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Field-theoretic techniques and critical dynamics. I. Ginzburg-Landau stochastic models without energy conservation

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Renormalization techniques of field theory are used to prove (order by order to all orders) dynamical scaling laws on a Ginzburg-Landau stochastic model studied by Halperin, Hohenberg, and Ma. The dynamical exponent is calculated to order ϵ^3 and so is the new exponent ω_b, which governs the vanishing of the imaginary part of the (renormalized) kinetic coefficient, and appears among the corrections to scaling. Difficulties of previous calculations taking a microscopic approach to the critical dynamics of a Bose system are commented upon.

I. INTRODUCTION

Wilson's[1,2] theory of static critical phenomena has been recently reformulated[3] using the techniques of field renormalization. This approach, although less general in scope, has the advantages that capitalizing upon the known results of field-renormalization theory, yields simple expressions for the renormalization-group equations and allows a simple and complete derivation of all scaling laws, of the critical exponent expressions and corrections thereof (irrelevant variables of Wilson[2] and Wegner[4]). We wish to show here that this approach can be extended to time-dependent Landau-Ginzburg stochastic models studied by Halperin, Hohenberg, and Ma[5,6] (HHM) yielding a derivation of dynamical scaling and an expression for the dynamical exponents. We exhibit the method by considering the simplest of the dynamical models discussed by HHM and others,[7] i.e., a model which contains no external (energy) conserved field. More precisely the (complex) order parameter $\varphi_k^{(\alpha)}(t)$ satisfies the Landau-Ginzburg time-dependent equation[8]

$$\frac{\partial}{\partial t}\varphi_k^{(\alpha)}(t) = -\Gamma_0(1+ib_0)\times\left(\frac{\delta\mathcal{H}}{\delta\varphi_k^*(t)}[\varphi,\varphi^*] - h_k^{(\alpha)}(t)\right) + \eta_k^{(\alpha)}(t) , \quad (1)$$

where the effective Ginzburg-Landau-Wilson Hamiltonian is as usual,

$$\mathcal{H}[\varphi,\varphi^*] = \sum_k (r_0+k^2)\vec{\varphi}_k\cdot\vec{\varphi}_k^* + \frac{g_0}{3!}\int d^dx\,[\vec{\varphi}(x)\cdot\vec{\varphi}^*(x)]^2 . \quad (2)$$

$\varphi_k^{(\alpha)}$ is the αth component[9] ($\alpha = 1, 2, \ldots, n$) of the order parameter of the external field and $\eta_k^{(\alpha)}$ of the Langevin noise source governed by a Gaussian probability distribution; $\Gamma_0(1+ib_0)$ is an inverse complex time scale.[10] Finally, r_0 is a bare "squared mass" (or a bare inverse susceptibility) and the *critical region* of interest is characterized by $k \ll \Lambda$, $\Gamma \ll \Lambda$, Λ being a cutoff and r the renormalized "squared mass." The space-time correlation function $G_k(t)$ of the order parameter is calculated by solving (1) for $\varphi_k^{(\alpha)}[\eta, h]$, averaging over η with a Gaussian weight and taking the linear dependence in h,

$$G_k(t) \equiv \frac{\delta}{\delta h_k^*(0)}\langle\varphi_k^{(\alpha)}(t)\rangle\Big|_{h=h^*=0} , \quad (3)$$

with

$$\langle\varphi_k^{(\alpha)}(t)\rangle = X^{-1}\int d\{\eta\}d\{\eta^*\}\times\exp\left(-(2\Gamma_0)^{-1}\sum_k\int dt\,\vec{\eta}_k(t)\cdot\vec{\eta}_k^*(t)\right)\varphi_k^{(\alpha)}(t) , \quad (4)$$

$$X = \int d\{\eta\}d\{\eta^*\}\exp\left(-(2\Gamma_0)^{-1}\sum_k\int dt\,\vec{\eta}_k(t)\cdot\vec{\eta}_k^*(t)\right) . \quad (5)$$

The use of standard renormalization techniques is difficult with the above equations, but it has been shown[11] that the system (1)–(3) is strictly equivalent to a standard Lagrangian system, in its classical limit, namely, we compute instead

$$G_k^{(\alpha)}(\omega) \equiv \langle\varphi_{k\omega}^{(\alpha)}\varphi_{k\omega}^{*(\alpha)}\rangle , \quad = \Xi^{-1}\int d\{\varphi\}d\{\varphi^*\}\exp(\mathcal{L}[\varphi,\varphi])\varphi_{k\omega}^{(\alpha)}\varphi_{k\omega}^{*(\alpha)} , \quad (6)$$

$$\Xi = \int d\{\varphi\}d\{\varphi^*\}\exp(\mathcal{L}[\varphi,\varphi^*]) , \quad (7)$$

with

$$-\mathcal{L}[\varphi,\varphi^*] = \sum_{\omega,k}\left(\frac{-i\omega}{\Gamma_0(1+ib_0)} + r_0 + k^2\right)\vec{\varphi}_{k\omega}\cdot\vec{\varphi}_{k\omega}^* + \frac{g_0}{3!}\sum_\omega\int d^dx\,(\vec{\varphi}_{\omega_1}(x)\cdot\vec{\varphi}_{\omega_2}^*(x))(\vec{\varphi}_{\omega_3}(x)\cdot\vec{\varphi}_{\omega_1-\omega_2+\omega_3}^*(x)) . \quad (8)$$

Here i stands for $i\,\mathrm{sgn}\,\mathrm{Im}(\omega)$. The frequency index ω takes the values $\omega = 2pi\pi$, $p=0, \pm 1, \pm 2, \ldots$. The field variable is now indexed by component (α), wave vector (k), and frequency (ω). The *classical limit* of Eq. (6) means that we neglect quantum effects, i.e., the smallest wavelength Λ^{-1} is supposed to remain much bigger than the thermal wavelength, with our units $|1+ib_0| \times \Gamma_0 \Lambda^2 \ll 1$. Knowing $G_k(\omega)$ at the discrete points $2pi\pi$, one has to take its analytic continuation $G_k(z)$ to be identified, as usual with the Laplace-Fourier transform of the retarded commutator of φ, φ^* (i.e., the linear response in h, h^*).

