Detailed steady-state approach to obtaining the electron distribution for semiconductors in quantizing magnetic fields and parallel electric fields*

David C. Rohlfing† and E. W. Prohofsky
Department of Physics, Purdue University, West Lafayette, Indiana 47907
(Received 24 February 1975)

In this paper we use detailed-steady-state conditions in the drifted Maxwellian approximation to determine separate drift velocities and electron temperatures for each Landau level, rather than using an over-all-steady-state (OSS) condition with a single-drift velocity and temperature. We find that the drift velocities and temperatures differ from level to level and from the results obtained when the OSS approach is used. We also find that the two approaches give different results for the electrical conductivity.

I. INTRODUCTION

The problem of calculating the electrical conductivity of a semiconductor with parallel electric and magnetic fields has been tackled by many people using many different methods in recent years.[1-8] One usually assumes either negligible electron-electron scattering, or the other limit of enough electron-electron scattering so that the drifted Maxwellian approximation is valid. Previous work with the drifted Maxwellian has assumed one electron temperature and drift velocity for the entire electron system.[3-5] However, when optical-phonon scattering is considered, the situation for the lowest level is quite different from that of the upper levels. For all levels the most important optical-phonon scattering process is the emission of a phonon and the consequent scattering of the electron to a lower energy state. In the lowest level this can happen only for electrons with energies greater than the optical-phonon energy $\hbar\omega_0$. In order for the rate of momentum loss by scattering to balance the rate of momentum gain from the field, a large number of electrons are necessary with kinetic energies above $\hbar\omega_0$. Therefore, a large drift velocity would be necessary for the lowest level. For the upper levels all electrons are able to emit phonons and scatter to a lower level if the magnetic field is large enough so that the spacing of the Landau levels, $\hbar\omega_c$, is greater than $\hbar\omega_0$. We might expect that the drift velocities of the upper levels need not be as large as that of the lowest level and still have sufficient scattering to achieve a momentum balance with the field. Because of this basic difference between the lowest level and the upper levels, we consider the drifted Maxwellian distribution with different drift velocities and electron temperatures for each Landau level.

To find out if different drift velocities and temperatures are indeed necessary, we solve the energy and momentum steady-state equations for each Landau level considered. We consider the case where electrons are scattered by both polar optical phonons and deformation-potential acoustic phonons. We also include the possibility of a distortion of the optical-phonon distribution. Ionized-impurity scattering and other types of phonon scattering are not included in this paper. The absence of ionized-impurity scattering limits the discussion to relatively pure materials at not too low a temperature.

Although we have used the physical parameters for InSb in our calculations we do not claim that the numerical results exactly describe the real material. By using different drift velocities and temperatures for each Landau level rather than a single drift velocity and temperature for all the levels, we hope to provide some insight into the electronic behavior in the intermediate quantum case, where several, but not many Landau levels are occupied.

II. THEORY

We consider an electron gas under steady-state conditions in a quantizing magnetic field H in the z direction and a parallel electric field E. The steady-state conditions for each Landau level are given by

$$\frac{d\langle p_z\rangle_n}{dt}=\frac{d\langle p_z\rangle_n}{dt}\bigg|_{\text{field}}+\frac{d\langle p_z\rangle_n}{dt}\bigg|_{\text{scattering}}=0 \tag{1}$$

and

$$\frac{d\langle\epsilon\rangle_n}{dt}=\frac{d\langle\epsilon\rangle_n}{dt}\bigg|_{\text{field}}+\frac{d\langle\epsilon\rangle_n}{dt}\bigg|_{\text{scattering}}=0\,, \tag{2}$$

where $n=0, 1, 2, \ldots$ indicates the Landau level.

When an electric field is applied, the electrons experience a force and the momentum of the electrons of each Landau level changes at a rate

$$\frac{d\langle p_z\rangle_n}{dt}\bigg|_{\text{field}}=-eEN_n\,, \tag{3}$$

where $-e$ is the electron charge, E is the electric field, and N is the number of electrons in the nth Landau level. If a Maxwellian-type distribution for the electrons is assumed, the distribution must be shifted away from the origin in order for there to be any momentum transfer due to scattering from the Landau systems. The amount of momentum transfer due to scattering is largely determined by the size of the shift, which is described by an electron drift velocity, $v_n = \langle p_z \rangle_n / m$.

If the electrons in a Landau level have an average nonzero drift velocity, energy is supplied to the electrons in that level by the electric field at a rate

$$\left.\frac{d\langle\epsilon\rangle_n}{dt}\right|_{\text{field}} = -eEN_n v_n \ . \tag{4}$$

We apply the drifted Maxwellian electron distribution approach to semiconductors in parallel or antiparallel magnetic and electric fields of arbitrary size. We assume that the carriers lie in parabolic energy levels, $\epsilon_n = (n+\frac{1}{2})\hbar\omega_c + \hbar^2 k_z^2/2m$, where ω_c is the cyclotron frequency, (eH/mc), and m is a scalar effective mass. Both acoustical and polar optical-phonon scattering are considered.

The electron-phonon interaction Hamiltonian is given by

$$H' = \sum_q (C_q a_q e^{i\vec{q}\cdot\vec{r}} + C_q^* a_q^\dagger e^{-i\vec{q}\cdot\vec{r}}) \ , \tag{5}$$

where a_q and $a_q^\dagger$ are phonon annihilation and creation operators, respectively. $\vec{q}$ is the phonon wave vector. For acoustic-phonon interaction via deformation potential coupling,

$$|C_q|^2_{ac} = E_1^2 \hbar q / 2\rho v_s V \ , \tag{6}$$

where E_1 is the deformation potential energy, ρ is the mass density of the crystal, v_s the constant velocity of sound, and V the volume of the crystal. The acoustical-phonon frequency is assumed to be

$$(\omega_q)_{\text{ac}} = v_s q \ .$$

For the polar optical-phonon interaction,

$$|C_q|^2_{\text{op}} = \frac{2\pi\hbar\omega_0 e^2}{V}\left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon_0}\right)\frac{1}{q^2} \ , \tag{7}$$

where ϵ_∞ and ϵ_0 are the dynamic and static dielectric constant of the crystal, respectively. The optical-phonon frequency is assumed to be independent of phonon wave vector

$$(\omega_q)_{\text{op}} \equiv \omega_0 = \text{const} \ .$$

We treat the electron-phonon collisions with acoustic phonons as elastic collisions since for the rather large magnetic fields considered here $\hbar v_s q \ll \epsilon_n$ even for most of the longer wave-vector phonons. Only those transitions involving the creation or annihilation of a single phonon are considered. Equilibrium acoustic-phonon distributions are used, but we do include the possibility of a distortion of the optical-phonon distribution.

