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Analyticity of critical temperatures in the large-n region

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The coefficient of $1/n$ in the expansion of the critical temperature for a classical n-component spin system is shown to be an analytic function of the dimension d near $d = 4$; this result holds both for the hypercubic-lattice model and for the continuous-space model.

I. INTRODUCTION

Contradictory statements have recently been reported[1,2] concerning the analyticity near $d=4$ of the critical temperature $T_c(n, d)$ for a classical n-component spin system in d-dimensional space. This question is not so academic as it looks: The determination of T_c for physical values of d by expansion[3] in powers of $1/d$ relies on the possibility of an analytic continuation through $d=4$.

Some arguments tend to support the existence of a singularity. It is now well established[4] that, considered as functions of the continuous variable d, the critical exponents are nonanalytic at $d=4$. It is thus tempting to speculate that the critical temperature presents a similar singularity, since its calculation involves essentially the same ingredients as the calculation of the critical exponents. Moreover,[1] a Padé analysis of the expansion of $T_c(1, d)$ in powers of $1/d$ seems to indicate a weak singularity near $d=4$.

On the other hand, however, the numerical convergence of this expansion[3] is good with a small number of terms for $d=3$ and even for $d=2$. Besides, $T_c(n, d)$ has been proved to behave analytically around $d=4$, both in the limit $n=-2$ for the continuous model[5] and in the limit $n=+\infty$ for the lattice model,[2,6] although in the latter case most critical exponents have a different analytic form for $d<4$ and $d>4$.

This argument in favor of analyticity at $d=4$ of $T_c(n, d)$ is not quite convincing, since many simplifications occur in the limit $n\to\infty$. In particular, the occurrence of two different analytic forms of critical exponents for $d<4$ and $d>4$ is not connected with a singularity, but is simply explained by a competition between two terms behaving differently. For instance, the two dominant contributions to the susceptibility χ near $d=4$ and $T=T_c$ have the form

$$\chi=\frac{a}{T-T_c}\frac{1}{4-d}\left[\left(\frac{b}{T-T_c}\right)^c-1\right], \tag{1}$$

where a, b, T_c, and $c=(4-d)/(d-2)$ are analytic functions of d; the resulting critical exponent γ switches from 1 for $d>4$, to $1+c$ for $d<4$. Such a trivial explanation does not hold for n finite, since critical exponents probably have a true singularity at $d=4$. This is indicated[7] by the poor convergence of their expansion in powers of $4-d$ for n finite, and also by the occurrence of a larger and larger number of singularities at positive rational values of $d-4$ in the diagrams contributing to the successive terms of the $4-d$ expansion. The $1/n$ contributions to critical exponents[8] are already nontrivial; in particular, the behavior of η [$\eta=0$ for $d>4$, $\eta\sim(4-d)^2/2n$ for $d<4$] cannot be traced to the occurrence of two competing analytic terms.

It is therefore of interest to ask whether the $1/n$ contribution to $T_c(n, d)$ is analytic at $d=4$ as $T_c(\infty, d)$, or not. We have investigated this point both for the continuous model and for the lattice model. The nonuniversal character of T_c is reflected in the differences between both calculations, but the final expressions are formally similar, and the conclusions are the same: to order $1/n$, T_c remains analytic at $d=4$. This result is not obvious, since T_c is obtained by equating the inverse susceptibility, χ^{-1}, to zero; the $1/n$ corrections to T_c and to γ come from the same expression, but behave quite differently around $d=4$.

The result supports the hypothesis concerning the analyticity of $T_c(n, d)$, but, of course, does not rule out the possibility of a singularity disappearing faster than $1/n$ for n large. It is likely, however, that if a singularity exists it is weak enough to allow a safe numerical evaluation of $T_c(n, 3)$ through expansion in powers of $1/d$.

