

Model for a multicomponent quantum system*

Bill Sutherland†

The Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794
(Received 3 March 1975)

In a recent paper, Lai introduced a lattice-gas model. In this paper we generalize Lai's model, making application to various systems such as dilute Heisenberg magnets, higher-spin systems, and a lattice of SU(3) triplets. By a careful consideration of general thermodynamic stability, and by variational arguments, we demonstrate Lai's solution to be incorrect, and in turn produce the correct solution in this case and in other cases including higher-dimensional models. The remaining cases we treat in one dimension by Bethe's ansatz, reducing the problem to coupled integral equations. We locate the singularities of the ground-state energy in the phase plane; and we explicitly calculate the absolute-ground-state energy, excitations above the absolute ground state, and the first correction to the absolute ground state for small concentrations of impurities.

I. INTRODUCTION

In a recent paper,¹ Lai introduced a model for a quantum lattice gas and presented a solution in the case of one dimension. However, his solution is incorrect for his most interesting example $\Delta = -1$ (our example B^2F), and thus his conclusion that two phases will coexist for some fixed concentration is invalid. This can be seen by noting that upon expanding the ground-state energy E/L as a function of M/L for fixed N/L , both of Lai's equations (15) and (20) give a positive correction. However, E/L for fixed N/L must be symmetric in M/L about $N/2L$ and by the requirement of thermodynamic stability, must be concave upward. Thus the first correction must be negative or zero.

The reason for Lai not obtaining the true ground state appears to be that neither of his assumptions for the distribution of quasimomentum in Bethe's ansatz corresponds to the ground-state distribution. In fact, it is unclear in what sense Bethe's ansatz is a solution in this case.

We have avoided this problem in the present paper by instead using general arguments of thermodynamic stability coupled with lower bounds on the ground-state energy from the variational principle. This method has a definite advantage over an explicit solution for the wave function and, in fact, gives results in higher dimension as well.

The other situation that Lai considered, his case $\Delta = +1$, corresponds to our BF^2 , and in this instance our equations agree.

Although our primary purpose is to discuss Lai's solution, we have in addition considerably generalized and rephrased the original problem. For instance, our Eq. (1) would be a natural choice for a prototypical Hamiltonian to exhibit the phenomena of mixing in a multicomponent system. In this formulation, we no longer have the arbitrary restriction of a coupling constant to particular values, i. e., $\Delta = \pm 1$. Finally, numerous applications are made to various branches of physics in which

multicomponent systems are of interest.

The final portion of our paper is devoted to explicit calculation of properties of interest and the demonstration of their dependence on the number of components.

II. STATEMENT OF THE PROBLEM

Consider a periodic one-dimensional lattice of N sites. Place N objects, or particles, on this lattice exactly one to a site. Let the operator $P_{j,k}$ permute whatever objects occupy sites j and k . We then wish to compute the eigenvalues and eigenfunctions of the Hamiltonian H ,

$$H = \pm \sum_{j=1}^N P_{j,j+1} \quad (N+1 = 1). \quad (1)$$

The problem is obviously invariant under the permutation group S_N . We shall also have occasion to consider the corresponding problem on a square or cubic lattice.

We must further specify the nature of the N objects. We assume P species, or components, denoted A, B, \dots, P . Objects of a given species A are identical, but may either be bosons $A = +1$, or fermions $A = -1$. This specification of species statistics we designate as type $\{P\} = \{A, B, \dots, P\}$. Further, if x of the species are bosons and y are fermions, $x + y = P$, then we shall often write the type $\{P\}$ as $B^x F^y$.

Let us define N_A as the number of objects of species A . The N_A are constants of motion, for the permutation operators do not create or annihilate particles. If we further define N_{AB} as the number of nearest-neighbor pairs of species A and B , then we have the following relationships:

$$\sum_A N_A = N, \quad (2a)$$

$$\sum_{A \geq B} N_{AB} = N, \quad (2b)$$

$$N_A = N_{AA} + \frac{1}{2} \sum_{B(\neq A)} N_{AB}$$

$$= \frac{1}{2} \left(N_{AA} + \sum_B N_{AB} \right). \quad (2c)$$

Corresponding quantities normalized to the total site number N we denote by lower case n , i. e., $n_A = N_A/N$, $n = 1$.

It is useful to define a restricted Hamiltonian $H(\{P\})$ which will be equivalent to H if we only consider states of type $\{P\}$,

$$H(\{P\}) = \pm \left(\sum_A AN_{AA} + \sum_{A>B} \phi_{AB} \right). \quad (3)$$

The operator ϕ_{AB} permutes only nearest neighbors of species A and B , otherwise it gives zero. This restricted Hamiltonian is invariant under $S_{N_A} \otimes \dots \otimes S_{N_P}$, not S_N .

If two operators are related by a unitary transformation, they have the same spectrum. In preparation for later discussion, we catalogue here the following unitary transformations, and their effect on H and $\{P\}$:

(a) Let us multiply all wave functions by the wave function completely antisymmetric in all objects. This operation we denote by L , and

$$LHL = -H, \quad L\{P\} = \{-P\}. \quad (4)$$

We see that the \pm sign in the original expression for H is unnecessary, if we vary the type. We henceforth choose the minus sign.

(b) Consider the following operations, denoted by J_A : Proceed around the lattice (N even) and at each even numbered site, multiply the wave function by $+1$ if no particle of species A is present, -1 if there is a particle of species A present. This operator then transforms H by

$$J_A H J_A = H + 2 \sum_{B(\neq A)} \phi_{AB}, \quad J_A \{P\} = \{P\}. \quad (5)$$

Thus J_A on H changes the sign of ϕ_{AB} , all $B \neq A$.

(c) The previous operations may be applied to higher dimension, as well as to the one-dimensional problem. However, there exists a transformation—the Jordan-Wigner transformation—applicable only to one dimension. This transformation may be used to change bosons into fermions and vice versa.

We first note the well-known lattice-gas analogy between hard-core bosons on a lattice, and two-component spins on a lattice: If $S_z = -\frac{1}{2}$, there is no boson; if $S_z = +\frac{1}{2}$, there is one boson. Then, in terms of the Pauli spin operators, the Jordan-Wigner transformation to fermion creation and annihilation operators C^\dagger , C is

$$C_j = \sigma_j^- \prod_{k=1}^{j-1} \sigma_k^z. \quad (6)$$

We denote such a Jordan-Wigner transformation on the species A by K_A . Then the effect of K_A is

$$K_A H K_A = H + 2\phi_{AA}, \quad K_A \{A, B, \dots, P\}$$

$$= \{-A, B, \dots, P\}. \quad (7)$$

That is, the transformation changes the sign of ϕ_{AA} . On the other hand, if we vary the signs of all the terms ϕ_{AA} in H , we need not vary the type $\{P\}$ at all.

The restricted Hamiltonian is invariant under K_A , but since the type of any wave function does change, we conclude that the eigenvalues of the restricted Hamiltonian are independent of the type.

