Onset of phase slip in superflow through channels*

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The energy required to change an initially uniform-flow field in a channel to a field containing a vortex is evaluated. It is found that this includes an energy impulse propagated in from the two ends of the channel. The result can be used to define the free energy of the vortex in an entirely unambiguous manner, permitting one to determine the amount of energy dissipated from the superfliud velocity field during an arbitrary phase-slip process. For the special case of a circular vortex ring propagating upstream, the free energy is found to have precisely the form assumed in current theories of intrinsic critical velocities. Further possible applications of the present more general result are discussed.

I. INTRODUCTION

One of the most fascinating properties of superfluid helium is that macroscopic flow will persist indefinitely without any detectable dissipation. Similarly, large flow rates can be maintained even through very small channels by vanishingly small pressure differentials. The existence of such nondissipative flows is at first sight surprising since intuitively one expects the fluid to slow down by converting its kinetic energy of motion into internal excitations of the fluid and the boundaries; but it may be qualitatively understood as arising from the fact that every wave function describing a possible state of the superfluid has the unusual property of spatial phase coherence. Because of this the probability amplitude of finding a particle at $\vec{r}, t$ will be a complex function $f(\vec{r}, t)\, e^{i\varphi(\vec{r}, t)}$, and there will be a corresponding macroscopic velocity field given by

$$\vec{v}_s(\vec{r}, t) = (\hbar/m)\nabla\varphi(\vec{r}, t)\,, \tag{1}$$

where m is the mass of a ^{4}He atom, and the time dependence represents the quantum-mechanically adiabatic response of the wave function to possible slow changes in the boundary conditions. At finite temperature, the superfluid will of course fluctuate through various of these quantum states, and in particular it can easily make transitions which preserve the macroscopic phase $\varphi(\vec{r}, t)$. Such fluctuations would, for example, result from the interactions of the phonon-roton elementary excitations with each other and with the boundaries. However, a transition in which φ (and hence $\vec{v}_s$) is changed over macroscopic distances while various excitations are produced involves two wave functions which differ so profoundly that the direct transition rate must be extremely small. One may therefore expect superfluid currents to be metastable in the sense that the fluctuations required to thermally degrade the velocity field $\vec{v}_s(\vec{r}, t)$ are unlikely to occur.[1]

Extensive observations have shown that in a given situation nondissipative behavior occurs only for characteristic flow velocities less than some *critical velocity* v_c, above which dissipative effects set in rapidly. These critical velocities have been studied in many experiments[2-6] involving a wide variety of channel geometries and observational procedures. While it has not always been easy to interpret the results of such experiments, or to reconcile one with the other, it has by now become clear that there are (at least) two distinct types of channel critical velocities, presumably involving somewhat different dissipation mechanisms.

At temperatures far below the λ point, it is generally found that v_c decreases as the characteristic channel size d becomes larger and that it does not depend strongly on the temperature. This has been called[7] the *extrinsic* regime. One might argue on dimensional grounds that $v_c \approx h/md$, and indeed there are many experiments which give critical velocities on this order. The agreement, however, is generally only very approximate, and it is in fact not even clear in trying to fit the observations to this phenomenological form whether one should let d represent the average size of the channel or the typical roughness of the channel walls.

In addition to being sensitive to the geometry of the channel, the onset of extrinsic dissipation appears to depend strongly on various experimental perturbations such as end effects, vibration, and contamination of the channel by dust particles. A most dramatic demonstration of this has been given by Hess,[7] who found that rouge superleaks placed in the flow channel above and below a 10-μm pinhole could raise v_c by a factor of 20 above the typical extrinsic value. It is perhaps useful to recall that the onset of turbulence in a classical fluid moving through a channel is in practice also found to depend greatly on the various nonideal features of the experimental situation. This is well understood

as arising from the fact that, although the classical flow is stable against infinitesimal perturbations up to very high velocities, it is unstable against small *finite* perturbations at much lower velocities. The analogy should not be pushed too far, but it serves to indicate that the observed extrinsic critical velocities reflect the response of the superflow to finite disturbances which may vary from one observation to the next. We will return to this idea later.

In recent years, in a series of very elegant experiments,[3] critical velocities near the λ point have been found to be essentially independent of the channel geometry and to vary strongly with temperature according to the simple power law $v_c \propto (T_\lambda - T)^{2/3}$. This second, *intrinsic* regime also differs from the extrinsic regime by the manner in which dissipation sets in. In contrast to the sudden, somewhat unpredictable instabilities seen at lower temperatures, one finds here that the channel velocity decays in the well-defined logarithmic manner characteristic of a thermally activated relaxation process. In this regime, v_c may be approximately defined as the velocity at which the dissipation becomes large in some sense. Since the intrinsic critical velocity increases rapidly as T is lowered, there is an eventual crossover to extrinsic behavior. The crossover typically occurs around 2°K, but the experiment of Hess[7] shows that if great care is taken to suppress external perturbations intrinsic behavior can be observed down to 1.2°K.

