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Calculation of critical exponents in two dimensions from quantum field theory in one dimension*

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We construct a relationship between the Baxter model in two dimensions and the Luttinger model in one, and use it to calculate critical exponents for the Baxter model from appropriate Luttinger-model correlation functions. An important part of this work involves the generalization of the Jordan-Wigner transformation to provide a representation for continuum spin operators. With this generalization, we are also able to calculate spin correlation functions for a continuum generalization of the spin-1/2 Heisenberg-Ising chain. We discuss the difference between the continuum and discrete lattice models, and with the help of a new scaling law, use previous results for the Baxter model to calculate new exponents for the Baxter and Heisenberg-Ising model on a lattice.

I. INTRODUCTION

Since the work of Schultz, Mattis, and Lieb,[1] the close relationship between the two-dimensional Ising model and the noninteracting fermion gas in one dimension has been well known. Recent solutions of the eight-vertex, or Baxter model,[2] which is a generalization of the Ising model, naturally raise the question about a possible one-dimensional fermion equivalent for this model as well. Indeed, the solutions of the Thirring[3] and Luttinger[4] models also exhibit nonuniversal behavior similar to the Baxter model, and it has been widely believed that these models are somehow related. This paper defines and explores the relationship between them.

We first discuss the spin-$\frac{1}{2}$ Heisenberg-Ising (HI) model in one dimension, for Sutherland[5] has shown it can be used to calculate correlation functions at the critical point in the Baxter model. A new feature of our work is the use of a continuum generalization for the HI model, and our construction of the continuum spin-$\frac{1}{2}$ algebra for this model. After these transformations, the HI Hamiltonian becomes the same as the Luttinger Hamiltonian, and the spin-$\frac{1}{2}$ operators are closely related to the boson representation of fermion operators, used previously in the calculation of the Luttinger-model correlation functions.[6] The calculation of these spin correlation functions is given, and related to the corresponding Baxter functions, thereby determining the electric critical exponents η and δ.

When the temperature in the Baxter model is not equal to the critical temperature, this equivalence leads to the appearance of a gap in the Luttinger-model excitation spectrum, analogous to a mass term in the Thirring model. Assuming homogeneity, we calculate the exponents which involve this mass term, thus completing the list of the electric critical exponents for the Baxter model.

For the special case of the Ising model, we easily reproduce the usual exponents. In addition, however, our method and a Wick rotation permit easy calculation of the asymptotic behavior of all higher spin functions as well. These are quite similar in form to the results found previously for the Thirring[3] and Luttinger[6] models because we find the spin-density operator is essentially the square root of a free-fermion operator. With the boson representation of these operators, calculation of the critical exponents is reduced to a trivial harmonic-oscillator problem.

The result found here for ν differs slightly from previous calculations[7] for the Baxter model on a lattice. This is to be expected because of our use of the continuum fermion model, rather than the lattice fermion model, for the transfer matrix. Comparing these two results for ν therefore provides a measure for the accuracy of the continuum fermion approximation for models of the interacting fermion gas on a lattice.

Because we have calculated all electric critical exponents, the coupling constant can be eliminated between any two, leading to an additional scaling law, $4\beta = 2\nu - 1$. This scaling law is argued to be universal, independent of the particular parametrization of the two-dimensional model, and is consistent with a conjecture for β by Baxter and Kelland[8] for the lattice model. The new law reduces the parameters needed to specify all exponents to one, and this one parameter is model dependent. The three different models, Thirring, Luttinger, and Baxter, differ on this additional parameter.

An intermediate step in our discussion involves solving the HI model, generalized to the continuum case. Results for correlation functions and their exponents are discussed. In addition, we calculate the exponents associated with basal-plane anisotropy. The exponents are found to depend on the ratio of longitudinal-to-transverse exchange. At the xy point, the exponent of the transverse-spin correlation function is $\frac{1}{2}$, in agreement with previous results[9] increasing to 1.06 at the Heisenberg point. As in the Baxter model, we can use the additional scaling law to calculate exponents for the lattice (HI) model. Doing this, gives the exponent 1 for the spin-$\frac{1}{2}$ Heisenberg model on the lattice, rather than the value 1.06 of the continuum version. The x-y result is the same in both lattice and continuum cases.

Our discussion uses the Ising representation of the Baxter model,[10] in which one views the system as two interpenetrating Ising lattices, with Ising variables μ_i and μ_i' attached to each site. These variables are related to the original Baxter variable α_i, through $\alpha_i = \mu_i \mu_{i+1}$, where $i+1$ is a particular nearest neighbor to i. In addition to the usual nearest-neighbor two-spin interactions, $-J\mu_i\mu_{i+1}'$ and $-J'\mu_i'\mu_{i+1}$, there is a four-spin interaction $J_4\mu_i\mu_{i+1}\mu_i'\mu_{i+1}'$ between four nearest-neighbor spins, which couples the two Ising lattices.

The calculation of correlation functions could be performed by diagonalization of the transfer matrix T. However, it is simpler to look for an associated Hamiltonian operator $\mathcal{H}$ which commutes with T. Since T and $\mathcal{H}$ then have the same eigenvectors, correlation functions can be calculated provided the proper choice of the eigenstate is made.

For the eight-vertex model, Sutherland[5] has shown that the one-dimensional spin-$\frac{1}{2}$ x-y-z Hamiltonian commutes with T:

$$\mathcal{H}_{xyz} = -\sum_i (J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z) \ . \quad (1)$$

Here we have chosen the coupling constants to be

$$J_x = 1 \ ,$$

$$J_y = \sinh 2K \sinh 2K' + \cosh 2K \cosh 2K' \tanh 2L \ , \quad (2)$$

$$J_z = \tanh 2L \ ,$$

with $K = J/k_BT$, $K' = J'/k_BT$, $L = J_4/k_BT$, and we have interchanged y and z in Sutherland's result for later convenience. In the case of vanishing four-spin coupling, this reduces to the familiar x-y model

$$\mathcal{H}_{xy} = -\sum_i (S_i^x S_{i+1}^x + \sinh 2K \sinh 2K' \, S_i^y S_{i+1}^y) \ . \quad (3)$$

According to Onsager's famous result, the phase transition of the two-dimensional problem takes place when

$$\sinh 2K \sinh 2K' = 1$$

and this is the point where $\mathcal{H}_{xy}$ becomes isotropic. Correlation functions for $\mathcal{H}_{xy}$ are therefore those of two noninteracting Ising models. For $J_z \neq 0$, the critical temperature is still determined by the condition that two coupling constants in $\mathcal{H}$ are equal. The Hamiltonian at the critical temperature is therefore of HI form

$$\mathcal{H} = -\sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z) \ . \quad (4)$$

Taking account of the $y \leftrightarrow z$ interchange, the correlation function along a line of the square lattice in the Baxter model $\langle \alpha_i \alpha_{i+n} \rangle$ is equal to $4\langle \theta | S_i^y S_{i+n}^y | \theta \rangle$, where $\langle \theta |$ is the ground state of H. Multioperator correlation functions $\langle \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_n} \rangle$ along this line are similarly given by $2^n \langle \theta | S_{i_1}^y S_{i_2}^y \cdots S_{i_n}^y | \theta \rangle$, a property which we use below. A calculation of these spin-$\frac{1}{2}$ correlation functions therefore determines the electric correlation functions in the Baxter model, and the corresponding electric critical exponents.

