Boltzmann-equation approach to harmonic generation in a magnetic field

Gautam Johri* and Harold N. Spector
Department of Physics, Illinois Institute of Technology, Chicago, Illinois 60616
(Received 27 January 1975)

Second-harmonic generation is investigated in the presence of a dc magnetic field transverse to the direction of propagation of the acoustic wave. The classical Boltzamann-equation approach is used with a constant-relaxation-time *Ansatz.* The flux at the second-harmonic frequency is calculated in terms of the flux at the fundamental. It is found that at high fields our results reduce to those of Spector. At low fields, the conditions for observing the effect of cyclotron resonance on the flux at the second-harmonic frequency are investigated.

I. INTRODUCTION

The effect of a magnetic field transverse to the direction of propagation of an acoustic wave on the linear gain or loss due to acoustoelectric interaction has been investigated both theoretically[1] and experimentally.[2,3] Recently, there has also been a revival of interest in frequency-mixing effects, mainly because of the role they play in the growth of domains of acoustic flux under conditions of acoustic amplification.[4–9] Most of the work involving nonlinear acoustoelectric interactions has been done either using a phenomenological theory or in the absence of external magnetic fields. Use of a phenomenological theory limits the validity of the calculation to situations where the sound wavelength is much greater than the average distance the carrier travels between collisions. In a magnetic field, this limits the validity of such a theory to the high-field regime where $qR \ll 1$, where R is the cyclotron radius of the carriers. Therefore, in the low-field regime $qR > 1$, a phenomenological theory is not applicable. In this region, cyclotron resonance greatly alters the acoustic attenuation due to the linear acoustoelectric interaction.[10] It is therefore of great interest to investigate the effects of a magnetic field on second-harmonic generation due to nonlinear acoustic interaction using a theory that does not limit the validity of the calculation to strong magnetic fields and long wavelengths.

In Sec. II we present the theory of second-harmonic generation in a piezoelectric semiconductor due to the acoustoelectric interactions between the ultrasound and the conduction electrons, in the presence of a dc magnetic field applied transverse to the direction of propagation of the acoustic wave. This is done using the classical treatment for a free-electron gas, obeying nondegenerate statistics, in a partially ionized background supporting the sound wave. This model is a simple approximation to an n-type impurity semiconductor. The constitutive equation giving the response of the electron gas to the electric field, the collision drag effect, and the electron density gradient accompanying the sound wave, is developed using the Boltzmann equation. The second-harmonic flux is calculated using the constitutive equation together with Maxwell's equations and the equation of state for a piezoelectric semiconductor. This approach is valid for both regimes $qR \ll 1$ and $qR \gg 1$, where q is the acoustic wave vector and R is the radius of the cyclotron orbit for the electrons. Finally, we present the discussion of our results in Sec. III.

II. THEORY

Following the same approach as in our previous paper,[11] we can write the ac current density in the form

$$J_i = \sigma_{ij} E_j + \Lambda_{ijk} E_j E_k \,, \tag{2.1}$$

where σ and Λ are the linear and nonlinear conductivity tensors.

The equation of motion of an elastic continuum[12] is

$$\rho \frac{\partial^2 \xi_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j} \,. \tag{2.2}$$

Here, ρ is the density, ξ_i is the displacement, and the stress tensor T_{ij} is determined by the equations of state for a piezoelectric semiconductor

$$T_{ij} = C_{ijkl} S_{kl} - \beta_{ijk} E_k \,, \tag{2.3}$$

$$D_i = \epsilon E_i + 4\pi \beta_{ijk} S_{jk} \,, \tag{2.4}$$

where

$$S_{jk} = \frac{1}{2}\left(\frac{\partial \xi_j}{\partial x_k} + \frac{\partial \xi_k}{\partial x_j}\right) \tag{2.5}$$

is the strain tensor, C_{ijkl} are the elastic constants, β_{ijk} are the piezoelectric constants, E is the electric field, D is the electric displacement, and ϵ is the static dielectric constant.

Supplementing Eqs. (2.1)–(2.5) with Maxwell's equations, we can solve for the displacement vectors ξ_i for the fundamental and the second harmonic, following the approach of Refs. 6, 11, and

13. Using the results of the above references, we have for the ratio of the flux in the second harmonic to the square of the initial flux in the fundamental

$$\frac{P_2}{P_1^2(0)}=A\,[e^{-2\alpha_2 z}+e^{-4\alpha_1 z}-2e^{-(2\alpha_1+\alpha_2)z}\cos(2q_1-q_2)z]\ , \qquad (2.6)$$

where

$$A=\frac{8}{\rho v_s^3}\left(\frac{2\pi\beta}{\epsilon}\right)^2\times\left|\frac{\Lambda_{zzz}(\omega)}{[1-(4\pi/i\omega\epsilon)\sigma_{zz}(\omega)][2\sigma_{zz}(\omega)-\sigma_{zz}(2\omega)]}\right|^2 . \qquad (2.7)$$

Here we have taken the wave vector of the acoustic wave to lie along the z axis. Also, in (2.6) and (2.7), α_i are the absorption (amplification) coefficients for the acoustic wave of frequency ω_i using the linear theory, v_s is the velocity of sound in the semiconductor, and β is the appropriate component of the piezoelectric tensor.

