Phonon dispersion in liquid He II

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The eigenfunctions of the linearized collision operator for the phonons in superfluid helium are determined numerically. The calculated sound velocity is in quantitative agreement with experimental data in a wide frequency and temperature range if an anomalous dispersion $\gamma = -10^{38}$ in cgs units is used.

I. INTRODUCTION

In recent years the frequency and temperature dependence of sound velocity and ultrasonic absorption in liquid helium II has attracted great interest. Extensive experimental work has been performed by the Argonne group.[1-3] The most remarkable feature of their results is the nonmonotonic frequency behavior of the sound velocity. Theoretical efforts to explain experimental data which were based either on Green's-function techniques[4,5] or on a relaxation-time treatment of the phonon Boltzmann equation[6,7] invariably yielded a monotonic increase of the velocity with frequency for all temperatures smaller than 0.5 K.

Concerning the ultrasonic attenuation, Maris and Massey[8] proposed that the discrepancy between measurement and theory in the same frequency and temperature domain could be resolved by assuming that the energies of the thermal phonons relevant for transport processes in the above temperature interval are given by

$$\omega_p = c_0 p(1 - \gamma p^2), \tag{1}$$

with a negative γ (anomalous dispersion). Using Eq. (1), Maris[9] then solved the full collision equation numerically by an iterative procedure and for the first time obtained values for the relative velocity shift $\Delta c/c_0$ in qualitative agreement with experiment. Encouraged by this success, two of the present authors[10] used a different method (the integral equation for the phonon density was approximated by a set of linear algebraic equations) and confirmed the findings of Maris.[9]

In a recent letter, Junker and Elbaum[11] reported new measurements of frequency and temperature dependence of the sound velocity in liquid helium II. Their data complement the earlier work done by Abraham et al.[1] and Roach et al.[2,3] Junker and Elbaum state that their results are in qualitative agreement with the predictions obtained from an iterative numerical solution of the Boltzmann equation by Maris[12] if an anomalous dispersion $\gamma = -15\times10^{37}$ cgs units is assumed.

In this work, we present results of a different method of solving the transport equation which, in contrast to Maris's work, is most effective for lower frequencies. Our results are in quantitative agreement with the measurements for a wide range of temperatures and frequencies, provided that we use $\gamma = -10\times10^{37}$. The theoretical framework is discussed in Sec. II. In Sec. III we present our results and compare them with the conclusions of other work.

II. OUTLINE OF THEORY

The phonon-collision equation, which was also used by Maris[12-16] in his investigations of various aspects of superfluid hydrodynamics, is of the form

$$[-i\Omega + i\vec{Q}\cdot\vec{v}(p)]\varphi(\vec{p},\vec{Q},\Omega) = \int d^3p' L(\vec{p},\vec{p}')\varphi(\vec{p}',\vec{Q},\Omega) + K(\vec{p},\vec{Q},\Omega). \tag{2}$$

Here, $\varphi(\vec{p},\vec{Q},\Omega)$ is the time and space Fourier transform of the density of phonons with wave vector $\vec{p}$ and group velocity $\vec{v}(p) \equiv \partial\omega_p/\partial\vec{p}$. L is the phonon-collision operator. Its form can be obtained by comparing the anharmonic terms in the quantum hydrodynamic model with the corresponding ones in lattice dynamics. For three-phonon collisions we get

$$\int d^3p'\, L(\vec{p},\vec{p}')\varphi(\vec{p}') = \frac{\hbar c_0}{64\rho\pi^2}\int d^3p' \int d^3p'' \left(\frac{m(p')m(p'')}{m(p)}\right)^{1/2} M(\vec{p},\vec{p}',\vec{p}'')$$
$$\times\{2\delta(\omega_p+\omega_{p'}-\omega_{p''})\delta(\vec{p}+\vec{p}'-\vec{p}'')[\varphi(\vec{p})+\varphi(\vec{p}')-\varphi(\vec{p}'')]$$
$$+\delta(\omega_p-\omega_{p'}-\omega_{p''})\delta(\vec{p}-\vec{p}'-\vec{p}'')[\varphi(\vec{p})-\varphi(\vec{p}')-\varphi(\vec{p}'')]\}, \quad (3)$$

with

$$M(\vec{p},\vec{p}',\vec{p}'') = p\,p'p''[2u-1+\cos(\vec{p},\vec{p}')+\cos(\vec{p},\vec{p}'')+\cos(\vec{p}',\vec{p}'')]^2. \quad (4)$$