II. THEORETICAL BACKGROUND

A. Role of vortices

As discussed earlier, it is not hard to understand qualitatively why dissipation-free flows can occur in superfluid helium. It is, however, much more difficult to identify the critical velocity mechanisms that limit this behavior, such a task requiring an examination of the likely ways in which $\varphi(\vec{r}, t)$ *can* be changed. Since the detailed microscopic description of superflow at finite temperatures becomes impossibly complicated, theoretical discussions of critical velocities have usually been based on the two-fluid model. In this approximation, the elementary excitations are assumed to act like a weakly interacting gas of wave packets with an effective mass density ρ_n, while the superfluid component of the motion is characterized by a density $\rho_s = \rho - \rho_n$ and a velocity field $\vec{v}_s(\vec{r}, t)$ which obeys the ideal-fluid equations

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \nabla)\vec{v}_s = -\frac{\nabla p}{\rho_s}, \tag{2a}$$

$$\nabla \cdot \vec{v}_s = 0 . \tag{2b}$$

Because of Eq. (1), the flow field must obey the additional quantum restrictions

$$\nabla \times \vec{v}_s = 0 , \tag{3}$$

$$\oint \vec{v}_s \cdot d\vec{l} = \frac{nh}{m} . \tag{4}$$

Both experiment and theory indicate that, sufficiently far below the λ point, the description of the superfluid velocity field provided by Eqs. (2)–(4) is useful down to a scale of 1 or 2 Å.[8] Normal-fluid excitations do not interact dissipatively with $\vec{v}_s(\vec{r}, t)$, provided $\vec{v}_s$ is uniform over a typical excitation wavelength.[9] Velocity fields that do vary significantly on this scale will scatter excitations, an effect which can be approximately included in Eq. (2a) by adding a nonconservative force field.[10]

Our form of the two-fluid equations is based on the assumption that the normal fluid remains in thermal equilibrium with the channel walls. Then the normal-fluid velocity $\vec{v}_n$ is zero, and the mass current density $\vec{j} = \rho_n(\vec{v}_n - \vec{v}_s) + \rho\vec{v}_s$ reduces to $\rho_s\vec{v}_s$. Equations (2a) and (2b) now simply represent Newton's law and the condition of mass conservation. In light of our later use of these equations, it should be noted that they adequately describe a situation in which $\vec{v}_s(\vec{r}, t)$ consists of the superposition of a macroscopic channel field $\vec{u}(\vec{r}, t)$ and a microscopically localized field $\vec{v}_l(\vec{r}, t)$ that is strongly nonuniform only over distances on the order of 1–10 Å. The macroscopic velocities may be assumed to vary slowly enough with time that the normal fluid stays in quasiequilibrium with the walls. The field $\vec{v}_l(\vec{r}, t)$, on the other hand, is so localized compared to the momentum-relaxation mean free path Λ of the excitations that its presence does not significantly affect their distribution function. Our discussion will not apply in its present form to situations involving appreciable counterflow, highly developed quantum turbulence, or temperatures very near the λ point where Λ is only a few angstroms.

Consider now a situation in which superfluid helium is flowing steadily through a long channel. In order for some change $\vec{u}(\vec{r}) \to \vec{u}(\vec{r}) + \vec{v}_l(\vec{r}, t)$ to constitute a dissipative relaxation process towards thermal equilibrium, it must result in a net transfer of energy from the superflow field into the other available degrees of freedom of the system. Of course, if the whole flow field $\vec{u}(\vec{r})$ were suddenly to slow down it would certainly release energy, but the rigidity of the superfluid wave function rules out such processes. Indeed, the most likely transitions should be those which produce the most localized modifications $\vec{v}_l(\vec{r}, t)$ possible. Since the velocity field is completely determined by Eqs. (2b) and (3), the only way in which a local change $\vec{v}_l$ can in fact come about is through the introduction of a discontinuity, such as a vortex line. The

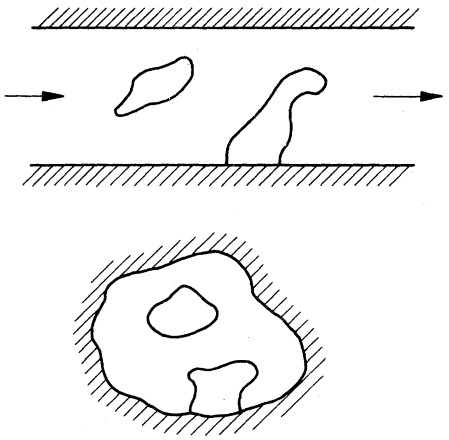

FIG. 1. Examples of typical loops and rings formed by vortex-line sigularities in a channel. Both a side view and an end-on view of the channel are given.

vortex line essentially makes the region occupied by the curl-free fluid multiply connected, and provides the extra freedom to specify the circulation about the line as part of the boundary conditions determining the velocity field. Vortex lines are indeed known to exist in superfluid helium, [2] and to be adequately described by Eqs. (2)–(4) down to a core radius of order 1 Å, inside of which the ideal-fluid description fails. By Eq. (4), the circulation about such a vortex line must be quantized in units $\kappa_0 = h/m$.