(d) We define $K \equiv \prod_A K_A$, and thus

$$K H K = \sum_A \phi_{AA} - \sum_{A>B} \phi_{AB}, \quad K \{P\} = \{-P\}. \quad (8)$$

(e) Finally, if we multiply K and L to give a transformation $J \equiv KL$, then

$$J H J = - \sum_A \phi_{AA} + \sum_{A>B} \phi_{AB}, \quad J \{P\} = \{P\}. \quad (9)$$

III. GENERALITIES

In this paper we shall consider the ground-state energy E_0 , or $\epsilon \equiv E_0/N$, for the Hamiltonian of Eq. (1) with the minus sign.² First, we shall derive some general properties of $\epsilon(\{P\}; n_A, \dots, n_P)$.

It is easy to verify the stability property that $\epsilon(\{P\}; n_A)$ considered as a function of the concentrations n_A is concave upwards: To derive an upper bound on ϵ at a concentration

$$n_A = a n_A^1 + (1-a) n_A^2 \quad (0 \leq a \leq 1), \quad (10)$$

we divide the system into two fractions; aN and $(1-a)N$. In the first portion, we take as a trial wave function the ground state at a concentration n_A^1 , and in the second, the ground state at a concentration n_A^2 . Then by the variational principle,

$$\epsilon(n_A) \leq a\epsilon(n_A^1) + (1-a)\epsilon(n_A^2). \quad (11)$$

This is precisely the statement that $\epsilon(n_A)$ concaves upwards as a function of n_A .

Suppose we have two species A and B which are both bosons; i. e., $A = B = +1$. Let us consider the reduced Hamiltonian,

$$H(\{P\}) = -(N_{AA} + N_{BB} + \phi_{AB})$$

$$- \sum_{C(\neq A, B)} (\phi_{AC} + \phi_{BC}) - \sum_{\substack{C \geq D \\ (\neq A, B)}} \phi_{CD}. \quad (12)$$

The minimum of the operator $-\phi_{AB}$ is equal to $-N_{AB}$. Thus, if we let the label S represent either A or B ,

$$H(\{P\}) \geq -N_{SS} - \sum_{C(\neq S)} N_{CS} - \sum_{\substack{C \geq D \\ (\neq S)}} \phi_{CD}. \quad (13)$$

If we consider eigenstates of $H(\{P'\})$ of type $\{P'\}$, where

$$\{P'\} = \{S=1, C, D, \dots, P\}, \quad (14)$$

then these states will serve as acceptable states for $H(\{P'\})$. We then choose the ground state $\psi_0(\{P'\})$ to minimize the right-hand side of Eq. (13) with the ground-state energy $\epsilon(\{P'\}; n_s, n_c, \dots, n_p)$. We finally conclude

$$\begin{aligned} \epsilon(\{1, 1, C, \dots, P\}; an, (1-a)n, n_c, \dots, n_p) \\ \geq \epsilon(\{1, C, \dots, P\}; n, n_c, \dots, n_p) \quad (0 \leq a \leq 1). \end{aligned} \quad (15)$$

But by the concavity property, the reverse must also be true, and hence we have an equality,

$$\begin{aligned} \epsilon(\{1, 1, C, \dots, P\}; an, (1-a)n, n_c, \dots, n_p) \\ = \epsilon(\{1, C, \dots, P\}; n, n_c, \dots, n_p) \quad (0 \leq a \leq 1). \end{aligned} \quad (16)$$

In words: If two components of a P -component system are bosons, then the ground-state energy is identical to a $(P-1)$ -component system.

Before restricting ourselves entirely to one dimension, we briefly summarize the few solutions to Eq. (1) which are known exactly in three dimensions. First, the one-component systems are obviously trivial. Second, by Sec. II, the ground-state energy for $B^x F^y$ is equivalent to BF^y . Thus for B^x , $\epsilon = -1$.

Third, let us consider BF . We use a representation where bosons are represented by vacancies on the lattice and the fermions hop about. Then the corresponding reduced Hamiltonian is

$$\begin{aligned} H &= N_{FF} - N_{BB} - \phi_{FB} \\ &= N_F - N_B - \phi_{FB}. \end{aligned} \quad (17)$$

Thus all eigenfunctions are given by the familiar expression

$$\Psi(\vec{X}_1, \dots, \vec{X}_{N_F}) = \det[\exp(i\vec{K}_j \cdot \vec{X}_i)], \quad (18)$$

where

$$\begin{aligned} K_{j\alpha} &= 2\pi n_{j\alpha} / N^{1/3}, \quad n_{j\alpha} \text{ are integers,} \\ j &= 1, \dots, N_F; \quad \alpha = 1, 2, 3. \end{aligned} \quad (19)$$

Then the energies are

$$\epsilon = n_F - n_B - \frac{2}{N} \sum \cos K_{j\alpha}. \quad (20)$$

For the ground state in one dimension,

$$\epsilon_0 = 2n_F - 1 - (2/\pi) \sin(\pi n_F). \quad (21)$$

Lastly, Eq. (20) or (21) also gives the ground-state energy for $B^x F$.

IV. EXAMPLES

In the remainder of this paper, we will consider only the case of one dimension. Before presenting

the exact solution, we will first give a few examples of systems to which the Hamiltonian of Eq. (1) would apply. For the case of two species $P=2$, it is natural to use the language of spin-1/2 systems. The reduced Hamiltonian is

$$H = -(AN_{AA} + BN_{BB}) - \phi_{AB}. \quad (22)$$

We consider each site as a spin $\frac{1}{2}$, with species A given by spin-up, species B given by spin-down. Then, in terms of the Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$, we have a representation of

$$\phi_{AB} = \sum_{nn'} \frac{1}{2} (\sigma_x \sigma'_x + \sigma_y \sigma'_y), \quad (23a)$$

and

$$AN_{AA} + BN_{BB} = \begin{cases} + \sum_{nn'} \frac{1}{2} (1 + \sigma_z \sigma'_z), & BB \\ - \sum_{nn'} \frac{1}{2} (1 + \sigma_z \sigma'_z), & FF \\ \pm \sum \sigma_z, & FB \end{cases} \quad (23b)$$

The expression $\sum_{nn'}$ represents a summation over nearest-neighbor pairs.

Since $\sum \sigma_z$ is a constant of the motion, equal to $N_A - N_B$, which may be fixed at will, we find the following cases:

(a) BB ,

$$H = -\frac{1}{2} \sum_{nn'} (1 + \vec{\sigma} \cdot \vec{\sigma}'). \quad (24)$$

Thus this type corresponds to a Heisenberg ferromagnet.

(b) FF ,

$$H = +\frac{1}{2} \sum_{nn'} (1 + \sigma_z \sigma'_z - \sigma_x \sigma'_x - \sigma_y \sigma'_y). \quad (25)$$

Let us transform this Hamiltonian by the operator J of Eq. (9), which changes the sign of ϕ_{AB} . Then the transformed H is

$$H = +\frac{1}{2} \sum_{nn'} (1 + \vec{\sigma} \cdot \vec{\sigma}'). \quad (26)$$

This type corresponds to the Heisenberg antiferromagnet.

(c) FB ,

$$H = -\frac{1}{2} \sum_{nn'} (\sigma_x \sigma'_x + \sigma_y \sigma'_y) \mp \sum \sigma_z. \quad (27)$$

This type corresponds to the $X-Y$ model.

