Renormalization-group approach to a Bose system

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The renormalization-group approach is extended to a Bose system within the framework of perturbation theory. Recursion relations are obtained for the parameters of the Hamiltonian and the correlation length index ν is evaluated. The values of ν and of the critical exponent η (evaluated in an earlier paper) agree with those derived for a classical system characterized by a two-component (real) order parameter. The agreement suggests that the universality hypothesis holds irrespective of the quantum or classical nature of a phase transition.

I. INTRODUCTION

The renormalization-group approach to critical phenomena initiated by Wilson[1] has led to a much better understanding of the behavior of many-body systems near the critical points. It has been used[2] to obtain expansions of critical exponents in powers of $\epsilon = 4 - d$, where d is the dimensionality of the system. The calculations thus far have been confined to classical systems,[3] such as the Ising and Heisenberg-like models of a ferromagnet. It is not obvious that the results obtained apply to phase transitions which are basically quantum mechanical in origin, such as the λ transition in a Bose system and the superconducting transition in a metal. In this paper we make an attempt to extend the renormalization-group approach to a system of bosons within the framework of perturbation theory. The calculation is similar in spirit to that of the classical S^4 model.[4] The purpose achieved by rescaling the spin variables in the S^4 model is achieved here by rescaling the mass of the particles, since it is not possible to rescale the field operators without violating commutation rules.

In Sec. II we derive recursion relations for the parameters characterizing a system of interacting bosons. The fixed point of the transformation and calculation of the critical index ν are discussed in Sec. III. In Sec. IV we compare the results for the Bose system with those for a classical system characterized by a two-component (real) order parameter and conclude that the universality hypothesis[5] appears to be valid irrespective of the classical or quantum-mechanical nature of a phase transition.

II. RENORMALIZATION-GROUP TRANSFORMATION AND RECURSION RELATIONS

For a system of bosons of mass $\frac{1}{2}m$, enclosed in a volume V and interacting via a two-body potential $v(r)$, the Hamiltonian in units such that $\hbar = 1$ is

$$H = \sum_q \frac{1}{m}(q^2 + r_0)\, a_q^\dagger a_q + \frac{1}{4V} \sum_{q \cdots q_4} [v(q_1 - q_3) + v(q_1 - q_4)] \times a_{q_1}^\dagger a_{q_2}^\dagger a_{q_3} a_{q_4}\, \delta_{\mathrm{Kr}}(q_1 + q_2 - q_3 - q_4)\ , \tag{1}$$

where $\mu = -r_0/m$ denotes the chemical potential, $v(q)$ is the Fourier transform of $v(r)$, and δ_{Kr} is the Kronecker symbol. Since momenta large compared to the thermal momentum $\lambda_T^{-1} = (4\pi\beta/m)^{-1/2}$ cannot be expected to play any role near the critical point, we shall replace the interaction term in (1) by the simpler form

$$\frac{u_0}{4V} \sum_{q_1 \cdots q_4} a_{q_1}^\dagger a_{q_2}^\dagger a_{q_3} a_{q_4}\, \delta_{\mathrm{Kr}}, \quad u_0 = 2v(0) \tag{2}$$

and restrict each q to the range $0 < |q| < p_c$, p_c being a cutoff of the order of λ_T^{-1}.

The grand-partition function Z is given by

$$Z = \mathrm{Tr} \exp[-\beta H(m, r_0, u_0)]\ . \tag{3}$$

In accordance with the basic idea of the renormalization group,[1,4] we attempt to construct a new Hamiltonian by integrating out the small wavelength modes. We write

$$a_q = a_{0q} + a_{1q}\ , \tag{4}$$

where

$$a_{0q} = \begin{cases} a_q, & q < p_0 \\ 0, & q > p_0 \end{cases}$$
$$a_{1q} = \begin{cases} 0, & q < p_0 \\ a_q, & q > p_0 \end{cases} \tag{5}$$

$$p_0 = p_c/\zeta, \quad \zeta \gg 1\ . \tag{6}$$

The Hilbert space of the system can accordingly be factorized as $(\mathfrak{h}_0 \otimes \mathfrak{h}_1)$, $\mathfrak{h}_0$ and $\mathfrak{h}_1$ being the subspaces on which a_{0q} and a_{1q}, respectively, operate. One can now write

$$Z = \mathop{\mathrm{Tr}}_{\mathfrak{h}_0} \mathop{\mathrm{Tr}}_{\mathfrak{h}_1} \exp(-\beta H[a_{0q}, a_{1q}])\ . \tag{7}$$