The idea that quantized vortex lines play a major role in the breakdown of persistent flow has several attractive features. First, it follows from what we have just said that vortex loops or rings (Fig. 1) are the only[11] allowed microscopically localized fluctuations of $\vec{v}_s$. Such vortex configurations need be no larger than a few angstroms across, and therefore represent a minimal disturbance of the phase $\varphi(\vec{r}, t)$. Second, once a vortex ring or loop has been introduced into $\vec{v}_s$, it provides a mechanism for coupling v_s dissipatively to the thermal degrees of freedom: Both the scattering of excitations from a vortex and the pinning of a vortex to a wall exert nonconservative forces on $\vec{v}_s$. Finally, it has long been recognized[12,13] that the growth and motion of vortices can give rise to a decrease in the phase difference between the two ends of a channel, thus providing a natural way for the macroscopic superflow to slow down, or to maintain a steady dissipative state when a pressure gradient is applied along the channel. This is most easily illustrated in terms of the two-dimensional geometry of Fig. 2. A general way of describing the fluid slowing down is to say that the phase difference

$$\varphi(+\infty) - \varphi(-\infty) = \frac{m}{\hbar} \int_\Gamma \vec{v}_s \cdot d\vec{l} \tag{5}$$

decreases, where the line integral is taken from one end of the channel to the other. If a pair of quantized vortex lines (the analog of a ring) is added to $\vec{u}$ and then moves across the channel as shown in Fig. 2, this phase difference will decrease by exactly 2π. A quantized vortex line which detaches itself from the boundary layer (the analog of a loop) and moves across the channel has the same effect. The elegance of this *phase-slip* description should not be allowed to obscure the fact that it does not in itself explain why a vortex would appear in the first place or why it should move so as to produce a decrease in the phase difference along the channel. It is these questions that are at the heart of the critical-velocity problem.

B. Extrinsic regime

One popular early approach[14–17] was to calculate the energy change when a vortex-ring field $\vec{v}_R$ is added to the channel field $\vec{u}$. According to the Landau criterion, if $\vec{u} + \vec{v}_R$ is of lower energy than $\vec{u}$, $\vec{v}_R$ is an allowed excitation. From the more modern point of view, one would say that the process $\vec{u} \to \vec{u} + \vec{v}_R$ releases energy from the flow field and is hence a thermal relaxation process. The central concern of this type of calculation is to evaluate

$$E_R = \tfrac{1}{2}\rho_s \int_{V^*} (2\vec{u} \cdot \vec{v}_R + v_R^2)\, dV\,, \tag{6}$$

where V^* is some volume large enough so that the integral is well defined. For a channel of uniform cross section, $\vec{u}$ is just a constant, $\vec{U}$, and the energy difference becomes

$$E_R(\vec{U}) = E_R(0) + \vec{U} \cdot \vec{P}_R(0)\,, \tag{7}$$

where $E_R(0)$ and $P_R(0)$ are the energy and momentum associated with the field v_R alone. Equation (7) implies that above some critical velocity

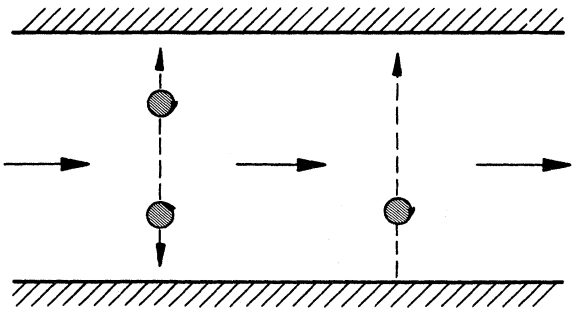

FIG. 2. Two kinds of vortex-line motion which can generate phase slip. The three-dimensional analogs of these are the growth of a vortex ring from some point in the fluid and the growth of a vortex loop from the boundary. Note the circulation direction which is required to decrease the phase difference between the two ends of the channel.

$[E_R(0)/P_R(0)]_{\min}$, $E_R(\vec{U})$ can be negative and thermal relaxation becomes possible. For a ring of radius $\frac{1}{2}d$, coaxial with a channel of circular cross section, it is found that

$$\left(\frac{E_R(0)}{P_R(0)}\right)_{\min} \approx \frac{\kappa_0}{2\pi d} f, \tag{8}$$

where f is of order one and depends on the details of the calculation. Even for less-idealized geometries, Eq. (8) gives roughly the lowest value of U for which a negative-energy ring will fit into the channel.

A serious objection to this kind of argument is the implausibility of describing the onset of dissipation in terms of the sudden appearance of macroscopically large vortex rings. Such a state would never appear as the result of a direct quantum-mechanical transition, and it is unlikely to evolve statistically from more localized vortex fluctuations since these have $E_R(U_c) > 0$. The most that can therefore be said for this type of calculation is that it establishes the *possibility* of dissipation for velocities above $(\kappa_0/2\pi d)f$.