II. CONTINUOUS MODEL

Consider a classical spin field $\vec{S}(x)$ with n components in a d-dimensional space, with an effective Hamiltonian of Wilson type,

$$\mathcal{H}=\frac{1}{2}\sum_p\left[p^2/\varphi(p)+r_0\right](\vec{S}_p\cdot\vec{S}_{-p}) + \tfrac{1}{8}u_0\int d^dx\,(\vec{S}^2(x))^2 . \tag{2}$$

The cutoff function $\varphi(p)$, behaving like $1+O(p^2)$ for p small and vanishing fast enough at infinity, has been introduced in momentum space in the quadratic part of the Hamiltonian; such a cutoff is essential for the evaluation of T_c, and will remain fixed throughout. The expansion for the susceptibility[8]

$$\chi \equiv (nT)^{-1}\langle \vec{S}_0^2 \rangle \equiv r^{-1}, \tag{3}$$

valid for T above T_c and n large, results from a diagrammatic expansion of the mass operator $\Sigma(p)$, and from the expansion of the full propagator

$$G(p) = [p^2/\varphi(p) + r + \Sigma(p)]^{-1}, \tag{4}$$

in powers of $\Sigma(p)$. The value of r is determined self-consistently via the relation

$$\Sigma(0) = 0. \tag{5}$$

At the critical temperature T_c, the inverse susceptibility r vanishes. The unperturbed propagator then reduces to

$$g(p) = \varphi(p)/p^2, \tag{6}$$

and Eq. (5) becomes, to first order in $1/n$,

$$r_0 + \frac{t_c}{2}\int d^dp\, g(p) + \frac{t_c}{n}\int d^dq\, \frac{g(q)}{1+t\Phi(q)}$$

$$-\frac{t_c^2}{2n}\int d^dp\, d^dq\, \frac{g^2(p)}{1+t_c\Phi(q)}\,[g(q-p) - g(q)] = 0, \tag{7}$$

where

$$t_c \equiv (2\pi)^{-d} n u_0 T_c, \tag{8}$$

measures the critical temperature. The integral

$$\Phi(q) = \frac{1}{2}\int d^dp\, g(p) g(q-p), \tag{9}$$

accounts for the screening of the interaction.

In the limit $n\to\infty$, the critical temperature

$$t_\infty = (-2r_0)\left(\int d^dp\, g(p)\right)^{-1}, \tag{10}$$

obviously has no singularity for $d>2$, as in the case of the lattice model.[2]

The correction of order $1/n$ to t_c resulting from (7) is then

$$\delta t = t_c - t_\infty = -2n^{-1}t_\infty + (-2r_0 n)^{-1} t_\infty^3 \times \int d^dq\, [1 + t_\infty \Phi(q)]^{-1}\Psi(q), \tag{11}$$

where

$$\Psi(q) \equiv \int d^dp\, g(p) \times [g(q)g(q-p) + g(p)g(q-p) - g(p)g(q)]. \tag{12}$$

The infrared behavior of $\Phi(q)$ is not the same for $d>4$ [$\Phi(0)$ finite] as for $d<4$ [$\Phi(q)\propto q^{d-4}$]. Moreover, $\Phi(q)$ is singular at $q=0$, and the denominator $1+t_\infty\Phi(q)$ in (11) might have singularities or zeros near the origin, which would yield after integration over q, singularities of δt as a function of the variable $d-4$. A careful analysis of the integral (11) in the infrared region is thus needed.

Let us first evaluate $\Psi(q)$ for q small near $d=4$. The dominant contribution is obtained for $2<d<6$ by suppressing the cutoff function $\varphi(p)$ in $g(p)$ which then reduces to p^{-2}. The integral (12) then yields

$$\Psi(q) \sim \frac{\pi^{(d+3)/2}(4-d)q^{d-6}}{2^{d-3}\sin[\frac{1}{2}\pi(4-d)]\Gamma(\frac{1}{2}(d-1))}. \tag{13}$$

Higher-order corrections behave as $q^{d-4}, q^{d-2}, \ldots, 1, q^2, \ldots$, and are cutoff dependent.

