Equation of state of a classical electron layer*†

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The equation of state of a dilute two-dimensional classical electron gas is calculated by the virial expansion in the ring approximation. Four corrections to ideal-gas behavior are obtained exactly; the central result, in terms of the small parameter $\epsilon^2 = 4\pi\rho e^4/(k_B T)^2$, is $P/\rho k_B T - 1 = \epsilon^2 \ln(2\epsilon)/4 - (1 - \gamma/2)\epsilon^2/4 - \epsilon^4 \ln^2(\epsilon/2)/4 + (1/2 - \gamma)\epsilon^4 \ln\epsilon/2 + O(\epsilon^4)$, where γ is Euler's constant. The lowest-order correction agrees with a Debye-Hückel calculation, which is also presented.

I. INTRODUCTION

The prediction and discovery of electron states bound above the free surface of liquid helium and other materials[1-5] have stimulated work on two-dimensional charged systems. Most previous attention has focused on the low-temperature region (also experimentally realizable by inversion layers in some semiconductors)[6,7] where the effects of quantum degeneracy are important. However, it has been pointed out that use of helium and (more favorably) neon bases may give the opportunity to observe surface states at such low dilution and high temperatures that the system behaves classically.[8,9] A calculation of the equation of state of a classical charged layer thus becomes desirable; such a calculation is presented here for a sheet of mobile charges imbedded in a uniform rigid neutralizing background.

Experience with the three-dimensional electron gas has revealed that systematic perturbation expansions for the equation of state contain terms which diverge because of the long-range nature of the interaction.[10-13] Infinite sequences of such divergent terms must be identified and summed to obtain convergent results; in consequence, the equation of state acquires nonanalytic dependence upon the expansion parameter. The lowest-order and simplest such sequence is the set of ring integrals. Our calculation is restricted to the ring integrals, whose evaluation allows a number of terms in the equation of state to be determined exactly.

This paper is divided into three parts. In Sec. II we determine the pressure via a linearized Debye-Hückel calculation of the screening; in three dimensions, this yields the correct equation of state to lowest order, but in two dimensions, a divergence is found unless a physically plausible cutoff is inserted; this underscores the need for a more careful calculation. In Sec. III we introduce the virial expansion and ring integrals, and transfer the calculation to Fourier space. We confirm the Debye-Hückel result and obtain a number of higher corrections, and show that refinements of the ring approximation yield corrections of higher order than ours. Section IV ends the paper with a discussion of the physical basis of our results.

II. DEBYE-HÜCKEL CALCULATION

We shall find the thermodynamic potential by determining the work done to assemble the screening charge around each electron.[14,15] Because the electrons are distinguishable particles in the classical limit, a given electron is regarded as a test charge inserted into the system. We begin by calculating the screened potential. If the mean electron density is ρ, the density $\rho(x)$ around a single electron is well approximated by the relation

$$\rho(x) = \rho e^{\beta e\varphi(x)} \quad (e = |e|),$$

where φ is the electrostatic potential of the test charge and the charge distribution screening it. The potential φ satisfies Poisson's equation; linearized in φ, this equation becomes

$$\nabla^2\varphi \simeq 4\pi e\delta(\vec{r})\delta(z) + 4\pi\beta e^2\rho\varphi(\vec{r},z)\delta(z). \quad (2.1)$$

The layer lies in the plane $z = 0$, and r denotes the radial coordinate in the xy plane. The solution of (2.1) is a solution of Laplace's equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\varphi(r,z) = 0,$$

with boundary conditions for the surface charge density,

$$-4\pi\sigma(r) = \frac{\partial\varphi}{\partial z}\bigg|_{z=-\epsilon}^{z=\epsilon} = 4\pi e\delta(\vec{r}) + 4\pi\beta e^2\rho\varphi(r,0). \quad (2.2)$$

We make the expansion

$$\varphi(r,z) = \int_0^\infty dk\, f(k)e^{-k|z|} J_0(kr), \quad (2.3)$$

where J_0 denotes a Bessel function. Substituting this expression into (2.2), we obtain

$$\int_0^\infty dk(k + k_D)f(k)J_0(kr) = -(e/r)\delta(r), \quad (2.4)$$

wherein we have defined the Debye wave number

$$k_D \equiv 2\pi\beta e^2\rho. \quad (2.5)$$

Inverting the Bessel transform, we find

$$f(k) = \frac{-e}{1 + k_D/k},$$

$$\varphi(r, z) = -e \int_0^\infty \frac{dk}{1 + k_D/k} e^{-k|z|} J_0(kr). \qquad (2.6)$$