Exact solutions have been found for all of these cases, and we will make use of these results in our later analysis of the three-component system. Lattice-gas interpretations for the above systems are also familiar in the literature.

We now consider the case of three components $P=3$. There are four distinct choices for the par-

ticle statistics. We discuss each in turn, again using the language of spin $\frac{1}{2}$: Place M spins on a lattice of N sites, one to a site—thus $M \leq N$. We then call an empty site, or vacancy, species A ; spin-up species B ; spin-down species C . In an obvious notation, $A=0$, $B=\uparrow$, $C=\downarrow$.

(a) *BBB*, $A=B=C=1$.

We write the Hamiltonian as

$$H = -(\phi_{0\uparrow} + \phi_{0\downarrow}) - N_{00} - (N_{\uparrow\uparrow} + N_{\downarrow\downarrow} + \phi_{\uparrow\downarrow}). \quad (28)$$

The first-term “hops” spins to nearby vacancies and thus represents the kinetic energy of the spins. We denote it by T . The last term may be written with Pauli matrices as

$$-(N_{\uparrow\uparrow} + N_{\downarrow\downarrow} + N_{\uparrow\downarrow}) = -\sum_{\mathbf{s}, \mathbf{p}_0} \frac{1}{2}(1 + \vec{\sigma} \cdot \vec{\sigma}'). \quad (29)$$

The symbol $\sum_{\mathbf{s}, \mathbf{p}_0}$ represents a summation over nearest-neighbor spin pairs only.

Using the relationships between the various N 's given in Eq. (2), we may rewrite N_{00} as

$$\begin{aligned} N_{00} &= N_0 - N_{\uparrow} - N_{\downarrow} + N_{\uparrow\uparrow} + N_{\downarrow\downarrow} + N_{\uparrow\downarrow} \\ &= N - 2M + \sum_{\mathbf{s}, \mathbf{p}_0} 1. \end{aligned} \quad (30)$$

The first two terms are constants of motion, which may be fixed. The last term is the number of spin pairs. If we combine this with Eq. (29), we obtain an interaction V between nearest-neighbor spins of the form

$$V = -\sum_{\mathbf{s}, \mathbf{p}_0} \frac{1}{2}(3 + \vec{\sigma} \cdot \vec{\sigma}') \quad (31)$$

Thus, if energies are measured with respect to separated spins, we find that spin pairs in a singlet state have zero energy, while spin pairs in a triplet state have energy -2 .

The final form for our Hamiltonian is

$$H = T + V - (N - 2M). \quad (32)$$

The obvious interpretation is as a dilute Heisenberg ferromagnet which prefers to form bound pairs.

The discussion of the other three types proceeds similarly, and we will present them with less detail.

(b) *FBB*, $-A=B=C=1$.

We find the Hamiltonian to be

$$H = T + V + N - 2M, \quad (33)$$

where T is the kinetic energy as before, but now the interaction energy V between spins is

$$V = \sum_{\mathbf{s}, \mathbf{p}_0} \frac{1}{2}(1 - \vec{\sigma} \cdot \vec{\sigma}'). \quad (34)$$

Again the interpretation is as a dilute Heisenberg ferromagnet, but this time there is no preference to form bound pairs over separated spins.

(c) *BFF*, $A=-B=-C=1$.

We find the Hamiltonian to be

$$\begin{aligned} H &= T - (N - 2M) \\ &= -\frac{1}{2} \sum_{\mathbf{s}, \mathbf{p}_0} (1 - \sigma_x \sigma'_x + \sigma_x \sigma'_x + \sigma_y \sigma'_y). \end{aligned} \quad (35)$$

We now apply a unitary transformation JJ_A from Eqs. (5) and (9) which has the effect of changing the sign of $\phi_{\uparrow\downarrow}$. Thus the final form for the transformed H is

$$H = T + V - (N - 2M), \quad (36)$$

with

$$V = -\frac{1}{2} \sum_{\mathbf{s}, \mathbf{p}_0} (1 - \vec{\sigma} \cdot \vec{\sigma}'). \quad (37)$$

The system is now a dilute Heisenberg antiferromagnet, inclined to form bound pairs of spins.

(d) *FFF*, $A=B=C=-1$.

Again we apply the transformation JJ_A to give a Hamiltonian

$$H = T + V + N - 2M, \quad (38)$$

with

$$V = \frac{1}{2} \sum_{\mathbf{s}, \mathbf{p}_0} (3 + \vec{\sigma} \cdot \vec{\sigma}'). \quad (39)$$

This is the case of a dilute Heisenberg antiferromagnet, now with no preference for bound spin pairs.

Thus our original Hamiltonian of Eq. (1) contains a wealth of particular models. For the examples just given, that of dilute magnets, we may expect both magnetic and electric behavior, with the possibility of singular behavior in the ground state.

Other interpretations are possible for three-component systems. The first that comes to mind might be a spin-1 system, with $S_z = +1, 0, -1$. But this is not the most natural, and we delay discussion for the moment. Instead, we return to the two-component system and note the Heisenberg magnets,

$$H = \pm \sum_{\mathbf{nn}} \vec{\sigma} \cdot \vec{\sigma}'. \quad (40)$$

We may consider the three Pauli spin matrices σ_α ($\alpha=1, 2, 3$) as the generators of $SU(2)$; H is similar in form to the Casimir operator. From this viewpoint, a natural generalization of Eq. 40 to the three-component case would be a Hamiltonian

$$H = \pm 2 \sum_{\mathbf{nn}} \sum_{\alpha=1}^3 F_\alpha F'_\alpha, \quad (41)$$

where F_α ($\alpha=1, \dots, 8$) are the generators of $SU(3)$.

The equivalence of Eq. (1) and (42) is most easily seen by using instead of F_α 's, the nine traceless 3×3 matrices A_μ .

$$(A_\nu^\mu)_{jK} = \delta_{\mu K} \delta_{\nu j} - \frac{1}{3} \delta_{\mu\nu} \delta_{jK}. \quad (42)$$

Then the Casimir operator is

$$2F^2 \equiv \sum A_\mu^\nu A_\nu^\mu. \quad (43)$$

We note the relationships

$$\sum_{nn} (A_\nu^\mu A_\mu^{\nu'} + A_\mu^{\nu'} A_\nu^\mu) = \phi_{\mu\nu} \quad (44)$$

($\mu \neq \nu$; no sum over μ, ν), and

$$\sum_{nn} A_\mu^\mu A_\mu^{\mu'} = \phi_{\mu\mu} - \frac{1}{3} 2N_\mu + \frac{1}{9} N. \quad (45)$$

Thus,

$$\sum_{nn} A_\nu^\mu A_\mu^{\nu'} = \sum_{\mu, \nu} \phi_{\mu\nu} - \frac{1}{3} N. \quad (46)$$

In fact, due to the extensive symmetry of the original problem, we may actually solve the more general problem of either a chain

$$\dots \{3\} \{3\} \{3\} \{3\} \dots, \quad (47a)$$

or

$$\dots \{3\} \{\bar{3}\} \{3\} \{\bar{3}\} \dots. \quad (47b)$$

$\{3\}$ and $\{\bar{3}\}$ are the two nonequivalent contragradient triplet representations of SU(3), often called quark and antiquark.