Consider now $\Phi(q)$. It is an analytic function of d near $d=4$ for q finite, but its dominant contribution for q small has a different form for $d>4$ and for $d<4$. If, however, we consider the first two terms of the expansion of $\Phi(q)$ near $q=0$, the same phenomenon as in Eq. (1) takes place. It is indeed easy to show from (9), that for $2<d<6$, one has

$$\Phi(q) = \frac{\pi^{(d+3)/2}q^{d-4}}{2^{d-2}\sin[\frac{1}{2}\pi(4-d)]\Gamma(\frac{1}{2}(d-1))} + \frac{\pi^{d/2}}{(4-d)\Gamma(\frac{1}{2}d)}\int_0^\infty p^{d-4}\,dp\,\frac{d}{dp}[\varphi(p)]^2 + \cdots. \tag{14}$$

The relative sizes of the two terms interchange when d crosses 4. Their coefficients diverge like $\pi^2(4-d)^{-1}$ and $\pi^2(d-4)^{-1}$, but the sum is continuous and equal to the value of $\Phi(q)$ for $d=4$. Subsequent terms would behave like $q^{d-2}, q^d \ldots, q^2, q^4 \ldots$.

Both the numerator and the denominator of the integrand in (11) are analytic for $q>0$, and singular at $q=0$, the integral being convergent for $d>2$. In order to exhibit a singularity of δt as a function of d after integration over q, we should look for singularities of the integrand in the complex q plane (besides the branch point $q=0$), tending to the real axis and in particular to the origin when $d\to 4$. The only possible singularities of this type might be poles, i.e., zeros of $1+t_\infty\Phi(q)$. From (14), this quantity has for q small the form

$$[A/(4-d)][(q/q_0)^{d-4} - 1] + B + \cdots. \tag{15}$$

It is not analytic at the point $q=0$, but obviously has neither singularities nor zeros elsewhere in the vicinity of $q=0$. Zeros of (15) for q small would indeed occur for

$$q/q_0 \approx [1 + (d-4)(B/A)]^{1/(d-4)},$$

a quantity which is not small, even for $d\to 4$ where it approaches $e^{B/A}$. We can therefore conclude that δt is an analytic function of $d>2$, in particular around $d=4$.

III. LATTICE MODEL

Let us now turn to the lattice model. At each site x_i of a hypercubic d-dimensional lattice with unit spacing, lies a classical n-component spin $\vec{S}(x_i)$ of length $n^{1/2}$. The Hamiltonian is

$$\mathcal{H} = -J \sum \vec{S}(x_i) \cdot \vec{S}(x_j) , \tag{16}$$

where the sum runs over the lattice bonds. The susceptibility

$$\chi \equiv (nT)^{-1} \sum_i \langle \vec{S}(x_i) \cdot \vec{S}(x_0) \rangle , \tag{17}$$

for $T > T_c$, is known[9] to expand to first order in $1/n$, as

$$(J\chi)^{-1} = s - \frac{1}{2n\Phi(0)} \int d^dp\, d^dq \frac{g^2(p)}{\Phi(q)} \times [g(q-p) - g(q)] , \tag{18}$$

where

$$[g(p)]^{-1} \equiv s + \sum_{k=1}^{d} (2\sin\tfrac{1}{2}p_k)^2 ; \tag{19}$$

the integrals here are to be carried over the Brillouin zone $|p_k| < \pi$; the quantity $\Phi(q)$ is again given by (9), and the parameter s is defined as a function of the temperature by

$$(2\pi)^d J T^{-1} \equiv t^{-1} = \int d^dp\, g(p) . \tag{20}$$

For $n = +\infty$, the results of the spherical model[2,6] are recovered. The value t_∞ of the critical temperature, where $\chi^{-1} = Js + O(n^{-1})$ goes to zero, is associated with $s = 0$ and is given by

$$t_\infty^{-1} = \int d^dp\, g(p)|_{s=0} = \frac{1}{2} \int_0^\infty dx\, [2\pi e^{-x} I_0(x)]^d . \tag{21}$$

