv}{d\rho}=W_2(\tilde u(\rho),\; v(\rho))\qquad v(1)=v, \tag{46b}$$

$$\rho\frac{d\lambda}{d\rho}=W_3(\tilde u(\rho),\; v(\rho),\; \lambda(\rho))\qquad \lambda(1)=\lambda, \tag{46c}$$

$$\rho\frac{d\zeta}{d\rho}=-\zeta(\rho)\,\eta_\Gamma(\tilde u(\rho),\; v(\rho),\; \lambda(\rho))\quad \zeta(1)=\zeta. \tag{46d}$$

Canonical dimensional analysis used together with Eq. (45) then gives

$$\Gamma_{\phi\phi}(-i\zeta;\; k;\; \tilde u, v, \lambda;\; \mu)=\rho^2\exp\left(-\int_1^\rho\frac{d\rho'}{\rho}\eta_\phi(\rho')\right)\times\Gamma_{\phi\phi}\left(-i\zeta(\rho)/\rho^2;\; \frac{k}{\rho};\; u(\rho),\; v(\rho),\; \lambda(\rho);\; \mu\right). \tag{47}$$

At this stage ρ is still arbitrary and we choose

$$\rho=k/\mu. \tag{48}$$

In the critical region where $k\ll\mu$, we are then led to study the differential system (46) in the limit $\rho\to 0$. When there is a stable fixed point $\tilde u^*$, v^*, and λ^* (the value $\lambda^*=\infty$ may be called by extension a fixed point) Eq. (47) then implies

$$\Gamma_{\phi\phi}(-i\zeta;\; k)\sim k^{2-\eta}\Phi(-i\zeta/k^3)\;, \tag{49}$$

with

$$\eta=\eta_\phi(\tilde u^*)\,, \tag{50}$$

$$z=2+\eta_\Gamma(\tilde u^*,\; v^*,\; \lambda^*). \tag{51}$$

We now solve successively Eqs. (46a)–(46d).

1. Coupling constant $\tilde u(\rho)$

We know from the standard theory that $\tilde u(\rho)$ approaches, when ρ goes to zero, its fixed-point value[4,8] $\tilde u^*$

$$\tilde u^*=\frac{6}{N+8}\epsilon\left(1+\epsilon\left(\frac{3(3N+14)}{(N+8)^2}-\frac{1}{2}\right)\right)+O(\epsilon^3), \tag{52}$$

as[8]

$$\tilde u(\rho)\underset{\rho\to 0}{=}\tilde u^*+C\rho^{+\omega}\,, \tag{53}$$

with

$$\omega=W_1'(\tilde u^*). \tag{54}$$

2. Coupling constant $v(\rho)$

Likewise Eq. (46c) yields that $v(\rho)$ approaches v^*, a solution of

$$W_2(\tilde{u}^*, v^*) = 0, \quad \frac{\partial W_2}{\partial v}(\tilde{u}^*, v^*) > 0. \tag{55}$$

If we return to Eqs. (29)–(31) we find two possible fixed points: (a)

$$v_1^* = 0, \quad \frac{\partial W_2}{\partial v}(\tilde{u}^*, 0) = -\frac{\alpha}{\nu}, \tag{56}$$

stable in the region I_a of the (N, d) plane, that is in the region $\alpha < 0$.

(b)

$$v_2^* = -\frac{\alpha/\nu}{B(\tilde{u}^*)}, \quad \frac{\partial W_2}{\partial v}(\tilde{u}^*, v_2^*) = \frac{\alpha}{\nu}, \tag{57}$$

stable in the complementary regions (I, II, and III) of positive α. A direct consequence of Eq. (29) and $v_2 \neq 0$ is

$$\eta_E(\tilde{u}^*, v^*) = -\alpha/\nu. \tag{58}$$

Let us note in passing that Eqs. (29)–(31) for W_2 yield a simple structure for the k-dependent energy static correlation function at $T = T_c$. Indeed, for $\alpha < 0$, $v_1^* = 0$, thus $\eta_E^* = 0$, and $\partial W_2/\partial v = -\alpha/\nu$, i.e., we have

$$\Gamma_{EE}^{-1} \simeq k^0(1 + Ak^{-\alpha/\nu}),$$

where k^0 is the leading power and $k^{-\alpha/\nu}$ arises from the correction power to scaling. Similarly for $\alpha > 0$, $v_2^* \neq 0$, thus $\eta_E^* = -\alpha/\nu$, and $\partial W_2/\partial v = +\alpha/\nu$, i.e., we have instead

$$\Gamma_{EE}^{-1} \simeq k^{-\alpha/\nu}(1 + A' k^{+\alpha/\nu}),$$

where now in turn, the constant term arises from corrections to scaling.