The passage from Eqs. (1)–(5) to Eqs. (6)–(8) will allow us to use, with appropriate changes, the field-renormalization methods already used in the static limit.

We shall first recall the general form of the perturbation series for the system (6)–(8) and in Sec. II apply the technique of field renormalization to derive the general functional form of the correlation function $G_k(z)$. Section III is devoted to the renormalization-group equations satisfied by $G_k(z)$ and the resulting scaling form, characterized by a dynamical exponent directly related to a renormalization factor, and by Wilson functions whose zeros are the fixed points of the theory. Section IV contains results to lowest order in ϵ and N^{-1} for the dynamic exponent already given by HHM, and the slope ω_b of the Wilson function at the fixed point (which together with the slope of the static Wilson function[3,4] governs corrections to scaling). Results of the so-called microscopic approach to ^{4}He are likewise recovered and shown, at any rate, not to be in correspondence with any attractive fixed point[12] (except for the $1/N$ expansion near dimension 2). Finally, Section V contains the next to lowest order results in ϵ.

II. RENORMALIZATION OF THE DYNAMIC CORRELATION FUNCTION

Let us first briefly recall the general form of the perturbation series for the equivalent Lagrangian system (6)–(8). It suffices to consider the inverse correlation function (one-irreducible two-point Green's function),

$$G_k^{-1}(z) \equiv \Gamma_B^{(2)}(z,k) = \frac{-iz}{\Gamma_0(1+ib_0)} + r_0 + k^2 - \Sigma(z,k) \;, \qquad (9)$$

where $\mathrm{Im}\, z > 0$. The mass operator $\Sigma(z,k)$ expanded in powers of g_0 is given a sum of diagrams $\mathfrak{D}$ whose detailed form is recalled in the Appendix and whose contribution can be written as (before summation over wave vectors and within standard weight factors),

$$\mathfrak{D} = \left[(-g_0)^n \prod_l \frac{1}{\epsilon_l^0} \right] \left[\prod_j \frac{\sum_{p,h\in v(j)} (\Delta_p + \Delta_h)}{-iz + \sum_{p,h\in j} (\Delta_p + \Delta_h)} \right] . \qquad (10)$$

Here the first bracket is the Wilson standard contribution, the product being over all lines l of the diagram and $\epsilon_l^0 = r_0 + l^2$. In the second bracket we have $\Delta_{p,h} \equiv \Gamma_0 \epsilon^0 (1 \pm ib_0)$ associated with particle and hole lines. In the denominators these $\Delta_{p,h}$ are summed over *all* particle and hole lines present in each interval j between two successive interaction vertices, in the numerator the summation is restricted to those lines directly connected to the earlier vertex of the interval.

The main point, however, is that the degree of superficial divergence is the same as for the static case, i.e., the explicit-frequency summation that transforms the Feynman perturbation expansion of (6)–(8) into (10) does not affect power counting. This result, which has been explicitly constructed for $\Gamma_B^{(2)}$ is also valid for the 1-irreducible n-point correlation functions $\Gamma_B^{(n)}$. It remains valid so long as the propagators $[-iz + \sum_{p,h} (\Delta_p + \Delta_h)]^{-1}$ cannot add singularities to power counting.[13]

Let us now separate the correlation functions $\Gamma_B^{(n)}$ into a static and a dynamic part

$$\Gamma_B^{(n)} = \Gamma_{B,\mathrm{st}}^{(n)} + \Gamma_{B,\mathrm{dyn}}^{(n)} \,. \qquad (11)$$

We know that the dynamic part must vanish when all the external frequencies go to zero. Given the structure of the $\Gamma_B^{(n)}$, $\Gamma_{B,\mathrm{dyn}}^{(n)}$ therefore acquires, perturbationwise, powers of internal momenta amounting to $(\Gamma_0 \epsilon^0)^{-1}$ (since external frequencies are homogeneous to $\Gamma_0\epsilon^0$). In the *conserved case* ($\Gamma_0 \sim k^2$) this yields four extra powers of internal momenta, which is enough to wipe out, in four dimensions, all superficial divergencies from $\Gamma_{B,\mathrm{dyn}}^{(n)}$ and make the *dynamics* of the conserved case trivial. In the *nonconserved case* ($\Gamma_0 \sim C$) it yields *two extra powers* of internal momenta. Thus, in this case, only $\Gamma_{B,\mathrm{dyn}}^{(2)}$ remains superficially divergent (like a logarithm) after we have made the subtractions implied by the (static) renormalization theory, and we therefore need only one extra subtraction to render the theory finite (i.e., in the Wilson recursion method, $\Gamma_{B,\mathrm{dyn}}^{(2)}$ leads to an additional marginal variable to be kept in the iterations). To this effect we write the intermediate Lagrangian $\mathfrak{L}_1$ (resulting from wave-function renormalization $\varphi \to Z_\varphi^{1/2}\varphi$) as

$$-\mathfrak{L}_1 = \sum_{k,\omega} \left(\frac{-i\omega Z_\varphi}{\Gamma_0 Z_\Gamma (1+ib)} + r + k^2 \right) \vec{\varphi}_{k\omega} \cdot \vec{\varphi}^*_{k\omega} + \mu^\epsilon \frac{u}{3!} \sum_\omega \int (\vec{\varphi} \cdot \vec{\varphi}^*)(\vec{\varphi} \cdot \vec{\varphi}^*) d^d x$$

$$+\sum_{k,\omega}[(r_0 Z_\varphi - r) + k^2(Z_\varphi - 1)]\vec{\varphi}_{k\omega}\cdot\vec{\varphi}^*_{k\omega} + \mu^\epsilon(Z_\varphi - 1)\frac{u}{3!}\sum_\omega\int(\vec{\varphi}\cdot\vec{\varphi}^*)(\vec{\varphi}\cdot\vec{\varphi}^*)d^d x - \sum_{k,\omega}\frac{i\omega Z_\varphi C}{\Gamma_0 Z_\Gamma(1+ib)}\vec{\varphi}_{k\omega}\cdot\vec{\varphi}^*_{k\omega}\ . \tag{12}$$