To determine the linear and nonlinear conductivity tensors, we have to determine the ac current density by solving a transport equation for the electron distribution function. The electronic current density in a piezoelectric semiconductor in presence of an external magnetic field $\vec{B}_0$ is given by

$$j_i=-e\int d\vec{v}\,v_i f(\vec{v})\ , \qquad (2.8)$$

where $f(\vec{v})$ is the electron distribution function in the piezoelectric semiconductor in the presence of the magnetic field $\vec{B}_0$ and the acoustic wave. It is determined by the Boltzmann equation

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\frac{\partial f}{\partial\vec{r}}-\frac{e}{m}\left(\frac{\vec{v}}{c}\times\vec{B}_0\right)\cdot\frac{\partial f}{\partial\vec{v}}-\sum_i\frac{e}{m}\left(\vec{E}_i+\frac{\vec{v}}{c}\times\vec{B}_i\right)\cdot\frac{\partial f}{\partial\vec{v}}=-\frac{f-f_s}{\tau}\ , \qquad (2.9)$$

where $\vec{E}_i$ and $\vec{B}_i$ are the self-consistent electric and magnetic fields induced in the piezoelectric semiconductor by a sound wave of frequency ω_i, f_s is the distribution to which the electrons relax in the presence of the acoustic wave but in the absence of external fields, and τ is the relaxation time, which is treated as a constant here. The distribution $f_s(\vec{v})$ is given by[14]

$$f_s(\vec{v})=f_0\left(\vec{v}-\frac{d\vec{\xi}}{dt},\ n_0+n_1+n_2+\cdots\right)\simeq f_0(\vec{v})-\frac{d\vec{\xi}}{dt}\cdot\frac{df_0}{d\vec{v}}+n_1\frac{df_0}{dn_0}+n_2\frac{df_0}{dn_0}\ , \qquad (2.10)$$

where $f_0(\vec{v})$ is the equilibrium distribution of the electrons, $\vec{\xi}$ is the amplitude of the acoustic wave, and n_i is the carrier density induced by ultrasound of frequency ω_i. The second and third terms on the right-hand side of (2.10) arise from the collision-drag effect and from the fact that scattering is local[14] and therefore does not change the electron density. The second term can be neglected in semiconductors where the electron-phonon interactions are either via the deformation potential coupling or piezoelectric coupling.[12] We can also take the self-consistent electric field induced by the ultrasound to be longitudinal, since the transverse field induced is smaller by a factor of $(v_s/c)^2$,[8] so that $\vec{B}_i=(c\vec{q}_i/\omega)\times\vec{E}_i=0$. The solution to (2.9) can be written

$$f=f_{dc}(\vec{v})+\sum_i g_i(\vec{v})\exp i(\vec{q}_i\vec{r}-\omega_i t)\ , \qquad (2.11)$$

where the first term represents the electron distribution function in presence of the dc magnetic field. But since the magnetic field alone does not change the electron distribution, $f_{dc}(\vec{v})$ is just the equilibrium Boltzmann distribution function. We can use nondegenerate statistics for the electrons since, in most cases where acoustic amplification is observed, the electron density is low enough for the electrons to obey classical statistics. $g_i(\vec{v})$ is the part of the distribution function induced by the acoustic wave of frequency ω_i. From Eqs. (2.9)–(2.11) we obtain the equations determining $g_1(\vec{v})$ and $g_2(\vec{v})$:

$$\left(\tau^{-1}+i(q_z v_z-\omega)+\frac{d}{ds}\right)g_1(\vec{v})=\left(-\frac{e}{k_B T}E_{1z}v_z+\frac{n_1}{n_0\tau}\right)f_0(\vec{v}) \qquad (2.12)$$

and

$$\left(\tau^{-1}+2i(q_z v_z-\omega)+\frac{d}{ds}\right)g_2(\vec{v})=\frac{e}{m}E_{1z}\frac{\partial g_1}{\partial v_z}+\left(-\frac{e}{k_B T}E_{2z}v_z+\frac{n_2}{n_0\tau}\right)f_0(\vec{v})\ , \qquad (2.13)$$

where the variable s is defined by

$$-\frac{e}{mc}(\vec{v}\times\vec{B}_0)\frac{d}{d\vec{v}}=\frac{d}{ds}\ . \qquad (2.14)$$

The solution of Eqs. (2.12) and (2.13) can be written

$$g_1(\vec{v})=\int_{-\infty}^{s}ds'\left(-\frac{e}{k_B T}E_{1z}v_z'+\frac{n_1}{n_0\tau}\right)f_0\times\exp[\tau^{-1}(s'-s)+iq_z(z'-z)-\omega(s'-s)] \qquad (2.15)$$

and

$$g_2(\vec{v})=\int_{-\infty}^{s}ds'\left[\frac{e}{m}E_{1z}\frac{\partial g_1}{\partial v_z'}+\left(-\frac{e}{k_B T}E_{2z}v_z'+\frac{n_2}{n_0\tau}\right)f_0\right]\times\exp[\tau^{-1}(s'-s)+2iq_z(z'-z)-2i\omega(s'-s)]\ . \qquad (2.16)$$

To solve for $g_1(\vec{v})$ and $g_2(\vec{v})$, we choose a coordinate system having the y axis in the direction of the magnetic field $\vec{B}_0$, which is orthogonal to the acoustic wave vector lying along the z axis. In this coordinate system the relation between $(\vec{r}, \vec{v})$ and $(\vec{r}', \vec{v}')$ is

$$\begin{aligned} v_x' &= v_0 w \sin[\omega_0(s'-s)+\phi] , \\ v_y' &= v_0 u , \\ v_z' &= v_0 w \cos[\omega_0(s'-s)+\phi] , \\ x' &= x-(v_0 w/\omega_0)\{\cos[\omega_0(s'-s)+\phi]-\cos\phi\} , \\ y' &= y+v_0 u(s'-s) , \\ z' &= z+(v_0 w/\omega_0)\{\sin[\omega_0(s'-s)+\phi]-\sin\phi\} , \end{aligned} \tag{2.17}$$

where ω_0 is the cyclotron frequency, ϕ is the polar angle, and w and u are the velocities in units of v_0, the mean thermal velocity, in the plane perpendicular to $\vec{B}_0$ and parallel to $\vec{B}_0$, respectively.