We obtain two equations per level, one from the steady-state momentum condition

$$\frac{d\langle p_z\rangle_n}{dt} = -eEN_n + \sum_{k_z} \hbar k_z \frac{\partial\rho_n(k_z)}{\partial t} = 0 \ , \tag{8}$$

and one from the steady-state energy condition

$$\frac{d\langle\epsilon\rangle_n}{dt} = -eEN_n v_n + \sum_{k_z}\left((n+\tfrac{1}{2})\hbar\omega_c + \frac{\hbar^2 k_z^2}{2m}\right)\frac{\partial\rho_n(k_z)}{\partial t} = 0 \ . \tag{9}$$

The number of electrons in the nth Landau level is

$$N_n = N_T \exp\left(\frac{-(2n+1)\hbar\omega_c}{2kT_n}\right) \times\left\{\sum_{l=0}^{L}\left[\exp\left(\frac{-(2l+1)\hbar\omega_c}{2kT_l}\right)\right]\right\}^{-1} \ , \tag{10}$$

where N_T is the total number of electrons in the sample, and $L+1$ is the number of Landau levels considered. The rate of change of the distribution function for the nth level due to scattering is

$$\frac{\partial\rho_n(k_z)}{\partial t} = \sum_{n',k_z'}[W(n,k_z;n',k_z')\rho_{n'}(k_z') - W(n',k_z';n,k_z)\rho_n(k_z)] \ , \tag{11}$$

where

$$\rho_n(k_z) = N_n\left(\frac{2\pi}{L_z}\right)\left(\frac{\hbar^2}{8\pi mkT_n}\right)^{1/2} \times\exp\left(\frac{-(\hbar k_z - mv_n)^2}{2mkT_n}\right) \tag{12}$$

is normalized to the total number of electrons. $W(n',k_z';n,k_z)$ is the usual golden rule transition probability for scattering from Landau level n to n' and from electron wave vector k_z to k_z', and is given by

$$W(n',k_z';n,k_z) = \sum_i\sum_q \frac{(2\pi)^2}{\hbar L_z}|f_{n'n}|^2 |C_q|_i^2 \times[(N_q)_i\delta(k_z'-k_z-q_z)\delta(\epsilon_{n'}-\epsilon_n-\hbar(\omega_q)_i) + (N_q+1)_i\delta(k_z'-k_z+q_z)\delta(\epsilon_{n'}-\epsilon_n+\hbar(\omega_q)_i)]. \tag{13}$$

The sum over i is over the various scattering mechanisms, and for $n' \geq n$,

$$|f_{n'n}|^2 = \frac{n!}{n'!} x^{n'-n} e^{-x} [L_n^{n'-n}(x)]^2 , \tag{14}$$

where $x = (\hbar^2 q_\perp^2/2m)/\hbar\omega_c$ and $L_n^{n'-n}(x)$ is the generalized Laguerre polynomial. (For $n \geq n'$, n and n' are interchanged everywhere they appear.) The sum over phonon wave vector q is changed to an integral

$$\sum_q \rightarrow \frac{V}{(2\pi)^3} \int d\theta \int dq_z \int_{q_\perp} q_\perp dq_\perp .$$

The integral over θ is simply 2π, and the integral over q_z is easily done using the δ function, with the result that q_z is replaced by either $(k_z' - k_z)$ or $(k_z - k_z')$.

In the case of acoustic-phonon scattering we assume that $(\hbar\omega_q)_{\text{ac}} = \hbar v_s q \ll kT_L$ (T_L is the lattice temperature), and make the approximation

$$(N_q)_{\text{ac}} \approx \frac{kT_L}{\hbar v_s q} \approx (N_q + 1)_{\text{ac}} . \tag{15}$$

We then have

$$|C_q|^2_{\text{ac}} (N_q)_{\text{ac}} \approx |C_q|^2_{\text{ac}} (N_q+1)_{\text{ac}} \approx E_1^2 kT_L / 2\rho v_s^2 V , \tag{16}$$

which is independent of q. In the energy δ function we drop $(\hbar\omega_q)_{\text{ac}}$, using the assumption that the acoustic-phonon scattering is elastic. The acoustic-phonon part of the transition probability is therefore equal to

$$2(E_1^2 kT_L / 2\rho v_s^2 \hbar L_z) \delta(\epsilon_{n'} - \epsilon_n) G^a_{n'n} , \tag{17}$$

where

$$G^a_{n'n} = \int_0^\infty q_\perp |f_{n'n}|^2 dq_\perp = \tfrac{1}{2} \left(\frac{2m}{\hbar^2} \right) \hbar\omega_c . \tag{18}$$

The factor of 2 arises from the fact that the absorption and emission terms are the same.

For polar-optical phonons ω_0 is a constant so that $(N_q)_{\text{op}}$ and $(N_q+1)_{\text{op}}$ may be taken out of the integral over q, as may the delta function over energy and ω_0 in the $|C_q|^2$ term. For the polar optical-phonon terms we then get

$$\frac{2\pi e^2 \omega_0}{L_z} \left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon_0} \right) G^{\text{op}}_{n'n}(B) \times [(N_q)_{\text{op}} \delta(\epsilon_{n'} - \epsilon_n - \hbar\omega_0) + (N_q+1)_{\text{op}} \delta(\epsilon_{n'} - \epsilon_n + \hbar\omega_0)] , \tag{19}$$

where for $n' \geq n$,

$$G^{\text{op}}_{n'n}(B) = \int \frac{|f_{n'n}|^2 q_\perp dq_\perp}{q_\perp^2 + (k_z' - k_z)^2} = \tfrac{1}{2} n! n'! \times \sum_{l,k=0}^{n} \frac{(-1)^{(l+k)} I_{m'}(B)}{(n'-n+l)!(n'-n+k)!(n-l)!(n-k)!l!k!} , \tag{20}$$

where

$$m' = n' - n + l + k ,$$

$$B = \frac{\hbar^2 (k_z' - k_z)^2 / 2m}{\hbar\omega_c} = \frac{\hbar^2 q_z^2 / 2m}{\hbar\omega_c} ,$$

$$I_{m'+1} = m'! - B I_{m'} ,$$

$$I_0 = -e^B Ei(-B) = \int_0^\infty \frac{e^{-y}}{y+B} dy = e^B E_1(B) .$$

$E_1(B)$ is the exponential integral. For $n \geq n'$, n and n' are interchanged.