The small-ρ behavior of $v(\rho)$ is thus given by

$$v(\rho) \underset{\rho \to 0}{=} v^* + C_1 \rho^{\omega} + C_2 \rho^{|\alpha/\nu|}. \tag{59}$$

3. Parameter $\lambda(\rho)$

We now have to distinguish between I_a ($v^* = 0$) and the rest of the (N, d) plane: (a) If $v^* = 0$,

$$W_3(\tilde{u}^*, 0, \lambda) = -\lambda \eta_\Gamma(\tilde{u}^*), \tag{60}$$

in which $2 + \eta_\Gamma(\tilde{u}^*)$ is the dynamical exponent for the system without conservation law calculated to third order in Ref. 2 and given by Eq. (12).

It follows from Eq. (59) and of the positivity of $\eta_\Gamma(\tilde{u})$ that $\lambda(\rho)$ goes to infinity as

$$\lambda(\rho) \simeq C\rho^{-\eta_\Gamma(\tilde{u}^*)}. \tag{61}$$

(b) If $v^* = v_2^*$ [Eq. (57)] we still have to distinguish between several situations according to whether $W_3(\lambda)$, the right-hand side (RHS) of Eq. (46) is negative[9] or zero.

Region I_b: in which the RHS of Eq. (45c) is negative, i.e.,

$$\eta_\Gamma(\tilde{u}^*, v^*, \infty) + \eta_E(\tilde{u}^*, v^*) > 0,$$

or equivalently

$$z - 2 - \alpha/\nu > 0; \tag{62}$$

$\lambda(\rho)$ is driven to infinity as

$$\lambda(\rho) \underset{\rho \to 0}{\simeq} C\rho^{-(z-2-\alpha/\nu)}. \tag{63}$$

Region II: the right-hand side of Eq. (46c) vanishes for a finite value λ^* of λ, which is the solution of the equation

$$\eta_\Gamma(\tilde{u}^*, v^*, \lambda^*) = \alpha/\nu. \tag{64}$$

The stability of this fixed point requires that the (positive) λ^* is such that

$$\omega_\Gamma = -\lambda^* \frac{\partial \eta_\Gamma}{\partial \lambda}(\tilde{u}^*, v^*, \lambda^*) > 0. \tag{65}$$

This stability requirement is fulfilled: (a) if $N < 2$, for which we have

$$\omega_\Gamma = \tfrac{1}{2}\epsilon(2 - N)[(4 - N)/(N + 8)] + O(\epsilon^2), \tag{66}$$

which is indeed positive. (b) If $N - 2$ is of order ϵ, ω_Γ is again positive but of order $\epsilon^2 \ln(1/\epsilon)$.
(c) If $N = 4 - \epsilon(4 + p)$, $\frac{1}{4}c < p < \frac{1}{2}c$ that is $N_2 < N < N_1$, we have

$$\omega_\Gamma = \tfrac{1}{12}\epsilon[(N_2 - N)(N - N_1)/(4 - 4\epsilon - N)] + O(\epsilon^3) = (\epsilon^2/12p)(p - \tfrac{1}{4}c)(\tfrac{1}{2}c - p) + O(\epsilon^3). \tag{67}$$

Region II has, therefore, the topology indicated on Fig. 2. Integration of Eq. (46c) thus leads to

$$\lambda(\rho) - \lambda^* \underset{\rho \to 0}{\simeq} C_1 \rho^{\omega} + C_2 \rho^{\alpha/\nu} + C_3 \rho^{\omega_\Gamma}. \tag{68}$$