The second line contains all the standard counter terms, and its last term affects the needed extra subtraction with

$$C = Z_\Gamma(1+ib)/(1+ib_0) - 1 \tag{13}$$

and

$$b_0 = bZ_b\ , \tag{14}$$

and, as in the static case

$$g_0 = \mu^\epsilon u Z_u/Z_\varphi^2\ . \tag{15}$$

Normalization conditions relate the bare and renormalized parameters, i.e., they yield Z_φ and Z_u as functions of u, and Z_Γ and Z_b as functions of u and b (and Z_φ, Z_u, Z_Γ, and Z_b as smooth functions of μ/Λ). These conditions are conveniently chosen in analogy with the static case and if we work *at the critical temperature* ($r=0$), as

$$\Gamma^{(2)}_{\rm st}\Big|_{k=0} = 0\ , \tag{16}$$

$$\frac{\partial}{\partial k^2}\Gamma^{(2)}_{\rm st}\Big|_{k^2=\mu^2} = 1\ , \tag{17}$$

$$\Gamma^{(4)}_{\rm st}\Big|_{\rm s.p.} = \mu^\epsilon u \tag{18}$$

(where now, the $\Gamma^{(n)}$ are the renormalized ones). Noting that perturbation expansion (10) implies, together with (12) that $\Gamma^{(2)}$ has the functional form

$$\Gamma^{(2)}(-i\zeta, k; ib, u, \mu, C)\ , \tag{19}$$

with

$$\zeta = zZ_\varphi/\Gamma_0 Z_\Gamma\ , \tag{20}$$

we choose the extra normalization condition as

$$\frac{\partial}{\partial i\zeta}\Gamma^{(2)}(-i\zeta, k=0)\Big|_{i\zeta=-\mu^2} = -\frac{1}{1+ib}\ . \tag{21}$$

With this choice, Z_Γ and Z_b are determined as *real* functions of u and b^2, and $\Gamma^{(2)}$ is a *real* function of $-i\zeta$ and ib, related to the original, bare correlation function $\Gamma^{(2)}_B$ by

$$\Gamma^{(2)}_B(-iz/\Gamma_0, k; ib_0, u_0, r_{0c}, \Lambda) = Z_\varphi^{-1}\Gamma^{(2)}(-i\zeta, k; ib, u, \mu)\ . \tag{22}$$

Indeed, once the functional form (22) is obtained, together with a proof that the μ/Λ dependence is smooth in the renormalized function $\Gamma^{(2)}$, scaling, closed form for critical exponents, etc., follow immediately. Thus everything depends upon the fact that the extra subtraction correctly does the job of removing the leftover superficial divergences and insertions thereof. This is clear from the fact that the frequency summations in the Feynman expansion do not change the structure of perturbation expansion in respect to power counting. For completeness, removal of these leftover divergencies is exhibited explicitly in the Appendix.

III. RENORMALIZATION-GROUP EQUATION FOR DYNAMIC CORRELATION FUNCTION

From Eq. (13) we immediately obtain the renormalization-group equation satisfied by $\Gamma^{(2)}$

$$\left[\left(\mu\frac{\partial}{\partial\mu} + W_u\frac{\partial}{\partial u} + iW_b\frac{\partial}{\partial ib} - \eta_\varphi\right) - (\gamma_\Gamma - \eta_\varphi)\zeta\frac{\partial}{\partial\zeta}\right]\times\Gamma^{(2)}(-i\zeta, k; ib, u, \mu) = 0\ , \tag{23}$$

where

$$W_u = \mu\frac{d}{d\mu}u\Big|_{u_0,\Lambda}\ , \tag{24}$$

$$W_b = \mu\frac{d}{d\mu}b\Big|_{u_0, b_0, \Lambda}\ , \tag{25}$$

and

$$\eta_\varphi = \mu\frac{d}{d\mu}\ln Z_\varphi = W_u\frac{\partial}{\partial u}\ln Z_\varphi\ , \tag{26}$$

$$\gamma_\Gamma = \mu\frac{d}{d\mu}\ln Z_\Gamma = W_u\frac{\partial}{\partial u}\ln Z_\Gamma + W_b\frac{\partial}{\partial b}\ln Z_\Gamma\ . \tag{27}$$

$Z_\Gamma(u, b^2; \mu/\Lambda)$ being defined by (12)–(14), (20) and (21). Note that static normalization conditions [Eqs. (16)–(18)] are enough to determine completely W_u and η_φ, which are thus *identical* with the corresponding static functions. If there exists an infrared-stable fixed point (u^*, b^*) such that

$$W_u(u^*) = W_b(u^*, b^*) = 0\ , \tag{28}$$

then the asymptotic solution of (23) is given by $(\eta_\varphi^* \equiv \eta)$,

$$\Gamma^{(2)}(-i\zeta, k) \simeq \mu^2(k/\mu)^{2-\eta_\varphi}\times\Phi[(-i\zeta/\mu)(\mu/k)^{2+\gamma_\Gamma^*-\eta_\varphi}]\ , \tag{29}$$

which establishes the scaling form[14] with the standard dynamical exponent

$$\omega_\varphi(k) = (k/\mu)^{z_\varphi},\quad z_\varphi = 2 + \gamma_\Gamma^* - \eta\ . \tag{30}$$