We now have

$$W(n', k_z'; n, k_z) = 2 \frac{E_1^2 kT_L}{2\rho v_s^2 \hbar L_z} \delta(\epsilon_{n'} - \epsilon_n) G^a_{n'n} + \frac{2\pi}{L_z} e^2 \omega_0 \left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon_0} \right) G^{\text{op}}_{n'n}(B) \times [(N_q)_0 \delta(\epsilon_{n'} - \epsilon_n - \hbar\omega_0) + (N_q+1)_0 \delta(\epsilon_{n'} - \epsilon_n + \hbar\omega_0)] . \tag{21}$$

When this expression is inserted into the momentum and energy steady-state equations we have sums over k_z and k_z' to evaluate. The sums are converted to integrals

$$\sum_{k_z, k_z'} \rightarrow \frac{L_z^2}{(2\pi)^2} \int dk_z dk_z' .$$

One of the integrations is performed by using the δ function over energy, and the other one is done numerically on the computer.

The distortion of the optical-phonon distribution is dealt with in the manner after Ferry.[9] The optical-phonon distribution is a Bose-Einstein distribution with a phonon quasitemperature T_p describing the number of phonons in the disturbed distribution:

$$\overline{N}_q = (e^{\hbar\omega_0 / kT_p} - 1)^{-1} . \tag{22}$$

The energy fed into the system by the field is transferred from the electrons to the lattice by the emission of optical phonons. The optical phonons that are emitted are either reabsorbed by the electrons or decay into acoustic modes with a characteristic relaxation time τ_p. The time rate of change of the optical-phonon density is given by

$$\frac{\partial \overline{N}_q}{\partial t} = \frac{\overline{N}_q - N_{q_0}}{\tau_p} - \left. \frac{\partial \overline{N}_q}{\partial t} \right|_{\text{scattering}} . \tag{23}$$

N_{q_0} is the equilibrium phonon density (at the lattice temperature) and $(\partial \overline{N}_q / \partial t)_{\text{scattering}}$ is the change in phonon density as a result of the electron-optical phonon scattering. τ_p can be written $C_p / \overline{N}_q$, where C_p is constant.[9] For a steady-state condition, $\partial \overline{N}_q / \partial t = 0$. The rate energy is supplied by the field is equal to the rate energy must be supplied by the electrons to the optical phonons so that

$$\left.\frac{d\langle\epsilon\rangle}{dt}\right|_{\text{field}} = -eE\sum_{n=0}^{m} v_n N_n = \sum_q \hbar\omega_0 \left.\frac{\partial \overline{N}_q}{\partial t}\right|_{\text{scattering}} = N\hbar\omega_0 \left.\frac{\partial \overline{N}_q}{\partial t}\right|_{\text{scattering}} . \tag{24}$$

The sum over q simply gives N, the number of atoms in the crystal, since ω_0 and $\overline{N}_q$ are taken to be independent of q for optical phonons. We have then the equation

$$\frac{\partial N_q}{\partial t} = 0 = \frac{\overline{N}_q - N_{q_0}}{C_p/\overline{N}_q} - \left(-eE\sum_{n=0}^{m} v_n N_n \Big/ N\hbar\omega_0\right), \tag{25}$$

which becomes

$$\overline{N}_q^2 - \overline{N}_q N_{q_0} - C_p(-eEN_T v_d/N\hbar\omega_0) = 0 \tag{26}$$

where $N_T v_d \equiv \sum_{n=0}^{m} N_n v_n$ and is a complex function of N_q. v_d might be called the average drift velocity of the electron system. The above equation must be added to the system of equations resulting from the steady-state momentum and energy equations. We have two variables per level, v_n and T_n, plus the phonon quasitemperature T_p. To obtain a solution we truncate the system to a finite number of levels, and solve the resulting system of equations numerically to obtain the phonon quasitemperature T_p, the electron temperatures T_n and the electron drift velocities v_n.

III. NUMERICAL RESULTS AND DISCUSSION

In this section we will present the results of numerical calculations. In the calculations we have used parameters for n-InSb, which are listed in Table I. The lattice temperature T_L is taken to be 77 K. We first show the dependence of $d\langle p_z\rangle/dt$ and $d\langle\epsilon\rangle/dt$ on the drift velocities and electron temperatures for the various scattering mechanisms, and on the field, for a one-level system. The case where there is only one level occupied is physically realistic in the case of a large magnetic field where $\hbar\omega_c$ is sufficiently larger than $\hbar\omega_0$. The results below are for $\hbar\omega_c = 2\hbar\omega_0$. We take the electric field to be in the $-z$ direction so that the distribution is drifted in the $+z$ direction.

Since the transition probability for scattering due to acoustical phonons is independent of q, and is essentially the same for both absorption and emission,

TABLE I. InSb parameters used for the calculation.

$m = 0.013m_e = 1.184\times10^{-29}$ g
$E_1 = 1.60\times10^{-11}$ erg $= 10$ eV
$\rho = 5.70$ g/cm^3
$v_s = 2.0\times10^5$ cm/sec
$\omega_0 = 3.457\times10^{13}$ sec^{-1}
$\epsilon_\infty = 15.68$
$\epsilon_0 = 17.88$
N_T/V = electron density $= 1.75\times10^{14}$ cm^{-3}
N/V = atomic density $= 1.41\times10^{22}$ cm^{-3}
$C_p = 5.36\times10^{-8}$ sec

$$\left.\frac{d\langle p_z\rangle_0}{dt}\right|_{\text{scattering}}^{\text{acoustic phonon}} \propto \int \frac{x}{|x|} e^{-(x-x_0)^2} dx = -2\int_0^{x_0} e^{-y^2} dy .$$

For small x_0 (drift velocity) the integral is approximately equal to x_0, and for $x_0 \to \infty$ the integral goes to a constant. Therefore, the momentum transfer rate increases linearly with x_0 for small x_0 and then saturates as x_0 becomes large, as shown in Fig. 1. For acoustic-phonon scattering