This yields[7-9]

$$\varphi(r, 0) = -(e/r)\{1 - \tfrac{1}{2}\pi k_D r [\overline{H}_0(k_D r) - Y_0(k_D r)]\}, \qquad (2.7)$$

where $\overline{H}_0$ and Y_0 are Struve and Neumann functions. The charge density obtained from (2.2) is

$$\sigma(r) = -e\delta(\vec{r}) - (ek_D/2\pi r)\{-1 + \tfrac{1}{2}\pi k_D r$$
$$\times [\overline{H}_0(k_D r) - Y_0(k_D r)]\}. \qquad (2.8)$$

The representation

$$\overline{H}_0(x) - Y_0(x) = \frac{2}{\pi} \int_0^\infty \frac{dt}{(1 + t^2)^{1/2}} e^{-xt}, \quad |\arg x| < \pi$$

may be used to show that the total surface charge vanishes, indicating that the test charge is completely screened at large distances.[9]

The work done by the system when each of the N electrons has its charge increased by $-de$ is

$$dW = N de\, \varphi_{\text{cloud}}(0),$$
$$\varphi_{\text{cloud}}(r) = ek_D \int_0^\infty \frac{dk}{k + k_D} J_0(kr). \qquad (2.9)$$

The potential φ_{cloud} of the induced charge diverges logarithmically near the origin, and (2.8) shows that the divergence is due to an infinite positive-induced charge density there. Since the density of the positive background is bounded in our model, the divergence arises because the linearized Poisson Eq. (2.1) is invalid near the test charge, for the induced charge density becomes greater than the density of the background. The distance r_c at which this happens may be estimated from (2.8), i.e., $\rho e \approx \sigma(r_c)$; r_c serves as a cutoff in (2.9). We have

$$r_c \approx k_D/\rho \approx \beta e^2 \quad (k_D r_c \ll 1),$$
$$\varphi_{\text{cloud}}(0) \approx ek_D |\ln(k_D r_c)| \approx -2\pi\beta e^3 \rho \ln(\beta^2 e^4 \rho). \qquad (2.10)$$

Identifying $-dW$ as the isothermal change in the Helmholtz free energy, we find that the contribution of the screening charge to the thermodynamic potential is

$$-\Delta P = \rho \int_0^e de\, (-2\pi\beta e^2 \rho)\ln(\beta^2 e^4 \rho)$$
$$\approx -\pi\rho(\beta e^2 \rho^{1/2})\ln(\beta e^2 \rho^{1/2}). \qquad (2.11)$$

The equation of state becomes

$$\beta P/\rho - 1 = \pi(\beta e^2 \rho^{1/2})^2 \ln(\beta e^2 \rho^{1/2}). \qquad (2.12)$$

We have confidence in the correction only to leading order; thus, we have retained no second-order terms (proportional to e^4).

The correction is nonanalytic in the density and coupling constant, as we expected for a long-range force. We recall, however, that in three dimensions, no cutoffs must be introduced into the linearized Debye-Hückel theory to obtain the correct equation of state.[14,15] The validity of (2.12) should be confirmed by a more formal calculation; furthermore, such a calculation is needed to determine corrections of higher order.

III. VIRIAL EXPANSION

The virial expansion for the equation of state is[16]

$$\beta P/\rho - 1 = -\sum_{m=1}^\infty \frac{m}{m+1} \beta_m \rho^m,$$

$$\beta_{m-1} = \lim_{A \to \infty} \frac{1}{A(m-1)!} \int d^2 r_1 \cdots d^2 r_m$$
$$\times \sum_{(S_m)} \prod_{S_m} f_{ij}, \quad (m > 2), \qquad (3.1)$$

$$\beta_1 = \lim_{A \to \infty} \frac{1}{A} \int d^2 r_1 d^2 r_2 f'_{12}.$$

The Mayer f functions are

$$f_{ij} = e^{-\beta V(r_{ij})} - 1, \quad f'_{ij} = f_{ij} + \beta V(r_{ij}),$$

with $V(r) = e^2/r$. The sum over S_m in (3.1) denotes a sum over all star configurations with m labeled points. The neutralizing background makes the first-order term in the coupling constant e^2 vanish for a Coulomb system, and hence, this term has already been subtracted in the above expansion.