In the same way that Eq. (40) is invariant under the total SU(2) group, Eq. (41) is invariant under the total SU(3) group. The conserved quantities—total third component of isospin and total hypercharge—are linear combinations of the particle numbers N_A .

Finally, we remark that a similar correspondence exists between the P -component system and an SU(P)-invariant interaction.

We now return to consider a spin-1 representation for our system. This is most easily done by writing the SU(3) generators in terms of the three spin operators S_μ ($\mu = 1, 2, 3$) and the six tensor operators $T_{\mu\nu} = S_\mu S_\nu + S_\nu S_\mu$. The six tensor operators are not independent, however, for

$$T_{\mu\mu} = 2S_\mu S_\mu = 4I. \quad (48)$$

Then we have the identity,

$$\sum_{nn} A_\nu^\mu A_\mu^{\nu'} = 2 \sum_{nn} (S_\mu S_\mu' + \frac{1}{2} T_{\mu\nu} T'_{\mu\nu}) - \frac{1}{3} 16N. \quad (49)$$

Thus

$$\sum_{\mu, \nu} \phi_{\mu\nu} = \sum_{nn} (2S_\mu S_\mu' + T_{\mu\nu} T'_{\mu\nu}) - 5N. \quad (50)$$

Considered as a model of a spin-1 magnet, we have not simply dipole interactions but also tensor interactions. Again, there are some nonequivalent choices of sign for the various terms in Eq. (50). The total spin components are individually conserved.

V. SOLUTION OF THE PROBLEM

We now present the exact solution of the one-dimensional problem. First, we restrict ourselves to types of either F^P or BF^P . If we have these solutions, then by the arguments of Sec. VII, we have the ground-state energy for all types. This is not to say the other types are not interesting, for we do not have the excited states. However, until the excited states of the Heisenberg ferromagnet are understood, we can not expect to proceed with the multicomponent cases.

We now order the species so that the first P species are fermions with $N_1 > N_2 > \dots > N_P$. We define the partial sums,

$$M_i = \sum_{j>i} N_j = N - \sum_{j \leq i} N_j. \quad (51)$$

We now consider the fermions of species 1 as a background through which the other M_1 objects move. We write the Hamiltonian as

$$H = N_{11} + T - \sum'_{A \geq B} \phi_{AB}. \quad (52)$$

However, we may use the relation

$$N_{11} = N - 2M_1 + \sum_{A \geq B} N_{AB} \quad (53)$$

to write

$$H = N - 2M_1 + T - \sum_{A > B} (\phi_{AB} - N_{AB}). \quad (54)$$

We seek solutions to this Hamiltonian as a wave function $\Psi(X_1, \dots, X_M)$, where X_1, \dots, X_M are the locations of the $M = M_1$ objects, of the form known as Bethe's ansatz,

$$\Psi_Q = \sum_P A(Q, P) \exp i \sum_{j=1}^M X_j K_{P_j}. \quad (55)$$

Here P is one of the $M!$ permutations of the K 's (not to be confused with the number of species of fermions), Q is one of the $M!$ permutations of the objects, and the X 's are ordered so that $X_1 < X_2 < \dots < X_M$. We may easily determine the eigenvalues of H , provided that such a wave function exists, by considering the particles to be separated. Then the final term of Eq. (54) gives zero, and we have

$$E = N - 2M - 2 \sum_{j=1}^M \cos K_j. \quad (56)$$

As is familiar in such verifications of Bethe's ansatz, we arrange the $M!$ coefficients $A(Q, P)$, for fixed P , as a column vector ξ_P . Then we find that the boundary conditions at $X_1 = X_M - 1$ can be satisfied provided

$$\xi \dots ij \dots = Y_{ji}^{45} \xi \dots ji \dots, \quad (57)$$

where

$$Y_{ij}^{\alpha\beta} = (X_{ij} + P_{\alpha\beta}) / (1 - X_{ij}). \tag{58}$$

Here $P_{\alpha\beta}$ interchanges Q_α and Q_β , and

$$\frac{1}{X_{ij}} = \frac{e^{iK_j} - e^{iK_i}}{(1 + e^{iK_i})(1 + e^{iK_j})}. \tag{59}$$

We now make a change of variables,

$$\alpha_j(K_j) = \frac{1}{2} \tan(\frac{1}{2}K_j), \tag{60}$$

so that

$$X_{ij} = i / (\alpha_i - \alpha_j). \tag{61}$$

However, this is precisely the form treated by Yang³ and more generally by Sutherland.⁴ Thus we immediately have that the Eqs. (57) are consistent, and that requiring the problem to be periodic imposes the following coupled algebraic equations on the K 's:

$$e^{iK(\alpha)N} = \prod_\beta \left(\frac{\alpha - \beta + \frac{1}{2}i}{\alpha - \beta - \frac{1}{2}i} \right) \prod_\alpha \left(\frac{\alpha - \alpha' - i}{\alpha - \alpha' + i} \right), \tag{62\alpha}$$

$$\prod_{\beta'} \left(\frac{\beta - \beta' - i}{\beta - \beta' + i} \right) = \prod_\alpha \left(\frac{\beta - \alpha - i/2}{\beta - \alpha + i/2} \right) \prod_\gamma \left(\frac{\beta - \gamma - \frac{1}{2}i}{\beta - \gamma + \frac{1}{2}i} \right) \tag{62\beta}$$

⋮

$$\prod_{\zeta'} \left(\frac{\zeta - \zeta' - i}{\zeta - \zeta' + i} \right) = \prod_\delta \left(\frac{\zeta - \delta - \frac{1}{2}i}{\zeta - \delta + \frac{1}{2}i} \right), \tag{62\zeta}$$

or

$$\prod_\delta \left(\frac{\zeta - \delta - \frac{1}{2}i}{\zeta - \delta + \frac{1}{2}i} \right) = 1, \tag{62\zeta}$$

Equation (62 ζ) presents the two alternate choices for the final equation, the first corresponding to F^P and the second to BF^P . All intermediate equations have the form of Eq. (62 β). In all, there are $P - 1$ equations for M_1 variables α , M_2 variables β, \dots, M_{P-1} variables ζ if F^P ; or P equations for M_1 variables α, \dots, M_P variables ζ if BF^P .

We now take the logarithm of these equations,

$$NK(\alpha) = 2\pi J_\alpha - \sum_{\alpha'} \theta(\alpha - \alpha') + \sum_\beta \theta(2\alpha - 2\beta), \tag{63\alpha}$$

$$\sum_{\beta'} \theta(\beta - \beta') = 2\pi J_\beta + \sum_\alpha \theta(2\beta - 2\alpha) + \sum_\gamma \theta(2\beta - 2\gamma), \tag{63\beta}$$

⋮

$$\sum_{\zeta'} \theta(\zeta - \zeta') = 2\pi J_\zeta + \sum_\delta \theta(2\zeta - 2\delta), \tag{63\zeta}$$

or

$$0 = 2\pi J_\zeta + \sum_\delta \theta(2\zeta - 2\delta), \tag{63\zeta}$$

Here $\theta(x) = -2 \tan^{-1}(x)$, and $J_\alpha, J_\beta, \dots, J_\zeta$ are integers (half odd integers) which arise from the logarithm of $+1(-1)$ and serve as quantum numbers.