Letting

$$\eta_\Gamma \equiv \gamma_\Gamma - \eta_\varphi\ , \tag{31}$$

$$\hat\eta \equiv \eta_{\varphi^2} - \eta_\varphi\ , \tag{32}$$

one may now vary the temperature in the vicinity of T_c, with[3]

$$t \equiv (T-T_c) Z_\varphi / Z_{\varphi^2} \tag{33}$$

[just like we had $\zeta \equiv z Z_\varphi / \Gamma_0 Z_\Gamma$, Eq. (20)], this yields

$$\left[\left(\mu \frac{\partial}{\partial \mu} + W_u \frac{\partial}{\partial u} + i W_b \frac{\partial}{\partial i b} - \eta_\varphi\right) - \eta_\Gamma \zeta \frac{\partial}{\partial \zeta} - \hat{\eta} t \frac{\partial}{\partial t}\right] \times \Gamma^{(2)}(-i\zeta, k; ib, u, t, \mu) = 0 \,, \tag{34}$$

whose asymptotic solution is

$$\Gamma^{(2)}(-i\zeta, k) \simeq \mu^2 \left(\frac{k}{\mu}\right)^{2-\eta} \times \Phi\left[\frac{t}{\mu^2}\left(\frac{\mu}{k}\right)^{2+\hat{\eta}^*}, \frac{-i\zeta}{\mu^2}\left(\frac{\mu}{k}\right)^{2+\eta_\Gamma^*}\right] \,. \tag{35}$$

Returning to the original bare correlation function, we obtain

$$\Gamma_B^{(2)}\left(\frac{-iz}{\Gamma_0}, k, T-T_c\right) \simeq k^2 \left(\frac{k}{\Lambda}\right)^{-\eta} \times \Phi\left[\frac{T-T_c}{\Lambda^2}\left(\frac{\Lambda}{k}\right)^{2+\hat{\eta}^*}, \frac{-iz}{\Gamma_0 \Lambda^2}\left(\frac{\Lambda}{k}\right)^{2+\eta_\Gamma^*}\right], \tag{36}$$

where the *full scaling form* is now established.

IV. CALCULATIONS TO LOWEST ORDER IN ϵ AND N^{-1}

Let us first consider the lowest-order terms already computed by HHM using the step-by-step Wilson-Kadanoff procedure.

A. ϵ expansion

In the ϵ expansion, we get to lowest order in u and ϵ

$$\Gamma^{(2)}(-i\zeta, k=0) - \Gamma^{(2)}(0,0) \equiv \Gamma^{(2)}_{\text{dyn}}(-i\zeta, k=0) = \frac{-i\zeta}{1+ib}\left[1 + \frac{N+2}{36}\left(\frac{-i\zeta}{\mu^2(1+ib)}\right)^{-\epsilon} \frac{u^2 D_0}{\epsilon}\right] - \frac{i\zeta}{1+ib} C(ib, u) \,, \tag{37}$$

with

$$D_0 = \frac{3-ib}{1-ib}\ln 2 - \frac{1}{2}\frac{3-ib}{1-ib}\ln(3-ib) - \frac{1}{2}\frac{1+ib}{1-ib}\ln(1+ib) \equiv R + iJ \,, \tag{38}$$

and where $C(ib, u)$ is determined by the normalization condition (21). Together with definitions (25) and (27) one finds, to lowest orders

$$\gamma_\Gamma + i\frac{W_b}{1+ib} = -W_u \frac{\partial}{\partial u} C(ib, u) \,, \tag{39}$$

that is

$$\gamma_\Gamma + i\frac{W_b}{1+ib} = \tfrac{1}{18}(N+2)u^2 [R(b) + iJ(b)] \,. \tag{40}$$

Now $J(b)$ has a *unique zero* on the real axis, with a *positive* slope. Near the origin we have

$$W_b \simeq \tfrac{1}{18}(N+2)(\ln\tfrac{4}{3})u^2 b \,.$$

Hence $b^* = 0$, together with the static value $u^* = 6\epsilon/(N+8)$, is an *attractive infrared fixed point* in (u, b) space.[15] At this *fixed point* which is *stable* because $\partial W_b / \partial b > 0$, one has

$$\gamma_\Gamma^* = \tfrac{1}{18}(N+2)u^{*2} R 0 = \epsilon^2[3(N+2)/(N+8)^2]\ln(\tfrac{4}{3}) \,, \tag{41}$$

which coincides with the value obtained by HHM and several authors,[7]

$$z_\varphi - 2 = \eta[6\ln(\tfrac{4}{3}) - 1] = 0.726\,09\,\eta \,. \tag{42}$$

At this point one may also recover a result common to several authors,[16] who have applied the matching Wilson-Fisher[1] method to the microscopic model of an interacting Bose system. Indeed, it is obtained by letting $\Gamma_0 = b_0^{-1} = 0$ on the model used here, and their pseudoexponent $\tilde{\gamma}_\Gamma$ is then given by

$$\tilde{\gamma}_\Gamma = \tfrac{1}{18}(N+2)u^{*2} R(+\infty) = \epsilon^2[(N+2)/(N+8)^2] 2\ln 2 \tag{43}$$

$$\tilde{z}_\varphi - 2 = \eta(4\ln 2 - 1) \,.$$

However, it is clear from the above that $b = +\infty$ is not a fixed point[17] and that Eq. (43) is meaningless.