II. CORRELATION FUNCTIONS FOR THE SPIN-½ *xyz* MODEL

In this section, we consider the calculation of the correlation functions for the spin-$\frac{1}{2}$ x-y-z model in one dimension. Our procedure involves relating the x-y-z model to the Luttinger model of an interacting one-dimensional spinless fermion gas, and using the solutions of this model to obtain the desired asymptotic properties. Customarily, the spin-$\frac{1}{2}$ x-y-z model is defined on a lattice and one problem that we must discuss is its continuum generalization, to permit application of the continuum fermion field theory.

It is convenient to write the Hamiltonian for the lattice model in the form

$$\mathcal{H}_{xyz} = -\sum_i [S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z + m_0 (S_i^x S_{i+1}^x - S_i^y S_{i+1}^y)] \ , \quad (5)$$

where $J_x = 1$ is the transverse exchange, J_z the longitudinal exchange, m_0 is the basal-plane anisotropy, $\vec{S}_i$ is a spin-$\frac{1}{2}$ operator, and the sum i runs over the N sites of a ring. Using the Jordan-Wigner transformation to fermion operators a_i

$$S_i^+ = S_i^x + iS_i^y = a_i^\dagger \exp\left(i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j \right), \ S_i^z = a_i^\dagger a_i - \tfrac{1}{2} \ ,$$

and Fourier transforming, the Hamiltonian can be written in the form

$$\mathcal{H}_{xyz} = -\sum_k \cos ks [a_k^\dagger a_k + J_z N^{-1} \rho(k) \rho(-k)] + m_0 \sum_k (e^{iks} a_k^\dagger a_{-k}^\dagger + \text{H.c.}) \ , \quad (6)$$

where s is the lattice constant, the sum over k is restricted to the first Brillouin zone

$$a_k = N^{-1/2} \sum_j e^{ikx_j} a_j \ ,$$

and $\rho(k)$ is the density operator

$$\rho(k)=N^{-1}\sum_j a_j^\dagger a_j e^{ikx_j} .$$

In zero external field, there is no net magnetization

$$\sum_i \langle S_i^z\rangle = 0 = \sum_i \langle a_i^\dagger a_i\rangle - \tfrac{1}{2}N ,$$

so that the mean fermion number is equal to $\frac{1}{2}N$, corresponding to a half-filled band. The pairing term, containing m_0, is seen from Eq. (3) to be proportional to $T-T_c$, which introduces a "mass gap" into the single-particle excitation spectrum.

This single-particle excitation spectrum for $J_z=0$ and $m_0=0$ is simply $-\cos ks$, with those states $-\frac{1}{2}\pi<ks<\frac{1}{2}\pi$ occupied. One expects the asymptotic behavior of correlation functions to be determined by low-lying excited states, those near the Fermi points at $k_F s=\pm\frac{1}{2}\pi$. It is therefore interesting to consider the related model with two linear single-particle spectra tangent to the $-\cos ks$ at the Fermi points. The group velocity is equal to plus or minus one and is exactly the single-particle spectrum of the Luttinger model,[4] and we may introduce $a_{1,k}$ ($a_{2,k}$) operators to describe the fermion particles with positive (negative) group velocity and the associated fields

$$\psi_1(x)=L^{-1/2}\sum_k a_{1,k}e^{ikx}, \quad \text{etc.},$$

where L is the length of the system.

We now proceed to relate the J_z interaction term in Eq. (6) to an appropriate interaction in the Luttinger model. The simplest procedure to follow requires that the matrix elements of this term for states near the Fermi points $k_F s=\pm\frac{1}{2}\pi$ be reproduced in the continuum model. This is accomplished first by replacing $\rho(k)$ by $\rho_1(k)+\rho_2(k)$, where

$$\rho_1(k)=\sum_p a_{1,p+k}^\dagger a_{1,p}$$

and

$$\rho_2(k)=\sum_p a_{2,p+k}^\dagger a_{2,p}$$

are the continuum fermion density operators. This replacement correctly reproduces the small momentum transfer part of the interaction term, but not the large momentum part with $k\simeq 2k_F=\pi s^{-1}$, which also involves states near the Fermi energy. These $2k_F$ processes are the backward scattering terms which have been discussed elsewhere, and can also be included in the Luttinger model. For the spinless fermion case, the result is simply

$$\sum_{k\simeq 2k_F}\rho(k)\rho(-k)\to-\sum_{k\simeq 0}\rho_1(k)\rho_2(-k) .$$

This can be easily seen using the configuration-space representation of the field operators $\psi_1(x)$ and $\psi_2(x)$ for the backward scattering term, which is

$$\int dx\,\psi_1^\dagger(x)\psi_2(x)\psi_2^\dagger(x)\psi_1(x) ,$$

and simply anticommutes into the form

$$-\int dx\,\psi_1^\dagger(x)\psi_1(x)\psi_2^\dagger(x)\psi_2(x) .$$

We also make use of $\psi_1^2(x)=0$, etc., to arrive at the result (with $m_0=0$, i.e., $T=T_c$)

$$\mathcal{H}=\sum_k k(a_{1,k}^\dagger a_{1,k}-a_{2,k}^\dagger a_{2,k}) - 4J_zN^{-1}\sum_k \rho_1(k)\rho_2(-k) . \quad (7)$$

When $m_0\neq 0$, the additional mass term must be also included. This problem has not previously been solved except for the free-particle case $J_z=0$ and we discuss below a solution for the asymptotic properties of the correlation functions near T_c for the case $J_z\neq 0$. With small m_0, states far away from the Fermi energy will not be affected by this term and we therefore may take

$$m_0\sum_k e^{iks}a_k^\dagger a_{-k}^\dagger+\text{H.c.}\to 2m_0 i\sum_{k>0}a_{1,k}^\dagger a_{2,-k}^\dagger+\text{H.c.}, \quad (8)$$

which correctly describes the mixing of the "1" states near k_F with the "2" states near $-k_F$. Together with Eq. (7) this completes the construction of the continuum fermion Hamiltonian, equivalent to the x-y-z model. We need now only determine the transformation of spin-density operators into this continuum representation, in order to calculate spin correlation functions.