Finally, we consider the ground state where the the J 's are dense about the origin and the variables

are smoothly distributed with densities $R_j(\alpha)$ between limits $\pm B_j$. These densities are normalized so that

$$\int_{-B_j}^{+B_j} R_j(\alpha) d\alpha = 2\pi \frac{M_j}{N} \equiv 2\pi m_j. \tag{64}$$

Then the Eqs. (63) become integral equations for the densities $R_j(\alpha)$. If we arrange the densities $R_j(\alpha)$ as the elements of a column vector \underline{R} , then the integral equations may be put in the concise matrix form

$$\underline{\xi} = \underline{R} + \underline{K} \underline{B} \underline{R}. \tag{65}$$

Here,

$$[\underline{\xi}]_j = \delta_{j1} \frac{dK}{d\alpha} = \frac{4}{1 + 4\alpha^2} \delta_{j1}, \tag{66}$$

$$[\underline{B} \underline{R}]_j = \begin{cases} R_j(\alpha), & |\alpha| < B_j \\ 0, & |\alpha| > B_j \end{cases}, \tag{67}$$

and \underline{K} is a matrix whose elements $[K]_{ij}$ are integral operators. Let us define K_n as the integral operator

$$K_n \psi(\alpha) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2n d\beta}{1 + n^2(\alpha - \beta)^2} \psi(\beta). \tag{68}$$

It has eigenvalues given by the Fourier transform $\tilde{K}_n(k)$

$$\tilde{K}_n(k) = e^{-|k|/n}. \tag{69}$$

We note the relation

$$K_n K_m = K_{nm/(n+m)} \tag{70}$$

We may now write out the matrix \underline{K} for the two cases: If F^P , R is of dimension $P - 1$, and

$$[K]_{ij} = \delta_{ij} K_1 - (\delta_{i,j+1} + \delta_{i,j-1}) K_2. \tag{71}$$

If BF^P , R is of dimension P , and

$$[K]_{ij} = (\delta_{i,j} - \delta_{iP} \delta_{jP}) K_1 - (\delta_{i,j+1} + \delta_{i,j-1}) K_2. \tag{72}$$

We may write the ground-state energy per particle ϵ also in matrix form as

$$\begin{aligned} \epsilon &= 1 - \frac{1}{2\pi} \underline{\xi}^\dagger \underline{B} \underline{R} \\ &\equiv 1 - \frac{1}{2\pi} \int_{-B_1}^{B_1} R_1(\alpha) \frac{dK}{d\alpha} d\alpha. \end{aligned} \tag{56'}$$

Finally, let us rewrite Eqs. (64) as matrix equations.

$$\underline{m} = \frac{1}{2\pi} \underline{\eta}^\dagger \underline{B} \underline{R}, \tag{64'}$$

where

$$[\underline{\eta}^\dagger \underline{B} \underline{R}]_j \equiv \int_{-B_j}^{+B_j} d\alpha R_j(\alpha). \tag{73}$$

To summarize, we collect in the very concise matrix form our basic Eqs. (65), (56'), and (64'):

$$\underline{\xi} = \underline{R} + \underline{K} \underline{B} \underline{R}, \quad (65)$$

$$2\pi \underline{m} = \underline{\eta}^\dagger \underline{B} \underline{R}, \quad (56')$$

$$\epsilon = 1 - \underline{\xi}^\dagger \underline{B} \underline{R} / 2\pi. \quad (64')$$

We are to solve for \underline{R} in terms of \underline{B} , calculate ϵ , \underline{m} as functions of \underline{B} , and then eliminate \underline{B} to find $\epsilon(\underline{m})$.

If all limits are finite, then Eq. (65) is a non-singular matrix Fredholm equation, and the solution is an analytic function of the limits B_j . Thus, in turn, the ground-state energy must be an analytic function of the concentrations. Thus the region $0 < B_j < +\infty$ we call the fundamental domain, and we now determine the region of concentrations to which it corresponds.

First, if $B_j = 0$, then $m_j = 0$, and we conclude

$$B_j = 0 \rightarrow n_k = 0, \quad k > j. \quad (74)$$

Second, suppose $B_j = \infty$; then we integrate the equation for R_j from $-\infty$ to $+\infty$ and find

$$\delta_{j1} = 2m_j - m_{j+1} - m_{j-1}, \quad j < P$$

or (75)

$$B_j = +\infty \rightarrow n_{j+1} = n_{j-1}, \quad j < P.$$

The P equation, however, gives

$$B_P = +\infty \rightarrow n_{P-1} = 0. \quad (76)$$

In this case, the problem is reduced from $F^P B$ to $F^{P-1} B$.

We finally conclude that the fundamental domain corresponds to

$$n_1 > n_2 > \dots > n_P, \quad (77)$$

and singularities occur only at the boundaries. By permuting the fermion concentrations, the fundamental domain is mapped onto the entire physical region of concentrations, and the ground-state energy obeys this permutation symmetry S_P . Figure 1 illustrates the surfaces of singularities in the phase plane for the two three-component systems F^3 and BF^2 ; the shaded region is the fundamental domain.

We note also, that if we are constrained to a surface of singularities, $B_j = +\infty$, then the corresponding R_j may be eliminated so that the resulting equations are again a nonsingular matrix Fredholm equation. We then conclude that restricted to a singular surface, singularities occur only at the intersection with another singular surface.

VI. ABSOLUTE GROUND STATE OF THE F^P PROBLEM

As an example, we may explicitly calculate the energy per particle $\epsilon(P)$ for the absolute ground

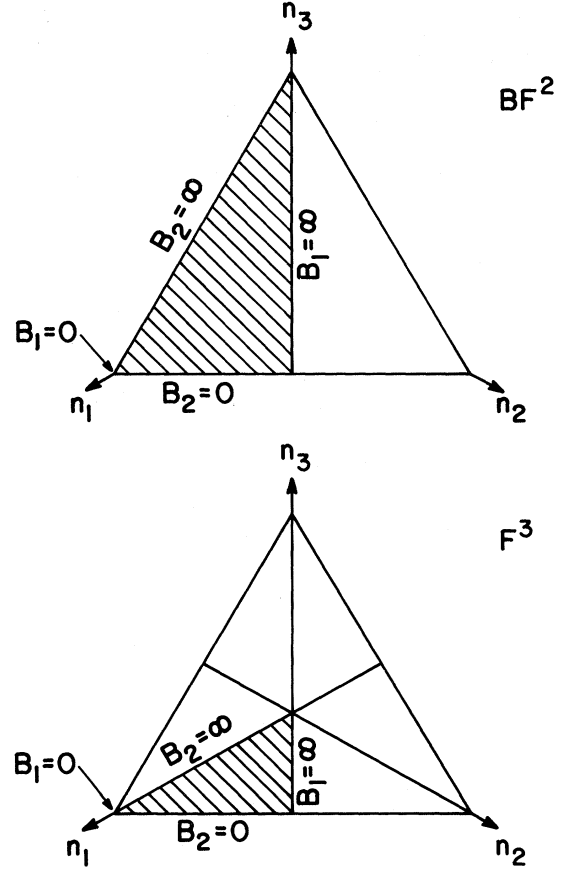


FIG. 1. We show the lines of singularities in the phase plane for the two three-component systems BF^2 and F^3 . The shaded region is the fundamental domain, and the ground-state energy is analytic within this region. The limits of the integral equations are here correlated with the boundaries of the fundamental domain.