Here ρ is the density, $u=(p/c)dc/d\rho$ stands for the Grüneisen parameter ($u\approx 2.84$), and

$$m(p) = [2\sinh(\hbar\omega_p/2k_BT)]^{-2}. \quad (5)$$

Three-phonon collisions are only possible if the dispersion is anomalous ($\gamma<0$), since otherwise energy and momentum conservation cannot be fulfilled simultaneously. K describes the effect of an ultrasonic wave on the thermal phonon system

$$K(\vec{p},\vec{Q},\Omega) = -i\Omega\frac{c}{\rho}\left(up+\frac{\vec{Q}\cdot\vec{p}}{Q}\right). \quad (6)$$

Observing the spacial isotropy of the problem, Maris[14] introduced eigenfunctions of L of the form

$$\psi_{nl}(\vec{p}) = R_{nl}(p)Y_l(\cos\theta), \quad (7)$$

where Y_l are spherical harmonics with argument

$$\cos\theta = (\vec{Q}\cdot\vec{p})/Qp \quad (8)$$

and R_{nl} is the "radial" part depending only on the magnitude p of $\vec{p}$. In the present work, we used the expansion

$$\varphi(\vec{p},\vec{Q},\Omega) = \sum_{nl} a_{nl}(\vec{Q},\Omega)R_{nl}(p)Y_l(\cos\theta) \quad (9)$$

in order to calculate $\Delta c/c_0$. This quantity is then determined by

$$\frac{\Delta c}{c_0} = \frac{\hbar^2}{2c_0k_BT}\frac{1}{4\pi^{5/2}}\int_0^\infty dp\,p^3m(p)$$
$$\times\sum_n\left(u\,\mathrm{Re}(a_{n0})R_{n0}(p)+\frac{1}{\sqrt{3}}\mathrm{Re}(a_{n1})R_{n1}(p)\right)$$
$$-\frac{\hbar^2}{2\rho k_BT}\int\frac{d^3p}{8\pi^3}m(p)p^2(u+\cos\theta)^2. \quad (10)$$

Numerically, this method turns out to be superior to the one used in Ref. 10, since it automatically guarantees the correct hydrodynamic limit for small Ω and $\vec{Q}$ and the results are less sensitive to the mesh in $\vec{p}$ space, provided that a sufficient number of ψ_{nl}'s are taken into account. Owing, in particular, to the almost linear phonon dispersion [Eq. (1)], which favors small angle scattering, a fairly large number of spherical harmonics must be used in order to describe the angular dependence of φ. We typically used 300 values for ψ_{nl}; e.g., $l=1,2,\ldots,30$, $n=1,2,\ldots,10$. This yielded fairly mesh-independent results for what may be termed "collision-

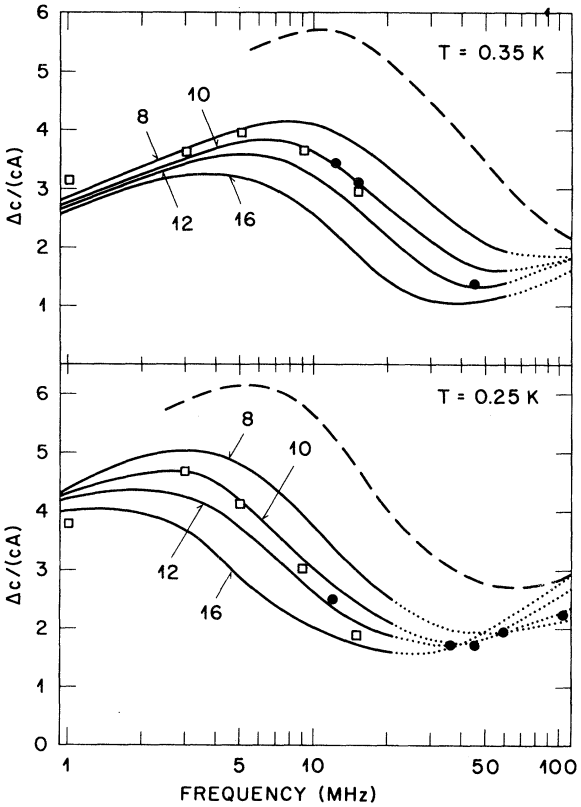

FIG. 1. Change in sound velocity, normalized according to Eq. (11), as a function of frequency for $T=0.25$ and 0.35 K. The curves showing the present results for $\gamma=-B\times10^{37}$ cgs units are labeled by the respective values of B. In the frequency region where the mesh dependence becomes appreciable, these curves are dotted representing averages over various reasonable meshes. The results obtained by Maris (Refs. 9 and 12) for $B=8$ are shown by dashed lines. Experimental points are the measurements of Roach *et al.* (Ref. 3), ●; and Junker and Elbaum (Ref. 11), □.

