Phase transitions in systems with a coupling to a nonordering parameter*

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The renormalization-group method is applied to the analysis of phase transitions in systems where the order parameter s is coupled to a nonordering additional variable y. A variety of critical and tricritical behaviors or first-order transitions is found as a function of the physical variables and possible macroscopic constraints imposed on the system. For s^2y coupling, the correlation function of y was found to be governed by a correlation length which is proportional to that of the order parameter, and by a critical index $\eta_y \cdot \eta_y = 2-2\alpha$; α here is the specific-heat exponent of the appropriate unconstrained system. The singular part of the susceptibility, χ_y, has a critical exponent equal to α, the true specific-heat exponent. When the coupling is s^2y^2, a weaker singularity of χ_y appears. The crossover between this behavior and the one typical to s^2y coupling is calculated. $\langle y\rangle$ has a singular part with an exponent $1-\alpha$, in the unconstrained case. The breakdown of the scaling law related to the correlation function of y in the constrained case is discussed.

I. INTRODUCTION

In every real system undergoing a phase transition, there exist couplings of the order parameter to other degrees of freedom which would not be critical by themselves. As examples for this, one may mention the coupling of the order parameter to the elastic degrees of freedom,[1–5] the coupling of an antiferromagnet to a ferromagnetic ordering[6,7] caused by a magnetic field or, in general, between different magnetic modes of ordering, coupling to electromagnetic or "gauge" fields,[8] coupling of, e.g., the critical point in a polar binary mixture to an electric field[9] or to the pressure, coupling between different crystallographic distortions, and many other examples.

Several new interesting phenomena are possible when these additional degrees of freedom can also become critical.[10,11] However, we shall restrict ourselves here to the case where those degrees of freedom remain noncritical and do not compete with the "primary" ordering parameter. We shall thus refer to these variables as "nonordering" parameters. A further important assumption that we shall make is that the nonordering parameters are in thermodynamic equilibrium and are therefore treated as usual dynamic variables in the partition function. The case where there exist "quenched" configurations of the nonordering parameter presents additional difficulties which will not be treated here.

The two main questions that one would like to answer are (a) How does the coupling to the nonordering parameter affect the critical properties of the primary order parameter? (b) What are the (presumably weak) critical singularities induced by the phase transition on the nonordering parameter?

In a theory based on extremely plausible assumptions, Fisher[12] answered the first question. He emphasized the importance of constraints placed on the nonordering parameter which lead to a well-defined renormalization of the critical exponents. The effect of constraints on the phase transition was further investigated in Ref. 13, where the possibility of a first-order transition and tricritical behavior induced by the constraints was treated. Griffiths and Wheeler[14] discussed the thermodynamics of the transition, including the singularities in the thermodynamic functions of the nonordering parameter. The possibility of obtaining a first-order transition and a tricritical point due to the reduction in the quartic Landau term in the free energy was pointed out by Ginzburg and Levanyuk.[15]

The purpose of this paper is to give a general treatment of the problem of coupling to a nonordering parameter using the renormalization-group (RG) approach.[16] This enables us not only to obtain a coherent picture of the whole problem but also to calculate directly the thermodynamics and correlation functions of both the ordering and nonordering parameters, with and without constraints. Nelson and Fisher[7] have already discussed, within the RG framework, the influence of the unconstrained nonordering parameter on the primary critical behavior. We believe that our results for the correlation function of the nonordering parameter will enable one to interpret experimental probes that couple directly to the nonordering parameter, but not to the order parameter. Examples for such experiments may be optical and light scattering measurements on ferromagnets[17] and near the λ point of ^{4}He and ^{3}He-^{4}He mixtures, and dielectric measurements near the gas-liquid critical point[18] and in a binary mixture.[9] We believe that from such experiments one could verify our results, identify the primary order parameter in some cases where this was not yet done, and also make an independent determination of the specific-heat critical exponent α.[14,18]

In Sec. II, we present the Hamiltonian which will be treated [see Eq. (2.1)]. We assumed a coupling of the form s^2y between the order parameter s and the nonordering parameter y. We then calculate the RG recursion relations of the general Hamiltonian to order ϵ, and represent four fixed points which exhibit different types of critical behaviors.

In Sec. III, we apply the results obtained in Sec. II to the calculation of the correlation function $\langle y_q y_{-q}\rangle$. In many cases, the lowest-order coupling will be, by symmetry, of the form $\int s^2y^2$, which is irrelevant to $O(\epsilon)$ $(\epsilon = 4-d)$ in the RG sense. But the introduction of the field E conjugate to y will result in a term proportional to $E\int ys^2$. An example for this is an antiferromagnet where s is the staggered magnetization, y is the magnetization, and E is the magnetic field.[6,7,19] A discussion of the behavior of χ_y, the susceptibility of the nonordering parameter, in a system with such coupling, with and without the field E, is given in Sec. IV. There we also calculate the crossover indices between different critical behaviors of χ_y as a function of E.