A second difficulty that arises in these discussions is the evaluation of $\vec{P}$ for a vortex in a channel. The actual momentum contained in the flow field of a vortex ring coaxial with the channel is, in fact, identically zero.[18] The common practice has been to replace P_R in Eqs. (7) and (8) by the impulse[19] of a vortex ring in an unbounded fluid. No entirely satisfactory justification or physical interpretation of this procedure has been given, although some of the ideas appearing in our later discussion are to be found in the interesting papers of Huggins[13] and Campbell.[20]

Gladerson and Donnelly[21] have offered a totally different interpretation of extrinsic behavior. Their idea is that a macroscopic pinned vortex which may initially be present in the stream $\vec{U}$ will simply develop a curvature such that its self-induced velocity will keep it stationary in the channel. For stream velocities greater than $(\kappa_0/2\pi d)f$ this is no longer possible, the pinned vortex then grows without limit, and dissipation sets in. The main objection to this model, and to later elaborations[22] of it, is that a pinned vortex will certainly not behave in the simple manner assumed. For example, if a straight vortex is placed across the channel with its ends pinned to the wall, the center part of the vortex will be swept downstream before anything else happens. It is obvious that an extremely complicated nonlinear development ensues which bears little resemblance to the simple pictures presented in Refs. 21 and 22. Nevertheless, the work of Glaberson and Donnelly contains the important idea that extrinsic critical velocities represent the response of the system to a large perturbation such as a pinned vortex of macroscopic length. As we discussed in Sec. I, there are reasons to believe that this is the proper interpretation.

C. Iordanskii-Langer-Fisher

The theory of critical velocities in the intrinsic regime appears to be in much better shape, mainly due to the work of Iordanskii[23] and of Langer and Fisher[24] (ILF). These authors assume that the *free* energy of a vortex ring in a uniform flow field is given by Eq. (7). If one considers specifically a circular ring with axis opposite to $\vec{U}$, approximates $E_R(0)$ as the energy of the ring in an unbounded fluid, and replaces $\vec{P}_R(0)$ by the impulse of such a ring in an unbounded fluid, one obtains the result that $E(R, U)$ varies as shown in Fig. 3. Free-energy curves similar to these are found in the theory of homogeneous nucleation of supersaturated vapors, and a model of phase-slip nucleation can be developed along analogous lines. Briefly, once a ring fluctuates into the region $R > R_c$, it has an overwhelming probability of continuing to grow until it annihilates at the walls. It is clear from our earlier discussion that one such event will decrease the phase difference between $x = \pm\infty$ by h/m. If the system has achieved quasiequilibrium with respect to the existence of very small vortex rings, the density of rings with $R = R_c$ is proportional to $e^{-E(R_c,U)/k_BT}$, and the rate per unit volume of phase-slip events is therefore $\nu_0 e^{-E(R_c,U)/k_BT}$. Here ν_0 contains the factors determining the absolute number of rings per unit volume, and the rate at which a ring with $R = R_c$ is kicked up to the next larger state.[25] Thus, the superfluid must on the average be slowing down at the rate

$$\frac{dU}{dt} = -\frac{hA\nu_0}{m} e^{-E(R_c,U)/k_BT}, \tag{9}$$

where A is the cross-sectional area of the channel.

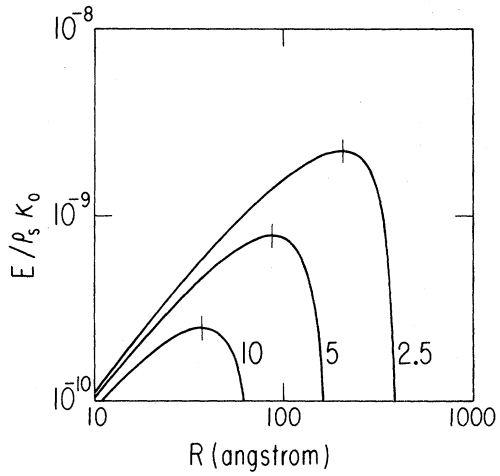

FIG. 3. Free energy of a circular vortex ring in an unbounded fluid, with propagation axis opposite to $\vec{U}$. The free energy is plotted in units of $\rho_s\kappa_0$ for $U = 2.5$, 5, and 10 msec^{-1}. The vertical lines show the appropriate values of R_c.

There is no doubt that the ILF theory is essentially correct. Langer and Reppy[26] discuss in detail how Eq. (9) at least qualitatively predicts the logarithmic decay of U, the value U_c at which the dissipation becomes large, and the variation of U_c with temperature near the λ point. However, as with the arguments discussed in Sec. II B, the physical meaning of some of the basic assumptions made in the ILF model remains obscure. Specifically, one should like to understand (a) how to use Eqs. (6) and (7) to represent a free energy associated with the vortex field, (b) how to treat the momentum term properly, and (c) how to generalize the results to describe arbitrarily complicated vortex loops and rings. It is the main purpose of our paper to address these particular questions, with the aim of perhaps throwing new light on the nature of both extrinsic and intrinsic critical velocities.

III. FREE ENERGY

The superfluid velocity field may be thought of as a system in weak thermal contact (but not equilibrium) with a very large reservoir composed of the normal-fluid excitations and the boundaries. Since this is a rather unusual thermodynamic system, we first discuss briefly the meaning of the free energy which appears in the ILF thermal-nucleation model. Let us again consider a change in which $\vec{v}_s$ goes from a uniform channel velocity $\vec{U}$ to $\vec{U}+\vec{v}_l$, where $\vec{v}_l$ represents the field of some microscopic vortex loop or ring. We shall shortly show that such a change will be accompanied by the extraction of a well-defined amount of energy $E(\vec{v}_l, U)$ from the thermal reservoir. Hence the reservoir will suffer an entropy change $\Delta S_R = -E(\vec{v}_l, U)/T_R$. If one for convenience pictures the reservoir as a large closed system, the number of microstates available to it will be $W_R = e^{(S_R/k_B)}$, so that

$$W_R(\vec{U}+\vec{v}_l)/W_R(\vec{U}) = e^{-E(\vec{v}_l, U)/k_B T_R} \quad . \tag{10}$$