We restrict ourselves to the ring graphs, so that each labeled point is connected by f functions to only two other labeled points (for $m > 2$). The integrations are transferred to Fourier space and a standard calculation yields

$$\int d^2 r_1 \ldots d^2 r_m \sum_{(S_m^{\text{ring}})} \prod_{S_m^{\text{ring}}} f_{ij}$$
$$= A\tfrac{1}{2}(m-1)! \int \frac{d^2 k}{(2\pi)^2} [f(k)]^m \quad (m > 2), \qquad (3.2)$$

$$f(k) = \int d^2 r\, e^{-i\vec{k} \cdot \vec{r}} f(r).$$

The numerical factor multiplying the integral comes from a summation over topologically equivalent graphs. The virial expansion becomes

$$\frac{\beta P}{\rho} - 1 = -\frac{1}{2\rho} \int \frac{d^2 k}{(2\pi)^2} \sum_{m=2}^\infty \left(1 - \frac{1}{m+1}\right)$$
$$\times [\rho f(k)]^{m+1} - \tfrac{1}{2}\rho \int d^2 r f'(r). \qquad (3.3)$$

Adding the term

$$0 = -\tfrac{1}{4}\rho \int d^2 r\, [f(r)]^2 + \tfrac{1}{4}\rho \int \frac{d^2 k}{(2\pi)^2} [f(k)]^2,$$

and performing the summations, we obtain the

equation of state

$$\frac{\beta P}{\rho} - 1 = -\frac{1}{2\rho} \int \frac{d^2 k}{(2\pi)^2} \left(\frac{\rho f(k)}{1 - \rho f(k)} + \ln[1 - \rho f(k)] \right)$$

$$+ \tfrac{1}{2}\rho \int d^2 r \{ \tfrac{1}{2}[f(r)]^2 - f'(r) \}. \qquad (3.4)$$

Evaluating the second integral gives

$$\frac{\beta P}{\rho} - 1 = -\frac{1}{2\rho} \left(1 - \rho \frac{\partial}{\partial \rho} \right) S - \pi \rho \beta^2 e^4 \ln 2,$$

$$S = \int \frac{d^2 k}{(2\pi)^2} \{ \ln[1 - \rho f(k)] + \rho f(k) \} . \qquad (3.5)$$

We determine $f(k)$ with an integration by parts and the formulas[17]

$$\int_0^a x^{n+1} J_n(x) \, dx = a^{n+1} J_{n+1}(a),$$

$$\int_0^\infty dx \, x^{-1} e^{-2a/x} J_n(x) = 2 J_n(2 a^{1/2}) K_n(2 a^{1/2}), \qquad (3.6)$$

where K_n is a modified Bessel function of the second kind,

$$f(k) = -(4\pi \beta e^2/k) J_1((2\pi \beta e^2 k)^{1/2}) K_1((2\pi \beta e^2 k)^{1/2}). \qquad (3.7)$$

A simple change of variables and integration by parts bring S to the form

$$S = \rho \epsilon^2 \int_0^\infty dx \left(x^4 J_1(x) K_1(x) \frac{d}{dx} [x^{-2} J_1(x) K_1(x)] \Big/ [x^2 + 2\epsilon^2 J_1(x) K_1(x)] \right) \equiv \rho \epsilon^2 I(\epsilon), \qquad (3.8)$$

where the dimensionless plasma parameter ϵ is defined by

$$\epsilon^2 \equiv 4\pi \rho \beta^2 e^4 = k_D^2 / \pi \rho. \qquad (3.9)$$

The quantity ϵ is small for a dilute system at high temperature; the virial expansion is seen to be actually an expansion in terms of the dimensionless parameter ϵ.