B. N^{-1} expansion

In the N^{-1} expansion in the lowest order Eq. (39) remains valid. At the fixed-point value for the coupling constant $u^* = 6/Na(\epsilon)$, it yields

$$\gamma_\Gamma(b) + i\frac{W_b(b)}{1+ib} = \frac{2}{N}\left(\frac{1}{a-D} - \frac{1}{a}\right) \,, \tag{44}$$

where

$$a(\epsilon) = \frac{1-\frac{1}{2}\epsilon}{\epsilon}\frac{\Gamma^3(1-\frac{1}{2}\epsilon)\Gamma(1+\frac{1}{2}\epsilon)}{\Gamma(2-\epsilon)} \,, \tag{45}$$

is a function related to the exponent η by $\eta \simeq 2\epsilon/N(4-\epsilon)a(\epsilon)$, and where

$$D(ib, \epsilon) = a(\epsilon) + \frac{\Gamma(2-\frac{1}{2}\epsilon)\Gamma(1+\frac{1}{2}\epsilon)}{\epsilon} \times \left[\left((1-ib)^{\epsilon-1}\int_1^{1-(1-ib)^2/2} - \int_{(1-ib)/2}^{1}\right)\frac{dx}{(1-x^2)^{\epsilon/2}}\right] \,. \tag{46}$$

Also

$$D(ib, 0) \equiv D_0 \,. \tag{47}$$

In *three dimensions*

$$D(ib, 1) = \tfrac{1}{4} i\pi \ln\tfrac{1}{2}[1 - ib - i(3 + 2ib + b^2)^{1/2}] \,, \tag{48}$$

and $W_b/(1+b^2)$ vanishes when the imaginary part of D vanishes or becomes infinite. It vanishes for $b^* = 0$, and in that vicinity

$$W_b \simeq \omega_b b, \quad \omega_b = (6\sqrt{3})/N\pi^3 \,, \tag{49}$$

is an infrared-stable fixed point. At that point, with (44) one has

$$\gamma_\Gamma^* = 4/N\pi^2 = \tfrac{3}{2}\eta \ , \tag{50}$$

i.e., as in HHM

$$z_\varphi - 2 = \tfrac{1}{2}\eta \ . \tag{51}$$

The imaginary part of D becomes infinite for $b = +\infty$. Switching to the variable b^{-1}, one has then in the vicinity of $b^{-1} = 0$,

$$\frac{W_{1/b}}{1 + 1/b^2} \simeq -\frac{4}{\pi \ln b^{-1}} , \tag{52}$$

that is a zero, but too weak to give rise to an attractive fixed point (besides the fact that for $b^{-1} = 0$, we return to the microscopic Lagrangian and its associated hydrodynamic singularities). For that value, one has,[18]

$$\tilde{\gamma}_\Gamma = -8/N\pi^2 = -3\eta \ , \tag{53}$$

$$\tilde{z}_\varphi - 2 = -4\eta \ , \tag{54}$$

again a meaningless result.

For *completeness* we quote the results for $0 \le \epsilon < 2$. For $b^* = 0$, one obtains

$$\omega_b = \frac{2}{N} [a(\epsilon) - D(0, \epsilon)]^{-2} \frac{\partial D}{\partial ib} (ib; \epsilon)\bigg|_{b=0} , \tag{55}$$

with

$$a(\epsilon) - D(0, \epsilon) = \frac{2\Gamma(2 - \frac{1}{2}\epsilon)\Gamma(1 + \frac{1}{2}\epsilon)}{\epsilon} \left(2^{-\epsilon} \frac{\Gamma^2(1 - \frac{1}{2}\epsilon)}{\Gamma(1 - \epsilon)} - (1 - \epsilon) \int_0^{\pi/6} (\cos\theta)^{1-\epsilon} d\theta \right) , \tag{56}$$

$$\frac{\partial D}{\partial ib}(ib, \epsilon)\bigg|_{b=0} = \frac{\Gamma(2 - \frac{1}{2}\epsilon)\Gamma(1 + \frac{1}{2}\epsilon)}{2\epsilon} \left(\frac{4}{3}\right)^{\epsilon/2} + \tfrac{1}{2}(1 - \epsilon)[a(\epsilon) - D(0, \epsilon)] , \tag{57}$$

hence $\gamma_\Gamma^*(\epsilon)$ with (44). Likewise for $b = +\infty$, we have[18] for $0 < \epsilon \le 1$,

$$a(\epsilon) - D(+\infty, \epsilon) = \Gamma(2 - \tfrac{1}{2}\epsilon)\Gamma(1 + \tfrac{1}{2}\epsilon) \left(\frac{2^{-\epsilon}\Gamma^2(1 - \frac{1}{2}\epsilon)}{\epsilon\Gamma(2 - \epsilon)} - \frac{i\pi 2^{-\epsilon}\Gamma(1 + \epsilon)}{2\Gamma^2(1 + \frac{1}{2}\epsilon)(1 - \epsilon)} \right) , \tag{58}$$

that is

$$\tilde{\gamma}_\Gamma + iW_{1/b} \quad (b^{-1} = 0) = \eta(\epsilon) \frac{4 - \epsilon}{\epsilon} (1 - 2^{\epsilon-1} - 2^{\epsilon-1} e^{i\pi\epsilon}) , \tag{59}$$

showing a zero of $W_{1/b}$ for $\epsilon = 1$ at $b^{-1} = 0$, the (soft) behavior around this zero being given by (52).