The specification of $\vec{v}_l$ does not involve any degree of randomness. Thus to the extent that the superfluid velocity field is in thermal equilibrium with respect to small vortex fluctuations, the relative probability of seeing the configuration $\vec{U}+\vec{v}_l$ is given by Eq. (10). One can conclude that the free energy appearing in the ILF theory and in the earlier Landau-Feynman arguments discussed in Sec. II B is just the energy $E(\vec{v}_l, U)$ extracted from the nonhydrodynamic degrees of freedom when $\vec{v}_s$ goes from $\vec{U}$ to $\vec{U}+\vec{v}_l$.

In order to find $E(\vec{v}_l, U)$ we begin by considering how to calculate the kinetic energy of a channel-flow field, which consists of a macroscopic flow moving from $x \to -\infty$ to $x \to +\infty$ (x being the axis of the channel), plus the localized velocity field $\vec{v}_l$

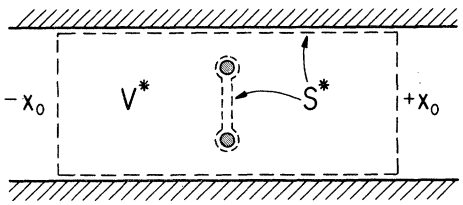

FIG. 4. One possible surface of integration S^* for a ring in a channel. Shown is a cut through the plane of the ring, and parallel to the axis of the channel.

arising from some configuration of quantized vortices (Fig. 1) in the region near $x = 0$. The discussion will be limited to channels the cross section of which does not change as one goes down the channel. The macroscopic velocity is then simply a constant, $U\hat{x}$, and the associated scalar potential is given by $\phi_u = Ux$. Let ϕ_l denote the scalar potential arising from the quantized vortex, including the effects of the boundaries. To find the total kinetic energy contained in some volume of fluid, one may make use of the transformation ($\nabla \cdot \vec{v} \equiv 0$)

$$\tfrac{1}{2}\rho_s \int \vec{v}\cdot\vec{v}\, dV = \tfrac{1}{2}\rho_s \int \nabla\cdot(\phi\,\vec{v}) dV$$

$$= \tfrac{1}{2}\rho_s \int_{S^*} \phi\,\vec{v}\cdot\hat{n}\, dS \ , \tag{11}$$

where S^* is the surface enclosing the volume under consideration. Since $\phi = \phi_u + \phi_l$ is multiple valued, this transformation is valid only if the surface is drawn so as to exclude the vortex-line singularity as, for example, in Fig. 4.

Equation (11) for the energy has a direct physical interpretation. Since the velocity field is irrotational in the entire volume under consideration, the energy-flux vector $\vec{v}_s(p + \frac{1}{2}\rho_s v^2)$ may be written in the form $\vec{v}_s(\text{const} - \rho_s \partial\phi/\partial t)$ by virtue of Bernoulli's theorem. If the velocity field changes from $\vec{v}_1(\vec{r}, t_1)$ to $\vec{v}_2(\vec{r}, t_2)$ by any process which does not involve the action of nonconservative forces within V^*, the energy change must be given by

$$E_2 - E_1 = \rho_s \int_{S^*}\int_{t_1}^{t_2} \frac{\partial\phi(\vec{r}, t)}{\partial t}\,\vec{v}(\vec{r}, t)\cdot\hat{n}\, dt\, dS \ . \tag{12}$$

If the process of going from $\vec{v}_1$ to $\vec{v}_2$ is imagined to occur very rapidly through the application of very large pressure fields acting across S^* for a very short time, one can make the usual impulse approximation of replacing $\vec{v}$ by its mean value and carrying out the time integration over ϕ:

$$E_2 - E_1 = \tfrac{1}{2}\rho_s \int_{S^*} (\phi_2 - \phi_1)(\vec{v}_2 + \vec{v}_1)\cdot\hat{n}\, dS \ . \tag{13}$$

For the case $\vec{v}_1 \equiv 0$, this reduces to the expression (11). Also, using $\nabla^2\phi_1 = \nabla^2\phi_2 = 0$ and Green's theorem, it is easily shown that Eq. (13) is equivalent to

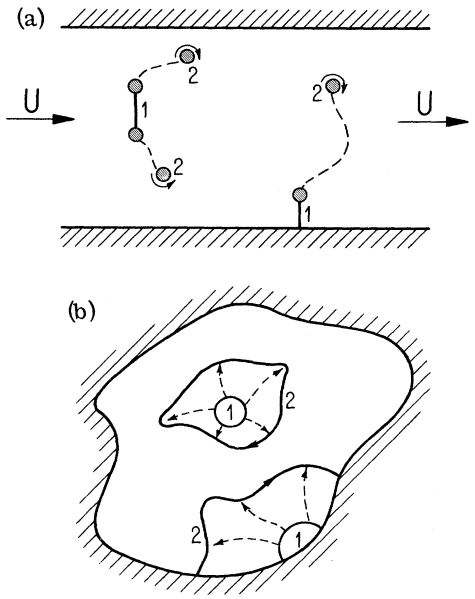

FIG. 5. (a) Three-dimensional phase-slip processes, side view of channel. The dashed lines show how the barrier is extended as the configurations evolve. (b) Three-dimensional phase-slip processes, end-on view of channel. The arrows on the line core show the required direction of the circulation vector if U is taken to be out of the plane of the figure.

$$E_2 - E_1 = \tfrac{1}{2}\rho_s \int_{S^*} (\phi_2 \vec{v}_2 - \phi_1 \vec{v}_1) \cdot \hat{n}\, dS \ . \qquad (14)$$

The energy transmitted into V^* through a given part of S^* during an impulsive change in the flow pattern must, however, be obtained from Eq. (13).