The final aim is to write

$$Z = \operatorname*{Tr}_{\mathfrak{h}_0} \exp(-\beta H'[a_{0q}]) , \tag{8}$$

where $H'[a_{0q}]$ denotes a new interaction having the same form as H.

Substituting (5) into (2), we can write

$$H = H_F(0) + H_F(1) + h , \tag{9}$$

where

$$H_F(0) = \sum_{|q|<p_0} \frac{1}{m}(q^2+r_0)\, a^\dagger_{0q} a_{0q} , \tag{10}$$

$$H_F(1) = \sum_{|q|>p_0} \frac{1}{m}(q^2+r_0)\, a^\dagger_{1q} a_{1q} , \tag{11}$$

and h denotes the interaction term.

The operator $e^{-\beta H}$ can be expanded as

$$e^{-\beta H} = \exp\{-\beta[H_F(0)+H_F(1)]\} S(\beta) , \tag{12}$$

$$S(\beta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta \cdots \int_0^\beta d\tau_1 \cdots d\tau_n [Th(\tau_1)\cdots h(\tau_n)] , \tag{13}$$

where

$$h(\tau) = e^{\tau[H_F(1)+H_F(0)]}\, h e^{-\tau[H_F(1)+H_F(0)]} \tag{14}$$

and T denotes the time-ordering operator. Taking the partial trace of (12) over the space $\mathfrak{h}_1$, we obtain

$$\operatorname*{Tr}_{\mathfrak{h}_1} e^{-\beta H} = Z_1 e^{-\beta H_F(0)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta \cdots \int_0^\beta d\tau_1 \cdots d\tau_n \langle Th(\tau_1)\cdots h(\tau_n)\rangle_1 , \tag{15}$$

where $\langle\ \rangle_1$ denotes thermodynamic average calculated with the unperturbed Hamiltonian $H_F(1)$ and Z_1 is a constant.

The first-order term in (15) gives a contribution similar to $H_F(0)$, viz.,

$$-u_0 \int_0^\beta d\tau \sum_{|q|<p_0} a^\dagger_{0q}(\tau) a_{0q}(\tau) \int_{p_c>|q|>p_0} \frac{d^d q}{(2\pi)^d} \{\exp[(1/m)\beta(q^2+r_0)]-1\}^{-1} . \tag{16}$$

It also gives a contribution similar to the interaction term in the original Hamiltonian, viz.,

$$-\frac{u_0}{4V} \int_0^\beta d\tau \sum_{q_1\cdots q_4} a^\dagger_{0q_1}(\tau) a^\dagger_{0q_2}(\tau)\, a_{0q_3}(\tau)\, a_{0q_4}(\tau)\, \delta_{\mathrm{Kr}} , \tag{17}$$

$a_{0q}(\tau)$ denoting operators in the interaction picture.

Introducing the dimensionless parameters

$$\upsilon = \frac{m^2 u_0}{\beta} p_c^{d-4} , \tag{18}$$

$$r = r_0 p_c^{-2} , \tag{19}$$

$$s = \beta p_c^2/m = (1/4\pi)(p_c^2/\lambda_T^{-2}) , \tag{20}$$

where d denotes the dimensionality of the system, (16) can be written

$$-\frac{s^2\upsilon}{\beta} f_1(s,r) \sum_{|q|<p_0} \int_0^\beta d\tau\, a^\dagger_{0q}(\tau) a_{0q}(\tau), \tag{16'}$$

and (17) as

$$-\frac{s^2\upsilon}{4V} \frac{p_c^{-d}}{\beta} \sum_{q_1\cdots q_4} \int_0^\beta d\tau\, a^\dagger_{0q_1}\cdots a_{0q_4}(\tau)\, \delta_{\mathrm{Kr}} . \tag{17'}$$

where

$$f_1(s,r) = \int_{1>|q|>\zeta^{-1}} \frac{d^d q}{(2\pi)^d} (e^{s(q^2+r)}-1)^{-1} . \tag{21}$$