Substituting from (2.17) to (2.15) and using the familiar relation[15]

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)\, e^{in\phi} , \tag{2.18}$$

where $J_n(z)$ is the Bessel function of order n, we get

$$g_1(\vec{v}) = \sum_{n=-\infty}^{\infty} f_0(e^{-ixw\sin\phi}) \times\left(-\frac{e\tau}{k_B T}\frac{E_{1z}v_0 n}{x}+\frac{n_1}{n_0}\right)\left(\frac{J_n(xw)\,e^{in\phi}}{P_n(\omega)}\right) , \tag{2.19}$$

where $x = qv_0/\omega_0$ and $P_n(\omega) = [1+i(n\omega_0-\omega)\tau]$.

Using Eqs. (2.17)–(2.19) in (2.16), we obtain

$$g_2(\vec{v}) = g_1(2\omega, 2q, E_{2z}, n_2) + g_2'(\vec{v}) , \tag{2.20}$$

where

$$g_2'(\vec{v}) = \frac{e}{m}E_{1z}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_0(e^{-2ixw\sin\phi}) \times\left(-\frac{e\tau}{k_B T}\frac{E_{1z}v_0 n}{x}+\frac{n_1}{n_0}\right)\left(\frac{e^{i(m+n)\phi}}{P_n(\omega)P_{m+n}(2\omega)}\right) \times\left(-\frac{2m}{v_0 x}J_m(xw)J_n(xw)+\frac{m}{v_0 w}J_m(xw)J_n'(xw) -\frac{n}{v_0 w}J_m'(xw)J_n(xw)\right) \tag{2.21}$$

and $J_n'(z) = d/dz\, J_n(z)$.

Substituting from (2.19) and (2.21) in (2.8), we can write the ac current densities induced by the fundamental and the second harmonic as

$$j_{1z} = \sigma_{zz}(\omega)E_{1z} - n_1 e v_s R_z(\omega) , \tag{2.22}$$

$$j_{2z} = \sigma_{zz}(2\omega)E_{2z} - n_2 e v_s R_z(2\omega) + \tau_{zzz}(\omega)E_{1z}^2 - n_1 e v_s S_{zz}(\omega)E_{1z} , \tag{2.23}$$

where

$$\sigma_{zz}(\omega) = \frac{4\sigma_0}{x^2}\sum_{n=-\infty}^{\infty}\frac{n^2}{P_n(\omega)}\int_0^\infty dw\, we^{-w^2}[J_n(xw)]^2 , \tag{2.24}$$

$$R_z(\omega) = \frac{2v_0}{xv_s}\sum_{n=-\infty}^{\infty}\frac{n}{P_n(\omega)}\int_0^\infty dw\, we^{-w^2}[J_n(xw)]^2 , \tag{2.25}$$

$$\tau_{zzz}(\omega) = \frac{2\sigma_0\mu}{x}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{n}{P_n(\omega)P_{m+n}(2\omega)}\left(-\frac{2m(m+n)}{v_0 x^2}\int_0^\infty dw\, we^{-w^2}J_m(xw)J_n(xw)J_{m+n}(2xw) + \frac{m(m+n)}{v_0 x}\int_0^\infty dw\, e^{-w^2}J_m(xw)J_n'(xw)J_{m+n}(2xw) - \frac{n(m+n)}{v_0 x}\int_0^\infty dw\, e^{-w^2}J_m'(xw)J_n(xw)J_{m+n}(2xw)\right) , \tag{2.26}$$

and

$$s_{zz}(\omega) = \frac{\mu v_0}{v_s}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{P_n(\omega)P_{m+n}(2\omega)}\left(-\frac{2m(m+n)}{v_0 x^2}\int_0^\infty dw\, we^{-w^2}J_m(xw)J_n(xw)J_{m+n}(2xw) + \frac{m(m+n)}{v_0 x}\int_0^\infty dw\, e^{-w^2}J_m(xw)J_n'(xw)J_{m+n}(2xw) - \frac{n(m+n)}{v_0 x}\int_0^\infty dw\, e^{-w^2}J_m'(xw)J_n(xw)J_{m+n}(2xw)\right) , \tag{2.27}$$

where σ_0 is the dc conductivity, and μ is the mobility of the carriers.