The particular process of interest here is one in which $\vec{v}_1 = \vec{U}$ and $\vec{v}_2 = \vec{U} + \vec{v}_l$. Then

$$\Delta E = \tfrac{1}{2}\rho_s \int_{S^*} \phi_l (2\vec{U} + \vec{v}_l) \cdot \hat{n}\, dS \ . \qquad (15)$$

The walls of the channel make no contribution to the surface integral, since both $\vec{U} \cdot \hat{n}$ and $\vec{v}_l \cdot \hat{n}$ are zero there. Thus the total kinetic-energy change of the fluid contained in V^* consists of a contribution transmitted in from the fluid at infinity, through the parts of S^* at $\pm x_0$ in the limit $x_0 \to \infty$, plus a contribution transmitted in through the barrier and the surface surrounding the vortex core. The contribution propagated in from infinity is provided by the superfluid field itself, and cannot be charged to the account of the local thermal reservoir. Therefore, the free energy $E(\vec{v}_l, U)$ absorbed from the nonhydrodynamic degrees of freedom equals the increase in the total kinetic energy of the fluid contained in V^* *minus the energy transmitted in through the part of* S^* *at* $\pm x_0$. $E(\vec{v}_l, U)$ is then just the term in Eq. (15) that arises from the integration over the two sides of the barrier and the surface surrounding the core of the vortex. It may be thought of as a local energy impulse, transmitted to the fluid by the impulsive pressures which these surfaces would have to exert on the fluid in order to generate the vortex.

While the contribution from the ends of the channel does not enter into our final expression for $E(\vec{v}_l, U)$, it will nevertheless prove informative to study it in greater detail. For a localized vortex configuration in a channel, $\vec{v}_l$ drops off faster than $|x|^{-1}$ as $x \to \infty$, while ϕ_l goes to a constant value. Hence the energy propagated in from infinity is just

$$\rho_s UA[\phi_l(x \to +\infty) - \phi_l(x \to -\infty)] \ ,$$

where A is the cross-sectional area of the channel. The important point is that, while we have so far considered only impulsive processes, this expression is in fact valid for *any* kind of process in which the fluid goes from $\vec{U}$ to $\vec{U} + \vec{v}_l$, provided only that the ideal-fluid equations hold at $x = \pm\infty$. This follows immediately from Eq. (12): Since $\vec{v}(\vec{r}, t)$ approaches $\vec{U}$ far away from $x = 0$, the velocity factor may be taken out of the integral. In particular, we may consider vortices which are created in some initial configuration 1 in Fig. 5 and then move to 2, generating the phase-slip barrier as they move. The change in the kinetic energy of the superfluid contained within V^* is of course independent of how the line was made and has traveled from 1 to 2, while we have just seen that the energy transmitted in from infinity also does not depend on the mechanism by which the new flow field was established. Hence one can assert quite generally that the *process* in which the vortex is created and moves from 1 to 2 will absorb an amount of energy from the thermal reservoir equal to the local energy impulse.

The local energy impulse can be expressed in a very simple form. The potential ϕ_l has a discontinuity $|\kappa_0| = h/m$ across the barrier, while $\vec{U}$ and $\vec{v}_l$ are continuous everywhere. Thus Eq. (15) reduces to an integral over the surface B defined by the barrier (Fig. 5)

$$E(B, U) = \rho_s \kappa_0 \int_B \vec{U} \cdot \hat{n}\, dS + \tfrac{1}{2}\rho_s \kappa_0 \int_B \vec{v}_l \cdot \hat{n}\, dS \ , \qquad (16)$$

where $\hat{n}$ is in the downstream direction and κ_0 is given a positive sign if the circulation is such as to move the fluid through B in the sense $\hat{n}$. The small corrections arising from the integration over the surface of the vortex core have been neglected, but could easily be included. Since the two integrals in Eq. (16) are proportional to the mass fluxes of the $\vec{U}$ and $\vec{v}_l$ fields through the barrier, $E(B, U)$ depends only on the boundaries defining the barrier (i.e., the initial and final positions of the vortex), and not on the details of how the line has moved to

its final position. It is also clear from the previous discussion that Eq. (16) can be interpreted in terms of a process: If the line moves so as to change the barrier from B_1 to B_2, the amount of energy that must be supplied by the thermal reservoir is $E(B_2, U) - E(B_1, U)$.