The second-order term in (15) gives contributions of the same form as (17′). They are indicated diagrammatically in Fig. 1. The dashed lines represent the operators a_{0q}, $a^\dagger_{0q}$ while the solid lines represent single-particle Green's functions G_0 corresponding to momenta larger than p_0. The contribution of diagram 1(a) is

$$\frac{s^3\upsilon^2}{8V} \frac{p_c^{-d}}{\beta} f_a(s,r) \int_0^\beta d\tau \sum_{q_1\cdots q_4} a^\dagger_{0q_1}(\tau)\cdots a_{0q_4}(\tau)\, \delta_{\mathrm{Kr}}, \tag{22}$$

where

$$f_a(s,r) = \int_{1>|q|>\zeta^{-1}} \frac{d^d q}{(2\pi)^d} (q^2+r)^{-1} (e^{-s(q^2+r)}-1)^{-2} \left(1 + \frac{e^{-2s(q^2+r)}-1}{2s(q^2+r)}\right). \tag{23}$$

In writing (22) we have ignored the effect of external momenta on the internal lines. This is justified since $\zeta \gg 1$. The contribution of diagram 1(b) can also be written in the form (22), the only difference being that f_a is replaced by f_b, where

$$f_b(s,r) = -4s \int_{1>|q|>\zeta^{-1}} \frac{d^d q}{(2\pi)^d} (e^{-s(q^2+r)} - 1)^{-1} (e^{s(q^2+r)} - 1)^{-1} . \tag{24}$$

We neglect for the moment the contributions of higher-order diagrams, postponing their discussion until Sec. IV. The grand-partition function then takes the form

$$Z = Z_1 \operatorname{Tr} e^{-\beta H'} ,$$

where the new Hamiltonian H' is given by

$$\beta H' = \sum_{|q|<p_c\zeta^{-1}} s[q^2 p_c^{-2} + r + s\upsilon f_1(s,r)] a_{0q}^\dagger a_{0q} + \frac{s^2 p_c^{-d}}{4V} \sum_{q_1 \cdots q_4} [\upsilon - \tfrac{5}{2} s\upsilon^2 f_2(s,r)] a_{0q_1}^\dagger \cdots a_{0q_4} \delta_{\mathrm{Kr}} , \tag{25}$$

$$f_2(s,r) = \tfrac{1}{5}[f_a(s,r) + f_b(s,r)] . \tag{26}$$

To bring H' into the same form as the original Hamiltonian, we rescale momenta according to

$$q' = \zeta q .$$

The density of states in momentum space accordingly changes by the factor ζ^{-d}. Unlike the case of classical spins, the operators a_{0q} cannot be rescaled without spoiling commutation rules. We may, however, rescale the mass of the particles according to

$$m' = \zeta^2 m . \tag{27}$$

Defining

$$b_q = a_{0q}\zeta^{-1} , \quad V' = \zeta^{-d} V , \tag{28}$$

we then obtain

$$\beta H' = \sum_{|q|<p_c} s'(q^2 p_c^{-2} + r') b_q^\dagger b_q + \frac{s'^2 p_c^{-d}}{4V'} \sum_{q_1 \cdots q_4} \upsilon' b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3} b_{q_4} \delta_{\mathrm{Kr}} , \tag{29}$$

where

$$s' = \beta p_c^2 / m' = \zeta^{-2} s , \tag{30}$$

$$r' = \zeta^2 [r + s\upsilon f_1(s,r)] , \tag{31}$$

$$\upsilon' = \zeta^{4-d} [\upsilon - \tfrac{5}{2} s\upsilon^2 f_2(s,r)] . \tag{32}$$

The use of the rescaled volume V' in (29) takes account of the changed density of states. Equations (30)–(32) are the recursion relations for the three parameters (s, r, υ) characterizing the Hamiltonian βH.