Using the results of the Appendix, we can write (2.24)–(2.27) as

$$\sigma_{zz}(\omega) = \frac{2\sigma_0(1-i\omega\tau)}{(ql)^2}\left(1-(1-i\omega\tau)\sum_{n=-\infty}^{\infty}\frac{e^{-x^2/2}I_n(\tfrac{1}{2}x^2)}{P_n(\omega)}\right) , \tag{2.28}$$

$$R_z(\omega) = -\frac{iv_0}{v_s ql}\left(1-(1-i\omega\tau)\sum_{n=-\infty}^{\infty}\frac{e^{-x^2/2}I_n(\tfrac{1}{2}x^2)}{P_n(\omega)}\right) , \tag{2.29}$$

$$\tau_{zzz}(\omega) = \frac{2\sigma_0\mu}{v_s(ql)^2}\left[-\frac{2}{iql}\left(\frac{3-4i\omega\tau}{4}-\frac{1-i\omega\tau}{2}(2-3i\omega\tau)\sum_{n=-\infty}^{\infty}\frac{e^{-x^2/2}I_n(\tfrac{1}{2}x^2)}{P_n(\omega)}+\tfrac{1}{2}(1-2i\omega\tau)\sum_{n=-\infty}^{\infty}\frac{e^{-x^2}I_n(x^2)}{P_n(2\omega)}\right.\right.$$

$$-\frac{(1-i\omega\tau)(1-2i\omega\tau)i\omega\tau}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\frac{e^{-3x^2/2}}{P_{n+m}(2\omega)P_{m+p}(\omega)}I_n(x^2)I_p(-x^2)I_p(-\tfrac{1}{2}x^2)+\frac{(1-i\omega\tau)^2}{iql}$$

$$\times\left(\frac{1}{1-i\omega\tau}-\sum_{n=-\infty}^{\infty}\frac{e^{-x^2/2}I_n(\frac{1}{2}x^2)}{P_n(\omega)}\right)+(1-i\omega\tau)(1-2i\omega\tau)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\frac{(p/x)\,e^{-3x^2/2}}{P_{m+n}(2\omega)P_{m+p}(\omega)}I_m(x^2)I_n(x^2)I_p(-\tfrac{1}{2}x^2)\Bigg], \tag{2.30}$$

$$s_{zz}(\omega)=\frac{\mu}{v_s(iql)}\Bigg[-\frac{2}{iql}\Bigg(\frac{1}{2}-\frac{2-3i\omega\tau}{2}\sum_{n=-\infty}^{\infty}\frac{e^{-x^2/2}I_n(\frac{1}{2}x^2)}{P_n(\omega)}+\frac{1-2i\omega\tau}{2}\sum_{n=-\infty}^{\infty}\frac{e^{-x^2}I_n(x^2)}{P_n(2\omega)}-\frac{(1-2i\omega\tau)\,i\omega\tau}{2}$$

$$\times\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\frac{e^{-3x^2/2}}{P_{m+n}(2\omega)P_{m+p}(\omega)}I_m(x^2)I_n(x^2)I_p(-\tfrac{1}{2}x^2)\Bigg)+\frac{1-i\omega\tau}{iql}\left(\frac{1}{1-i\omega\tau}-\sum_{n=-\infty}^{\infty}\frac{e^{-x^2/2}I_n(\frac{1}{2}x^2)}{P_n(\omega)}\right)$$

$$+(1-2i\omega\tau)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\frac{(p/x)\,e^{-3x^2/2}}{P_{m+n}(2\omega)P_{m+p}(\omega)}I_m(x^2)I_n(x^2)I_p(-\tfrac{1}{2}x^2)\Bigg]\;. \tag{2.31}$$

We can rewrite (2.22) and (2.23) in the form (2.1):

$$j_{1z}=\sigma'_{zz}(\omega)E_{1z}\;, \tag{2.32}$$

$$j_{2z}=\sigma'_{zz}(2\omega)E_{2z}+\Lambda_{zzz}(\omega)E_{1z}^2\;, \tag{2.33}$$

where

$$\sigma'_{zz}(\omega)=\frac{\sigma_{zz}(\omega)}{1-R_z(\omega)}\;, \tag{2.34}$$

$$\Lambda_{zzz}(\omega)=\frac{\tau_{zzz}(\omega)}{1-R_z(2\omega)}+\frac{S_{zz}(\omega)\sigma'_{zz}(\omega)}{1-R_z(2\omega)}\quad. \tag{2.35}$$

A. High-field limit

In the high-magnetic-field limit, $\omega_0 \gg qv_0$, the conductivity tensors (2.28)–(2.31) can be evaluated using the small-argument limit of the Bessel functions. Using the results of the Appendix, we obtain for the conductivity tensors

$$\sigma'_{zz}(\omega)=\frac{\sigma_0}{b+i\omega/\omega_D}\;, \tag{2.36}$$

$$\Lambda_{zzz}(\omega)=-\frac{\mu\sigma_0}{v_s}\frac{1}{(b+i\omega/\omega_D)(b+2i\omega/\omega_D)}, \tag{2.37}$$

where $\omega_D=v_s^2/D$ is the diffusion frequency, D is the diffusion coefficient and $b=1+(\omega_0\tau)^2$ is a parameter which measures the effectiveness of the dc magnetic field in reducing the mobility of the carriers.

Substituting for the conductivity tensors from (2.36) and (2.37) into (2.7), we obtain the ratio of the flux generated in the second harmonic to the square of the flux in the fundamental

$$A=\frac{2(4\pi\beta\mu/\epsilon v_s)^2[b^2+(\omega/\omega_D)^2]}{\rho v_s^3[b^2+(\omega_c/\omega)^2(1+\omega^2/\omega_c\omega_D)^2][b^2+9(\omega/\omega_D)^2]}\;, \tag{2.38}$$

where ω_c is the dielectric relaxation frequency.

This result is the same as the one obtained by Spector using a phenomenological theory in Ref. 13. The effect of the magnetic field on the second harmonic flux generated is discussed in that reference. The magnetic field changes the magnitude of the intensity and downshifts the frequency at which the second-harmonic generation is a maximum. Also, the second-harmonic generation becomes independent of the magnetic field for frequencies much higher than the frequency of maximum gain, $\omega_m=(\omega_c\omega_D)^{1/2}$.