The Jordan-Wigner transformation for the spin operators on the lattice has the obvious generalization to the continuum situation, as discussed in the Appendix, which is

$$i\pi\sum_{m=1}^{n-1}a_m^\dagger a_m\to i\pi\int_0^{x_n-s}dy[\rho_1(y)+\rho_2(y)+(2s)^{-1}]\equiv N(x_n) ,$$
$$a_i\to(\tfrac{1}{2}s)^{+1/2}[\psi_1(x)+\psi_2(x)] . \quad (9)$$

The resulting representation for continuum spin operators is

$$S^-(x)=(2s)^{-1/2}[\psi_1(x)+\psi_2(x)]e^{-N(x)} ,$$
$$S^+(x)=[S^-(x)]^\dagger , \quad (10)$$
$$2S^z(x)=\rho_1(x)+\rho_2(x)+\psi_1^\dagger(x)\psi_2(x)+\psi_2^\dagger(x)\psi_1(x) ,$$

where the total uniform density is $N/2L$ appropriate for the half-filled band, corresponding to zero-average magnetization. It can be verified that these operators satisfy the desired algebra for the continuum spin-$\frac{1}{2}$ field, in the limit $s\to 0$. As in the lattice model, this follows from the relations

$$[\psi_1(x)+\psi_2(x),\ e^{-N(x')}]_-=0\ ,\qquad x>x'-\tfrac{1}{2}s\ ,$$
$$[\psi_1(x)+\psi_2(x),\ e^{-N(x')}]_+=0\ ,\qquad x<x'-\tfrac{1}{2}s\ ,\qquad (11)$$
$$[N(x),\ N(x')]_-=0\ ,$$

where $-(+)$ means (anti) commutator, which are readily verified using the Luttinger-model operator algebra.[4]

The continuum limit is essential for our calculations, which concentrate on the asymptotic behavior for distances much larger than a lattice constant. Contributions to this asymptotic region can occur from backward scattering at the Fermi energy, with momentum transfer $2k_F=\pi/s$, and k_F diverges if we naively set $s=0$ here. Also, the density operators used by Lieb and Mattis are defined to have $\langle\rho(k=0)\rangle=0$, rather than $\langle\rho(k=0)\rangle=N/2L=\pi k_F=1/2s$. We will use the former, which introduces the phase factor, $(2s)^{-1}$ in Eq. (9), into the "ordering" operator $e^{N(x)}$. The continuum model limit which is appropriate for our calculations keeps these factors of k_F fixed, corresponding to fixed uniform density, and uses the continuum spin algebra.

These relations permit the spin correlation functions to be expressed as correlation functions in the continuum fermion problem. The result is

$$-4s^2\langle S^y(x,t)S^y(0)\rangle=\langle[\psi(x,t)e^{-N(x,t)}-\psi^+(x,t)e^{+N(x,t)}][\psi e^{-N}-\psi^\dagger e^{+N}]\rangle\ ,$$
$$4\langle S^z(x,t)S^z(0)\rangle=\langle\rho_1(x,t)\rho_1\rangle+\langle\rho_2(x,t)\rho_2\rangle+\langle\psi_1^\dagger(x,t)\psi_2(x,t)\psi_2^\dagger\psi_1\rangle+\langle\psi_2^\dagger(x,t)\psi_1(x,t)\psi_1^\dagger\psi_2\rangle\ ,\qquad (12)$$

where the time evolution and averaging is in the Hamiltonian of Eq. (7) plus the mass term, Eq. (8), and we have used the shorthand notation $\psi(x)\equiv\psi_1(x)+\psi_2(x)$.

These functions can be readily evaluated because the density operators satisfy boson algebra. Using the boson representation[6,11] for the $\psi_1(x)$ and $\psi_2(x)$ operators, we find

$$\psi(x)e^{-N(x)}=(2\pi\alpha)^{-1/2}\left(e^{3\phi_2(x)/2-\phi_1(x)/2+2ik_Fx+\tau}+e^{\phi_1(x)/2+\phi_2(x)+\tau}\right),\qquad (13)$$

where

$$\phi_{1,2}(x)=\mp 2\pi L^{-1}\sum_k \rho_{1,2}(k)\,e^{-ikx}k^{-1}\,e^{-\alpha|k|/2}\ ,$$

the upper (lower) sign goes with the $\rho_1(\rho_2)$ operator, α is a cutoff parameter which is of order the bandwidth $\sim s$, and $2\tau=\phi_2(0)-\phi_1(0)$. It has been implicitly assumed that $J_x>0$. If $J_x<0$, the "1" and "2" single-particle branches are interchanged and we must take $k_F\to -k_F$ in the definition of the field operators. As a result, the e^{2ik_Fx} phase factor cancels out in the first term of Eq. (13), but appears in the second term. Finally, we use the Lieb and Mattis result that Eq. (7) can be written entirely in terms of boson density operators:

$$\mathcal{H}=2\pi L^{-1}\sum_k[\rho_1(-k)\rho_1(k)+\rho_2(-k)\rho_2(k)]-4J_zL^{-1}\sum_k\rho_1(k)\rho_2(-k)\ ,\qquad (14)$$

where

$$[\rho_1(-k),\ \rho_1(k')]=[\rho_2(k'),\ \rho_2(-k)]=kL\delta_{kk'}/2\pi\ .$$

This Hamiltonian is bilinear in bosons and the operators in the correlation functions are either bosons or exponentials of bosons, consequently, they may be trivially evaluated.