state of the F^P problem. The limits are all $+\infty$, the concentrations are all equal, and thus if we Fourier transform, we obtain a matrix equation. The Fourier transform of \underline{K} is a $(P-1) \times (P-1)$ matrix,

$$[\underline{I} + \underline{K}]_{ij} = \delta_{ij}(1 + X^2) - (\delta_{i,j+1} + \delta_{i,j-1})X \quad (78)$$

where $X \equiv e^{-ik/2}$. The Fourier transform of the resolvent $(\underline{I} + \underline{J})$ is the inverse of this matrix and may be calculated by first finding the eigenvectors,

$$\psi_n(j) = (2/P)^{1/2} \sin(\pi nj/P) \quad (n = 1, \dots, P-1), \quad (79)$$

and eigenvalues

$$\lambda_n = 2X \cosh \frac{1}{2} k - \cos(\pi n/P). \quad (80)$$

Then we have

$$[\underline{I} + \underline{J}]_{jk} = \sum_{n=1}^{P-1} \lambda_n^{-1} \psi_n(j) \psi_n^*(k). \quad (81)$$

We then may use this expression to solve for the Fourier transform of the densities,

$$\tilde{R}_j(k) = \frac{2\pi}{P} \sum_{n=1}^{P-1} \frac{\sin(\pi j n/P) \sin(\pi n/P)}{\cosh(\frac{1}{2}k) - \cos(\pi n/P)}. \quad (82)$$

To sum this series, we Fourier transform back and obtain

$$R_j(\alpha) = \frac{4\pi}{P} \frac{1}{\sinh 2\pi\alpha} \times \sum_{n=1}^{P-1} \sin(\pi j n/P) \sinh[2\pi\alpha(1 - n/P)]. \quad (83)$$

The summation now is simply four finite geometric series, and we have

$$R_j(\alpha) = \frac{2\pi}{P} \frac{\sin(\pi j/P)}{\cosh(2\pi\alpha/P) - \cos(\pi j/P)}. \quad (84)$$

The Fourier transform of this expression is

$$\tilde{R}_j(k) = 2\pi \frac{\sinh \frac{1}{2}k(P-j)}{\sinh \frac{1}{2}Pk}. \quad (85)$$

A simple check is to calculate the concentrations m_j by

$$m_j = (1/2\pi) R_j(0) = (P-j)/P. \quad (86)$$

Thus all $n_j = 1/P$.

Using the evaluation of Eq. (85), we may return and evaluate the Fourier transform of the resolvent, a symmetric matrix,

$$[\tilde{I} + \tilde{J}]_{jl} = e^{lK/2} \frac{\sinh \frac{1}{2}K(P-j) \sinh \frac{1}{2}Kl}{\sinh \frac{1}{2}PK \sinh \frac{1}{2}K} \quad (j \geq l). \quad (87)$$

The energy is given as

$$\begin{aligned} \epsilon(P) &= 1 - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{4d\alpha}{1+4\alpha^2} R_1(\alpha) \\ &= 1 - \int_{-\infty}^{+\infty} dK e^{-lK/2} \frac{\sinh \frac{1}{2}k(P-1)}{\sinh \frac{1}{2}kP}. \end{aligned} \quad (88)$$

Let us define a new variable,

$$y \equiv e^{-Pk}. \quad (89)$$

We may rewrite $\epsilon(P)$ as

$$\epsilon(P) = 1 - \frac{2}{P} \int_0^1 dy \frac{y^{1/P-1} - 1}{1-y}. \quad (90)$$

Such an integral may be rewritten in terms of Euler's digamma function,

$$\psi(x) \equiv \frac{d}{dx} \ln \Gamma(x). \quad (91)$$

We find

$$\begin{aligned} \epsilon(P) &= 1 - (2/P) [\psi(1) - \psi(1/P)] \\ &= -1 + (2/P) [\psi(1 + 1/P) - \psi(1)]. \end{aligned} \quad (92)$$

Typical values are

$$\psi(1) = -C = -0.577215\dots, \quad (93)$$

where C is Euler's constant, which cancels out in the explicit expressions

$$\begin{aligned} \psi(1) - \psi(\frac{1}{2}) &= 2 \ln 2, \\ \psi(1) - \psi(\frac{1}{3}) &= \pi/2\sqrt{3} + 3 \ln \frac{3}{2}, \\ \psi(1) - \psi(\frac{1}{4}) &= \pi/2 + 3 \ln 2. \end{aligned} \quad (93a)$$

Finally, an expression useful for large P is

$$\epsilon(P) = -1 + 2 \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{P^k}. \quad (94)$$

Here $\zeta(k)$ is Riemann's zeta function,

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}. \quad (95)$$

The behavior of $\epsilon(P)$ is shown in Fig. 2.

As for any problem which is solved by Bethe's ansatz, the low-lying excited states are obtained from the original algebraic equations by making alternate choices for the quantum numbers. Thus, we select one quantum number from any of the $P-1$ equations, and change it; the removal creates a hole and the new choice creates a particle. The necessary manipulations of the integral equations are by now familiar and lead to the following expressions for the energy $\Delta\epsilon$ and momentum Δk :

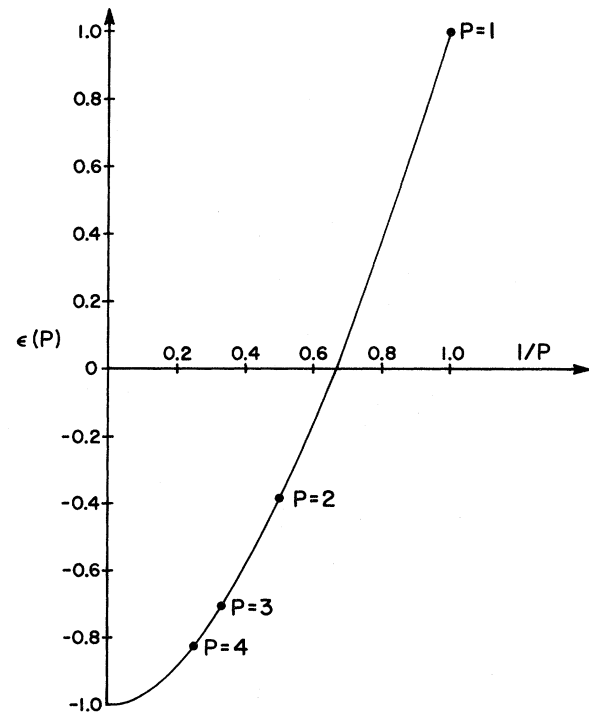


FIG. 2. Absolute ground-state energy per particle for the P -component system F^P is shown as a function of the number of components.

$$\Delta\epsilon = \sum R_i(B_i),$$

$$\Delta k = \sum \int^{B_i} d\alpha_i R_i(\alpha_i). \quad (96)$$

The variables B_i are to be eliminated between the two equations.