The details of the RG results for the general Hamiltonian (2.1), interrelations among the various fixed points, and their stabilities in the most general parameter space, are discussed in the Appendix. Different types of critical and tricritical behaviors are obtained and discussed. Similar RG results for particular restricted order-parameter spaces which allowed constraints to be put only on the energy density of the order parameter were given in Refs. 3 and 20. The case where y is equal to the energy density s^2 was considered, with time dependence, in Ref. 21. The larger parameter space used here enables us to handle the correlation functions of y, treat higher-order terms, and find the crossover in the critical behavior of χ_y discussed in Sec. IV.

The results are summarized in the Sec. V, where we also obtain the singular behavior of $\langle y\rangle$ at a constant E when $T\to T_c$ both above and below T_c.

II. GENERAL HAMILTONIAN AND RG PROCEDURE FOR s^2y COUPLING

We shall study the partition function z, based on the general Hamiltonian (in units of k_BT)

$$-\mathcal{H}=\frac{1}{2}\int_q (r+q^2)s_q s_{-q}+u\int_q\int_{q'}\int_{q''} s_q s_{q'} s_{q''} s_{-q-q'-q''} +\mu\int_q'\int_{q'} y_q s_{q'} s_{-q-q'}+\tfrac{1}{2}\beta\int_q' y_q y_{-q}+\frac{1}{2}\frac{\beta_0}{\Omega}y_0^2 +\frac{v}{\Omega}\int_q\int_{q'} s_q s_{-q} s_{q'} s_{-q'}+\frac{\mu_0}{\Omega}y_0\int_q s_q s_{-q}\,. \tag{2.1}$$

The integrals $\int_q$ are taken over $|q|<1$ in a $d=(4-\epsilon)$-dimensional space. By $\int_q'$ we mean that the integral is over $q\neq 0$, and Ω is the volume of the system.

Equation (2.1) is a typical field-theory Hamiltonian for the ordering parameter s, with a coupling of the type $\int s^2y\,d\vec{x}$ to the nonordering parameter y. This coupling signifies a modulation of the local values of T_c due to the field $y(\vec{x})$. This kind of Hamiltonian was extensively used in the literature,[2,3,5,7,8,10,11] and was derived for a specific microscopic model in Ref. 7, on the same level as the usual Landau-Ginzburg-Wilson Hamiltonian for a single-order parameter. The four-spin coupling constant v enables us to impose macroscopic constraints[2,3,5,20] on the system. We single out the $q=0$ component of y and $(s^2)_q$, the reason being that the constraint will affect only these components. It should be kept in mind that it is possible to eliminate some of the terms in Eq. (2.1) by shifting the variables y_q and performing the Gaussian integrals over them. This will result in a number of physically equivalent Hamiltonians, and RG fixed points. The reason why we shall eventually deal with the general form (2.1) is that it gives us an easy handle on the correlation function of the y_q's. Using the general $\mathcal{H}$ of Eq. (2.1), one could also add higher-order "anharmonic" terms, which will be argued below to be irrelevant. The large parameter space of Eq. (2.1) will permit us to deal with higher-order coupling terms such as $\int s^2y^2$, and to calculate the crossover between several different critical behaviors of the nonordering parameter y. The general form (2.1) includes several models with and without coupling and the possibility of placing constraints either on y or on the energy of the system.[20,21]

Before discussing the RG transformation on Eq. (2.1), we note that by adding terms on the form $h_{-q}y_q$ to Eq. (2.1) and taking the appropriate derivatives before and after eliminating the y_q's, one easily finds that

$$\langle y_q y_{-q}\rangle=(\mu/\beta)^2\langle (s)^2_q(s)^2_{-q}\rangle\,, \tag{2.2}$$

i.e., the correlation function of y is proportional to that of the energy density; the case where the order parameter s is coupled to the energy density was treated including the dynamic effects in Ref. 21. For $q\neq 0$, $\langle y_q\rangle = 0$ as $h_q\to 0$ and $\partial\langle y_q\rangle/\partial h_{-q} = \langle y_q y_{-q}\rangle$. For $q=0$, one is interested in the susceptibility $\chi_y=\langle(\Delta y_0)^2\rangle=\langle y_0^2\rangle-\langle y_0\rangle^2$, which is proportional to the specific heat, and its singular contribution behaves like $t^{-\alpha}$. Thus the critical exponent of χ_y is

$$\gamma_y = \alpha \,, \qquad (2.3)$$

where α is the specific-heat exponent of the system. We remark that correlation functions of the type $\langle y_q s^2_{-q} \rangle$ are also easily obtainable.

When a constraint is imposed on the value of y_0, $y_0 = \theta$, $\langle (\Delta y_0)^2 \rangle = 0$. One may replace the term $\mu_0 y_0 \int s^2$ by $\mu_0 \int \theta s^2$, which will shift the critical temperature by $\Delta r_0 = \mu_0 \theta / \Omega$. The inverse susceptibility χ_y^{-1} will be given by $\partial^2 \ln z / \partial \theta^2$, and one immediately finds that $\gamma_y = \alpha_R$. Here, α_R is the specific-heat exponent of the constrained system. According to Fisher,[12] we expect the following renormalization for α,

$$\alpha_R = -\alpha/(1-\alpha) \,, \qquad (2.4)$$

which is consistent with our RG results.