Instead, for $1 \le \epsilon < 2$, $[a(\epsilon) - D(+\infty, \epsilon)]^{-1} = 0$ and[18]

$$\tilde{\gamma}_\Gamma + iW_c \quad (c = 0) = -\eta \frac{4 - \epsilon}{\epsilon}, \quad \tilde{z}_\varphi - 2 = -(4/\epsilon)\eta \ . \tag{60}$$

Again, $W_{1/b}$ $(b^{-1} = 0)$ vanishes, but now as $(1/b)^{\epsilon-1}$. Only in the two-dimension limit would this zero become attractive enough for scaling, a limit in which the whole procedure used here breaks down anyway. Actually, this limit (and the whole region $1 < \epsilon \le 2$) deserves better attention. Ikami[19] has computed directly the leading logarithms in the $1/N$ expansion up to order $(1/N)^3$, and verified exponentiation with the power $\tilde{z}_\varphi$ of Eq. (60), a power identical to the one obtained with a multicomponent Bose model by Halperin.[20] If these results are indeed identical, it means that the singularities associated with hydrodynamic modes are less harmful than anticipated, below three dimensions, for the $1/N$ series.[21]

V. CALCULATION TO NEXT ORDER IN ϵ

To the next order in the ϵ expansion, Eq. (39) remains valid and one finds

$$\gamma_\Gamma + i \frac{W_b}{1 + ib} = \frac{N+2}{18} u^2 \left(D - \epsilon(D_0 - E_0) + \frac{N+8}{6} u (D_0 - E_0 + D_0^2) - \frac{u}{6} \frac{4ibD_0^2}{1 + ib} \right) , \tag{61}$$

where $D(ib, \epsilon)$ given as above by (45) and (46) should be retained to first order in ϵ, and D_0 is $D(ib, 0)$ from (38) and (47), also

$$E_0 = \int_1^\infty \frac{dx}{x} \frac{d(ib, x) - d(ib, 1)}{x - 1} , \tag{62}$$

with

$$d(ib, x) = \frac{1 + x}{2} \ln(1 + x) - \left(\frac{x(1 + A) + A}{2A} \right) \ln[x(1 + A) + A] + \frac{x(1 + A)}{2A} \ln[x(1 + A)] + \frac{x + A}{2A} \ln(x + A) , \tag{63}$$

$$A = \frac{1 - ib}{1 + ib} . \tag{64}$$

At the fixed point value

$$u^* = \frac{6\epsilon}{N+8} \left[1 + \epsilon \left(\frac{3(3N + 14)}{(N+8)^2} - \frac{1}{4} \right) \right] ,$$

one has

$$\gamma_\Gamma^* + i \frac{W_b}{1 + ib} = \frac{N+2}{18} u^{*2} \left[D + \epsilon D_0^2 \left(1 - \frac{4ib}{(N+8)(1 + ib)} \right) \right] . \tag{65}$$

The fixed point $b^* = 0$ is unchanged and remains attractive, with a slope

$$\omega_b = \epsilon^2 \frac{N+2}{(N+8)^2} 2\ln(\tfrac{4}{3}) \times \left[1+\epsilon\left(\frac{6(3N+14)}{(N+8)^2} - \frac{9\ln\frac{4}{3}}{N+8} - 1.531\right)\right] , \quad (66)$$

the corresponding value of γ_Γ^* is

$$\gamma_\Gamma^* = \epsilon^2 \frac{N+2}{(N+8)^2} 3\ln(\tfrac{4}{3}) \times \left[1+\epsilon\left(\frac{6(3N+14)}{(N+8)^2} - 1.475\right)\right] , \quad (67)$$

which yields

$$\gamma_\Gamma^* = \begin{cases} \epsilon^2 \times 3.20\times 10^{-2}(1-0.216\epsilon)+O(\epsilon^4) & \text{for } N=1 \\ \epsilon^2\times 3.45\times 10^{-2}(1-0.275\epsilon)+O(\epsilon^4) & \text{for } N=2 . \end{cases} \quad (68)$$

The parametrization used by HHM, i.e., $z_\varphi - 2 = \eta c(\epsilon)$, is clearly only possible to lowest and next to lowest order, since, in general, c ought to depend on both ϵ and N. To the order considered here, this parametrization reflects the poor convergence character of the ϵ series for η,

$$c(\epsilon) = 0.7261(1-1.687\epsilon)+O(\epsilon^2) . \quad (69)$$

Comparison with "experiment" can be attempted for the Ising system ($N=1$) (with a dynamic *à la* Glauber[22] equivalent to the HHM model considered). Using high-temperature series, Yahata[23] has obtained, in three dimensions

$$\Delta_Y \equiv \nu z_\varphi = 1.413 . \quad (70)$$

If one takes this value seriously, it implies a γ_Γ^* ten times as large as given above [Eq. (68)], an improbable result. Representation of dynamical effects through the exponent Δ is not very convenient, since it is insensitive to the dynamics contained in γ_Γ^*, and if one tries to get $z_\varphi - 2$ from (70) the result is very sensitive to errors on Δ_Y. For completeness we give $\Delta = \nu z_\varphi$ in powers of ϵ

$$\Delta = 1+\tfrac{1}{6}\epsilon + 0.0931\epsilon^2 - 0.167\epsilon^3 + O(\epsilon^4) , \quad (71)$$

that is $\Delta = 1.243$ in three dimensions.

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APPENDIX

In this appendix we first recall the detailed rules for the perturbation series expansion of $\Gamma^{(2)}$ defined by Eqs. (6)–(9). We then show how the extra counter term introduced in Eq. (12) effectively removes the internal (logarithmic) divergences appearing along with subdiagrams (insertions).

Let us complete the rules given with Eq. (10). Diagrams are time-ordered, time grows from left to right, the creation vertex of the mass operator $\Sigma(t,k)$ is at $\theta=0$, the annihilation vertex at $\theta=t$. Its Fourier-Laplace transform $\Sigma(z,k)$ is, thus, the sum of all distinct time-ordered diagrams, and with each diagram is associated a contribution $\mathfrak{D}$, which is calculated within standard weight factors, with the following rules.