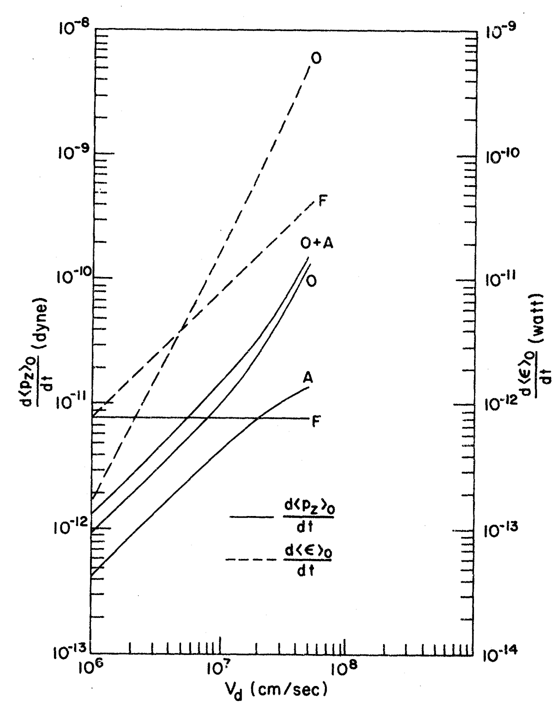

FIG. 1. $d\langle p_z\rangle_0/dt$ and $d\langle\epsilon\rangle_0/dt$ per electron as a function of drift velocity for a one-level case. The field terms (F) and the scattering terms (A, acoustic-phonon scattering; O, optical-phonon scattering) are of opposite sign. The electron temperature T_0 is taken equal to the lattice temperature, $T_L = 77$ K. The magnetic field is such that $\hbar\omega_c = 2\hbar\omega_0$.

within a level,

$$\frac{d\langle\epsilon\rangle_0}{dt}\bigg|_{\text{scattering}}^{\text{acoustic phonon}} = 0 ,$$

since we are considering the collisions to be elastic.

The $1/q^2$ optical-phonon interaction gives rise to the q_z dependence of Eq. (20). Because of this q_z dependence electrons will tend to scatter more readily to the same side of the energy level that they are initially on than they will to the opposite side, although both processes do occur. With the electron distribution drifted in the $+z$ direction, the scattering due to absorption of an optical phonon will result in a net gain of momentum and energy for the level. Emission of an optical phonon will result in a net loss of momentum and energy for the level. In order for the level to lose energy and momentum by scattering we see that the optical phonon emission terms must be larger than the absorption terms. The transition probability for emission is greater than that for absorption since for the temperature under consideration, $N_q + 1 \gg N_q$. However, the lower energy states are generally more likely to be occupied than the higher energy states so that the absorption and emission contributions are about the same size. In addition to the competition between emission and absorption, for small drift velocities there is considerable cancellation of net momentum transfer because the effect of processes on one side of k space are opposite to that of the other. An increase in the drift velocity will result in an increase in the ratio of momentum loss to momentum gain because the number of electrons with energies above the phonon energy increases, and because the increase is on one side of k space only, so that there is less cancellation. Since cancellation effects are not present for energy transfer, the scattering processes involving optical phonons are more efficient for energy transfer than for momentum transfer when the drift velocities are small.

Figure 1 shows that $(d\langle\epsilon\rangle_0/dt)_{\text{op}}$ increases approximately as the square of the drift velocity while $(d\langle p_z\rangle_0/dt)_{\text{op}}$ varies approximately linearly with drift velocity for $v_0 < 10^7$. For larger drift velocities, more transitions occur at smaller phonon wave vectors, and for smaller phonon wave vectors the scattering rate and the momentum transfer increase more rapidly.

For the smaller values of drift velocity an increase in the electron temperature will have only a small effect on the ratio of momentum loss to momentum gain due to cancellation from the two sides. If the drift velocity is quite large there is less cancellation and temperature can become more important. Because there are no cancellations from the two sides for energy scattering, the energy scattering rate is strongly dependent on the electron temperature. The drift velocity is therefore the important parameter for achieving momentum balance and the electron temperature is the important parameter for achieving energy balance.

With two or more levels the situation is complicated by the exchange of momentum and energy between different levels. The effect on the lower level(s) of adding one more level depends primarily on the spacing of the levels compared to the optical phonon energy and the number already included. In the example under consideration, that is, $\hbar\omega_c = 2\hbar\omega_0$, the effect is small since the occupation of the second level is very small. As $\hbar\omega_c$ gets smaller, adding an additional level will effect a larger and larger change in the parameters of the lower levels. We decide how many levels are needed at a given magnetic field, or for a range of magnetic fields, on the basis of how much the temperatures and drift velocities of the lower levels change when we add one more level. Table II shows what happens when we consider 1–5 levels at a magnetic field such that $\hbar\omega_c = 2\hbar\omega_0$.

The electric field dependence of the temperatures and drift velocities for a four-level case at the same magnetic field are shown in Figs. 2 and 3, respectively. Results for the one-level case are shown for comparison. As E approaches zero the electron temperatures approach the lattice temperature and the drift velocities approach zero as expected. The drift velocities increase with increasing field, and then level off at high fields, because at high fields the number of electrons with enough energy to emit an optical phonon increases quite rapidly, and because more of the phonons emitted have smaller wave vectors. As anticipated in the introduction, v_0 is much larger than the $v_{n\neq 0}$'s, and the velocities for the upper levels have similar values since they all have at least one level below them to which electrons can scatter.

For the lower fields the drift velocities necessary to achieve a momentum balance tend toward an excess of energy loss due to scattering in $n = 0$ because of the greater efficiency of the energy transfer process. Since the energy scattering terms are responsive to electron temperature changes while the momentum terms are not, the electron temperature T_0 decreases until both an energy balance and a momentum balance are achieved. For the larger electric fields the drift velocities are larger and the efficiency of the momentum transfer process increases. The larger velocities also increase the rate energy is gained from the field so that the rate of energy gain from the field tends to increase faster than the rate of

TABLE II. Effect of including 1–5 Landau levels at high magnetic fields; $\hbar\omega_c = 2\hbar\omega_0$, $E = 5.0$ V/cm, $T_L = 77$ K.

No. of levels	Electron temperature (K)					
	T_0	T_1	T_2	T_3	T_4	
1	76.52					
2	76.55	77.70				
3	76.55	77.70	78.10			
4	76.55	77.70	78.10	78.36		
5	76.56	77.70	78.10	78.36	78.56	
No. of levels	Electron drift velocities (10^6 cm/sec)					
	v_0	v_1	v_2	v_3	v_4	v_d
1	5.8911					5.8911
2	5.8015	5.0699				5.8007
3	5.8014	5.0429	4.9775			5.8005
4	5.8015	5.0429	4.9557	4.9479		5.8006
5	5.8013	5.0428	4.9556	4.9282	4.9122	5.8005

momentum loss due to scattering. As a consequence the electron temperature T_0 must rise to maintain the energy balance, and for large enough electric fields rises above the lattice temperature.