The RG operation on the Hamiltonian transforms $\mathcal{H}$ into $\mathcal{H}'$, where $\mathcal{H}'$ is obtained by integrating out all variables y_q and s_q with momenta such that $b|q| > 1$, where $b > 1$. By rescaling the momentum variables by b, and the variables $y_{q\neq 0}$, y_0, and s_q by c, c_0, and ζ, respectively,[7,21] $\mathcal{H}'$ will be of the same form as $\mathcal{H}$. We notice that c_0 may differ from c, which is related to the possible special role of y_0. The rescaling factor ζ $(s_q = \zeta s'_{bq})$ is chosen to keep the coefficient of q^2 equal to unity. To order ϵ it has the usual value[16] $\zeta = b^{3-\epsilon/2}$. This rescaling factor is appropriate to s being the critical order parameter. We restrict ourselves to cases where y does not have a critical behavior, thus c and c_0 are chosen[7] so that $\beta' = \beta$ and $\beta'_0 = \beta_0$. This causes the gradient y term to be irrelevant, and is equivalent to the assumption that the fluctuations of y_q are not critical. It will also follow that higher-order terms which do not appear in Eq. (2.1), such as $\int y^2 s^2$, $\int y^n$ $(n \geq 3)$, are irrelevant in the RG sense.

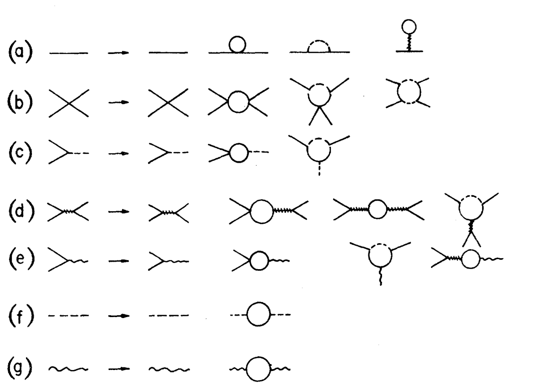

FIG. 1. (a)–(g) are graphs contributing to order ϵ to the recursion relations of r, u, z, v, w, β, and β_0, respectively.

The RG recursion relations for β and β_0 [Figs. 1(f) and 1(g)] to order ϵ are

$$\beta' = \beta c^2 b^{-d}(1 - 2A_2 z) \,, \quad \beta'_0 = \beta_0 c_0^2 b^{-d}(1 - 2A_2 w) \,,$$

where

$$z = \frac{\mu^2}{\beta} \,, \quad w = \frac{\mu_0^2}{\beta_0} \,, \quad \text{and} \quad A_l = \int_{|q|>1/b} \frac{d^d q}{(r+q^2)^l} \,.$$

The rescaling factors that make $\beta' = \beta$ and $\beta'_0 = \beta_0$ are

$$c = b^{2-\epsilon/2}(1 + A_2 z) \,, \quad c_0 = b^{2-\epsilon/2}(1 + A_2 w) \,. \qquad (2.5)$$

The other RG recursion relations to order ϵ are (Fig. 1)

$$\begin{aligned} u' &= b^\epsilon (u - 36 A_2 u^2 + 24 A_2 z u - 4 A_2 z^2) \,, \\ z' &= b^\epsilon z (1 - 24 A_2 u + 10 A_2 z) \,, \\ v' &= b^\epsilon v (1 - 24 A_2 u + 8 A_2 z - 4 A_2 v) \,, \qquad (2.6) \\ w' &= b^\epsilon w (1 - 24 A_2 u + 8 A_2 z - 8 A_2 v + 2 A_2 w) \,, \\ r' &= b^2 (r + 12 A_1 u - 4 A_1 z + 4 A_1 v - 2 A_1 w) \,. \end{aligned}$$

In the last equation, we included a term $-2A_1 w$ for r. This term arises from the shift $y_0 \to y_0 - c_0 \mu_0 A_1 / \beta_0$, which is made to eliminate a linear term $c_0 \mu_0 A_1 y_0$ which appears in the renormalized Hamiltonian.[7]

We shall restrict ourselves in the rest of this section, for the sake of clarity, to the Hamiltonian which describes a linear coupling between a scalar nonordering parameter and a critical parameter

$$-\mathcal{H} = -H(r, u; s) + \mu \int_q' (s^2)_{-q} y_q + \frac{\mu_0}{\Omega} y_0 \int s_q s_{-q} + \tfrac{1}{2}\beta \int_q y_q y_{-q} + E \int_x y(\vec{x}) \qquad (2.7)$$

ignoring higher-order irrelevant terms. E is the field conjugate to y, and $-H(r, u; s)$ is the uncoupled usual Ising Hamiltonian.[16] E can be eliminated by shifting $y(x)$, causing a shift of T_c,

$$-\mathcal{H} = -H\left(r - \frac{E\mu}{\beta}, \ u; \ s\right) + \mu \int_q' (s^2)_{-q} y_q + \frac{\mu_0}{\Omega} y_0 \int_q s_q s_{-q} + \tfrac{1}{2}\beta \int y_q y_{-q} - \frac{E^2 \Omega}{2\beta} \,. \qquad (2.8)$$