III. FIXED-POINT CORRELATION LENGTH INDEX

By repeating the above procedure one gets a sequence of parameters (s_l, r_l, υ_l) corresponding to a sequence of effective Hamiltonians H_l. Equation (30) implies that after many repetitions the parameter s_l becomes very small not withstanding the fact that to begin with p_c is of order λ_T^{-1}. This circumstance enables us to write the functions f_1 and f_2 in the simpler form

$$f_1(s,r) = s^{-1} g_1(r) , \quad f_2(s,r) = s^{-1} g_2(r) , \tag{33}$$

$$g_1(r) = \int_{1>|q|>\zeta^{-1}} \frac{d^d q}{(2\pi)^d} (q^2 + r)^{-1} , \tag{34}$$

$$g_2(r) = \int_{1>|q|>\zeta^{-1}} \frac{d^d q}{(2\pi)^d} (q^2 + r)^{-2} . \tag{35}$$

The recursion relations (31), (32) after several iterations consequently become

$$r_{l+1} = \zeta^2 [r_l + \upsilon_l g_1(r_l)] , \tag{36}$$

$$\upsilon_{l+1} = \zeta^{4-d} [\upsilon_l - \tfrac{5}{2} \upsilon_l^2 g_2(r_l)] . \tag{37}$$

A nontrivial fixed point (r^*, υ^*) of this transformation is given by

$$r^*(1 - \zeta^2) = \zeta^2 \upsilon^* g_1(r^*) , \tag{38}$$

$$\upsilon^* = (1 - \zeta^{d-4}) \tfrac{2}{5} g_2^{-1}(r^*) . \tag{39}$$

At a fixed point, the correlation length becomes infinite,[4] i.e., it corresponds to a critical point of the system. Assuming r^* to be small,[6] we may expand the g's as

$$g(r^*) = g(0) + r^* g'(0) + \cdots . \tag{40}$$

υ^* is then seen to be small for d close to 4. The perturbation calculation can therefore be relied upon. Defining

$$\epsilon = 4 - d \tag{41}$$

the fixed point is approximately given by

$$\upsilon^* = \frac{2\epsilon \ln\zeta}{5 g_2(0)} , \quad r^* = -\frac{2}{5} \frac{g_1(0)}{g_2(0)} \frac{\zeta^2 \epsilon \ln\zeta}{\zeta^2 - 1} . \tag{42}$$

The correlation length index ν can be calculated

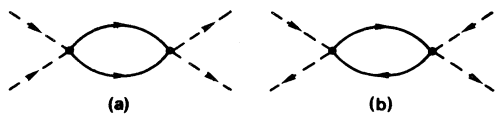

FIG. 1. Diagrams representing the second-order term in (15). The dashed lines represent the operators a_{0q}, $a_{0q}^\dagger$, while the solid lines represent single-particle Green's functions G^0.

from the linearized form of the recursion relations. In fact ν is given by

$$\nu = \ln\zeta/\ln\lambda_1 , \tag{43}$$

where λ_1 is the largest eigenvalue of the matrix formed by the coefficients of the linearized recursion relations. For details we refer the reader to Ref. 4. The linearized version of (36), (37) is

$$r_{l+1} - r^* = \zeta^2[1 + \upsilon^* g_1'(0)](r_l - r^*) + \zeta^2 g_1(0)(\upsilon_l - \upsilon^*), \tag{44}$$

$$\upsilon_{l+1} - \upsilon^* = -\tfrac{5}{2}\zeta^6 g_2'(0)\upsilon^{*2}(r_l - r^*) + \zeta^6[1 - 5\upsilon^* g_2(0)](\upsilon_l - \upsilon^*). \tag{45}$$