A. Dynamic part (between 0 and t)

(*i*) With each vertex associate

$$-g_0 \sum_{p,h} (\Delta_p + \Delta_h) ,$$

where the sum is over holes and particles lines incoming to or outgoing from the vertex and at the left of it, $\Delta_{p,h} = \Gamma_0 \epsilon^0 (1\pm ib_0)$.

(*ii*) With each interval between successive vertices associate the propagator

$$\left(-iz + \sum_{p,h} (\Delta_p + \Delta_h)\right)^{-1} ,$$

where the sum is now over each hole and particle present in the interval.

B. Static parts

Defined as each part of the diagram which can be separated by cutting any number of lines at a time θ_0, such that θ_0 can be freely moved to $\theta_0 = -\infty$ (keeping fixed the creation vertex $\theta=0$, the annihilation vertex $\theta=t$, and the relative order of vertices internal to each static part). These static parts are to be looked at as Feynman-Wilson diagrams with their associated rule:

(*i*) vertex: $(-g_0)$

(*ii*) line l: $(\epsilon_l^0)^{-1}$.

If we work with the intermediate Lagrangian $\mathcal{L}_1$, Eq. (12), then iz is replaced by izZ_φ, $\Delta_{p,h}$ by $\Gamma_0 Z_\Gamma \epsilon^0 (1\pm ib)$, and ϵ_l^0 now means $r+l^2$, that is, the effect of the main terms [first line of Eq. (12)] of $\mathcal{L}_1$ is to give rise to contributions to $\Gamma^{(2)}$,

$$\mathfrak{D}_1 = -\left((-u\mu^\epsilon)^n \prod_l \frac{1}{\epsilon_l^0}\right) \prod_j \frac{\sum_{\epsilon v(j)} \epsilon^0(1\pm ib)}{-i\zeta + \sum_{\epsilon j} \epsilon^0 (1\pm ib)} ,$$

with $\zeta = zZ_\varphi/\Gamma_0 Z_\Gamma$, the summations being as explained above. The effect of counter terms [last term of Eq. (12), besides the standard static ones] is obtained by replacing $(1+ib)$ by $(1+ib)/(1+C)$, and $(1-ib)$ by $(1-ib)/(1+C^*)$, and expanding in powers of C and C^*, C is defined by the normalization condition (21). In the following, to abbreviate we write $\epsilon_j \equiv \epsilon_j^0(1+ib)$.

Let us now look at divergences and how they are compensated by counter terms. As an example,

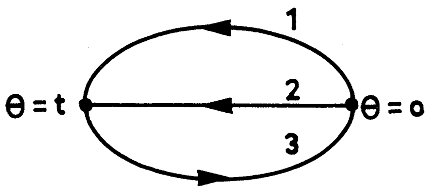

FIG. 1. Second-order diagram contributing to the mass operator.

we work out in detail the case of second-order insertions in the second-order diagram (Fig. 2).

1. Second order (Fig. 1)

Main term: We have for the contribution to $\Gamma^{(2)}$

$$\mathcal{D}_1(1) = -\mu^{2\epsilon}u^2\frac{N+2}{18}\int d1\,d2 \times\left(1+\frac{i\zeta}{-i\zeta+\epsilon_1+\epsilon_2+\epsilon_3^*}\right)\frac{1}{\epsilon_1^0\epsilon_2^0\epsilon_3^0}$$
$$\equiv -\Sigma^{(2)}(-i\zeta)\equiv -\left[\Sigma^{(2)}_{\mathrm{st}}+\Sigma^{(2)}_{\mathrm{dyn}}(-i\zeta)\right]. \qquad \text{(A1)}$$

The divergent term (in four dimensions) of the dynamic part is thus [Eq. (37)]

$$\Sigma^{(2)}_{\mathrm{dyn}}(-i\zeta)\simeq -\frac{i\zeta}{1+ib}\delta\ , \qquad \text{(A2)}$$

with

$$\delta = u^2\,\frac{N+2}{36}\,\frac{R+iJ}{\epsilon}\ . \qquad \text{(A3)}$$

Counter term: $-i\zeta C/(1+ib)$ and the normalization (21) imposes

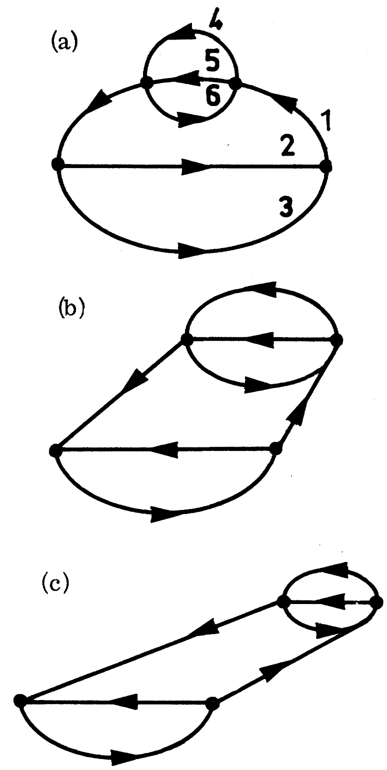

FIG. 2. Fourth-order diagrams with a second-order insertion.

$$C = -\delta\ , \qquad \text{(A4)}$$

thus compensating the divergent term.