When the coupling constant e^2 vanishes, the integral in (3.8) has a logarithmic singularity due to the divergence of the integrand at the origin. By gauging the strength of this singularity, one can confirm that the leading correction to the equation of state is given correctly by the Debye-Hückel result. We do not pause to do this but continue directly to a more systematic and accurate evaluation.

We subtract from and add to $I(\epsilon)$ another integral which has the same divergence in ϵ as $I(\epsilon)$ itself; the new integral is chosen to be expressible in terms of tabulated functions. We obtain

$$S = \tfrac{1}{2}\rho \epsilon^2 \int_0^\infty dx \frac{x e^{-x}}{x^2 + \epsilon^2} - 2\rho \epsilon^2 \int_0^\infty dx \, x \left(\frac{[J_1(x) K_1(x)]^2}{x^2 + 2\epsilon^2 J_1(x) K_1(x)} - \frac{\tfrac{1}{4} e^{-x}}{x^2 + \epsilon^2} \right) + \rho \epsilon^2 \int_0^\infty dx \, J_1(x) K_1(x) \frac{d}{dx} [J_1(x) K_1(x)]$$

$$+ \rho \epsilon^2 \int_0^\infty dx \, J_1(x) K_1(x) \frac{d}{dx} [J_1(x) K_1(x)] \left(\frac{x^2}{x^2 + 2\epsilon^2 J_1(x) K_1(x)} - 1 \right) . \qquad (3.10)$$

The last three integrals above remain convergent if we set ϵ to zero in the integrands; doing so gives S to order ϵ^2. We also can display terms of higher order by expanding the denominator $[x^2 + 2\epsilon^2 J_1(x) K_1(x)]^{-1}$ in powers of $(x^2 + \epsilon^2)^{-1}$,

$$S = -\tfrac{1}{2}\rho \epsilon^2 \int_0^\infty dx \frac{x e^{-x}}{x^2 + \epsilon^2} - 2\rho \epsilon^2 \left(\int_0^\infty \frac{dx}{x} \{ [J_1(x) K_1(x)]^2 - \tfrac{1}{4} e^{-x} \} - 2\epsilon^2 \int_0^\infty \frac{dx}{x} \frac{[J_1(x) K_1(x)]^3 - \tfrac{1}{8} e^{-x}}{x^2 + \epsilon^2} \right.$$

$$\left. - 2\epsilon^4 \int_0^\infty \frac{dx}{x} \frac{[J_1(x) K_1(x)]^3 [1 - 2 J_1(x) K_1(x)]}{(x^2 + \epsilon^2)^2} - 2\epsilon^6 \int_0^\infty \frac{dx}{x} \frac{[J_1(x) K_1(x)]^3 [1 - 2 J_1(x) K_1(x)]^2}{(x^2 + \epsilon^2)^3} + \cdots \right)$$

$$+ \rho \epsilon^2 \int_0^\infty \left[dx \, J_1(x) K_1(x) \frac{d}{dx} [J_1(x) K_1(x)] - 2\epsilon^2 \int_0^\infty dx \left([J_1(x) K_1(x)]^2 \frac{d}{dx} [J_1(x) K_1(x)] \Big/ (x^2 + \epsilon^2) \right) + \cdots \right] . \qquad (3.11)$$

We evaluate this expansion to all orders lower than $\rho \epsilon^4$. The integrations are performed in the Appendix, yielding

$$S = \tfrac{1}{2}\rho \epsilon^2 \ln \epsilon + [\tfrac{1}{4}\gamma + \ln 2 - \tfrac{3}{4}]\rho \epsilon^2 - \tfrac{1}{4}\rho \epsilon^4 \ln^2(\tfrac{1}{2}\epsilon)$$

$$+ [\tfrac{3}{8} - \tfrac{1}{2}\gamma]\rho \epsilon^4 \ln \epsilon + O(\rho \epsilon^4), \qquad (3.12)$$

which leads to the equation of state

$$\beta P/\rho - 1 = \tfrac{1}{4}\epsilon^2 \ln(2\epsilon) - \tfrac{1}{4}(1 - \tfrac{1}{2}\gamma)\epsilon^2$$

$$- \tfrac{1}{4}\epsilon^4 \ln^2(\tfrac{1}{2}\epsilon) + \tfrac{1}{2}(\tfrac{1}{2} - \gamma)\epsilon^4 \ln \epsilon + O(\epsilon^4), \qquad (3.13)$$

where $\gamma \simeq 0.577$ is Euler's constant. This corroborates the Debye-Hückel result obtained previously.