To order ϵ, the eigenvalues of the matrix of the coefficients are

$$\lambda_1 = \zeta^2\left(1 + \frac{2}{5}\frac{g_1'(0)}{g_2(0)}\epsilon\ln\zeta\right), \quad \lambda_2 = 1 - \epsilon\ln\zeta. \tag{46}$$

Substituting for λ_1 in (43), and noting that $g_1'(0) = -g_2(0)$, we get

$$\nu = \tfrac{1}{2} + \tfrac{1}{10}\epsilon. \tag{47}$$

IV. DISCUSSION

The recursion relations (30)–(32) in Sec. II have been derived by ignoring the contributions of higher-order diagrams. We now wish to discuss the effect of these diagrams on the results obtained. The matter is important because we must make sure that the inclusion of the neglected terms of the perturbation expansion will not upset the simple results derived in Secs. II and III.

First of all it is not difficult to see that contributions to the kinetic-energy term from diagrams of second and higher order, and to υ' of diagrams of third and higher order, depend upon the external momenta so that, in general, we must write $r'(k)$ and $\upsilon'(k_1, \ldots, k_4)$ in place of the constants r' and υ'. More important is the fact that the expansion (15) gives rise to interaction terms involving six operators, eight operators, etc., with appropriate vertex functions $u_6(k_1, \ldots, k_6)$, $u_8(k_1, \ldots, k_8)$, etc., respectively. Successive iterations will lead to recursion relations for r, υ, u_6, u_8, etc. The exact fixed point consequently is a set of fixed functions $r^*(k)$, $\upsilon^*(k_1, \ldots, k_4)$, $u_6^*(k_1, \ldots, k_6)$, etc., which are solutions of an infinite set of functional equations.

What one wants to know is whether the infinite set of functional equations has a solution that, up to first order in ϵ, reduces to the constants r^* and υ^* given by (42). Since it is not possible to solve the set of functional equations directly, the best one can do is to assume that, to first order in ϵ, the fixed functions reduce to two constants r^* and υ^* and then check the consistency of this approximation with respect to the set of functional equations. For the Ising model, such an exercise has been carried out by Wilson and Kogut.[4] We have performed a similar analysis for the Bose system and have checked that the above assumption is consistent with the set of functional equations provided r^* and υ^* have the values given by (42). An important role in this analysis is played by the irrelevant nature of the parameter s [Eq. (30)] which enables us to replace all Bose distribution factors $(e^{s\epsilon(p)} - 1)^{-1}$ by $[s\epsilon(p)]^{-1}$, leading to a great simplification of the set of functional equations.

It is interesting to compare the results for a Bose system with those obtained for a classical spin system. From (47), we have for the correlation length index

$$\nu = \tfrac{1}{2} + \tfrac{1}{10}\epsilon, \tag{48}$$

while from the calculation of an earlier paper[7] we have for the critical exponent η the result

$$\eta = \tfrac{1}{50}\epsilon^2. \tag{49}$$

This result was obtained in Ref. 7 by the Feynman-graph method together with the assumption that close to the critical point only propagators of zero frequency were important. The approach of this paper provides justification for that assumption by revealing the irrelevant character of the variable s. We note that the result (49) for η can be obtained directly from the renormalization-group approach by calculating the contribution of second-order diagrams to $r'(k)$.

For the n-component classical Heisenberg model, Wilson[8] has obtained the expansions

$$\nu = \frac{1}{2} + \frac{(n+2)\epsilon}{4(n+8)} + O(\epsilon^2), \tag{50}$$

$$\eta = \frac{(n+2)\epsilon^2}{2(n+8)^2} + O(\epsilon^3). \tag{51}$$

For $n = 2$,[9] (50) and (51) agree with (48) and (49), respectively. This agreement is rather remarkable in view of the fact that the transition in a Bose system is basically quantum mechanical in nature. A moment's reflection shows that this "universality" is brought about essentially by the rescaling of the mass of the particles which means that after a large number of iterations the mass becomes so large that quantum effects disappear.

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