2. Fourth order (Fig. 2)

Main term: The contribution of Fig. 2(a) can be written

$$\mathcal{D}_1(2\mathrm{a}) = \left(-\frac{N+2}{18}\mu^{2\epsilon}u^2\int d1\,d2\,\frac{\epsilon_1+\epsilon_2+\epsilon_3^*}{(-i\zeta+\epsilon_1+\epsilon_2+\epsilon_3^*)^2}\,\frac{1}{\epsilon_1^0\epsilon_2^0\epsilon_3^0}\right)\Sigma^{(2)}(-i\zeta+\epsilon_2+\epsilon_3^*)\ . \qquad \text{(A5)}$$

The quantity $\Sigma^{(2)}$ gives rise to a static part whose divergence is removed, as usual, and a dynamic part which as in (A1)–(A3) gives rise to the divergent contribution

$$(i\zeta-\epsilon_2-\epsilon_3^*)\delta. \qquad \text{(A6)}$$

The contribution of Fig. 2(b) is

$$\mathcal{D}_1(2\mathrm{b}) = \left(-\frac{N+2}{18}\mu^{2\epsilon}u^2\int d1\,d2\,\frac{\epsilon_2+\epsilon_3^*}{-i\zeta+\epsilon_1+\epsilon_2+\epsilon_3^*}\,\frac{1}{\epsilon_1^0\epsilon_2^0\epsilon_3^0}\right)\overline{\Sigma}^{(2)}(-i\zeta+\epsilon_2+\epsilon_3^*) \qquad \text{(A7)}$$

with

$$\overline{\Sigma}^{(2)}(-i\zeta) = \left(\frac{N+2}{18}u^2\mu^{2\epsilon}(1+ib)\int d4\,d5\,\frac{1}{-i\zeta+\epsilon_4+\epsilon_5+\epsilon_6^*}\,\frac{1}{\epsilon_4^0\epsilon_5^0\epsilon_6^0}\right)\ . \qquad \text{(A8)}$$

Again using (A1)–(A3) one may write the divergent part of $\overline{\Sigma}^{(2)}(-i\zeta)$ as δ. That is, besides a factor identical to the first bracket of (A6), one obtains for (A7)

$$(\epsilon_2+\epsilon_3^*)\delta\frac{-i\zeta+\epsilon_1+\epsilon_2+\epsilon_3^*}{\epsilon_1+\epsilon_2+\epsilon_3^*}\ .$$

Finally, the diagram of Fig. 2(c) contains no dynamic divergent part. Putting them all together, we get the first bracket in (A5) multiplied by

$$+i\zeta\,\frac{\epsilon_1\delta}{\epsilon_1+\epsilon_2+\epsilon_3^*}\ . \tag{A9}$$

Counter terms: they are obtained by expanding $\mathfrak{D}(1)$, where ϵ_1 is replaced by $\epsilon_1/(1+C)$, to first order in C. We again get the same common factor as in (A5) multiplied by

$$+i\zeta\,\frac{\epsilon_1 C}{\epsilon_1+\epsilon_2+\epsilon_3^*}\ , \tag{A10}$$

that is, with (A4) a term cancelling exactly, (A9).

3. General term

The above cancellation of divergent parts is easily shown to operate for any number of insertions made on the same line. When those insertions are made on two or more lines, cancellations are more tedious to display directly. A general proof is, as usual, made easy by working line by line, and more precisely, on Fourier-Laplace transforms of line propagators. Let, $g_p(-iz_1)$, be that transform for the particle line 1, we write

$$g_p(-iz_1)=\Gamma(-iz_1)\Phi^+(-iz_1)\ , \tag{A11}$$

with[24]

$$\Gamma(-iz)=-iz+\frac{\epsilon}{1+C}-\frac{1+ib}{1+C}\Sigma^{(2)}(-iz)\ , \tag{A12}$$

$$\Phi^\pm(-iz)=1\pm\frac{\epsilon}{2(1+C)}+\frac{1+ib}{1+C}\overline{\Sigma}^{(2)}(-iz)\ . \tag{A13}$$

For the hole line 3, we would have

$$g_h(-iz_3)\equiv g_p^*(+iz_3^*)\ . \tag{A14}$$

Finally, the global contribution for any number of insertions made on the lines of Fig. 1, can be written

$$\mathfrak{D}_1=-\frac{N+2}{18}u^2\mu^{2\epsilon}\int d1\,d2\int_0^\infty e^{i\zeta t}\,dt\int dz_1dz_2dz_3\,e^{-it(z_1+z_2+z_3)}$$
$$\times\Gamma(-iz_1)\Gamma(-iz_2)\Gamma^*(+iz_3^*)[\Phi^+(-iz_1)\Phi^+(-iz_2)\Phi^{+*}(+iz_3^*)-\Phi^-(-iz_1)\Phi^-(-iz_2)\Phi^{-*}(+iz_3^*)]\ , \tag{A15}$$

where in the last bracket, only the terms *linear* in ϵ should be retained (terms independent of and quadratic in ϵ drop out, cubic terms in ϵ are not retained in accordance with rule [a(i)] above).

Using (A2), that is,

$$\Sigma^{(2)}(-iz)=izC/(1+ib)+R^{(2)}(iz)\ , \tag{A16}$$

$$\overline{\Sigma}^{(2)}(-iz)=-C+\overline{R}^{(2)}(iz)\ , \tag{A17}$$

where $R^{(2)}$, and $\overline{R}^{(2)}$ are regular parts, we have

$$\Gamma(-iz)=\frac{1}{1+C}\left[-iz+\epsilon-(1+ib)R^{(2)}(iz)\right]$$

$$\equiv\frac{1}{1+C}\Gamma_R(-iz)\ , \tag{A18}$$

$$\Phi^\pm(-iz)=\frac{1}{1+C}\left(1\pm\tfrac{1}{2}\epsilon+(1+ib)\overline{R}^{(2)}(iz)\right)$$

$$\equiv\frac{1}{1+C}\Phi_R^\pm(-iz)\ . \tag{A19}$$

It is then obvious that, in (A15), it is correct to replace everywhere the Γ's and the $\Phi^\pm$'s by the *corresponding regular expressions* Γ_R and $\Phi_R^\pm$.

The above procedure is trivially extended to any type of insertion or iteration thereof, proving thus, that the extra counter term of Eq. (12) is enough to renormalize $\Gamma^{(2)}_{\rm dyn}$.

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