Region III: the right-hand side of Eq. (46c) again vanishes but for the value $\lambda = 0$. One should note from Eqs. (42) and (43) that when λ vanishes, W_3 vanishes also but contains a term proportional to $\lambda \ln \lambda$. Therefore $\lambda = 0$ is not a stable fixed point in the usual sense. Actually there is a nonuniformity in performing the limits λ and ϵ going to zero. Equation (43) has been derived when it is assumed that ϵ goes to zero first for fixed λ. In fact, if λ is fixed at the value zero, a separate study is needed. New divergences and hence new counterterms are present; the dynamical renormalization constant Z_Γ acquires a logarithmic dependence in the frequency ζ. The problem is, of course, that we should use the RG equation (46c) starting from some generic value $\lambda(1)$. Then integration for ρ small leads to smaller and smaller values of λ, where the λ, ϵ nonuniformity begins to be felt.

Indeed, to order v^2, we have in Eq. (43) for η_Γ

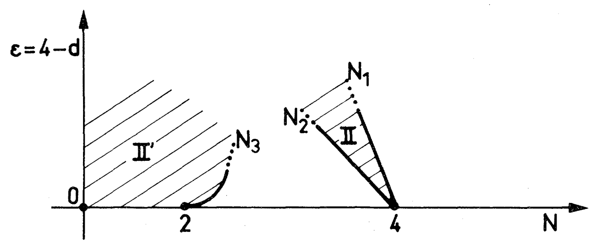

FIG. 2. Region II is near $d=4$. The boundary near $N=2$ is tangent (in fact all the derivatives vanish). Whether II′ and II are connected is beyond the reach of the ϵ expansion.

a term $-v^2\ln\lambda$ if λ is nonzero, whereas if λ is fixed at zero, the renormalization of the theory to the same order yields the singular term $-2v^2\ln(-i\zeta/\mu^2)$. Integration of the RG equations (46c) and (46d) thus yields a relaxation rate of the order parameter

$$\omega(k)\sim k^z(1+A(\ln k)^2), \tag{69}$$

with

$$z-2=2\epsilon(4-N)/N(N+8)+O(\epsilon^2),\quad 2<N<4. \tag{70}$$

Whether Eq. (69) should be interpreted as a sum of powers whose combination generates the log-squared terms or rather—and more likely—as a breakdown of scaling is beyond our present understanding.

Note, finally, that the exponent z is continuous across the boundaries of the various regions and that $z_{\mathrm{I}_a}-2>0>\alpha/\nu$, $z_{\mathrm{I}_b}-2>\alpha/\nu$, $z_{\mathrm{II}}-2=\alpha/\nu$, and [if Eqs. (69) and (70) are meaningful] $z_{\mathrm{III}}-2<\alpha/\nu$.

D. Renormalization group equations for T near T_c

Away from T_c two modifications appear in Hamiltonian (2): r_0 and s_0 are no longer fixed to their critical values r_{0c}, s_{0c}. In the renormalized Hamiltonian (21), we thus have the following extra terms

$$\delta\mathcal{H}(x)=t_1Z_\phi\hat{Z}\vec{\phi}(x)\vec{\phi}^*(x)+\mu^{-\epsilon/2}t_2Z_E^{1/2}E(x), \tag{71}$$

where the two parameters

$$t_1=(r_0-r_{0c})\hat{Z}^{-1},\qquad t_2=(s_0-s_{0c})Z_E^{1/2}\mu^{\epsilon/2}, \tag{72}$$

are proportional to t, which is a measure of $(T-T_c)$. $\hat{Z}$ is the renormalization constant necessary to renormalize the $\vec{\phi}\cdot\vec{\phi}^*$ insertions (Appendix A).

From the relation between bare and renormalized correlation functions

$$\Gamma^B_{\phi\phi}(r_0-r_{0c},\ s_0-s_{0c})=Z_\phi^{-1}\Gamma_{\phi\phi}((r_0-r_{0c})\hat{Z}^{-1},\ (s_0-s_{0c})\mu^{\epsilon/2}Z_E^{1/2}), \tag{73}$$

we obtain through the $\mu(d/d\mu)$ differentiation (with fixed bare parameters) the complete RG equation

$$\left(\mu\frac{\partial}{\partial\mu}+W_1\frac{\partial}{\partial\tilde{u}}+W_2\frac{\partial}{\partial v}+W_3\frac{\partial}{\partial\lambda}-\eta_\phi-\hat{\eta}t_1\frac{\partial}{\partial t_1}+\tfrac{1}{2}(\eta_E+\epsilon)t_2\frac{\partial}{\partial t_2}-\eta_\Gamma\zeta\frac{\partial}{\partial\zeta}\right)\Gamma_{\phi\phi}=0. \tag{74}$$