Finally, we show that the terms in the virial expansion which we neglected are of order ϵ^4. In the first step beyond the ring approximation, another infinite sequence of divergent terms, the watermelon graphs, must be summed.[18,19] It follows directly from the results of Ref. 19 that the change in S produced by incorporating the watermelon in-

tegrals is

$$\Delta S = -\rho^2 \int d^2x \{[e^{h(x)} - 1 - h(x)] e^{-\beta V(x)} - \tfrac{1}{2} h^2(x)\},$$

$$h(x) = \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot\vec{x}} \frac{\rho f^2(k)}{1 - \rho f(k)} . \tag{3.14}$$

We make the approximation $f(k) \simeq -\beta V(k)$, finding

$$h(x) = (k_D^2/4\rho)[\bar{H}_0(k_D x) - Y_0(k_D x)], \tag{3.15}$$

and it follows directly that

$$\Delta S = O(\rho\epsilon^4). \tag{3.16}$$

We are justified in evaluating only the ring integrals for the analysis of terms of lower order than ϵ^4.

IV. DISCUSSION

We now discuss the physical basis of our results. From Eq. (3.4) it is evident that the virial series in ρ for the pressure diverges ever more strongly term by term; at long wavelengths, a convergent result is secured only by summing the series, thereby taking account of screening of the Coulomb interaction at large distances. In addition, we note that expanding the Mayer function to lowest order in the coupling constant e^2, i.e.,

$$f(r) \simeq -\beta V(r),$$
$$f(k) \simeq -\beta V(k) = \lim_{h\to 0} f(k), \quad V(k) = 2\pi e^2/k, \tag{4.1}$$

makes the momentum integral in (3.4) diverge at short wavelengths, mirroring the formal divergence in the Debye-Hückel calculation; as we anticipated in the Debye-Hückel case, retaining the full Mayer function cuts off the divergence at $k^{-1} = \beta e^2$, where the Landau length βe^2 is roughly the classical distance of closest approach of two electrons of energy $k_B T$. However, for the three-dimensional electron gas, no divergence appears in the Debye-Hückel treatment, and the approximation in Eq. (4.1) correctly describes the leading deviation from ideal-gas behavior.

Since the electron layer is assumed dilute and at high temperature, we have treated it by the methods of classical statistics. But for an electron assembly there is a unit of energy, the rydberg (1 Ry = $\tfrac{1}{2}m e^4 \hbar^{-2}$), which becomes infinitely large in the strict classical limit $\hbar \to 0$. That the rydberg sets a high-energy scale in terms of Planck's constant indicates that the system behaves quantum mechanically at sufficiently large temperatures.[20,21]

The characteristic length for quantum correlations in the layer is the thermal wavelength $\lambda_{th} = (2\pi\beta\hbar^2/m)^{1/2}$. Below this distance, the picture of the particles as classical point scatterers is no longer valid, and they must be described quantum mechanically[20,21]; thus, the thermal wavelength is

an alternative choice for a short-distance cutoff in the Debye-Hückel calculation. For temperatures much smaller than a rydberg (and much larger than the Fermi temperature), the Landau length βe^2 is large compared to a thermal wavelength; the electrons are rarely close enough for quantum effects to be significant, and a classical treatment like ours is valid. When the temperature is raised far above a rydberg, the thermal wavelength becomes much larger than the classical distance of closest approach, and quantum correlations dominate the short-distance behavior of the system. Setting $r_c = \lambda_{th}$ in Eq. (2.10) of our Debye-Hückel treatment yields

$$\beta P/\rho - 1 \approx \tfrac{1}{8}\epsilon^2 \ln(\epsilon x),$$
$$x \equiv (2\pi\beta\hbar^2\rho/m)^{1/2}, \tag{4.2}$$

which is the result of a previous calculation by Fetter.[9]