IV. APPLICATIONS AND SPECULATIONS

Within the limitations of the two-fluid approximation, Eq. (16) provides a very general and powerful expression for the free energy of a vortex fluctuation. Its physical origin is clear and it applies to arbitrary processes leading to arbitrarily complicated vortex configurations. Furthermore, it is simple in form with each term having a straightforward physical interpretation. The first term on the right-hand side is just $\rho_s \kappa_0 UB_\perp$ where $B_\perp$ is the area of the barrier projected onto the plane perpendicular to $\vec{U}$. That is, it is κ_0 times the mass flux carried by U through the phase-slip barrier. The second term is just the kinetic energy associated with the flow field $\vec{v}_l$ in the absence of $\vec{U}$. It is positive definite. We note that, while we have neglected the integral over the core surface in Eq. (16) and while the core radius is also relatively unimportant in the first term of Eq. (16), the core radius does enter into the second term in a crucial way: Near the center of the line $v_l \to \kappa_0/2\pi r$, and the integral of $\vec{v}_l \cdot \hat{n}$ over the barrier will diverge logarithmically as the edge of the barrier approaches the line center. A phenomenological cutoff at $r = a_0$ must therefore be introduced as usual.

In order for any phase-slip process to be part of the dissipative thermal relaxation of the superflow, it must give $E(B, U) < 0$. Since the second term in Eq. (16) is positive definite, this can only occur if κ_0 is negative in the sense defined earlier. From Fig. 5 one sees that the line motion which defines the barrier must then be such as to decrease $\int_\Gamma \vec{v}_s \cdot d\vec{l}$. Thus the statistically driven motion of the line does in fact satisfy the requirement of the phase-slip picture in that it leads to a decrease in the superfluid phase difference $\varphi(+\infty) - \varphi(-\infty)$.

We now turn to more specific matters, recalling that the particular vortex fluctuation assumed to be important in the ILF theory is a circular vortex ring of radius R heading upstream. If boundary contributions are neglected Eq. (16) gives

$$E(R, U) = -\pi\rho_s \kappa_0 UR^2 + \tfrac{1}{2}\rho_s \kappa_0^2 R[\ln(8R/a_0) - 2] \quad (17)$$

for such a configuration. Equation (17) is precisely the form of the free energy assumed by the authors of the ILF model, and one may therefore view the discussion of the Sec. III as providing a sound physical basis for this particular ansatz.

The generality of Eq. (16), however, also makes it useful for discussing more complicated fluctuations. It is of course rather difficult to find $\vec{v}_l$ for an arbitrary vortex loop or ring, particularly when the boundary effects are important. But for any reasonably simple vortex configuration, one can write the useful approximate relation

$$E(B, U) \approx -\rho_s |\kappa_0| UB_\perp + \rho_s(\kappa_0^2/4\pi) L \ln(2\delta/a_0) , \quad (18)$$

where L is the length of the vortex and δ is a characteristic length. For a circular ring or a half-circle loop on a plane surface, δ would equal $4R$. For a pair of vortex lines or a vortex line parallel to a plane, δ would have the meaning shown in Fig. 6. Since δ has a weak logarithmic effect, Eq. (18) should be fairly accurate for geometries intermediate between these extremes.

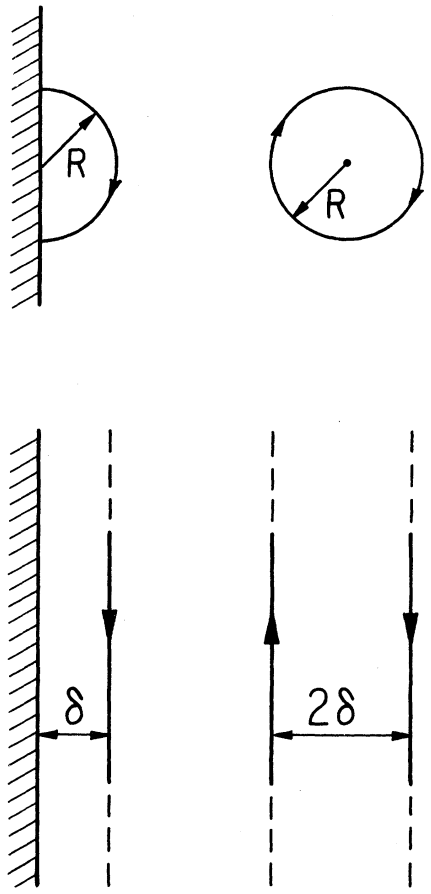

FIG. 6. Inhomogeneous versus homogeneous nucleation for two highly idealized cases. Again, the arrows denote the required direction of the circulation vector if U is taken to be out of the plane of the figure.

Equation (18) still exhibits the characteristic free-energy peak (Fig. 3), provided $B_\perp$ is assumed to vary as L^α, $\alpha > 1$. The curves for the *inhomogeneous* nucleation of a loop from a wall may, however, have a different shape and be much lower than those for homogeneously nucleated vortex rings. This is already obvious for the highly idea-

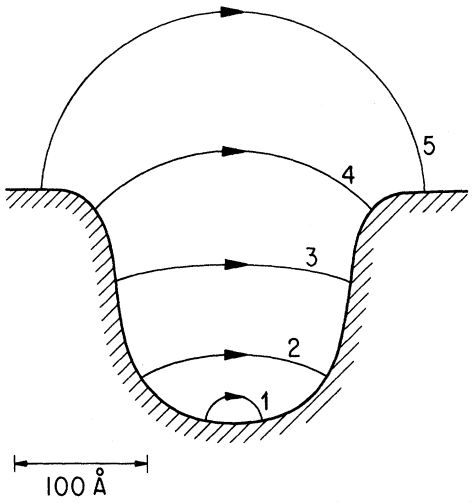

FIG. 7. Inhomogeneous nucleation from a boundary imperfect on a scale small compared to the channel size.