The discussion of the fixed points is the same as before. The variables t_1 and t_2 evolve under dilation as

$$\rho\frac{dt_1}{d\rho}=-\hat{\eta}(\tilde{u})t_1(\rho), \tag{75}$$

$$\rho\frac{dt_2}{d\rho}=\tfrac{1}{2}(\eta_E(\tilde{u},v)+\epsilon)t_2(\rho). \tag{76}$$

Consequently from Eq. (75) we obtain

$$t_1(\rho)\underset{\rho\to 0}{\sim}t\rho^{2-1/\nu}, \tag{77}$$

and from Eq. (76),

$$t_2(\rho)\underset{\rho\to 0}{\sim}t\rho^{1/2[\epsilon+\eta_E(\tilde{u},v)]}=\begin{cases}t\rho^{2-1/\nu}, & \alpha>0\\ t\rho^{\epsilon/2}, & \alpha<0\,.\end{cases} \tag{78}$$

Integration of Eq. (74) replaces Eq. (47) by the expected scaling equation (when there is a stable fixed point)

$$\Gamma_{\phi\phi}(-i\zeta;\ k,t;\ u,v,\lambda;\ \mu)\simeq k^{2-\eta}\Gamma_{\phi\phi}(-i\zeta/k^2;\ tk^{-1/\nu};\ u^*,v^*,\lambda^*;\ \mu). \tag{79}$$

In the case of a negative α, Eq. (78) leads to a correction to the main divergence of the correlation length $t^{-\nu}$, proportional to $t^{-\nu+|\alpha|/d}$.

IV. CONCLUSION

Our calculation confirms and refines upon the HHM theory on the following points: (a) It provides to all orders in ϵ a derivation of the dynamical scaling laws (except for the singular region III). (b) Region I is split into I_a and I_b governed by identical values of the critical exponent z but different leading corrections to scaling [due to the different small-ρ behavior of $v(\rho)/\lambda(\rho)$]. (c) In region I_a and I_b, z is given up to order ϵ^3; in region II the scaling result $z=2+\alpha/\nu$ is derived to all orders in ϵ, and (d) stability of the fixed points is characterized and the association correction-to-scaling exponents are computed, introducing a new dynamical correction exponent ω_Γ.

The authors thank Dr. B. Halperin for helping them to correct an error in Eq. (44).

APPENDIX A: RENORMALIZATION OF THE STATIC CORRELATION FUNCTIONS

In this section we discuss the renormalization of the static correlation functions when we choose not to integrate over the energy field (theory I). Clearly the theory has connections with the theory II in which the E field is integrated out; the E field of theory I is, in a sense which will be made precise below, similar to the $\vec{\phi}^2$ field of theory II. However, the RG equations that we are dealing with hold for the one-particle irreducible (1-PI) vertices both on the ϕ and the E lines, and this creates a few differences. For short we indicate the number of ϕ fields by a superscript.

The primitive divergences are: (a) a quadratic divergence of the two-ϕ inverse correlation function $\Gamma^{(2)}$, leading to mass and field-strength renormalization; (b) a logarithmic divergence of the four-ϕ vertex $\Gamma^{(4)}$, leading to the renormalization of the coupling constant g_0; (c) a logarithmic divergence of the E-$\phi\phi$ vertex $\Gamma_E^{(2)}$ and the associate renormalization of the coupling constant γ_0; (d) a logarithmic divergence of the two-E inverse correlation function Γ_{EE}, and the corresponding E-field strength renormalization; and (e) a logarithmic divergence of the $\langle\phi^2\phi\phi\rangle$ 1-PI function and the corresponding ϕ^2 renormalization.