The steep increase in the temperatures of the upper levels is a consequence of electrons entering the upper levels from lower levels. These electrons bring with them a large amount of energy but most bring very little momentum. The electrons that are scattered by acoustic phonons have an equal probability of being scattered to the $+k_z$ or $-k_z$ side of the energy level so they contribute no

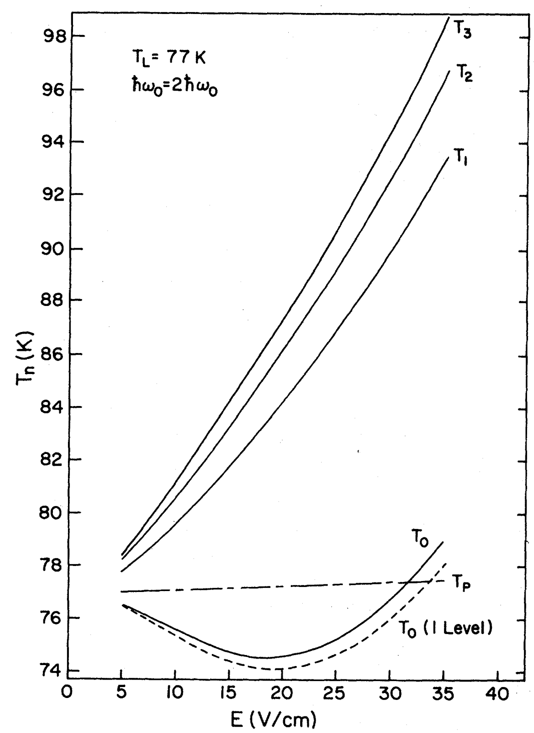

FIG. 2. Electron temperature vs electric field for a four-level case. T_0 for the one-level case is shown for comparison. The optical-phonon pseudotemperature T_p is also shown.

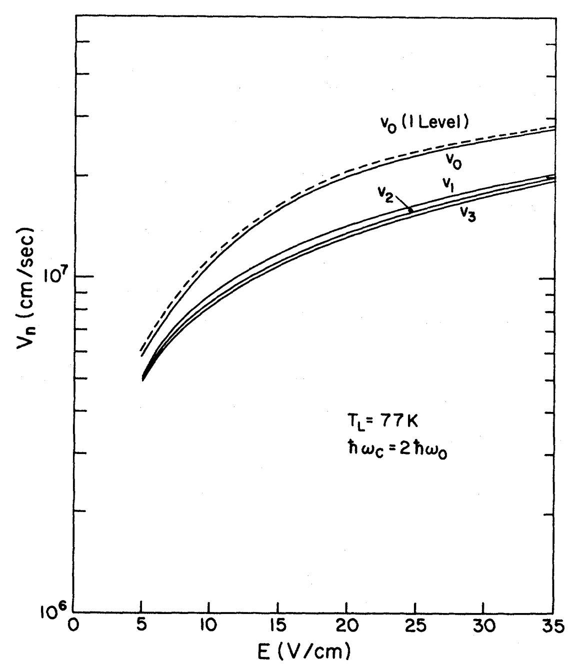

FIG. 3. Electron drift velocities vs electric field for a four-level case. v_0 for the one-level is shown for comparison.

net momentum gain to the level. Most of the electrons that are scattered by optical phonons will come in around $k_z = 0$ so again the net momentum gain is small. The drift velocity needed to achieve momentum balance is not enough to get the needed energy loss by scattering, so the electron temperature must rise rather sharply.

Figure 2 also shows the optical-phonon temperature T_p as a function of electric field. The approximate dependence of T_p on the electric field can be seen from Eq. (26), which has the "solution,"

$$\overline{N}_q = \frac{N_{q_0}}{2}\left[1 + \left(1 - \frac{4C_p eEN_T v_d}{N_{q_0} N\hbar\omega_0}\right)^{1/2}\right].$$

(v_d is a function of N_q, so the solution is not so straightforward.) The quantity

$$4C_p eEN_T v_d / N_{q_0} N\hbar\omega_0 \equiv 4KEv_d$$

is much less than 1 so that

$$\overline{N}_q \cong N_{q_0}(1 - KEv_d).$$

Expressing the inverse of this equation in terms of T_p we have

$$e^{\hbar\omega_0/kT_p} - 1 \approx (e^{\hbar\omega_0/kT_L} - 1)(1 + KEv_d).$$

Solving for T_p we obtain the approximate expression

$$T_p \approx T_L[1 - (kT_L/\hbar\omega_0)KEv_d].$$

If we remember that E and v_d have opposite signs, we see that T_p should increase with increasing E, directly from the E term, and also from the v_d term, which also increases with E.

The increase in T_p is small, indicating that the optical-phonon distribution is not too disturbed at the electric fields we are considering here. The calculations have been done using the equilibrium optical-phonon distribution, and the differences in the electron temperatures and drift velocities have been small.

The direct magnetic field dependence enters the steady-state equations as the cyclotron energy $\hbar\omega_c$. By replacing T_n with the lattice temperature T_L and limiting discussion to magnetic fields such that $\gamma_c \equiv \hbar\omega_c/kT_L$ is of the order of unity or larger, we may approximate Eq. (10) by

$$N_{n\neq 0} \approx N_T e^{-n\gamma_c}$$

and

$$N_0 \approx N_T(1 - e^{-\gamma_c}).$$

The fractional changes in these quantities are

$$\Delta N_{n\neq 0}/N_{n\neq 0} \approx -n\Delta\gamma_c$$

and

$$\Delta N_0/N_0 \approx e^{-\gamma_c}\Delta\gamma_c.$$

The large fractional change in $N_{n\neq 0}$ compared to the small fractional change in N_0 for a small change in γ_c is a dominant factor in determining the magnetic field dependence of the temperatures and drift velocities.

The transition probability for scattering by acoustic phonons is directly proportional to the magnetic field as seen from Eq. (18). The interesting behavior comes from the magnetic field dependence of the transition probability for scattering by optical phonons, and is more complicated. $G^{\text{op}}_{n,n'}(B)$ is a function of $B = (x - x')^2/\gamma_c$, where $x^2 = \hbar^2 k_z^2/2mkT_L$ and $x'^2 = \hbar^2 k_z'^2/2mkT_L$. B depends on the magnetic field through γ_c in the denominator, and also on the $(x - x')^2$ term, since the (x, x') pair is determined from the energy conservation conditions. The transition probability is therefore a complicated function of the magnetic field.

When one intergrates over either k_z or k_z' by using the energy δ function, the electron density of states for that level is introduced. By exercising care in choosing which integration (k_z or k_z') is to be performed by using the delta function, and which will be done numerically, it is possible to avoid any singularities arising from the electron density of states term except for the case of vertical transitions, that is, whenever $\gamma_0 = m\gamma_c$, where m is an integer. For the momentum equations no problem arises as the density-of-states term is exactly cancelled by the electron momentum. This is not the case for the energy equations, and there are real singularities in the energy scattering terms whenever $\gamma_0 = m\gamma_c$.