The leading asymptotic behavior of the transverse equal-time correlation function is given by

$$4\pi s^2\langle S^+(x)S^-(0)\rangle=\langle e^{-\phi_1(x)/2-\phi_2(x)/2}e^{+\phi_1/2+\phi_2/2}\rangle\ ,\qquad (15)$$

where averages are calculated using Eq. (14). Using the usual procedure for evaluating such functions,[6] we must first diagonalize the Hamiltonian of Eq. (14) with the transformation

$$\phi_1(x)\to\phi_1(x)\cosh\varphi-\phi_2(x)\sinh\varphi\ ,$$

where

$$\tanh 2\varphi=2J_z/\pi\ .$$

Consequently,

$$\phi_1(x)+\phi_2(x)\to e^{-\varphi}[\phi_1(x)+\phi_2(x)]\ ,$$

and the Hamiltonian is the sum of the "1" and "2" boson energies. As a result, Eq. (15) can be written

$$\exp(\tfrac{1}{2}\theta\langle\phi_1^2-\phi_1(x)\phi_1\rangle+\tfrac{1}{2}\theta\langle\phi_2^2-\phi_2(x)\phi_2\rangle=\exp\left(\tfrac{1}{2}\theta\sum_{k>0}2\pi(kL)^{-1}(e^{ikx}-1)+\text{H. c.}\right)\sim\left(\frac{s}{x}\right)^\theta,\qquad (15')$$

where $2\theta=e^{-2\varphi}=[(\pi-2J_z)/(\pi+2J_z)]^{1/2}$, we have used Eq. (13) with $\alpha\sim s$, and assumed $x\gg s$. The other correlation functions are similarly calculated, giving the results

$$s^2\langle S^y(x)S^y\rangle\sim(s/x)^\theta+\cdots\ ,$$
$$s^2\langle S^z(x)S^z\rangle\sim(s/x)^2+e^{2ik_Fx}(s/x)^{\theta^{-1}}+\cdots\ ,\qquad (16)$$

where we have only retained dimensional factors. For the free-particle case, $J_z=0$ and $m_0=0$, the problem is exactly the isotropic xy model, studied by Lieb, Schultz, and Mattis,[12] and McCoy,[9] and our results are in agreement with theirs.

The multispin correlation functions in the asymptotic region can be calculated, including the time dependence as well, using well-known results for operators of this form. The static functions involve only the exponent θ, because of the property that averages of exponentiated free-boson operators in the free-density matrix can be written

$$\langle e^{-p\phi_1(x_1)} e^{p\phi_1(x_2)} e^{-p\phi_1(x_3)} \cdots e^{p\phi_1(x_{2n})}\rangle_1 = \exp \sum_{i<j}^{2n} (-)^{i-j} p^2 \ln\left(\frac{x_j - x_i}{s}\right), \tag{17}$$

which is proved by repeated application of the Baker-Hausdorff formula. In this expression, all $x_j - x_i$ are much larger than s, and the ordering $x_1 < x_2 < x_3 \cdots x_n$ is assumed. We will make use of this property below to derive the form of higher-order correlation functions in these models.

The correlation functions for $m_0 \neq 0$ can be calculated using similar methods. It is first helpful to consider the simple physical picture appropriate for this situation. The noninteracting-particle case $J_z = 0$ has been discussed by McCoy.[9] A gap (or mass) is introduced into the fermion spectrum, BCS-like in nature, and consequently a gap appears in the density-density correlation functions used to calculate correlation functions as in Eq. (15′).

For the general case, the momentum sum in Eq. (15′) will be modified when the length scale is of order m^{-1}, since the k summation is modified for $k < m$, where m is the renormalized gap or mass. This, in turn, causes the correlation function to alter its behavior in the length scale $m^{-\theta}$, where θ is the coefficient multiplying the log in Eq. (15′). The assumption of homogeneity then implies $\langle S^+(x)S^-\rangle \propto m^{-\theta}$ as $x \to \infty$, if long-range order exists. For the xy model with $J_z = 0$, we have $\theta = \frac{1}{2}$ and $m = m_0$ since there are no interactions to modify the free-particle gap.

With interactions present, m is not proportional to m_0, but $m \sim m_0^\nu$. This can be established in several ways. The simplest assumes the $\langle \rho_1(x)\rho_1\rangle$ functions are of the scaling form $\sim x^{-2} f(mx)$, where f is an unspecified function. Then perturbation theory in m_0 gives terms of the form

$$\begin{aligned}\langle \rho_1(x)\rho_1\rangle = \sum_n m_0^n \int_0^\infty dt_1 \cdots \int_0^{t_{n-1}} dt_n \int dx_1 \cdots \int dx_n x \\ \times \langle \rho_1(x)\rho_1 (e^{-\phi_1(t_1,x_1)-\phi_2(t_1,x_1)} + \text{H.c.}) \\ \cdots (e^{-\phi_1(t_n,x_n)-\phi_2(t_n,x_n)} + \text{H.c.})\rangle\end{aligned} \tag{18}$$

which can be evaluated by patient application of Eq. (17). The integrals can be made dimensionless by scaling out x, with the resulting series

$$x^{-2} \sum_n (m_0 x^y)^{2n} C_n ,$$

where the C_n are finite constants when all short-distance cutoffs are included. Dimensional analysis of the $2n$-point correlation functions in Eq. (18) gives $y = 2(1-\theta)$ and the scaling form requires $m \sim |m_0|^{1/y}$ and therefore $2\nu = (1-\theta)^{-1}$.

With this value for ν, we can determine the long-range-order exponent β, defined as $x \to \infty$ by $\langle S^+(x)S^-\rangle \sim \langle (S^y)^2\rangle \sim |m_0|^{2\beta}$. Since $\langle S^+(x)S^-\rangle$ approaches m^θ we have $2\beta = \theta(1-\theta)^{-1}$. These exponents will obviously be related to critical exponents in the Ising or Baxter models, and are discussed in Sec. III.

It should be realized that the model we have solved here is a continuum generalization of the xyz model on a lattice. It is not obvious that the exponents for the lattice and continuum models should be identical, for we have insisted only that matrix elements of the two-body interaction terms be equal at the Fermi energy. If intermediate states far from the Fermi energy do contribute, the present procedure will not necessarily treat them correctly. It is possible to determine indirectly whether such processes do contribute by using these results to compute exponents in the Baxter model and compare them with those already known. This is carried out in Sec. III, and we postpone a further discussion of the HI model on the lattice until these contributions have been discussed. However, it should be obvious that the asymptotic behavior of the noninteracting-particle problem $J_z = 0$ is given exactly by these results because there is no interaction and consequently no microscopic length scale in these exponents. This corresponds to the Ising problem.