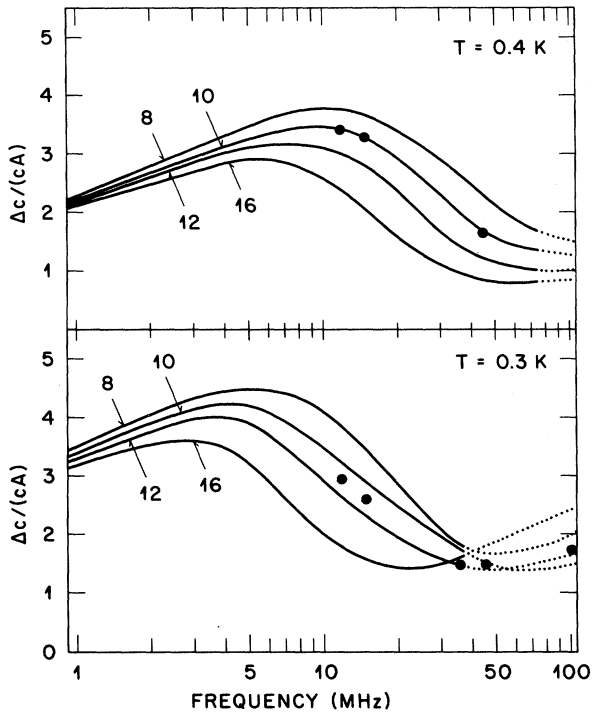

FIG. 2. Change in sound velocity, normalized according to Eq. (11), as a function of frequency for $T = 0.3$ and 0.4 K. The notation is the same as in Fig. 1.

dominated" frequency domain, i.e., $\Omega/2\pi \leq 20$ MHz for $T \approx 0.2$ K, and $\Omega/2\pi \leq 100$ MHz for $T \approx 0.4$ K. On the other hand, the values of $\Delta c/c_0$ showed a significant mesh dependence in the high-frequency region where the number of eigenfunctions used here is still insufficient.

III. RESULTS AND COMPARISON WITH OTHER WORK

In Figs. 1 and 2, we present our theoretical results for

$$\delta c = \Delta c/Ac_0 = [c(T,\Omega) - c_0]/Ac_0, \tag{11}$$

for various values of temperature T and dispersion coefficient γ. A is given by[17]

$$A = \frac{\hbar\pi^2(u^2+2u+\frac{1}{3})}{30\rho c_0}\,\frac{k_B T^4}{\hbar c_0}. \tag{12}$$

Our plots demonstrate that the peculiar behavior of the sound velocity as a function of frequency, found experimentally, is indeed reproduced. The maxima and minima of δc are shifted towards higher Ω with increasing temperature in a systematic way. This is expected if we interpret the decrease of δc with Ω as being due to the transition from the hydrodynamic ($\Omega\tau<1$) to the high-frequency ($\Omega\tau>1$) domain, τ representing some mean relaxation time.[18] Quantitatively, it is evident that a value of $\gamma = -10\times10^{37}$ cgs units yields good agreement for all temperatures shown in the figures. The values for δc obtained by Maris[12] are consistently higher than ours in the relevant frequency domain. The present result for γ lies well within the range of values deduced from measurements of specific heat C and viscosity η. The original analysis of $C(T)$ by Phillips, Waterfield, and Hoffer[19] yielded $\gamma = -4.1\times10^{37}$ cgs, whereas Maris,[9] by reexamining their data, found $\gamma \approx -8\times10^{37}$. Maris,[15] working with the same eigenfunctions of the collision operator as used here, calculated η and found agreement with experiment for $\gamma \approx -10\times10^{37}$. Thus our γ, extracted from ultrasonic data, deviates considerably less from these values than $\gamma = 15\times10^{37}$ given by Junker and Elbaum.[11]

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[18]In this connection, the following observation may be interesting: if we define $\tau = 1/\Omega_{max}$ by means of the frequency Ω_{max} for which δc reaches its maximum, we find that $\tau \propto T^{-3}$. This is the same temperature dependence as that of the normal process relaxation time found in second-sound experiments in solid He [C. C. Ackermann and R. A. Guyer, Ann. Phys. (N.Y.) 50, 128 (1968)] and NaF [H. Beck, Z. Phys. B 21, 209 (1975)].

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