B. Cyclotron resonance and the low-field limit

We expect the cyclotron resonance effects to occur when the sound-wave frequency is of the order of the cyclotron frequency, i.e., $\omega\simeq\omega_0$. In this case, the frequency denominators in the conductivity tensors (2.28)–(2.31) can become small, giving rise to the possibility of oscillatory behavior. Under this condition x will become much greater than unity, since $x=qv_0/\omega_0=\omega/\omega_0\times v_0/v_s$. Using the results of the Appendix, we get, for the conductivity tensors in the limit of low fields and short acoustic wavelengths $(ql\gg 1)$,

$$\sigma_{zz}(\omega)=\frac{2\sigma_0(1-i\omega\tau)}{(ql)^2}\left[1-\frac{\pi^{1/2}(1-i\omega\tau)}{ql}\coth\!\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\right], \tag{2.39}$$

$$R_z(\omega)=\frac{v_0}{v_s iql}\left[1-\frac{\pi^{1/2}}{ql}(1-i\omega\tau)\coth\!\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\right], \tag{2.40}$$

$$\tau_{zzz}(\omega)=\frac{i\sigma_0\mu}{v_0(ql)^3}(1-2i\omega\tau)\Bigg[1-\frac{2\pi^{1/2}}{ql}(1-i\omega\tau)\times\coth\!\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi+\frac{\pi^{1/2}}{2ql}\coth\!\left(\frac{1-2i\omega\tau}{\omega_0\tau}\right)\pi-\frac{\pi(1-i\omega\tau)i\omega\tau}{(ql)^2}\coth\!\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\coth\!\left(\frac{1-2i\omega\tau}{\omega_0\tau}\right)\pi\Bigg], \tag{2.41}$$

$$S_{zz}(\omega)=\frac{\mu}{v_s(ql)^2}(1-2i\omega\tau)$$

$$\times\left[-\frac{\pi^{1/2}}{ql}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi+\frac{\pi^{1/2}}{2ql}\coth\left(\frac{1-2i\omega\tau}{\omega_0\tau}\right)\pi\right.$$
$$\left.-\frac{i\omega\tau\pi^{1/2}}{2(ql)^2}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\coth\left(\frac{1-2i\omega\tau}{\omega_0\tau}\right)\pi\right]. \tag{2.42}$$

As the magnetic field approaches zero, $\coth[(1-in\omega\tau)/\omega_0\tau]\pi$ approaches unity, and taking this fact into account we obtain, for $\omega\tau>1$,

$$\sigma'_{zz}(\omega)=-\frac{i\omega\epsilon}{4\pi}\left(\frac{q_d}{q}\right)^2\left[1+i\pi^{1/2}\frac{v_s}{v_0}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\right] \tag{2.43}$$

$$\Lambda_{zzz}(\omega)=\frac{\epsilon}{4\pi}\frac{ev_s}{mv_0^2}\left(\frac{q_d}{q}\right)^2\left[1+2i\pi^{1/2}\frac{v_s}{v_0}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\right.$$
$$\left.-\pi\left(\frac{v_s}{v_0}\right)^2\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\coth\left(\frac{1-2i\omega\tau}{\omega_0\tau}\right)\pi\right], \tag{2.44}$$

where ϵ is the static dielectric constant of the piezoelectric semiconductor, and $q_d=(4\pi n_0e^2/\epsilon k_BT)^{1/2}$ is the electron Debye wave vector. Using (2.43) and (2.44) in (2.7), we get the flux in the second harmonic

$$A=A_0\left|\left[1+2i\pi^{1/2}\frac{v_s}{v_0}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi-\pi\left(\frac{v_s}{v_0}\right)^2\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\coth\left(\frac{1-2i\omega\tau}{\omega_0\tau}\right)\pi\right]\right.$$
$$\times\left\{\left[1+\left(\frac{q^2}{q^2+q_d^2}\right)i\pi^{1/2}\frac{v_s}{v_0}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\right]\right.$$
$$\times\left[1+\tfrac{4}{3}i\pi^{1/2}\frac{v_s}{v_0}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\right.$$
$$\left.\left.\left.-\tfrac{1}{3}i\pi^{1/2}\frac{v_s}{v_0}\coth\left(\frac{1-2i\omega\tau}{\omega_0\tau}\right)\pi\right]\right\}^{-1}\right|^2, \tag{2.45}$$

where

$$A_0=\frac{2}{9\rho v_s^3}\left(\frac{8\pi\beta}{\epsilon}\right)^2\frac{e^2}{(mv_0^2)^2}\frac{q^2}{(q^2+q_d^2)^2}. \tag{2.46}$$

The cyclotron resonance in the flux of the second harmonic will occur for $2\omega=n\omega_0$. When $\omega_0\tau>1$, we will see the cyclotron resonance, while when $\omega_0\tau<1$, the cyclotron resonance will be damped out. Therefore, the best candidate for observing cyclotron resonance would be semiconductors with relatively long relaxation times. For semiconductors like n-InSb and GaAs, with $\tau\simeq10^{-12}$ sec, the condition $\omega_0\tau>1$ pushes the frequency of the acoustic wave above the region of interest. However, in germanium at 10 K we have a τ of 10^{-10} sec from the cyclotron-resonance experiments[16] and an n_0 of 10^{13} cm^{-3} is obtainable,[17] and the velocity of sound is $v_s=10^5$ cm/sec. From (2.45) we see that for the cyclotron-resonance effect to be appreciable, $\pi^{1/2}v_s/v_0\coth[(1-i\omega\tau)/\omega_0\tau]\pi$ must be of the order of unity. This implies a cyclotron-resonance frequency of 10^{11} Hz for Ge at 10 K. Going down in temperature lowers the ratio of v_s/v_0 and hence the cyclotron frequency needed to make the resonant term observable. However, in Ge the acoustoelectric interaction is via the deformation-potential coupling and our theory has to be modified accordingly. This would change the frequency dependence of A_0 given by the Eq. (2.46), and the maxima would no longer occur at $q=q_d$.[8]