This Hamiltonian corresponds to the Hamiltonian (2.1) with the initial restrictions on the parameters: $v = 0$, $\beta = \beta_0$, $z \neq 0$. Note that we did not require $w = z$; this includes the possibility $z \neq 0$, $w = 0$, which corresponds to a constraint on y. In the latter case $y_0 = \theta$ and the initial term $(\mu_0 \theta / \Omega) \int s^2$ can be treated as a shift of r, resulting in $w_{\text{eff}} = 0$.

There are four fixed-point solutions of Eq. (2.6) which satisfy these initial conditions. Each of them has a different critical behavior; these fixed points are

TABLE I. Fixed points with the initial condition $v=0$, $\mu \neq 0$. Their critical indices, ν; their crossover indices ϕ_i, and the corresponding eigenvectors $\vec{\Phi}_i$ of the RG recursion relations.[a]

	u^*	z^*	v^*	w^*	r^*	ν	ϕ_1	$\vec{\Phi}_1$	ϕ_2	$\vec{\Phi}_2$	ϕ_3	$\vec{\Phi}_3$	ϕ_4	$\vec{\Phi}_4$
RI1	$\frac{1}{9}\bar{\epsilon}$	$\frac{1}{6}\bar{\epsilon}$	0	0	$-\frac{2}{3}\rho$	$\frac{1}{2}+\frac{1}{6}\epsilon$	$-\epsilon$	(2, 3, 0, 0)	$-\frac{1}{3}\epsilon$	(1, 2, 0, 0)	$-\frac{1}{3}\epsilon$	(0, 0, 1, 0)	$-\frac{1}{3}\epsilon$	(0, 0, 0, 1)
I1	$\frac{1}{9}\bar{\epsilon}$	$\frac{1}{6}\bar{\epsilon}$	0	$\frac{1}{6}\bar{\epsilon}$	$-\frac{1}{3}\rho$	$\frac{1}{2}+\frac{1}{12}\epsilon$	$-\epsilon$	(2, 3, 0, 3)	$-\frac{1}{3}\epsilon$	(1, 2, 0, 2)	$-\frac{1}{3}\epsilon$	(0, 0, 1, 2)	$\frac{1}{3}\epsilon$	(0, 0, 0, 1)
S1	$\frac{1}{4}\bar{\epsilon}$	$\frac{1}{2}\bar{\epsilon}$	0	0	$-\rho$	$\frac{1}{2}+\frac{1}{4}\epsilon$	$-\epsilon$	(1, 2, 0, 0)	$-\epsilon$	(0, 0, 1, 0)	$-\epsilon$	(0, 0, 0, 1)	ϵ	(1, 3, 0, 0)
G1	$\frac{1}{4}\bar{\epsilon}$	$\frac{1}{2}\bar{\epsilon}$	0	$\frac{1}{2}\bar{\epsilon}$	0	$\frac{1}{2}$	$-\epsilon$	(1, 2, 0, 2)	$-\epsilon$	(0, 0, 1, 2)	ϵ	(1, 3, 0, −3)	ϵ	(0, 0, 0, 1)

[a] $\bar{\epsilon}=(\ln b/A_2)\epsilon$, $\rho=A_1\bar{\epsilon}/(b^2-1)$, $\vec{\Phi}=(u, z, v, w)$, $\phi_i=\ln\lambda_i/\ln b$.

RI1: constrained (renormalized) Ising-like behavior;
I1: Ising-like behavior (Ref. 7);
S1: spherical (renormalized Gaussian)-like behavior;
G1: Gaussian-like behavior.

The fixed-point parameter values, their critical behaviors, and their stability properties are listed in Table I. The Hamiltonian flow trajectories in the w-z plane with the corresponding crossover exponents are given in Fig. 2.

The most unstable fixed point is G1. It corresponds to an unconstrained ($w\neq 0$) tricritical point,[7] and is stable only if $z=2u$. The two trajectories from it to I1 and S1 represent crossover to unconstrained Ising behavior (I) or to the constrained tricritical behavior (S). The crossover indices are equal to α_G ($\alpha_G=2-\frac{1}{2}d$) to order ϵ. From I1 and S1, there is a crossover to the stable constrained Ising behavior RI1. The crossover index from I1 to RI1 is again α, while the crossover index from S to RI1 is the unconstrained Gaussian α_G ($=\frac{1}{2}\epsilon$). RI1 will also turn out to be the only stable fixed point in the general parameter space of Eq. (2.1). One may describe this situation by regarding every system as being constrained, except for the particular case where the effect of the constraint disappears.