Let us first note that there is in the Hamiltonian a source linear in the E field which is such that at the critical point the expectation value of E vanishes. Finally, the corresponding renormalization conditions, and the relations between the bare and renormalized quantities are

$$\Gamma^{(2)}\big|_{p^2=0}=0, \tag{A1}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}\big|_{p^2=\mu^2}=1, \tag{A2}$$

$$\Gamma^{(n)}=Z_\phi^{n/2}\,\Gamma^{(n)B}, \tag{A3}$$

$$\Gamma_{EE}\big|_{p^2=(4/3)\mu^2}=1, \tag{A4}$$

$$\Gamma_{EE}=Z_E\,\Gamma_{EE}^B, \tag{A5}$$

$$\Gamma^{(4)}\big|_{p_ip_j=(1/3)\mu^2(4\delta_{ij}-1)}=u\mu^\epsilon, \tag{A6}$$

$$g_0=\mu^\epsilon(uZ_u/Z_\phi^2), \tag{A7}$$

$$\Gamma_E^{(2)}(q;\,p_1,\,p_2)\big|_{p_1^2=p_2^2=(3/4)q^2=\mu^2}=(\mu^\epsilon v)^{1/2}, \tag{A8}$$

$$\gamma_0^2=\mu^\epsilon(vZ_v^2/Z_EZ^2), \tag{A9}$$

$$\Gamma_{\phi^2}^{(2)}(q;\,p_1,\,p_2)\big|_{p_1^2=p_2^2=(3/4)q^2=\mu^2}=1, \tag{A10}$$

$$\Gamma_{\phi^2}^{(2)}=Z_{\phi^2}Z_\phi\Gamma^{(2)B}, \tag{A11}$$

whereas for theory II, which depends only on $\tilde{u}=u-3v$, they are

$$\Gamma^{(2)}\big|_{p^2=0}=0, \tag{A12}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}\big|_{p^2=\mu^2}=1, \tag{A13}$$

$$\Gamma^{(n)}=Z^{n/2}(\tilde{u})\,\Gamma^{(n)B}, \tag{A14}$$

$$\Gamma^{(4)}\big|_{p_ip_j=(1/3)\mu^2(4\delta_{ij}-1)}=\mu^\epsilon u, \tag{A15}$$

$$g_0=\mu^\epsilon\tilde{u}\,Z_{\tilde{u}}/Z^2, \tag{A16}$$

$$\Gamma_{\phi^2}^{(2)}\big|_{p_1^2=p_2^2=(3/4)q^2=\mu^2}=1, \tag{A17}$$

$$\Gamma_{\phi^2}^{(n)}=\hat{Z}(\tilde{u})\,Z^{n/2}\,\Gamma_{\phi^2}^{(n)B}, \tag{A18}$$

$$\Gamma_{\phi^2,\phi^2}\big|_{p^2=\mu^2}=0, \tag{A19}$$

$$\Gamma_{\phi^2,\phi^2}=\hat{Z}^2\Gamma^B_{\phi^2\phi^2}+\mu^{-\epsilon}A(\tilde{u}). \tag{A20}$$

Expressing that the (bare) correlations of the field ϕ in both theories are the same we obtain

$$Z_\phi(\tilde{u},\,v)=Z(\tilde{u}). \tag{A21}$$

The same result holds for the ϕ^2 renormalization

$$Z_{\phi^2}(\tilde{u},\,v)=\hat{Z}(\tilde{u}). \tag{A22}$$

Then we write the bare correlation function $\langle E\phi\phi\rangle$ with truncated legs on the ϕ lines alone, that is $(\Gamma_{EE}^B)^{-1}(\Gamma_E^{(2)B})$. These diagrams coincide with those of the 1-PI $\langle\phi^2\phi\phi\rangle$ diagrams of theory II within a factor γ_0,

$$\gamma_0\,\Gamma_{\phi^2}^{(2)B}=(\Gamma_{EE}^B)^{-1}\,\Gamma_E^{(2)B}. \tag{A23}$$

If we express this relation in terms of the renormalized vertices of theories I and II and choose the normalization point $p_1^2=p_2^2=\frac{3}{4}q^2=\mu^2$, we obtain

$$Z_v(\tilde{u},\,v)=Z_E(\tilde{u},\,v)\,Z(\tilde{u})\,\hat{Z}(\tilde{u}). \tag{A24}$$