The conductivity tensors are also evaluated in the Appendix for the case when the magnetic field goes to zero. In this limit our results agree with those of our previous calculation[11] for both the short, and the long-wavelength regimes. They are also in agreement with the results of Conwell and Ganguly[6] for $ql\ll1$, and of Wu and Spector[8] for $ql\gg1$.

III. DISCUSSION

In this paper we have presented the calculation of second-harmonic generation in a piezoelectric semiconductor in presence of a dc magnetic field transverse to the direction of propagation of the acoustic wave. This is done following the Boltzmann-equation approach with a constant relaxation time. This approach excludes the region of very high magnetic fields, where quantum effects may become important. For strong magnetic fields ($\omega_0\gg qv_0$), our results are in agreement with those of Spector,[13] whose calculations were done using the phenomenological approach.

In the low-magnetic-field region ($\omega_0\ll qv_0$), we expect to see cyclotron resonance in semiconductors with relatively long relaxation times ($\tau\simeq10^{-10}$ sec). The cyclotron resonance is damped out when the condition $\omega_0\tau>1$ is not met. For piezoelectric semiconductors like n-InSb and GaAs, with $\tau\simeq10^{-12}$ sec, the condition $\omega_0\tau>1$ pushes the frequency of the acoustic wave above the region of interest. From cyclotron resonance[16] and mobility[17] data at low temperatures, we find that values of τ as long as 10^{-10} sec are now available for very pure Ge samples. However, in Ge the acoustoelectric interaction is via the deformation-potential coupling and our theory has to be modified accordingly.

This modification will change the frequency dependence of A_0 in (2.45) but will leave the oscillating part of (2.45) unchanged. For the cyclotron-resonance effect in the flux of the second harmonic to be appreciable, the condition $\pi^{1/2}v_s/v_0\coth[(1-i\omega\tau)/\omega_0\tau]\pi\simeq1$ must be met. For the Ge sample at 10 K, this implies a cyclotron-resonance frequency of 10^{11} Hz. Going down in temperature lowers the ratio v_s/v_0 and hence lowers the cyclotron-resonance frequency needed to observe the cyclotron resonance. Even in Ge, the effect of the cyclotron resonances on the harmonic generation would be much less striking than the effect on the linear gain or loss. This is because the resonant term here will not be much greater than the nonresonant term due to the factor of v_s/v_0.

As the magnetic field goes to zero, our results reduce to those of our earlier calculation[11] and of Conwell and Ganguly[6] in the limit $ql \ll 1$, and of Wu and Spector[8] in the limit $ql \gg 1$.

The calculations presented in this paper have been done in the absence of a dc electric field. The presence of a dc electric field can drastically alter the absorption coefficient, changing a linear loss to a linear gain. However, the coefficient A in Eq. (2.45) is only a very weak function of such a drift field except at very low frequencies [compare Eq. (3.35) of Ref. 11 to Eq. (26) of Ref. 13]. Also, extending the calculations of Ref. 8 to the case where a drift field is present (unpublished), we find that the effect of the drift field on the second-harmonic amplitude is small in the short-wave-length limit ($ql \gg 1$) as long as the drift velocity is smaller than the thermal velocity of the carriers. Therefore, although the presence of a dc drift field will change the rate at which the second harmonic will grow or decay with position, it will not greatly alter our calculation of the amplitude A.

APPENDIX

The conductivity tensors (2.24)–(2.27) can be rewritten using the relations

$$\sum_{n=-\infty}^{\infty} J_n(v) J_{m\pm n}(u) = J_m(u \pm v) \tag{A1}$$

together with the recursion relations for Bessel functions[15]

$$\sigma_{zz}(\omega) = \frac{4\sigma_0(1-i\omega\tau)}{(ql)^2}\left[\tfrac{1}{2} - (1-i\omega\tau)A(x,\omega)\right], \tag{A2}$$

$$R_z(\omega) = -(2iv_0/v_s ql)\left[\tfrac{1}{2} - (1-i\omega\tau)A(x,\omega)\right], \tag{A3}$$

$$\begin{aligned}\tau_{zzz}(\omega) = {}&[2\sigma_0\mu/v_0(ql)^2]\{-(2/iql)[\tfrac{1}{2}(1-i\omega\tau) - (1-i\omega\tau)^2 A(x,\omega) + \tfrac{1}{4}(1-2i\omega\tau) - \tfrac{1}{2}(1-2i\omega\tau)^2 A(2x,2\omega)\\ &-(1-i\omega\tau)(1-2i\omega\tau)A(x,\omega) + (1-i\omega\tau)(1-2i\omega\tau)A(2x,2\omega) - (1-i\omega\tau)(1-2i\omega\tau)i\omega\tau C(x,\omega)]\\ &-2(1-i\omega\tau)^2 B(x,\omega) - (1-i\omega\tau)(1-2i\omega\tau)qlD(x,\omega)\},\end{aligned} \tag{A4}$$