III. CALCULATION OF CRITICAL EXPONENTS

From a knowledge of the asymptotic behavior of the $\langle S^y(x)S^y\rangle$ correlation function found in Sec. II, the so-called electric critical exponents of the Baxter model can be calculated. As discussed previously this function gives the long-range behavior of the $\langle \alpha_i\alpha_0\rangle$ correlation function, where $\alpha_i = \mu_i\mu_{i+1}$ in the original Ising variable (μ_i) language From Eq. (16), and the discussion following Eq. (18), we have $\eta = \theta$, $2\nu = (1-\theta)^{-1}$, and $4\beta = \theta(1-\theta)^{-1}$, where the order parameter is taken to be $\langle\alpha\rangle$, the electric order parameter. Fisher's argument then yields $2\gamma = 2\nu(2-\eta) = (2-\theta)(1-\theta)^{-1}$. When the four-spin interaction equals zero, $J_z = 0$, the two Ising lattices do not interact. Consequently, $\langle \alpha_i\alpha_0\rangle = \langle \mu_i\mu_0\rangle^2$, $\theta = \frac{1}{2}$, and the usual Ising exponents $\eta_I = \frac{1}{2}\eta = \frac{1}{4}$, $\beta_I = \frac{1}{2}\beta = \frac{1}{8}$, and $\nu = 1$ result. For general J_z, these magnetic exponents, defined for the μ variables, have not yet been calculated with our method.

The calculation of the exponent δ requires some additional discussion, for the behavior of the correlation functions along one line does not determine it, in contrast to the above. We seek therefore the form of the general $2n$-point function at T_c, using the result that the equal-time correlation function $\langle S^y(x_1)S^y(x_2) \cdots S^y(x_n)\rangle$ determines the multioperator correlation function $\langle \alpha_1\alpha_2 \cdots \alpha_n\rangle$ along a particular line, as discussed in the Introduction.

One might expect that the time-evolution operator $e^{i\mathcal{H}t_1}S^y(x_1)e^{-i\mathcal{H}t_1}$ could be used to determine $S^y(x_1, it_1)$ which would in turn determine the general n-point function after a Wick rotation $it_1 \to y_1$. But this fails due to the presence of the τ operator in Eq. (13), which is different for equal-time functions than for equal-space functions. This difficulty can be circumvented by noting this τ operator actually makes no contribution to the equal-time correlation function. Therefore, if the operator $\hat{S}^y$ is defined to be the operator S^y of Eq. (13), with τ dropped, the equal-space correlation functions will be the same as the equal-time functions after it $\to y$. Since the equations of motion are

$$\hat{S}^y(x,t) = e^{i\mathcal{H}t}e^{iTx}\hat{S}^y e^{-iTx}e^{-i\mathcal{H}t},$$

where T is the total momentum operator $[\mathcal{H}, T] = 0$, and both the equal-space and equal-time correlation functions are correct, it follows that arbitrary space-time points are reached in this manner, with $x_i^2 - t_i^2$ invariants. Then Wick rotation in this correlation function produces the desired Euclidean invariance.

For example, the above procedure applied to the four-spin correlation function leads to the leading asymptotic behavior[20]:

$$\begin{aligned}\langle S^y(\vec{x}_1)S^y(\vec{x}_2)S^y(\vec{x}_3)S^y(\vec{x}_4)\rangle \\ \sim \left(\frac{|\vec{x}_1-\vec{x}_3|\,|\vec{x}_2-\vec{x}_4|}{|\vec{x}_1-\vec{x}_2|\,|\vec{x}_2-\vec{x}_3|\,|\vec{x}_3-\vec{x}_4|\,|\vec{x}_4-\vec{x}_1|}\right)^\theta \\ +(2 \longleftrightarrow 3) \\ +(1 \longleftrightarrow 4)\,, \end{aligned} \tag{19}$$

where $\vec{x}_1 = (x_1, y_1)$ represents the continuation $t_1 \to iy_1$ of (x_1, t_1), using the operator $s\hat{S}^z(x,t)$, and it is understood that the expectation value is computed before the continuation. The electric exponent δ can be determined from scaling. This assumes the susceptibility $\chi \sim m^{-\gamma/\nu} f(Em^z)$, where E is the field which couples to the order parameter α. But Eq. (19) gives the correction to χ of order E^2 by integrating over the space coordinates. Scaling out factors of m to make the integral dimensionless gives $2z = \theta - 4$, and as $m \to 0$, $\chi \sim E^{\nu/\gamma z}$, giving $\delta = \theta(4-\theta)^{-1}$. In the corresponding Ising variables. μ and H, and realizing that Eq. (19) is the square of an Ising correlation function, this gives $\delta_I = 15$ for the $J_z = 0$ case.

We now wish to make contact with known exact results on the Baxter model. At present, the two exponents α and ν have been obtained, the latter for a vertical correlation function. If we compare the two results

$$\frac{1}{\nu} = 2 - \left(\frac{\pi - 2J_z}{\pi + 2J_z}\right)^{1/2} \qquad \text{continuum model,} \tag{20}$$

$$\frac{1}{\nu} = 1 + \frac{2}{\pi}\arcsin J_z \qquad \text{lattice model,} \tag{21}$$

with $J_z = \tanh 2L_c$. Obviously, they do not coincide, but agree to first order in J_z:

$$1/\nu = 1 + 2J_z/\pi + \cdots$$

and both functions show qualitatively the same behavior. The difference between the two results is due to the use of the continuum Luttinger model with its strictly linear dispersion relation. The introduction of bandwidth effects would alter the parameters in the Fermion model we have used.

However, we can still determine the exponents for the lattice case, because the parameter θ can be eliminated between any two equations for these exponents. As a result, all exponents can be expressed in terms of any one, for example, ν. Using Eq (21) then gives the appropriate lattice exponents. This requires the choice

$$\theta = \tfrac{1}{2} - \pi^{-1}\arcsin J_z \tag{21'}$$

in the table of exponents in Table I, which, together with Baxter's result $\alpha = (1-2\theta)(1-\theta)^{-1}$, completes the list of electric exponents. It can be verified that the scaling law $\alpha + 2\beta + \gamma = 2$, is satisfied. In addition, the result $4\beta = 2\nu - 1$ is consistent with a previous conjecture of Baxter and Kelland.[8]

Stated in terms of a "universality" picture, it is argued that these results demonstrate the existence of an additional scaling law, for example, $4\beta = 2\nu - 1$. Together with the other scaling laws, all exponents are determined by one parameter. Different models in two dimensions with discrete broken symmetries differ in their parameter. However, the additional relation should be the same, as demonstrated by comparison of our rigorous results for the Luttinger parametrization with the previous work on the lattice model.