Its expression, analytically continued in d, is analytic for $d > 2$. The relation between s and t, valid for arbitrary n, namely,

$$t - t_\infty = \tfrac{1}{2} t t_\infty \int_0^\infty dx (1 - e^{-xs/2}) [2\pi e^{-x} I_0(x)]^d , \tag{22}$$

reduces for s small to

$$t - t_\infty \sim \tfrac{1}{4} s t_\infty^2 \int_0^\infty x\, dx [2\pi e^{-x} I_0(x)]^d , \tag{23a}$$

for $d > 4$, and to

$$t - t_\infty \sim 2 s^{(d-2)/2} t_\infty^2 \pi^{d/2} (d-2)^{-1} \Gamma[\tfrac{1}{2}(4-d)] , \tag{23b}$$

for $d < 4$, in agreement with the value of the critical exponent γ_∞ for $n = +\infty$ {namely $\gamma_\infty = \max[1, 2/(d-2)]$}.

The determination of the $1/n$ correction, $\delta t = t_c - t_\infty$, to the critical temperature requires, however, more care than for the continuous model. The expansion (18) of the inverse susceptibility, established by the steepest descents method, is valid for s (or t) fixed, and $n \to \infty$; it does not hold when $s \to 0$ at the same time as $n \to \infty$. Since the value s_c, of s at t_c, as given by (23), is small like a power of $1/n$, we cannot calculate it to lowest order simply by requiring that the first two terms of the expansion of χ^{-1} vanish.

In order to determine δt without letting t approach too close to t_c in (18), we should rather expand[9] the form

$$\chi^{-1} \sim A(t - t_c)^\gamma , \tag{24}$$

in powers of $1/n$, and identify the resulting first terms with (18). For s small, but finite as $n \to \infty$, this yields

$$\frac{\gamma s \delta t}{t - t_\infty} \sim \frac{1}{2n\Phi(0)} \int d^dp\, d^dq \frac{g^2(p)}{\Phi(q)} \times [g(q-p) - g(q)] , \tag{25}$$

where the various factors should be evaluated to lowest order. In particular, $\gamma = \gamma_\infty$ has a singularity for $d = 4$, which at first sight might be expected to reflect on δt. We note, however, by differentiating (20) and by using (23), that the factor

$$\Phi(0) = \frac{1}{2t^2} \frac{dt}{ds} , \tag{26}$$

combines with $\gamma_\infty s/(t - t_\infty)$, so that (25) yields

$$\delta t = n^{-1} t_\infty^2 \int d^dp\, d^dq [\Phi(q)]^{-1} g^2(p) \times [g(q-p) - g(q)] , \tag{27}$$

in which the integral is to be calculated with $s = 0$. We may finally rewrite (27) as

$$\delta t = -2n^{-1} t_\infty + n^{-1} t_\infty^2 \int d^dq [\Phi(q)]^{-1} \Psi(q) , \tag{28}$$

where $\Phi(q)$ and $\Psi(q)$ are defined by the same equations (9) and (12) as for the continuous model, the only difference being that $g(p)$ is now given by (19) with $s = 0$.

Although the critical temperature is model-dependent, we have exhibited a formal analogy between its expression (10) and (11) for the continuous model, and its expression (21) and (28) for the lattice model. The lack of isotropy would make it difficult to perform an explicit analytic continuation of (28) over d. Since, however, we are interested only in possible infrared singularities, we may expand $g(p)$ around its dominant isotropic part, p^{-2}, and analytically continue each resulting term. Once angular integrations are performed, the argument of Sec. II applies to the radial integration of (28) over q, since the denominator $\Phi(q)$ again has the behavior (15). The critical temperature of the lattice model is therefore also an analytic function of $d>2$.

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