We note that there are more fixed-point solutions to the RG [Eq. (2.6)]. They do not represent, except for subtle effects discussed in Sec. IV, any new critical behavior of the Hamiltonian (2.1), and are due to the various subspaces in the parameter space which are spanned by the initial Hamiltonian. Such subspaces can be obtained by integrating over some Gaussian degrees of freedom, as discussed in the Appendix. We shall use some of these fixed points in Sec. IV to represent a crossover between different singular behaviors of the susceptibility χ_y in a more general model. A detailed discussion of these multiple fixed points and the interrelations among them is given in the Appendix.

III. CORRELATION FUNCTION $\langle y_q y_{-q}\rangle$ IN $\int ys^2$ COUPLING

It is straightforward to obtain the correlation function $\langle y_q y_{-q}\rangle$ from the appropriate $O(\epsilon)$ graphical analysis, which will be done below for $\langle y_0 y_0\rangle$. However, before doing that, it is useful to note that most of the behavior of $\langle y_q y_{-q}\rangle$ can be obtained from general scaling arguments and dimensional analysis. The behavior of the Hamiltonian near a fixed point is governed by the largest RG eigenvalue. This sets the relation between the length scale and the temperature scale in a way which is independent of which correlation function is considered. Thus the correlation function $\langle y_q y_{-q}\rangle$ has, in the case where y is coupled to the order parameter, a characteristic decay length which is proportional to the order-parameter correlation length $\xi\sim t^{-\nu}$ [t is the usual reduced temperature, $t=(T-T_c)/T_c$]. This argument was already used for the n spin-correlation functions, which are all characterized by the same length ξ.[16] Physically, the spatial correlations of y are transmitted by its nonzero coupling to s^2. In addition to its correla-

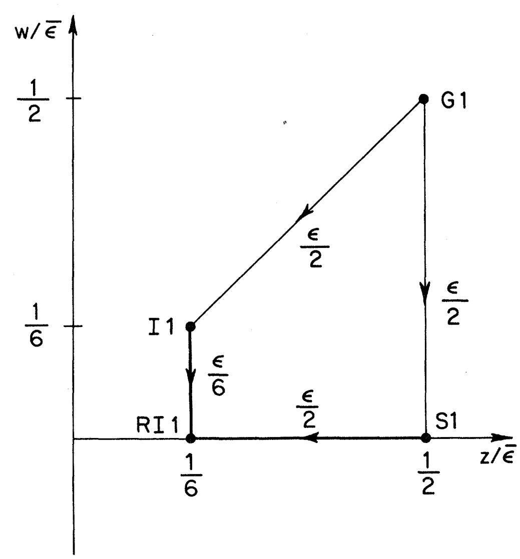

FIG. 2. The Hamiltonian flow, and the crossover indices between the fixed points of Eq. (2.7) in the (z, w) plane, with $z=\mu^2/\beta$ and

$$w=\begin{cases}\mu^2/\beta, & \text{unconstrained system ,}\\ 0, & \text{constrained system.}\end{cases}$$

The portions $w>z$ and that to the right of the line $z=\frac{1}{2}\bar{\epsilon}$ correspond to the runaway of the RG transformation.

tion length, $\langle y_q y_{-q}\rangle$ is characterized by an anomalous dimension, related to the critical index which we shall denote by η_y, i.e., at T_c, $\langle y_q y_{-q}\rangle \sim q^{-(2-\eta_y)}$.

η_y is computed from the rescaling factor c. Dimensional analysis leads to[16,21,22]

$$c^2 = b^{d+2-\eta_y} . \tag{3.1}$$

Using Eq. (2.5), we obtain, to order ϵ,

$$2-\eta_y = 2A_2 z^*/\ln b, \tag{3.2}$$

where z^* is the fixed-point value of z. The values of η_y for the different critical behaviors are

$$2-\eta_y \begin{cases} \frac{1}{3}\epsilon = 2\alpha_I , & \text{Ising and renormalized Ising cases} \\ \epsilon = 2\alpha_G , & \text{Gaussian and spherical cases ,} \end{cases} \tag{3.3}$$

where we used the z^* values from Table I. These results for the unconstrained case are the same as those of Ref. 21, where $y(x)=\epsilon(x)$, the energy density.

The $q=0$ correlation function has the asymptotic singular part, as $t\to 0$,

$$\langle y_0 y_0\rangle \sim t^{-\gamma_y} \sim \tilde{r}^{-\gamma_y/\gamma_s} , \tag{3.4}$$

where $\tilde{r}$ is the true inverse susceptibility of s, that goes to zero when $t\to 0$ like $\tilde{r}\sim t^{\gamma_s}$. The Dyson equation for $\langle y_0 y_0\rangle$ is

$$\langle y_0 y_0\rangle^{-1} = \beta_0 - \Sigma . \tag{3.5}$$

The only graph that contributes to Σ to order ϵ is the one denoted by g in Fig. 1,

$$\Sigma = \frac{2\mu_0^*}{(2\pi)^d}\int_0^\infty d^d p\,[p^2(1+p^2)^2+\tilde{r}]^{-2} . \tag{3.6}$$