IV. TIME-DEPENDENT CORRELATION FUNCTIONS OF THE HEISENBERG-ISING CHAIN

We now return to the problem of the spin-$\frac{1}{2}$ linear chain. The time dependence has been discussed previously, for example, in Eq. (19). Here

TABLE I. Exponents for the Baxter model.

β	γ	ν	η	θ
$\frac{1}{4}\frac{\theta}{1-\theta}$	$\frac{1}{2}\frac{2-\theta}{1-\theta}$	$\frac{1}{2}\frac{1}{1-\theta}$	θ	$\frac{4-\theta}{\theta}$

Continuum model: $2\theta = \left(\frac{\pi - 2J_z}{\pi + 2J_z}\right)^{1/2}$

Discrete lattice: $2\theta = 1 - (2/\pi)\arcsin J_z$

we wish to examine the results in more detail, and consider the question of next leading terms in the asymptotic expansion. The boson representation for spin operators, used to calculate these next leading terms, is discussed in the Appendix.

Using Eq. (A6), and following the usual procedure for evaluating correlation functions of this type,[6] gives the answer

$$2\pi^2\alpha^2\langle S^z(x,t)S^z\rangle = \cos 2k_F x \times\left(\frac{\alpha^2}{x^2-c^2t^2}\right)^{\theta^{-1}/2} - \frac{\alpha^2(x^2+c^2t^2)}{x^2-c^2t^2}, \tag{21}$$

where $c=[\theta+(4\theta)^{-1}]^{-1}$ is the renormalized Fermi velocity. It is easy to perform a Fourier transformation of these functions which leads to the spectral representation of the form

$$\langle S^zS^z\rangle_{k,\omega} \sim [\omega^2 - c^2(k-2k_F)^2]^{\theta^{-1}/2-1} + (k_F \longleftrightarrow -k_F) + \delta(\omega^2 - c^2k^2). \tag{22}$$

The first expression shows the power-law behavior near the thresholds $\omega = \pm c(k-2k_F)$ and $\omega = \pm c(k+2k_F)$ which is characteristic of Luttinger correlation functions. The relation of these functions to those in the Luttinger model is now obvious. The first terms are the $\pm 2k_F$ spectral density, while the last is the long-wavelength spectral density.

In the regions $|\omega| > c|k-2k_F|$ and $|\omega| > c|k+2k_F|$ there is spectral weight for excitations via the operator S^z, while outside these regions the imaginary part vanishes (at $T=0$). For finite temperature, the sharp distinction between the two regions is blurred since the correlation functions develop exponential tails, as discussed in recent work on the Luttinger model.[6]

The real part of the susceptibility $\chi^{zz}(k,\omega)$ has a singularity at $\omega=0$, and $k\to 2k_F$ given by

$$\mathrm{Re}\chi^{zz}(k,\omega) \sim (k-2k_F)^{\theta^{-1}-2}, \tag{23}$$

as is familiar from the Luttinger model with repulsive interactions. For the case $J_z<0$, this signals an instability of the system against application of a staggered field in the z direction. The mechanism here is the same as in the Peierls transition: the system can lower its energy under a perturbation by opening a gap at the Fermi momentum. The possibility of such instabilities in magnetic chains has been discussed recently by several authors.[13] For the ferromagnetic case $J_z>0$, the correlation function is not divergent and the susceptibility remains finite.

In calculating the transverse correlation function, a subtle point has to be considered. The diagonalization of the lattice fermion Hamiltonian is different for even and for odd numbers of fermions present. *A priori* there is no simple relationship between the eigenfunctions in the two cases. Now the operators S^+ just change the fermion number by one, in contrast to the operator for S^z. As pointed out by McCoy *et al.*,[14] difficulties can then arise. These authors also gave a recipe to avoid this problem. Instead of calculating the two-spin function directly, one can obtain it via a factorization property at large distances from an appropriate four-spin function. But this means, in our approach, that the τ operators, cf. Eq. (13), cancel pair-wise and therefore do not appear. This further justifies dropping τ in correlation functions, as argued in the previous Sec. III. Having established this result, we do not have to take the detour via the four-spin function but can compute the transverse correlation directly. We then obtain from Eq. (12) and Eq. (A8) the result

$$2\pi^2\alpha^2\langle S^y(x,t)S^y\rangle = \left(\frac{x^2+c^2t^2}{x^2-c^2t^2}\right)^{\theta/2} - \cos(2k_F x) \times \frac{x^2+c^2t^2}{x^2-c^2t^2}\left(\frac{\alpha^2}{x^2-c^2t^2}\right)^{(\theta+\theta^{-1})/2}. \tag{24}$$

The Fourier transformation is straightforward, as before. The imaginary part of the susceptibility χ^{+-} is nonzero in the regions $|\omega| \geq c|k|$ and shows the power-law singularities at the borderlines. Correspondingly, the real part of the transverse susceptibility diverges at $k=0$. For antiferromagnetic coupling in the basal plane the cosine factor in Eq. (24) moves to the first term, as discussed in the paragraph following Eq. (13). The instability is then shifted to $k=\pm 2k_F$ and therefore is similar to the one in the longitudinal function.

An interesting special case is the antiferromagnetic Heisenberg chain, corresponding to $J_z=-1$. For this problem it is known that the ground state is a singlet and nondegenerate, hence longitudinal and transverse correlation functions have to be equal. Now according to our original identification $J_z=-1$ gives $2\theta=(\pi+2)^{1/2}(\pi-2)^{-1/2}$. Obviously the exponents of the correlation functions are not equal for this value of θ. However, if we make the same identification as was used for the Baxter model, $2\theta = 1-2\pi^{-1}\arcsin J_z$, then $J_z=-1$ leads to $\theta=1$. For this value of θ the two correlation functions indeed are equal, after including the shift of the cosine factor, discussed above, which is appropriate for the antiferromagnetic case. The functions in this limit decay as x^{-1} or t^{-1}, respectively, and the change of sign from site to site is consistent with numerical results of Bonner and Fisher on a ring of 10 spins.[15]

Using the Luttinger-model results for calculating correlation functions at finite temperature[6] leads to the result that the staggered susceptibility diverges as T^{-1}, therefore $\gamma=1$. At $T=0$, the correlation function behaves as x^{-1}, giving $\eta=2$. In addition, the exponent δ for the staggered magnetization is easily calculated using the procedure of Sec. II. Here we use the formal expression for the staggered magnetization M_s as a perturbation expansion in the external staggered field H_s and assume scaling. The result is

$$\frac{\partial M_s}{\partial H_s} \sim \int d^2x\, f(H_s^2 x^\omega) x^{-\theta} , \tag{25}$$

where

$$f(H_s^2, x^\omega) = \sum_n c_n H_s^{2n} x^{n\omega}$$

is the result of a dimensional analysis of the formal perturbation theory, with c_n finite constants and $\omega = 4 - \theta$. From this we determine the dimension of H_s relative to x, and thereby calculate the exponent $\theta\delta = 4 - \theta$, which gives $\delta = 3$ for the antiferromagnetic case.