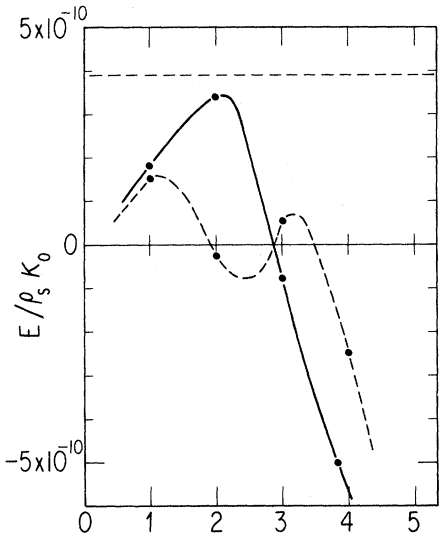

FIG. 9. Semiquantitative free-energy curves for the nucleation events shown in Fig. 7 (solid curve) and Fig. 8 (dashed curve). U is taken to equal 5 msec^{-1}. The horizontal dashed line gives $E(R_c, U)/\rho_s k_0$ for the idealized half-ring shown in Fig. 6.

lized vortex loop shown in Fig. 6, for which the peak height would be exactly half the height for a circular ring. The effect should be even more drastic for the more realistic geometries shown in Figs. 7 and 8. Figure 9 summarizes the various free-energy curves estimated from Eq. (18). One may conclude that if the channel walls are rough on the scale of R_c, inhomogeneous vortex-loop nucleation could well be the dominant dissipative process, particularly in very small channels where the surface-to-volume ratio is high. In this connection it is interesting to note that perhaps the only serious difficulty with the ILF model is that it predicts an activation energy which is quite a bit larger than that determined experimentally. Obviously, one worthwhile extension of our ideas would be to carry out a quantitative calculation of the inhomogeneous nucleation rates, to see whether these will explain the experimental observations.

Finally, we would like to suggest that Eq. (18), when combined with the Glaberson-Donnelly concept of considering the response of the superfluid to a large "external" perturbation, also throws some light on the nature of extrinsic critical velocities. Suppose for example that the channel walls are rough on a scale Δ, and assume that this nonideal feature generates vorticity within a distance Δ of the wall. Such an assumption is entirely speculative, but the fluid velocity around a sharp projection can be very high and may serve as a region of anomalous inhomogeneous thermal nucleation,[27] or vibration of the channel walls may cause local projections to act as impulse-delivering surfaces in the sense discussed earlier. One may idealize the resulting perturbation as a half-ring (Fig. 6) with R equal to Δ. The response of the fluid to such a finite disturbance will depend on whether the half-ring finds itself on the "uphill" or "downhill" side of the free-energy curve. This depends on U, the critical value being approximately given by

$$U_c \approx \frac{|\kappa_0|}{4\pi\Delta} \ln \frac{8\Delta}{a_0} \ . \tag{19}$$

If $U > U_c$, the loop will be on the downhill side and will continue to grow, thus completing the phase-slip process. If $U < U_c$, it will annihilate at the wall with no resulting phase slip.

It is clear that there is a critical velocity above which the imagined process can lead to phase slip. This critical velocity is similar in form to Eq. (8),

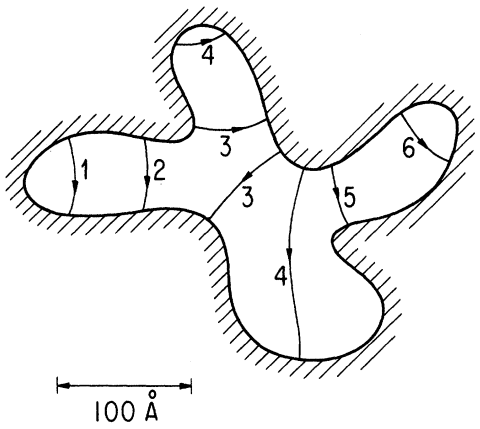

FIG. 8. Inhomogeneous nucleation in a channel with a highly irregular cross section.

but it is based on a quite different physical picture in which the scale of roughness Δ of the channel walls is the dominant characteristic distance. Although we have derived Eq. (19) by considering an idealized process which injects a half-ring, a very similar result is obtained for vortex loops of arbitrary shape. The simple physical point is that if some *ex cathedra* mechanism creates vortex lines at a characteristic distance Δ from the wall of the channel, there exists a critical velocity $U_c \approx (\kappa_0/4\pi\Delta)f$ [f is $O(1)$] above which the lines can move out into the channel and lead to phase slip. While this interpretation of extrinsic behavior is at present no more than an interesting speculation, it appears to be at least as attractive as other explanations that have been offered. To test its ultimate validity will probably require more detailed experimental investigations of the influence of channel roughness, etc., than have been possible up to now.

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