(We have used here the soft-momentum cutoff.[16]) At the limit $\tilde{r}\to 0$, we obtain

$$\Sigma \sim -\beta_0[w^*/2(2\pi)^2]\ln\tilde{r} . \tag{3.7}$$

Using the value $A_2 = \ln b/8\pi^2$ and substituting Eqs. (3.5)–(3.7) in Eq. (3.4), one gets for γ_y

$$\gamma_y = \gamma_s(A_2/\ln b)\, w^* , \tag{3.8}$$

$\gamma_s = 1+O(\epsilon)$, and $w^* = O(\epsilon)$, and thus to order ϵ one gets

$$\gamma_y = (A_2/\ln b) w^* . \tag{3.9}$$

In Table I, w^* is given in terms of $\bar{\epsilon}\equiv(\ln b/A_2)\epsilon$; hence,

$$\gamma_y = \begin{cases} \frac{1}{6}\epsilon = \alpha_I , & \text{Ising case} \\ \frac{1}{2}\epsilon = \alpha_G , & \text{Gaussian case} \\ -\frac{1}{6}\epsilon = -\alpha_I = \alpha_{RI} , & \text{constrained Ising case} \\ -\frac{1}{2}\epsilon = -\alpha_G = \alpha_S , & \text{spherical case .} \end{cases} \tag{3.10}$$

When γ_y is negative, one has to add constant terms which were neglected in Eq. (3.7). Hence γ_y describes the singular part, not necessarily the largest part, of χ_y. Note that to evaluate γ_y, one usually needs the second-order terms in the expansion of $\chi_y^{-1} = A + Bt^{\gamma_y}$ in $\gamma_y \ln|t|$. In the unconstrained case, we use the fact that $A=0$ due to the divergence of χ_y. In the constrained case, to establish that $\gamma_{y,\,\text{constrained}} = -\gamma_{y,\,\text{unconstrained}}$, we have to analyze the $O(\epsilon^2)$ terms, which only change sign due to the constraint. To evaluate the constrained value of χ_y we used the fixed-point values of points RI3 and S3. In the Appendix, we show that the correlation function calculated in these cases is not $\langle y_0 y_0\rangle$, which is of course meaningless when $\int y$ is constrained, but the correlation function $\langle (s^2)_0 (s^2)_0\rangle$ to which the singular part of χ_y in the constrained case is proportional.

In all cases, the singular behavior of χ_y is identical to that of the specific heat.[14] The results for the critical exponents of y are summarized in Table II. We find that the usual scaling law

$$\gamma_y = (2-\eta_y)\,\nu \tag{3.11}$$

holds only for the Gaussian and Ising critical behavior, and *not* for the constrained cases (spherical and renormalized Ising). This is due to the fact that $\gamma_y = \alpha$, and the Fisher renormalization of α involves a minus sign $[\alpha\to -\alpha/(1-\alpha)]$ which does not appear in the renormalization of, e.g., ν $[\nu \to \nu/(1-\alpha)]$, and at the same time η_y (as well as η) is not renormalized at all. This could lead one to some worry, since the scaling law (3.11) usually follows simply from the scaling of the correlation function. The reason for its failure here is because χ_y is not equal to $\lim_{q\to 0}\langle y_q y_{-q}\rangle$. The constraint in fact affects only the Fourier component y_0 and it almost does not affect the $q\neq 0$ part of the correlation function (except for the renormalization of ν).

IV. NONORDERING SUSCEPTIBILITY IN $\int s^2 y^2$ COUPLING

There are systems in which the coupling between the nonordering and the ordering parameters is of the form $\lambda\int_x s^2(x) y^2(x)$. Examples are the coupling between an antiferromagnet to a ferromagnetic ordering,[6,7] or a binary liquid mixture with differ-

TABLE II. Critical indices characterizing the correlation function of y near and at T_c, and the specific-heat exponent.

Critical behavior	$2-\eta_y$	γ_y	α
Ising	$\frac{1}{3}\epsilon$	$\frac{1}{6}\epsilon$	$\frac{1}{6}\epsilon$
Gaussian	ϵ	$\frac{1}{2}\epsilon$	$\frac{1}{2}\epsilon$
Renormalized Ising	$\frac{1}{3}\epsilon$	$-\frac{1}{6}\epsilon$	$-\frac{1}{6}\epsilon$
Spherical	ϵ	$-\frac{1}{2}\epsilon$	$-\frac{1}{2}\epsilon$

ing polar components near the miscibility critical point, coupled to the polarization field.[9] The Hamiltonian of such systems, in the presence of an applied field conjugate to the nonordering density, is

$$-\mathcal{H}=-H(r,\ u;\ s)+\lambda\int_x y^2(x)s^2(x)+\frac{\beta}{2}\int_x y^2(x)+E\int_x y(x)\ . \tag{4.1}$$

After elimination of the linear term by shifting $y(x)$, we get

$$-\mathcal{H}=-H\left(r+\frac{\lambda E^2}{\beta^2}\ ,\ u;\ s\right)-2\lambda\frac{E}{\beta}\int_x ys^2+\frac{\beta}{2}\int_x y^2-\frac{E^2}{2\beta}\Omega\ . \tag{4.2}$$

(We have not written the term $\int x^2y^2$, which is irrelevant as discussed in Sec. II.)