Therefore if we take the logarithmic derivative of this relation as well as of Eq. (A9) with respect to μ, keeping the bare parameters fixed, we obtain

$$W_2=-v[\eta_E(\tilde{u},\,v)+2\hat{\eta}(\tilde{u})+\epsilon], \tag{A25}$$

in which $\hat{\eta}(\tilde{u})=1/\nu(\tilde{u})-2$ is given by the usual static theory II.[8] Similarly the bare correlation functions of two E fields satisfy the relation

$$(\Gamma_{EE}^B)^{-1}=1+\gamma_0^2\,\Gamma^B_{\phi^2\phi^2}, \tag{A26}$$

from which we deduce

$$Z_E^{-1}=1+vA(\tilde{u}), \tag{A27}$$

and thus

$$\eta_E(\tilde{u},\,v)=-v\,B(\tilde{u}), \tag{A28}$$

which shows in particular that $\eta_E(\tilde{u},v)$ vanishes with v.

APPENDIX B: TWO-LOOP CALCULATIONS OF THE DYNAMICAL RENORMALIZATION GROUP

It has been established in Secs. I–III and Appendix A that the dynamical properties of the theory are entirely determined by the static functions and by one new quantity which is $\eta_\Gamma(\tilde{u},v,\lambda)$ defined by Eq. (41). This function is computed from the dynamical renormalization constant Z_Γ whose magnitude is fixed by the normalization conditions (37) and (38). Therefore, we have to compute the dynamical correlation $\Gamma_{\phi\phi}$ as a function of ζ and λ, taking into account the counterterms (35) and fix the value of Z_Γ by the condition (38). This calculation has been done up to order v^2, $v\tilde{u}$, and $\tilde{u}^2$ (the u^3 term is known from previous work[2]). The result is

$$\ln Z_\Gamma = -\frac{v(1-\frac{1}{2}\epsilon+\frac{1}{2}\epsilon\ln(1+\lambda))}{\epsilon(1+\lambda)} - \frac{N+2}{24\epsilon}(1-\epsilon)(\tilde{u}+3v)^2\ln\frac{4}{3} - \frac{Nv^2}{4\epsilon^2(1+\lambda)^2}\left(1+\frac{\epsilon}{2}-\frac{3\epsilon}{2}\lambda(\lambda+2)\ln\frac{4}{3}\right)$$
$$-\frac{N+2}{6\epsilon^2}\frac{\tilde{u}v}{(1+\lambda)^2}\left(1+\frac{\epsilon}{2}-\frac{3\epsilon}{2}\lambda\ln\frac{4}{3}\right) - \frac{v^2}{2\epsilon^2(1+\lambda)^3}$$
$$\times\left(\lambda-3\epsilon\lambda(1+\lambda)\left(1+\frac{\lambda}{2}\right)\ln\frac{4}{3}-\frac{\epsilon}{2}(\lambda+2)\ln2\lambda-\epsilon\left(1+\frac{\lambda}{2}\right)\ln\left(1+\frac{\lambda}{2}\right)-\frac{\epsilon}{2}(\lambda^2+4\lambda-1)+2\epsilon(1+\lambda)\ln(1+\lambda)\right), \quad \text{(B1)}$$

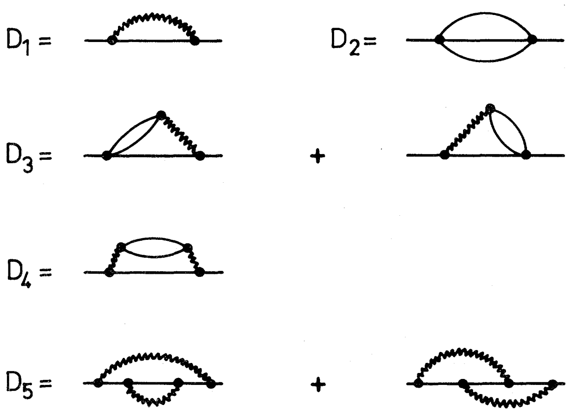

FIG. 3. Diagrams from which (B1) is derived.