$$\begin{aligned}S_{zz}(\omega) = {}&[\mu/v_s(iql)]\{-(2/iql)[\tfrac{1}{2} - (1-i\omega\tau)A(x,\omega)\\ &-(1-2i\omega\tau)A(x,\omega) + (1-2i\omega\tau)A(2x,2\omega)\\ &-(1-2i\omega\tau)i\omega\tau C(x,\omega)] - 2(1-i\omega\tau)B(x,\omega)\\ &-(1-2i\omega\tau)qlD(x,\omega)\},\end{aligned} \tag{A5}$$

where

$$A(x,\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{P_n(\omega)} \int_0^\infty dw\, w e^{-w^2}[J_n(xw)]^2, \tag{A6}$$

$$B(x,\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{i\omega_0\tau}\frac{1}{P_n(\omega)} \int_0^\infty dw\, e^{-w^2} J_n(xw) J_n'(xw), \tag{A7}$$

$$\begin{aligned}C(x,\omega) = {}&\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{P_n(\omega)P_{m+n}(2\omega)}\\ &\times \int_0^\infty dw\, w e^{-w^2}[J_m(xw)J_n(xw)J_{m+n}(2xw)],\end{aligned} \tag{A8}$$

$$\begin{aligned}D(x,\omega) = {}&\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{x\omega_0\tau}\frac{1}{P_n(\omega)P_{m+n}(2\omega)}\\ &\times \int_0^\infty dw\, e^{-w^2} J_{m+n}(2xw)\,[\,mJ_m(xw)J_n'(xw)\\ &- nJ_m'(xw)J_n(xw)]\end{aligned} \tag{A9}$$

Equations (A6)–(A9) can be evaluated by expressing $[P_n(\omega)]^{-1}$ as

$$\frac{1}{P_n(\omega)} = \int_0^\infty ds\, e^{-(1-i\omega\tau+in\omega_0\tau)s} \tag{A10}$$

and using Graf's summation theorem for Bessel functions[18]

$$J_\nu(w)\, e^{i\nu\chi} = \sum_{k=-\infty}^{\infty} J_{\nu+k}(u) J_k(v)\, e^{ik\alpha}, \tag{A11}$$

with

$$w^2 = u^2 + v^2 - 2uv\cos\alpha, \tag{A12a}$$

$$w\cos\chi = u - v\cos\alpha, \tag{A12b}$$

$$w\sin\chi = v\sin\alpha. \tag{A12c}$$

Substituting (A10)–(A12) into (A6)–(A9), we get

$$A(x,\omega) = \frac{1}{2}\int_0^\infty ds\, e^{-(1-i\omega\tau)s - (x^2/2)(1-\cos\omega_0\tau s)}, \tag{A13}$$

$$\begin{aligned}B(x,\omega) = {}&-\frac{1}{2iql}\int_0^\infty ds\, e^{-(1-i\omega\tau)s}\\ &\times\left(1 - e^{-(x^2/2)(1-\cos\omega_0\tau s)}\right),\end{aligned} \tag{A14}$$

$$\begin{aligned}C(x,\omega) = {}&\frac{1}{2}\int_0^\infty ds\, e^{-(1-2i\omega\tau)s}\int_0^\infty ds'\, e^{-(1-i\omega\tau)s'}\\ &\times\left(\exp -\frac{x^2}{2}\left[3 - 2\cos\omega_0\tau s\right.\right.\\ &\left.\left. - 2\cos\omega_0\tau(s+s') + \cos\omega_0\tau s'\right]\right)\end{aligned} \tag{A15}$$

$$\begin{aligned}D(x,\omega) = {}&-\frac{1}{2}\int_0^\infty ds\, e^{-(1-2i\omega\tau)s}\\ &\times\int_0^\infty ds'\, e^{-(1-i\omega\tau)s'}\frac{i\sin\omega_0\tau s'}{\omega_0\tau}\end{aligned}$$

$$\times\left(\exp-\frac{x^2}{2}\left[3-2\cos\omega_0\tau s\right.\right.$$
$$\left.\left.-2\cos\omega_0\tau(s+s')+\cos\omega_0\tau s'\right)\right] \tag{A16}$$

Using the well-known relation[19]

$$e^{z\cos\theta}=\sum_{n=-\infty}^{\infty} I_n(z)\,e^{-in\theta}, \tag{A17}$$

where $I_n(z)$ are hyperbolic Bessel function of order n.

We get for (A13)–(A16),

$$A(x,\omega)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{e^{-x^2/2}}{P_n(\omega)}I_n(\tfrac{1}{2}x^2) \tag{A18}$$

$$B(x,\omega)=-\frac{1}{2iql}\left(\frac{1}{1-i\omega\tau}-\sum_{n=-\infty}^{\infty}\frac{e^{-x^2/2}}{P_n(\omega)}I_n(\tfrac{1}{2}x^2)\right), \tag{A19}$$

$$C(x,\omega)=\frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\frac{e^{-3x^2/2}}{P_{n+m}(2\omega)P_{m+p}(\omega)}$$
$$\times I_m(x^2)I_n(x^2)I_p(-\tfrac{1}{2}x^2) \tag{A20}$$

$$D(x,\omega)=-\frac{1}{qlx}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\frac{pe^{-3x^2/2}}{P_{m+n}(2\omega)P_{m+p}(\omega)}$$
$$\times I_m(x^2)I_n(x^2)I_p(-\tfrac{1}{2}x^2)\,. \tag{A21}$$