We first discuss the magnetic field dependence of the electron drift velocities temperatures for a two-level system since it is easier to understand and contains most of the physics of the problem. The average momentum gain from the field is proportional to the electron population of $n = 0$, which decreases very slowly with decreasing magnetic field. The largest contribution to the scattering momentum loss is optical phonon scattering within the $n = 0$ level. This scattering decreases much more rapidly than does the field term because in addition to the affect of the decreasing electron population there is the decrease in the transition probability as the argument B increases with the decrease of γ_c. Since the scattering momentum loss tends to decrease more rapidly than the field momentum gain decreases, the drift velocity v_0 must generally increase to maintain momentum balance as seen in Fig. 4(a).

The interesting inflection of the drift velocity just before the resonance point at $\alpha = 1$ reflects the contribution to the scattering momentum loss by optical-phonon transitions between the $n = 0$ and $n = 1$ levels. The scattering momentum loss tends

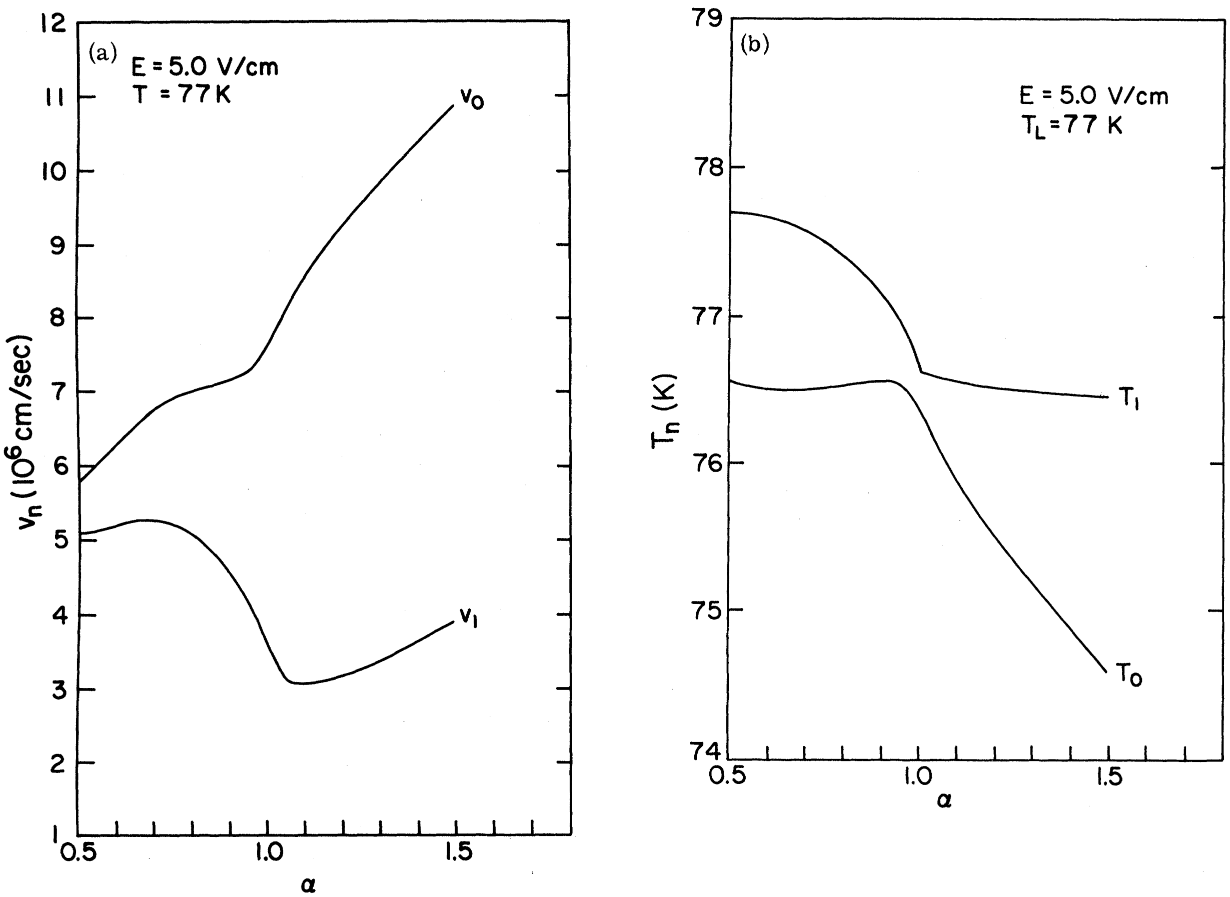

FIG. 4. Electron drift velocities (a) and temperatures (b) vs $\alpha \equiv \hbar\omega_0/\hbar\omega_c$ for the two-level case.

to increase initially as α increases due to the increasing optical phonon transition probability and to the fact that more electrons are able to absorb a phonon and scatter up from $n=0$ to $n=1$. Although the optical phonon transition probability has a maximum at $\alpha=1$, it also tends to decrease as $\hbar\omega_c$ gets smaller, which makes for a curve that rises more slowly to the maximum than it falls off after the maximum. This fact, plus the ever present decreasing population of $n=0$ and increasing population of $n=1$, turn the curve around before the resonance point is reached, which would result in a maximum in the scattering momentum loss before $\alpha=1$ if temperature and drift velocity where held constant. The fact that the momentum loss due to optical-phonon scattering between the two levels tends to rise before the resonance point means that such a steep rise in drift velocity is not needed and v_0 can flatten out somewhat. After resonance the decreasing transition probability and decreasing population of $n=0$ would lead to a rapid decrease of the scattering momentum loss so it is necessary for v_0 to increase more rapidly in order to maintain a momentum balance.

The decrease in v_1 to a minimum just beyond $\alpha=1$ as the magnetic field decreases reflects the effect of the optical-phonon transitions between the two levels on the $n=1$ level's momentum loss. In addition to the increasing momentum loss from $n=1$ due to the increase in the number of electrons in $n=1$ and the accompanying decrease in the number in $n=0$, there is also the increase in the optical-phonon transition probability as $\alpha=1$ is approached. The scattering loss therefore tends to increase more rapidly than the field gain and the drift velocity must decrease in order to maintain a steady state. After the resonance point the phonon interaction term decreases. So does the number of electrons eligible to emit a phonon and scatter down to $n=0$. There is still the increasing contribution from the number of electrons in the $n=1$ level, but the other two contributions largely succeed in overcoming that and the curve flattens out at some point beyond resonance. As the scattering tends to level off, the drift velocity must increase again to keep the scattering even with the increasing field term.