From the equivalence of our Hamiltonian to the Luttinger model, we also conclude that the leading contribution to the low-temperature specific heat is $C_v = (2\pi c^{-1}) L T$.

There are still other results known for this antiferromagnetic case. Des Cloiseaux and Pearson[16] determined the triplet excitations of the chain with the dispersion relation $\omega(k) = \pi/2 |\sin k|$. Near $k \approx 2k_F = \pi$ this determines a region $\omega \geq \pi/2 |k - 2k_F|$ in the $\omega - k$ plane where spectral density is nonzero. Although we have a similar region in our calculation $\omega \geq c|k - 2k_F|$, we produce the value $\frac{4}{5}$ for the corresponding velocity. This implies that the lattice model has a different dependence of the Fermi velocity on interaction strength than the continuum. One should probably not expect the velocities to be the same in this intermediate coupling region, because prefactors certainly can be model dependent.

V. DISCUSSION

The steps involved in relating the two-dimensional Baxter model to the continuum field theory appear complicated because of the intermediate steps involving the spin-$\frac{1}{2}$ x-y-z Hamiltonian. It may be possible to overlook the ease of calculation which these results establish. The calculation of the exponent η in the Ising model is illustrative of this simplicity. From Eq. (4) and Eq. (15), the Ising correlation function is seen to be

$$S^2\langle \mu(x)\mu\rangle = \langle e^{\phi_1(x)/2} e^{-\phi_1/2}\rangle_0 , \tag{26}$$

where the "0" means an expectation value in the free-particle Luttinger model. One finds immediately the behavior $x^{-1/4}$, using Eq. (15′). Similarly, the calculation of the other exponents can be stripped to essentially trivial operator algebra, following the procedures of Sec. III. From this viewpoint, the reason that η equals $\frac{1}{4}$, is essentially dimensional: the spin-density operator is the square root of a free-Fermion operator, which can be defined with the boson representation.

With this connection in mind, it is then most plausible that critical exponents in the Baxter model will depend continuously on the interaction parameter, because the corresponding Fermi-model exponents do. If an interaction term were added to the Luttinger model which caused a gap in the $2k_F$ spectral density, corresponding to the $\langle S^z S^z\rangle$ function, this would presumably remove the continuous dependence on coupling constant. In this way, one could map back onto the statistical mechanic problem and determine a type of "Baxter model" which still exhibits Ising exponents.

A further application of our mapping of statistical mechanics problems onto field-theory problems involves the massive Luttinger model defined by Eqs. (7) and (8). Clearly, it is possible to determine the correlation functions in the Baxter model which are equal to Green's functions in the field theory. In particular, the homogeneity properties of the Baxter model imply the same homogeneity properties of these Green's functions. Consequently, we can trivially determine the set of "critical" exponents which appear in these finite-mass theories.

One concludes that the two-point functions satisfy

$$G(x) \sim x^{-\eta} f(mx)$$

for equal times, where $f(mx)$ is an undetermined function, η the zero-mass exponent, and $m \sim m_0^\nu$, where m_0 is the bare mass and ν is given in Table I. Based on physical arguments and on a variety of calculations, one believes the Baxter model is a one-length theory, consequently, the massive Luttinger model is also. It follows that higher-order Green's functions satisfy a similar homogeneity condition, with the exponent ν relating the correct length scale to the bare mass. Calculations of the Baxter model functions for $T \neq T_c$ therefore can be used to determine vacuum expectation values of these massive field theories.

Finally, we wish to remark on the question of the difference in exponents between lattice fermion theories and the continuum theories. This problem is, in fact, solved by the calculations summarized in Table I. Fermion correlation functions all involve the quantity θ, and we have established, with these calculations, that all exponents can be related to just one, for example, ν. The Baxter model on a lattice is rigorously the lattice fermion problem, Eq. (6). Using the lattice parametrization for θ therefore leads to expressions for fermion correlation function exponents which are correct for the lattice model with $m_0 = 0$. The differences between lattice and continuum exponents are small for coupling constant in the intermediate range of interest in many problems of the quasi-one-dimensional conductors, $J_z \approx 1$. We take this evidence as support for the position that the Luttinger model, and its extension to include spin and backward scattering, are excellent approximations for the fermion lattice problems as well.

It is interesting to consider the calculation which must be performed to determine exponents when the bandwidth is finite. As independently recognized by Ferrell,[17] there is an analogy to the x-ray threshold problem.[18] Indeed the phase factor $N(x)$ in Eq. (15) can be viewed as a potential of strength π turned on at the origin and off at x, while in the x-ray problem such a switching takes place in time. We offer the biased prejudice that our solution here bears the same relation to the lattice Baxter model as the Tomonaga-model calculation[19] does to the x-ray threshold problem.

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APPENDIX: CONTINUUM SPIN-½ ALGEBRA IN ONE DIMENSION

The purpose of this appendix is to discuss the ambiguities involved in the construction of operators which satisfy the algebra

$$\begin{aligned}&[S^+(x),\ S^-(x')] = 2S^z(x)\delta(x-x') ,\\ &[S^\pm(x),\ S^z(x')] = \mp S^\pm(x)\delta(x-x') .\end{aligned} \tag{A1}$$