When $E\neq 0$, we get a coupling[7] $\int s^2y$ with a coefficient $-2\lambda E/\beta$, which was discussed in Sec. II and found to be relevant. The fixed-point values are of the variable $(2\lambda E)^2/\beta^3$ and thus allow a negative sign of the interaction $-2\lambda E/\beta$, and the sign of E is unimportant. χ_y behaves like the specific heat and thus diverges at T_c. A similar effective interaction

$$\frac{\theta}{\Omega}\int_q' (s^2)_q y_{-q}$$

is also obtained in the constrained case ($\int y=\theta$, $\theta\neq 0$). There, we obtain $\chi_y\sim\text{const}+\text{const}'\times t^{\alpha_{\rm I}}$, as found in Sec. III.

In the case where $E\to 0$, which may be of experimental interest, the relevant coupling term $\mu\int s^2y$ vanishes. The coupling $\int s^2y^2$ will not enter the lowest order RG calculation. We can find the singular behavior of χ_y by an exact thermodynamic calculation, similar to the one used in Sec. II:

$$\chi_y=\frac{\partial\langle y\rangle}{\partial E}=\left\langle\frac{\Omega}{\beta}-\frac{2\lambda}{\beta^2}\int s^2\right\rangle+E^2\left\langle\left[\Delta\left(\frac{2\lambda}{\beta^2}\int s^2-\frac{2\lambda}{\beta}\int ys^2\right)\right]^2\right\rangle, \tag{4.3}$$

$$\chi_y(E=0)=\frac{\Omega}{\beta}-\frac{2\lambda}{\beta^2}\int s^2=\text{const}+\text{const}'\times t^{1-\alpha}\ . \tag{4.4}$$

Thus for $E=0$, χ_y has the relative weak singularity

TABLE III. The fixed points, their critical index ν, the eigenvalues and the corresponding eigenvectors of the general RG recursion relations.