Figure 4(b) shows the magnetic field dependence of the electron temperatures for the two level system. T_0 shows a slight rise to a maximum just before $\alpha=1$, and then decreases rather sharply. The fact that $T_0<T_L$ reflects the fact that energy transfer is more efficient than momentum transfer. In the region before $\alpha=1$ where the drift velocity v_0 rose less sharply, the temperature was forced to rise just slightly so the rate of energy loss due to scattering was large enough to balance the rate energy was gained from the field. The sharper increase in v_0 beginning just before the resonance point that was necessary to obtain momentum balance also tended to increase the energy scattering term too much. In order to achieve energy balance it was necessary for the temperature to decrease.

The decrease in v_1 that is necessary to achieve momentum balance in level $n=1$ also reduces the energy lost by scattering, but not enough to achieve an energy balance with the field. The singularity in the density of states leads to a large contribution to the energy loss of level $n=1$ at resonance, and in order to maintain an energy balance with the field it is necessary for the temperature T_1 to decrease quite sharply as the resonance point is approached. Once the resonance point is reached the temperature continues to decrease, but only slightly. Since the density of states term is now decreasing it might be expected that T_1 should increase again. When more levels are included it does, but for the two level case the decrease in T_0 brings about a larger than usual increase in the population of $n=1$. This leads to the increase in scattering energy loss that is necessary to achieve an energy balance.

We have used the relatively simple two-level system so that it is easier to see what is happening physically. However, two levels are not sufficient to give an accurate description of the real system at the lower fields we have used in the two-level system ($\hbar\omega_0/\hbar\omega_c=\frac{3}{2}$). Table III shows what happens to the temperatures and velocities when two, three, four, and five levels are included in the calculations at slightly lower fields than used in the two-level examples.

Figure 5 shows the temperature results for four levels over a slightly larger range of magnetic fields than in the two level case. The basic features of the two level case are still present. T_0 does not fall off so rapidly after $\alpha=2$, and then decreases again. The behavior of T_0 around $\alpha=2$ is very much like its behavior around $\alpha=1$. The dip in T_0 at $\alpha=1$ is caused by the resonant ($q_z=0$) exchange of optical phonons between $n=0$ and $n=1$, whereas the dip in T_0 around $\alpha=2$ is caused by the resonance exchange of optical phonons between $n=0$ and $n=2$. As mentioned earlier, T_1 rises slightly after resonance instead of decreasing slightly. It then continues to rise to a maximum just before $\alpha=2$, and then decreases. Once $\hbar\omega_c<\hbar\omega_0$ the $n=1$ level will have no further resonant exchanges with $n=0$, so that at the second resonance point the behavior of T_1 is similar to the behavior of T_0 at the first (and second) resonance point. The dip in T_1 at $\alpha=2$ is caused by the resonant exchange between the $n=3$ and $n=1$ levels. The two additional temperatures, T_2 and T_3 behave much like T_1 before the $\alpha=1$ resonance, but rise sharply again after resonance. Near $\alpha=1$ the main exchanges are between adjacent levels, so that the $n>1$ levels are not influenced so strongly by the $n=0$ level. The occupation of both levels between which the dominant scattering takes place increases with decreasing magnetic field. There-

TABLE III. Effect of including 2–5 Landau levels at moderately high magnetic fields; $\hbar\omega_c=0.47\hbar\omega_0$, $E=5.0$ V/cm, $T_L=77$ K.

No. of levels	Electron temperatures (K) T_0	T_1	T_2	T_3	T_4	
2	72.78	76.26				
3	74.46	76.51	76.30			
4	74.96	76.76	76.45	77.32		
5	75.00	76.78	76.50	77.32	77.29	
No. of levels	**Electron drift velocities (10^6 cm/sec) v_0**	**v_1**	**v_2**	**v_3**	**v_4**	**v_d**
2	13.5927	6.0950				12.3196
3	11.5507	5.4785	3.7080			10.3118
4	10.8147	5.1610	3.5051	3.3560		9.6166
5	10.6824	5.0495	3.4377	3.2789	3.0743	9.4810

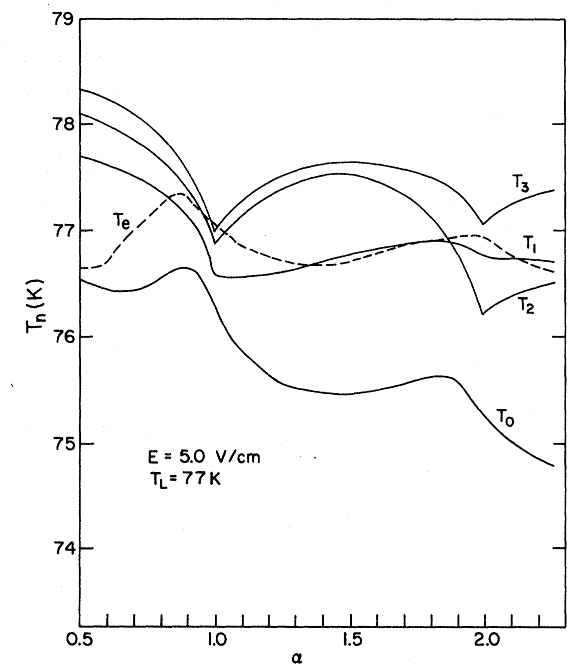

FIG. 5. Electron temperatures vs α for a four-level case. T_e is the electron temperature obtained by using the over-all steady-state approach.

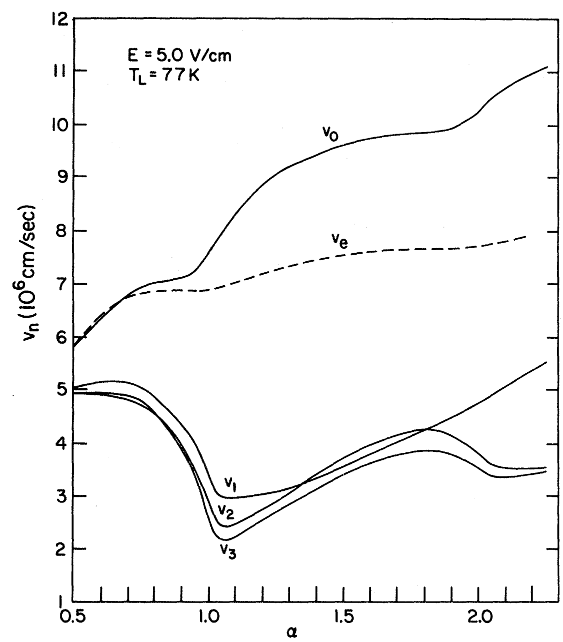

FIG. 6. Electron drift velocities vs α for a four-level case. v_e is the drift velocity obtained by using the over-all steady-state approach.

fore, the mechanism that allowed T_1 to level off after $\alpha = 1$ (that is, the rapid decrease in T_0) is not present here. T_2 and T_3 must therefore rise sharply after $\alpha = 1$. The upper temperatures also have a dip at the second resonance point. Although the $n = 2$ level now has the resonant exchange with the $n = 0$ level, the temperature in $n = 0$ does not decrease fast enough that T_2 does not need to rise again. Hence the increase in T_2 after the resonance point.