These relations, together with $[S^+(x)]^2 = [S^-(x)]^2 = 0$, and $0 = [S^+(x),\ S^+(x')]$, etc., are taken to define the $S=\frac{1}{2}$ algebra. What we desire is a relation between these spin operators and continuum fermi field operators, equivalent to the Jordan-Wigner transformation for lattice operators,

$$\begin{aligned}&S_i^+ = a_i^\dagger \exp\left(i\pi \sum_{j=i}^{i-1} a_j^\dagger a_j \right) ,\\ &S_i^- = (S_i^+)^\dagger ,\\ &S_i^z = a_i^\dagger a_i - \tfrac{1}{2} ,\end{aligned} \tag{A2}$$

where the a_i operators are fermions, and the S_i satisfy the spin-$\frac{1}{2}$ algebra. There turn out to be several possibilities for the continuum algebra, differing only in the definition of $S^z(x)$ in terms of the underlying fermi field operators. An obvious choice is

$$S^z(x) = \psi_1^\dagger(x)\psi_1(x) + \psi_2^\dagger(x)\psi_2(x) - (s)^{-1} , \tag{A3}$$

where ψ_1 and ψ_2 refer to fermion operators satisfying the operator algebra of the Luttinger model, and we assume $\langle\psi_1^\dagger\psi_1\rangle = \langle\psi_2^\dagger\psi_2\rangle = (2s)^{-1}$, consequently,

$$S^z(x) = \rho_1(x) + \rho_2(x) , \tag{A4}$$

where ρ_1 and ρ_2 are canonical density operators in the Luttinger model. The complete spin algebra for this case is then

$$\begin{aligned}&S^+(x) = (2s)^{-1/2}[\psi_1^\dagger(x)\, e^{N(x)} + \psi_2^\dagger(x)\, e^{N'(x)}] ,\\ &S^-(x) = S^+(x)^\dagger ,\end{aligned} \tag{A5}$$

with

$$N(x) = i\pi \int_0^{x-s/2} dy[\rho_1(y) + \rho_2(y)]$$

and

$$N'(x) = N(x+s).$$

Keeping s finite, and using the algebra of Eq. (11), leads to the conclusion that this representation satisfies Eq. (A1) in the limit $s \to 0$.

However, this definition of $S^z(x)$ does not correspond to the intuitive definition which is suggested by the Jordan-Wigner representation. The operator

$$a_i^\dagger a_i = L^{-1} \sum_{p,q} a_{p+q}^\dagger a_p\, e^{-iqR_j}$$

has matrix elements between states at the Fermi energy not only near $q \approx 0$, but near $q \approx \pm 2k_F$. This leads to the definition

$$S^z(x) = \rho_1(x) + \rho_2(x) + \psi_1^\dagger(x)\psi_2(x) + \psi_2^\dagger(x)\psi_1(x) , \tag{A6}$$

which has the correct matrix elements, in the Luttinger model, both near $q=0$ and $\pm 2k_F$. We use Eq. (A6) to define the continuum spin representation of $S^z(x)$ with fermi operators.

It is now necessary to construct the proper transverse spin operators. The choice given in Eq. (10) satisfies this spin algebra, as can be directly verified. However, these transverse operators suffer from an asymmetry. Consider the representation for $S^-(x)$ as written out in Eq. (13). It is seen that the combination $e^{3\phi_2/2-\phi_1/2}$ occurs but not $e^{3\phi_1/2-\phi_2/2}$, which is simply a $1 \leftrightarrow 2$ interchange. The absence of this term is not important for $|J_z| < 1$, because it is not even a next leading singularity. At $J_z = -1$, however, it is of dimension two, and is necessary for the next leading singularities in the transverse functions to be the same as the longitudinal, as discussed in Sec. IV. We conclude that the fermion representation for the spin algebra of Eq. (10) correctly reproduce the leading terms, but cannot reproduce next leading contributions at $J_z = -1$.

Correcting this flaw requires a more symmetric choice for the "ordering" operator $e^{N(x)}$. Obviously any linear combination of the form $\alpha e^{N(x)} + \beta e^{-N(x)}$ will satisfy the $[S^+(x),\ S^z(x)]$ algebra, provided $[N(x),\ S^z(x')] = 0$. We are then led to propose the symmetric combination $2^{-1/2}(e^{N(x)} + e^{-N(x)})$ as the replacement for $e^{N(x)}$. With this definition,

$$2S^+(x) = s^{-1/2}(e^{N(x)} + e^{-N(x)})[\psi_1(x) + \psi_2(x)] \; , \qquad S^-(x) = [S^+(x)]^\dagger \; , \tag{A7}$$

together with Eq. (A6) defines the fermion problem which we solve in Sec. IV. It should be noted, however, that the commutator $[S^+(x), S^-(x)]$ leads to an operator which differs from Eq. (A6), giving $S^z(x)(1 + e^{2N(x)} + e^{-2N(x)})$ instead of $S^z(x)$, differing by the operators $e^{2N(x)}$ of dimensional behavior s^2x^{-2}. This algebra is therefore not exact, but because these higher-order operators do not contribute to leading or next leading terms, and vanish as $s \to 0$, the algebra is adequate for our purposes. We therefore discard these additional operators

As a consequence of these e^{2N} operators which appear in S^z, there will be corresponding spurious operators in the transverse functions. These also contribute only to weaker singularities, vanish as $s \to 0$, and can be discarded for these reasons. The leading contributions are correctly given by Eq. (15), while the next leading contributions are given by

$$(2\pi\alpha s)\langle S^y(x, t)S^y\rangle' = \langle e^{3\Phi_1(x,t)/2 - \phi_2(x,t)/2 + 2ik_F x} \times e^{-3\Phi_1/2 + \phi_2/2}\rangle + 1 \longleftrightarrow 2 \; , \tag{A8}$$

where $1 \leftrightarrow 2$ means the interchange $\Phi_1 \leftrightarrow \phi_2$ and $k_F \leftrightarrow -k_F$, and the prime indicates that this is to be added to the result of Eq. (15). It should also be noted that the $2ik_F x$ phase factors for these ϕ_1 and ϕ_2 operators are chosen for the case $J_x > 0$, as in Eq. (13). For $J_x < 0$, the antiferromagnetic case, the $2ik_F x$ phase factor moves over to the leading singularity term, as discussed following Eq. (13), and does not appear in Eq. (A8).

It is necessary to introduce the cutoff parameter α to define the momentum summation in the operators $\phi_1(x)$ and $\phi_2(x)$, as in Eq. (13), which are the Fourier transforms of the objects $2N_1(x)$. This momentum cutoff α^{-1} can naturally be taken as the Brillouin-zone boundary momentum, and we have therefore chosen $s = \pi\alpha$ in Eq. (24).

In arriving at this result for the spin-algebra representation to be used in calculating correlation functions, it is clear that difficulties have been pushed into yet higher terms in the asymptotic series. It might be desirable to have an expression which is correct for all terms in this series, but for the calculation of exponents in leading or next leading terms this is not necessary. We postpone this problem for future work.

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