Let us consider the low-lying excited states about the absolute ground state. Then $R_i(\alpha)$ are given by Eq. (96). We decompose $\Delta\epsilon$ and Δk as a sum over single excitation dispersion curves and calculate each to be

$$\epsilon_j = \frac{2\pi}{P} \frac{\sin(\pi j/P)}{\cosh(2\pi B/P) - \cos(\pi j/P)},$$

$$k_j = 2 \tan^{-1} [\cot(\pi j/2P) \tanh(2\pi B/P)] - (\pi - \pi j/P). \quad (97)$$

Eliminating B , we obtain

$$\epsilon_j(k) = \frac{2\pi}{\sin(\pi j/P)} [\cos(\pi j/P - |k|) - \cos(\pi j/P)] \quad (98)$$

for $|k| < 2\pi j/P$. The function is periodic with period $2\pi j/P$.

We see that, in general, there are $P-1$ distinct branches. However, at $k=0$, all branches have a common slope, with

$$\epsilon_j(k) \approx \frac{2\pi}{P} |k|, \quad (99)$$

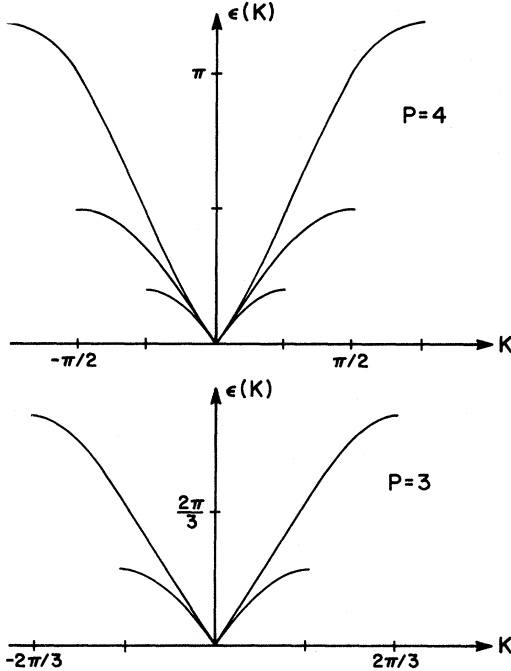


FIG. 3. $P-1$ dispersion curves for the low-lying excitations above the absolute ground state of the F^P system are shown for the cases $P=4$ and $P=3$.

and thus a common velocity of sound. In Fig. 3 we show the cases of $P=3$ and $P=4$.

Suppose we have a system F^P which is at the absolute ground state and replace some of the particles with $\delta n \ll 1$ impurities. Then the equations for the $P-1$ original densities become

$$\underline{\xi} + \underline{\xi}' = (\underline{I} + \underline{K}) \underline{R}, \quad (100)$$

where

$$[\underline{\xi}']_j = \delta_{j,P-1} \delta n \frac{4}{(1+4\alpha^2)}. \quad (101)$$

Therefore our solution is given by

$$\underline{R} = \underline{R}^0 + \underline{R}', \quad (102)$$

where \underline{R}^0 is the previous solution, and the Fourier transform of \underline{R}' is

$$\tilde{R}'_j(k) = 2\pi \frac{\sinh \frac{1}{2} j k}{\sinh \frac{1}{2} P k} \delta n. \quad (103)$$

Then we verify that

$$m_j = (P-j)/P + \delta n j/P,$$

or

$$n_j = (1 - \delta n)/P \quad (j=1, \dots, P). \quad (104)$$

We then calculate the energy as

$$\epsilon = \epsilon(P) - \frac{\delta n}{2\pi} \int_{-\infty}^{+\infty} \frac{4 d\alpha R'_j(\alpha)}{1+4\alpha^2} \quad (105)$$

or

$$\Delta\epsilon = -\delta n \int_{-\infty}^{+\infty} dk e^{-|k|/2} \frac{\sinh \frac{1}{2} k}{\sinh \frac{1}{2} P k}. \quad (106)$$

Again, we define a variable as in Eq. (89) and write

$$\Delta\epsilon = -\frac{2\delta n}{P} \int_0^1 \frac{y^{-1/2} - y^{1/2} + 1/P}{1-y} \quad (107)$$

This may be written in terms of the digamma function as

$$\Delta\epsilon = -\frac{2\delta n}{P} \left[\psi\left(\frac{1}{2} + \frac{1}{P}\right) - \psi\left(\frac{1}{2}\right) \right]. \quad (108)$$

We have the special values

$$\frac{d\epsilon}{dn} = \begin{cases} -4, & P=1 \\ -2 \ln 2, & P=2 \\ -\frac{1}{4}\pi + \frac{1}{2} \ln 2, & P=4. \end{cases} \quad (109)$$

In Fig. 4 we show the general curve. We note the limiting form

$$\frac{d\epsilon}{dn} \sim -\left(\frac{\pi}{P}\right)^2. \quad (110)$$

We remark that this is the correct form no matter what the nature of the impurities—fermions, bosons, or mixed.

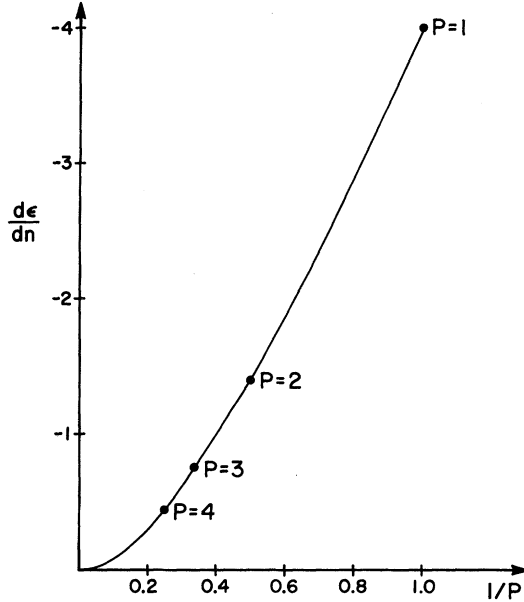


FIG. 4. We show the first correction to the absolute ground-state energy of the F^P system as a small number of particles are replaced by impurities. The correction is shown by the first derivative of the ground-state energy with respect to impurity concentration at zero-impurity concentration, as a function of the number of components P .