and this yields by the differentiation Eq. (41) the result (43) for η_Γ. The fact that η_Γ is finite in four dimensions is a test for the dynamical renormalizability of the theory. We list in Fig. 3 the results for the diagrams from which (B1) is derived. Diagrams D_3, D_4, and D_5 stand in fact for a sum of all possible time orderings allowed by the rules of Ref. 2, Appendix A. They are evaluated for zero external wave vector and external frequency ζ. The function Z_Γ is then determined by their derivative [Eq. (38)],

$$\mathring{D} \equiv \frac{d}{d(-i\zeta)} D(-i\zeta)\Big|_{-i\zeta=\mu^2}.$$

For instance we have

$$\mathring{D}_1 = \mu^\epsilon v \frac{d}{d(-i\zeta)}\Big|_{\mu^2} \int \frac{d^dq\,(1+\lambda)}{q^2(1+\lambda)-i\zeta}$$
$$= -\frac{v}{\epsilon(1+\lambda)}\left(1-\frac{\epsilon}{2}+\frac{\epsilon}{2}\ln(1+\lambda)+O(\epsilon^2)\right). \quad \text{(B2)}$$

Likewise we have

$$\mathring{D}_2 = -\tfrac{1}{24}(N+2)[(\tilde{u}+3v)^2/\epsilon](1-\epsilon)\ln\tfrac{4}{3}, \quad \text{(B3)}$$
$$\mathring{D}_3 = +[(N+2)/12\epsilon^2][(\tilde{u}+3v)v/(1+\lambda)]$$
$$\times(1-\tfrac{1}{2}\epsilon+\epsilon\ln(1+\lambda)+\tfrac{3}{2}\epsilon\lambda\ln\tfrac{4}{3}), \quad \text{(B4)}$$
$$\mathring{D}_4 = +(N/4\epsilon^2)[v^2/(1+\lambda)^2]$$
$$\times\left(1+2\lambda+\epsilon(1+2\lambda)\ln(1+\lambda)-\frac{\epsilon}{2}(1+4\lambda)+\frac{3\epsilon}{2}\lambda^2\ln\frac{4}{3}\right), \quad \text{(B5)}$$
$$\mathring{D}_5 = -[v^2/2\epsilon^2(1+\lambda)^3]\,(3+4\lambda+2\lambda^2$$
$$-\tfrac{1}{2}\epsilon(\lambda+2)\ln[\lambda(\lambda+2)]+\tfrac{3}{2}\epsilon\lambda^2(1+\lambda)\ln\tfrac{4}{3}$$
$$-\tfrac{1}{2}\epsilon(3\lambda^2+6\lambda+5)+\epsilon(2\lambda^2+5\lambda+5)\ln(1+\lambda)). \quad \text{(B6)}$$

[1]B. I. Halperin, P. C. Hohenberg, and S. K. Ma, Phys. Rev. B 10, 139 (1974).

[2]C. De Dominicis, E. Brézin, and J. Zinn-Justin, Saclay report (unpublished).

[3]As long as Γ_0 is real, there is no difference between a system of $\frac{1}{2}N$ complex fields and N real fields. We may recall that in Ref. 2, it has been shown that if we take a complex Γ_0 in the scaling region, its imaginary part does not play any role.

[4]K. G. Wilson and J. Kogut, Phys. Rep. C 12, 75 (1974).

[5]C. De Dominicis, Nuovo Cimento Lett. 12, 567 (1975).

[6]In Lagrangian (19) $i\omega$ is a real number, and as in Ref. 2, i stands for $i\,\mathrm{sgn}(\mathrm{Im}\omega)$.

[7]In particular they are summarized in Appendix A of Ref. 2 where it suffices to know that the dynamic, $\Gamma_0(r_0+k^2)$, and the static, r_0+k^2, energies associated with the order-parameter field are replaced, respectively, by $\Lambda_0 k^2$ and 1 for the energy field. The systematics of the iteration of Eq. (1a) and (1b) has obviously been known to HHM (yet to appear) and is detailed for the case of ferromagnets by S. Ma and E. Mazenko, Phys. Rev. B 11, 4077 (1975).

[8]E. Brézin, J-C. Le Guillou, and J. Zinn-Justin, *Field Theoretical Approach to Critical Phenomena*, edited by C. Domb and M. Green (to be published), Vol. VI.

[9]A negative sign for W_3/λ corresponds to a stable fixed point at $\lambda^*=\infty$ as easily seen from changing $\lambda\to\lambda^{-1}$.