The work of Ref. 9 is expressed in the language of finite-temperature quantum-field theory; when the sequence of Feynman diagrams chosen there is evaluated for the three-dimensional electron gas, to leading order it corresponds to the classical ring approximation with the simplification of Eq. (4.1).[13] Just as the approximation (4.1) leads to a divergence in two dimensions, so the corresponding set of Feynman diagrams must diverge when the classical limit $\hbar \to 0$ is taken; and indeed, the pressure in (4.2) is divergent in this limit. To obtain the classical limit by field-theoretical techniques, it is necessary to incorporate ladder-like diagrams describing the interaction of the electrons at short distances; the rearrangement of the Feynman perturbation series which reduces to the classical Mayer cluster expansion is given by Smith.[22]

For many-electron surface states above neon, a characteristic temperature is $T \approx 10\,°\mathrm{K}$ and $k_B T \approx 10^{-4}$ Ry; we conclude by stressing the observation[9] that these states may give the possibility of observing a classical two-dimensional electron gas.

ACKNOWLEDGMENTS

APPENDIX

We evaluate the integrals appearing in S, considering only terms of order lower than $\rho\epsilon^4$. Our first integral is expressible in terms of tabulated functions,[23]

$$\int_0^\infty dx \, \frac{x e^{-x}}{x^2 + \epsilon^2} = - \text{Ci}(\epsilon) \cos\epsilon - \text{Si}(\epsilon) \sin\epsilon \,,$$

$$\text{Ci}(x) = \gamma + \ln x + \int_0^x dt \, (\cos t - 1)/t \qquad (A1)$$

$$(\gamma \text{ is Euler's constant} \simeq 0.57),$$

$$\text{Si}(x) = -\frac{\pi}{2} + \sum_{k=1}^\infty \frac{(-)^{k+1} x^{2k-1}}{(2k-1)\Gamma(2k)} \,.$$

Therefore

$$\int_0^\infty dx \, \frac{x e^{-x}}{x^2 + \epsilon^2} = - \ln\epsilon - \gamma + \tfrac{1}{2}\pi\epsilon$$
$$+ \tfrac{1}{4}\epsilon^2 \ln\epsilon + O(\epsilon^2).$$

Next, we consider

$$I = \int_0^\infty \frac{dx}{x} \{ [J_1(x)K_1(x)]^2 - \tfrac{1}{4} e^{-x} \}$$

$$= \lim_{\delta \to 0} \int_0^\infty \frac{dx}{x^{1-\delta}}$$

$$\times \{ [J_1(x)K_1(x)]^2 - \tfrac{1}{4} e^{-x} \} \,. \qquad (A2)$$

We set up this limiting form so that each term of the integral, not just their difference, is well defined before the limit is taken. In what follows, δ will be treated as an infinitesimal, and terms of order δ will be dropped even before formally passing to the limit.

The integration involving the Bessel functions may be carried out by use of the identities[24]

$$J_1^2(x) = \sum_{m=0}^\infty \frac{(-)^m \Gamma(2m+3)}{\Gamma(m+1)\Gamma(m+3)[\Gamma(m+2)]^2}$$

$$\times (\tfrac{1}{2}x)^{2m+2} \,, \qquad (A3)$$

$$K_1^2(x) = \int_0^\infty K_2(2x \cosh t) \, dt \,,$$

and integrals[23]

$$\int_0^\infty K_p(x) x^{q-1} dx = 2^{q-2} \Gamma(\tfrac{1}{2}(q-p))$$

$$\times \Gamma(\tfrac{1}{2}(q+p)) \qquad (p^2 < q^2) \,, \qquad (A4)$$

$$\int_0^\infty \frac{\cosh(2px)}{\cosh^2 qx} dx = 2^{2(q-1)} \frac{\Gamma(q+p)\Gamma(q-p)}{\Gamma(2q)} \,,$$

$$(\text{Re} q > |\text{Re} p|) \,.$$

A little algebra yields

$$\int_0^\infty \frac{dx}{x^{1-\delta}} [J_1(x)K_1(x)]^2 = \tfrac{1}{8}\Gamma(\tfrac{1}{2}\delta)$$

$$+ \tfrac{1}{4} \sum_{m=1}^\infty \frac{(-)^m}{(m+2)(m+1)m}$$

$$= \tfrac{1}{8}\Gamma(\tfrac{1}{2}\delta) + \tfrac{1}{4} \int_0^{-1} dx \int_0^x dy$$

$$\times \int_0^y \frac{dz}{1-z} = \tfrac{1}{8}\Gamma(\tfrac{1}{2}\delta) - \tfrac{1}{2}\ln 2 + \tfrac{5}{16} \,. \qquad (A5)$$