In the previous discussion, we have assumed that the point $B_j = \infty$ or $n_j = 1/P$ is the absolute ground state of the F^P system. This is certainly reasonable, and we may indeed verify the claim by examining the correction to the ground-state energy in the vicinity of $B_j = \infty$. We first transform the integral equations over the domain B to equations over the domain $(I - B)$ by multiplying our basic Eq. (65) by the resolvent of Eq. (87) to give

$$\begin{aligned} \underline{R}^0 &\equiv (\underline{I} + \underline{J}) \underline{\xi} \\ &= \underline{R} + \underline{J}(\underline{I} - \underline{B})\underline{R}. \end{aligned} \quad (111)$$

We may then write Eqs. (74) and (73) as

$$\begin{aligned} \Delta\epsilon &\equiv \epsilon - (1 - \underline{\xi}^\dagger \underline{R}^0 / 2\pi) \\ &= (1/2\pi) \underline{R}^{0\dagger} (\underline{I} - \underline{B})\underline{R}, \end{aligned} \quad (112)$$

and

$$\begin{aligned} -[(\underline{I} + \underline{K})](0) \underline{\Delta m} &= -[(\underline{I} + \underline{K})_{(0)}] (\underline{r} - \underline{\eta}^\dagger \underline{R}^0 / 2\pi) \\ &= \underline{\eta}^\dagger (\underline{I} - \underline{B})\underline{R} / 2\pi. \end{aligned} \quad (113)$$

We note that the components on the left-hand side are

$$\begin{aligned} 2\pi(n_j - n_{j+1}) &= [\underline{\eta}^\dagger (\underline{I} - \underline{B})\underline{R}]_j \\ &= (\Delta n_j - \Delta n_{j+1}) 2\pi \geq 0. \end{aligned} \quad (114)$$

If all B 's are large—that is, we are in the vicinity

of $n_j = 1/P$ —then we may approximate the inhomogeneous term \underline{R}^0 of Eq. (111), as given in Eq. (84), by expanding

$$\underline{R}_j^0(\alpha) \approx 4\pi \sin(\pi j/P) e^{-2\pi\alpha/P} / P. \quad (115)$$

We see that to this order,

$$\underline{R}_j(\alpha) = \underline{R}_{P-j}(\alpha). \quad (116)$$

Further, we see by looking at the explicit formula for the resolvent $\underline{I} + \underline{J}$ given in Eq. (87) that

$$[\underline{I} + \underline{J}]_j, l = [\underline{I} + \underline{J}]_{P-j, P-l}. \quad (117)$$

Thus to the lowest order in Δn_j ,

$$\begin{aligned} \epsilon(1/P + \Delta n_1, 1/P + \Delta n_2, \dots, 1/P + \Delta n_P) \\ = \epsilon(1/P - \Delta n_P, \dots, 1/P - \Delta n_2, 1/P - \Delta n_1). \end{aligned} \quad (118)$$

But since the energy is symmetric in permutations of the P particles, we have

$$\begin{aligned} \epsilon(1/P + \Delta n_1, 1/P + \Delta n_2, \dots, 1/P + \Delta n_P) \\ = \epsilon(1/P - \Delta n_1, 1/P - \Delta n_2, \dots, 1/P - \Delta n_P). \end{aligned} \quad (119)$$

Thus we conclude that the ground-state energy is, to the lowest order in n_j , symmetric about the point $n_j = 1/P$.

However, by the basic concavity property of Eq. (11), we conclude then that $\Delta n_j = 0$ is a local minimum of the ground-state energy, and hence, the absolute ground state (with no restriction on the concentrations).

We note that this symmetry requires that for low-lying states, conjugate representations of $SU(P)$ must be degenerate.

We have been unable to calculate explicitly the first-order correction to the absolute ground-state energy for general variations of concentration n_j . However, in two instances we may make a calculation:

(a) If all B_j 's are equal,

$$\underline{R}_j^0 = 2\pi(2/P)^{1/2} e^{-2\pi B/P} e^{-2\pi\sigma/P} \psi_1(j). \quad (120)$$

We have defined a new variable σ by $\alpha = B + \sigma$. If we also define

$$\underline{\zeta} = (2/P)^{1/2} e^{-2\pi B/P}, \quad (121)$$

then we approximate Eq. (111) by

$$\begin{aligned} 2\pi \underline{\zeta} e^{-2\pi\sigma/P} \underline{\psi}_1 \\ = \underline{S}(\sigma) + \int_0^\infty d\tau \underline{J}(\sigma - \tau) \underline{S}(\tau). \end{aligned} \quad (122)$$

Let us make an eigenvector decomposition of \underline{S} ,

$$\underline{S} = \sum S_n(\sigma) \underline{\psi}_n \quad (123)$$

We then find $S_n = 0$, $n \neq 1$, and

$$S_1(\sigma) + \int_0^\infty d\tau J_1(\sigma - \tau) S_1(\tau) = 2\pi\zeta e^{-2\pi\sigma/P}, \quad (124)$$

with

$$\tilde{I} + \tilde{J}_N + \frac{1}{\lambda_n} = \frac{\frac{1}{2} e^{ikl}}{2[\cosh \frac{1}{2}k - \cos \pi n/P]}. \quad (125)$$

This is a Wiener-Hopf equation, and as in Ref. 2, we have

$$\Delta\epsilon = [(2\pi/P)\Delta q]^2 (1 - \cos \pi/P), \quad (126)$$

with

$$\Delta q = - \sum_{j=1}^{P-1} \sin \frac{\pi j}{P} \Delta m_j. \quad (127)$$

We have the restriction on Δm_j of

$$\Delta m_j = - \sin \left(\frac{\pi j}{P} \right) \frac{2\Delta q}{P}. \quad (128)$$

(b) In the second case, suppose that all $B_l = \infty$, except $B_j = B$. In this case we define

$$\zeta_j = (2/P) \sin(\pi j/P) e^{-2\pi B/P} \quad (129)$$

The equation we are to solve is

$$2\pi\zeta_j e^{-2\pi\sigma/P} = S_j(\sigma) + \int_0^\infty d\tau J_{jj}(\sigma - \tau) S_j(\tau). \quad (130)$$

Again, it is a Wiener-Hopf equation, and we find

$$\Delta\epsilon = (\pi\Delta n)^2 j(P-j)/P^2, \quad (131)$$

where

$$\Delta n_l = \Delta n_{l+1}, \quad l \neq j \quad (132)$$

$$\Delta n_j - \Delta n_{j+1} = \Delta n.$$

We finally remark that all of these examples are consistent with

$$\Delta\epsilon = \frac{\pi^2}{P} \sum_{j=1}^P (\Delta n_j)^2, \quad (133)$$

and conjecture that this is in fact the exact form.

And thus, as a consequence, we conjecture that all second derivatives of the ground-state energy with respect to the concentrations are continuous within the whole phase space.

ACKNOWLEDGMENTS

We would like to thank Professor C. N. Yang for suggesting the original problem, and for the hospitality of the Institute for Theoretical Physics, where this paper was in large part written. Also, we very much benefited from conversations with Y. Dothan.

*Work supported in part by National Science Foundation Grant No. CP 38905.

†Permanent address: Physics Department, University of Utah, Salt Lake City, Utah 84112.

¹C. K. Lai, J. Math. Phys. 15, 1675 (1974).

²Our treatment in this section, and in all later sections,

owes much to the treatment of the Heisenberg-Ising problem by C. N. Yang and C. P. Yang [Phys. Rev. 147, 303 (1966); 150, 321 (1966); 151, 258 (1966)].

³C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967).

⁴B. Sutherland, Phys. Rev. Lett. 20, 98 (1968).