Figure 6 shows the four-level results for drift velocities. As in the case of the temperatures, the results are similar to the two-level case. v_0 continues to increase while undergoing changes in slope near the resonance points. The drift velocities of the three upper levels are grouped together and all have a minimum just after the resonance point at $\alpha = 1$. At the second resonance point v_1 leaves the pack and continues to rise while v_2 and v_3 decrease to a minimum a little past $\alpha = 2$. v_1 continues to rise because most electrons in the $n = 1$ level no longer have a large probability of emitting phonons and scattering to the lower $n = 0$ level. Levels $n = 2$ and $n = 3$ still have major exchanges with levels $n = 0$ and $n = 1$, respectively, so they retain the character they possessed at the first resonance point.

This pattern is expected to continue as higher-order resonance points are passed. Of course the lower the magnetic field used, the more levels that will have to be included to get meaningful results. The number of levels included is limited only by restrictions of computer time and memory. Since the basic features have been shown with only four levels, we do not include any more levels here. In an application of this electron distribution it may be necessary to use more levels in order to obtain results for lower fields.

The magnetic field dependence of T_p is small, arising from the magnetic field dependence of v_d. v_d increases roughly by a factor of 2 over the range of magnetic fields in the four level example. Therefore, the deviation of the phonon temperature from the lattice temperature increases by a similar factor of 2.

IV. TRANSPORT CALCULATIONS

In this section we use the distribution function discussed in Secs. II and III to calculate the electrical conductivity. We compare these results with the results obtained when a single drift velocity and temperature are used for all levels.

The conductivity σ is defined by the relation

$$j = \sigma E ,$$

where E is the applied electric field and j, the current density, is given by

$$j = -2e \sum_{n,k_z} \frac{\hbar k_z}{m} \frac{\rho_n(k_z)}{V} .$$

Changing the sum over k_z to an integral and substituting the expression for $\rho_n(k_z)$ [Eq. (12)] we integrate over k_z and obtain

$$j = -e \sum_n \frac{N_n v_n}{V} .$$

Earlier we defined v_d by

$$N_T v_d = \sum_n N_n v_n ,$$

so that we can write j as

$$j = -e n_0 v_d ,$$

where $n_0 = N_T/V$ is the electron density. For the conductivity we then obtain

$$\sigma = -e n_0 v_d / E .$$

This same result holds whenever only one drift velocity and temperature is assumed. The single drift velocity is then what appears in the equation for the conductivity.

Figure 7 compares the calculations of the conductivity as a function of $\alpha \equiv \hbar\omega_0/\hbar\omega_c$ for the cases of a single drift velocity and temperature and for n drift velocities and temperatures. The number of levels included in both calculations is four. As would be expected, at the higher magnetic fields the two calculations yield essentially the same result since only the $n = 0$ level is appreciably occupied. As the field decreases and the upper levels become more occupied the differences between the two methods become more pronounced. Not only is the conductivity greater when different drift velocities and temperatures are used, but the oscillations near the resonance points are more pronounced as well.

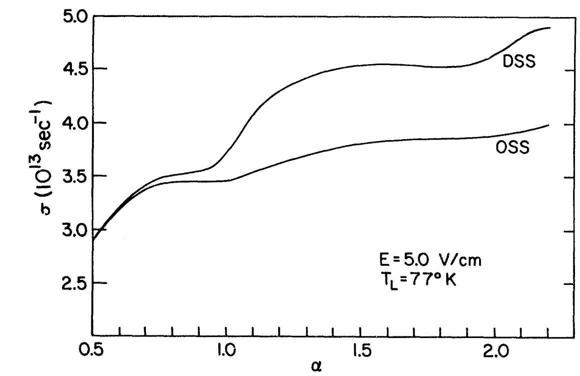

FIG. 7. Electrical conductivity vs α obtained from the detailed-steady-state approach (DSS), and from the overall-steady-state approach (OSS).

The differences arise because we have relaxed the restriction that all the physics must be expressed in two parameters. The single drift velocity and temperature must in some way reflect both the situation of the lower level and of the upper levels, and as a result the finer details of both are lost (see Figs. 4 and 5). With the additional parameters the differences between levels become apparent.

More important, the differences also show up in the conductivity calculation. It is mostly the $n = 0$ drift velocity that determines the average drift velocity since it is considerably larger than the drift velocities of the upper levels, and also because the population of the lower level is the largest. The average drift velocity does deviate from v_0 as the magnetic field decreases, but not to the extent that the single drift velocity does.

---

*Work supported by the Advanced Research Projects Agency (IDL Program DAHC-0213) and by the National Science Foundation (MRL Program GH33574).

†Present address: Dept. of Physics, Midland Lutheran College, Fremont, Neb. 68025.

[1]V. L. Gurevich and Yu. A. Firsov, Zh. Eksp. Teor. Fiz. 47, 734 (1964) [Sov. Phys.—JETP 20, 489 (1965)].

[2]G. I. Kharus and I. M. Tsidil'kovskii, Sov. Phys.-Semicond. 5, 534 (1971).

[3]R. L. Peterson, Phys. Rev. B 2, 4135 (1970).

[4]R. L. Peterson, B. Magnusson, and P. Weissglas, Phys. Status Solidi B 46, 729 (1971).

[5]R. L. Peterson, Phys. Rev. B 5, 3994 (1972).

[6]N. Kotera, E. Yamado, and K. F. Komatsubara, J. Phys. Chem. Solids 33, 1311 (1972).

[7]B. Magnusson, Phys. Status Solidi B 52, 361 (1972).

[8]B. Magnusson, Phys. Status Solidi B 56, 269 (1973).

[9]D. K. Ferry, Phys. Rev. B 8, 1544 (1973).