We recognize the other piece of I as a Γ function, and we have

$$I = \tfrac{5}{16} - \tfrac{1}{2}\ln 2 + \lim_{\delta \to 0} [\tfrac{1}{8}\Gamma(\tfrac{1}{2}\delta) - \tfrac{1}{4}\Gamma(\delta)]. \qquad (A6)$$

Evaluating the limit, we find the result

$$\int_0^\infty \frac{dx}{x} \{ [J_1(x)K_1(x)]^2 - \tfrac{1}{4} e^{-x} \} = \tfrac{1}{8}\gamma - \tfrac{1}{2}\ln 2 + \tfrac{5}{16} \,. \qquad (A7)$$

Our next integral is straightforward,

$$\int_0^\infty dx \, J_1(x)K_1(x) \frac{d}{dx} [J_1(x)K_1(x)] = -\tfrac{1}{8} \,. \qquad (A8)$$

The functions of ϵ defined by the remaining integrals need not be determined exactly; we desire only their leading behavior in ϵ, and it suffices to examine the integrals in the neighborhood of the origin.[9] The variable of integration x is confined to the region $0 \le x \le \delta$, where the dummy parameter δ is chosen so that $0 \ll \epsilon \ll \delta \ll 1$; this is justifiable since the gas is assumed dilute and at high temperature. Using small-argument expansions for the Bessel functions appearing in the integrals, we obtain

$$4\rho\epsilon^4 \int_0^\infty \frac{dx}{x} \frac{[J_1(x)K_1(x)]^3 - \tfrac{1}{8} e^{-x}}{x^2 + \epsilon^2} = \tfrac{1}{4}\pi\rho\epsilon^3 - \tfrac{3}{8}\rho\epsilon^4 \ln^2(\tfrac{1}{2}\epsilon) + \tfrac{1}{4}(\tfrac{13}{4} - 3\gamma)\rho\epsilon^4 \ln\epsilon + O(\rho\epsilon^4), \qquad (A9)$$

$$4\rho\epsilon^6 \int_0^\infty \frac{dx}{x} \frac{[J_1(x)K_1(x)]^3 [1 - 2J_1(x)K_1(x)]}{(x^2 + \epsilon^2)^2} = -\tfrac{1}{8}\rho\epsilon^4 \ln\epsilon + O(\rho\epsilon^4) \,, \qquad (A10)$$

$$4\rho\epsilon^8 \int_0^\infty \frac{dx}{x} \frac{[J_1(x)K_1(x)]^3 [1 - 2J_1(x)K_1(x)]^2}{(x^2 + \epsilon^2)^3} = O(\rho\epsilon^6 \ln^2\epsilon) \,, \qquad (A11)$$

$$-2\rho\epsilon^4 \int_0^\infty dx \, [J_1(x)K_1(x)]^2 \frac{d}{dx} [J_1(x)K_1(x)] \Big/ (x^2 + \epsilon^2) = \tfrac{1}{8}\rho\epsilon^4 \ln^2(\tfrac{1}{2}\epsilon) + \tfrac{1}{4}(\gamma - \tfrac{1}{4})\rho\epsilon^4 \ln\epsilon + O(\rho\epsilon^4), \qquad (A12)$$

$$-2\rho\epsilon^6 \int_0^\infty dx \, \frac{[J_1(x)K_1(x)]^2[\,1 - 2J_1(x)K_1(x)\frac{d}{dx}[J_1(x)K_1(x)]}{(x^2+\epsilon^2)^2} = O(\rho\epsilon^6 \ln^3 \epsilon) \, . \tag{A13}$$

The integrals which vanish to order ϵ^4 indicate the point at which to cut off the expansion of $[x^2 + 2\epsilon^2 J_1(x)K_1(x)]$ in terms of $(x^2+\epsilon^2)^{-1}$.

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