In the high-field limit, $x\ll 1$, (A18)–(A21) can be evaluated using the small-argument limit of Bessel functions. For small x,[20]

$$I_n(x)=(\tfrac{1}{2}x)^n/\Gamma(n+1)\,. \tag{A22}$$

We obtain

$$A(x,\omega)=\tfrac{1}{2}(1-i\omega\tau)^{-1}, \tag{A23}$$

$$B(x,\omega)=-\frac{x^2}{2iql}\left(\frac{1-i\omega\tau}{(1-i\omega\tau)^2+(\omega_0\tau)^2}\right), \tag{A24}$$

$$C(x,\omega)=\tfrac{1}{2}(1-i\omega\tau)^{-1}(1-2i\omega\tau)^{-1}, \tag{A25}$$

$$D(x,\omega)=-\frac{i}{2(1-2i\omega\tau)}\left(\frac{1}{(1-i\omega\tau)^2+(\omega_0\tau)^2}\right). \tag{A26}$$

In the low-field limit, $x\gg 1$, we can evaluate terms of the type

$$Q=\sum_{n=-\infty}^{\infty}\frac{1}{P_n(\omega)}e^{-x^2/2}I_n\left(\frac{x^2}{2}\right) \tag{A27}$$

using the asymptotic form of the hyperbolic Bessel functions. For large x,

$$I_n(x)=e^x/(2\pi x)^{1/2}\,. \tag{A28}$$

This is valid only when $x>n$. When n exceeds x, the hyperbolic Bessel functions become small. Hence, if we use (A28) to evaluate (A27), we make an error of the form of the final term in the following equation:

$$\sum_{n=-\infty}^{\infty}\frac{e^{-(x^2/2)}I_n(\frac{1}{2}x^2)}{1+i(n\omega_0-\omega)\tau}=\frac{1}{\pi^{1/2}x}\left[\sum_{n=-\infty}^{\infty}\frac{1}{1+i(n\omega_0-\omega)\tau}\right.$$
$$\left.-O\left(\sum_{n=x^2/2}^{\infty}\frac{2(1-i\omega\tau)}{(1-i\omega\tau)^2+(n\omega_0\tau)^2}\right)\right]. \tag{A29}$$

The last term may be evaluated by replacing the summation by an integration over n, and the term is found to be of the order of $1/x(ql)^2$, whereas the first term is of the order $1/ql$. Therefore, for $\omega\tau\gg 1$, we may take ql to be very large and neglect the last term. The first term can be evaluated directly by using

$$\sum_{n=-\infty}^{\infty}\frac{1}{b+in}=\pi\coth b\pi\,. \tag{A30}$$

Therefore, we have from (A27),

$$Q=\frac{\pi^{1/2}}{ql}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\,. \tag{A31}$$

Substituting from (A31) in (A18)–(A21), we have, for low fields and $ql\gg 1$,

$$A(x,\omega)=\frac{\pi^{1/2}}{2ql}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi, \tag{A32}$$

$$B(x,\omega)=-\frac{1}{2iql}\left[\frac{1}{1-i\omega\tau}-\frac{\pi^{1/2}}{ql}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\right], \tag{A33}$$

$$C(x,\omega)=\frac{\pi}{4(ql)^2}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\coth\left(\frac{1-2i\omega\tau}{\omega_0\tau}\right)\pi, \tag{A34}$$

$$D(x,\omega)=\frac{i\pi}{(ql)^3}\frac{1}{x}\coth\left(\frac{1-i\omega\tau}{\omega_0\tau}\right)\pi\coth\left(\frac{1-2i\omega\tau}{\omega_0\tau}\right)\pi$$
$$\times\sum_{p=-\infty}^{\infty}e^{x^2/2}pI_p(-\tfrac{1}{2}x^2)\,. \tag{A35}$$

Therefore, for nonzero magnetic fields, D goes to zero since p is an odd function of p and I_p is an even function of p.

For zero magnetic field, A, B, C, and D can be evaluated exactly by putting $\omega_0=0$ in (A13)–(A16). We get

$$A(x,\omega)=\frac{\pi^{1/2}}{2ql}w\left(-\frac{1-i\omega\tau}{iql}\right), \tag{A36}$$

$$B(x,\omega)=-\frac{1}{2iql}\left[\frac{1}{1-i\omega\tau}-\frac{\pi^{1/2}}{ql}w\left(-\frac{1-i\omega\tau}{iql}\right)\right], \tag{A37}$$

$$C(x,\omega)=\frac{\pi^{1/2}}{2ql}\left[w\left(-\frac{1-2i\omega\tau}{2iql}\right)-w\left(-\frac{1-i\omega\tau}{iql}\right)\right], \tag{A38}$$

$$D(x,\omega)=\left[\frac{1}{iql}+\frac{(1-i\omega\tau)i\pi^{1/2}}{(ql)^2}w\left(-\frac{1-i\omega\tau}{iql}\right)\right.$$
$$\left.-i\pi^{1/2}w\left(-\frac{1-i\omega\tau}{iql}\right)+i\pi^{1/2}w\left(-\frac{1-2i\omega\tau}{2iql}\right)\right], \tag{A39}$$

where the function $w(z)$ is a function related to the complementary error function[21]

$$w(z)=e^{-z^2}\operatorname{erfc}(-iz)\,. \tag{A40}$$

*Submitted in partial satisfaction of the requirements for the degree of Ph.D. in physics, Illinois Institute of Technology, Chicago, Ill. 60616.

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