			(I) $\nu=\frac{1}{2}+\frac{1}{12}\epsilon$					$r^*=-\frac{1}{3}A_1\bar{\epsilon}/(b^2-1)$				
	u^*	z^*	v^*	w^*	ϕ_1[a]	$\vec{\Phi}_1$	ϕ_2	$\vec{\Phi}_2$	ϕ_3	$\vec{\Phi}_3$	ϕ_4	$\vec{\Phi}_4$
I1	$\frac{1}{9}\bar{\epsilon}$	$\frac{1}{6}\bar{\epsilon}$	0	$\frac{1}{6}\bar{\epsilon}$	$-\epsilon$	(2, 3, 0, 3)	$-\frac{1}{3}\epsilon$	(1, 2, 0, 2)	$-\frac{1}{3}\epsilon$	(0, 0, 1, 2)	$\frac{1}{3}\epsilon$	(0, 0, 0, 1)
I2	$\frac{1}{9}\bar{\epsilon}$	$\frac{1}{6}\bar{\epsilon}$	$-\frac{1}{12}\bar{\epsilon}$	0	$-\epsilon$	(4, 6, 3, 0)	$-\frac{1}{3}\epsilon$	(1, 2, −1, 0)	$\frac{1}{3}\epsilon$	(0, 0, 1, 0)	$\frac{1}{3}\epsilon$	(0, 0, 0, 1)
I3	$\frac{1}{36}\bar{\epsilon}$	0	$\frac{1}{12}\bar{\epsilon}$	$\frac{1}{6}\bar{\epsilon}$	$-\epsilon$	(1, 0, 3, 6)	$-\frac{1}{3}\epsilon$	(0, 0, 1, 2)	$\frac{1}{3}\epsilon$	(1, 2, −1, 0)	$\frac{1}{3}\epsilon$	(0, 0, 0, 1)
I4	$\frac{1}{36}$	0	0	0	$-\epsilon$	(1, 0, 0, 0)	$\frac{1}{3}\epsilon$	(1, 2, 0, 2)	$\frac{1}{3}\epsilon$	(0, 0, 1, 0)	$\frac{1}{3}\epsilon$	(1, 2, 0, 1)
			(RI) $\nu=\frac{1}{2}+\frac{1}{6}\epsilon$					$r^*=-\frac{2}{3}A_1\bar{\epsilon}/(b^2-1)$				
RI1	$\frac{1}{9}\bar{\epsilon}$	$\frac{1}{6}\bar{\epsilon}$	0	0	$-\epsilon$	(2, 3, 0, 0)	$-\frac{1}{3}\epsilon$	(1, 2, 0, 0)	$-\frac{1}{3}\epsilon$	(0, 0, 1, 0)	$-\frac{1}{3}\epsilon$	(0, 0, 0, 1)
RI2	$\frac{1}{36}\bar{\epsilon}$	0	$\frac{1}{12}\bar{\epsilon}$	0	$-\epsilon$	(1, 0, 3, 0)	$-\frac{1}{3}\epsilon$	(0, 0, 1, 0)	$-\frac{1}{3}\epsilon$	(0, 0, 1, 0)	$\frac{1}{3}\epsilon$	(1, 2, −1, 0)
RI3	$\frac{1}{9}\bar{\epsilon}$	$\frac{1}{6}\bar{\epsilon}$	$-\frac{1}{12}\bar{\epsilon}$	$-\frac{1}{6}\bar{\epsilon}$	$-\epsilon$	(4, 6, −3, −6)	$-\frac{1}{3}\epsilon$	(1, 2, −1, 2)	$-\frac{1}{3}\epsilon$	(0, 0, 0, 1)	$\frac{1}{3}\epsilon$	(0, 0, 1, 2)
RI4	$\frac{1}{36}\bar{\epsilon}$	0	0	$-\frac{1}{6}\bar{\epsilon}$	$-\epsilon$	(1, 0, 0, −6)	$-\frac{1}{3}\epsilon$	(0, 0, 0, 1)	$\frac{1}{3}\epsilon$	(1, 2, 0, 2)	$\frac{1}{3}\epsilon$	(0, 0, 1, 2)
			(G) $\nu=\frac{1}{2}$					$r^*=0$				
G1	$\frac{1}{4}\bar{\epsilon}$	$\frac{1}{2}\bar{\epsilon}$	0	$\frac{1}{2}\bar{\epsilon}$	$-\epsilon$	(1, 2, 0, 2)	$-\epsilon$	(0, 0, 1, 2)	ϵ	(1, 3, 0, 0)	ϵ	(0, 0, 0, 1)
G2	$\frac{1}{4}\bar{\epsilon}$	$\frac{1}{2}\bar{\epsilon}$	$-\frac{1}{4}\bar{\epsilon}$	0	$-\epsilon$	(1, 2, −1, 0)	ϵ	(1, 3, 0, 0)	ϵ	(0, 0, 1, 0)	ϵ	(0, 0, 0, 1)
G3	0	0	$\frac{1}{4}\bar{\epsilon}$	$\frac{1}{2}\bar{\epsilon}$	$-\epsilon$	(0, 0, 1, 2)	ϵ	(1, 3, 0, 0)	ϵ	(0, 1, 1, 0)	ϵ	(0, 0, 0, 1)
G4	0	0	0	0	ϵ	(1, 0, 0, 0)	ϵ	(0, 1, 0, 0)	ϵ	(0, 0, 1, 0)	ϵ	(0, 0, 0, 1)
			(S) $\nu=\frac{1}{2}+\frac{1}{4}\epsilon$					$r^*=-A_1\bar{\epsilon}/(b^2-1)$				
S1	$\frac{1}{4}\bar{\epsilon}$	$\frac{1}{2}\bar{\epsilon}$	0	0	$-\epsilon$	(1, 2, 0, 0)	$-\epsilon$	(0, 0, 1, 0)	$-\epsilon$	(0, 0, 0, 1)	ϵ	(1, 3, 0, 0)
S2	0	0	$\frac{1}{4}\bar{\epsilon}$	0	$-\epsilon$	(0, 0, 1, 0)	$-\epsilon$	(0, 0, 0, 1)	ϵ	(1, 0, −3, 0)	ϵ	(1, 3, 0, 0)
S3	$\frac{1}{4}\bar{\epsilon}$	$\frac{1}{2}\bar{\epsilon}$	$-\frac{1}{4}\bar{\epsilon}$	$-\frac{1}{2}\bar{\epsilon}$	$-\epsilon$	(1, 2, −1, 2)	$-\epsilon$	(0, 0, 0, 1)	ϵ	(0, 0, 1, 2)	ϵ	(1, 3, 0, 0)
S4	0	0	0	$-\frac{1}{2}\bar{\epsilon}$	$-\epsilon$	(0, 0, 0, 1)	ϵ	(1, 3, 0, 0)	ϵ	(0, 1, 1, 0)	ϵ	(0, 1, 0, 2)

[a] $\phi_i=\ln\lambda_i/\ln b$

$t^{1-\alpha}$, while nonzero values of E yield, as found before, a divergence like $t^{-\alpha}$, with an amplitude of order E^2. This is consistent[14] with the results found by Fisher[6] for the singular behavior of the magnetic susceptibility of a two-dimensional decorated Ising antiferromagnet.

We now discuss the above result from the RG viewpoint. The possible different critical behaviors of χ_y are (see also the Appendix and Table III)

RI1: $\chi_y \sim \text{const} + \text{const}' \times t^{\alpha_I}$, coupled constrained Ising behavior;

I1: $\chi_y \sim t^{-\alpha_I}$, coupled unconstrained Ising behavior;

I4: $\chi_y \sim \text{const} + \text{const}' \times t^{1-\alpha_I}$, Ising behavior with "irrelevant" coupling.

In Fig. 3(a) we plotted these points and marked the crossover exponents that characterize the change of the critical behavior.

As a function of E, we can change the initial effective value of u (see the Appendix) to get a runaway (first-order transition) or tricritical point (for a special limiting value of E). The fixed points corresponding to the possible tricritical points are

S1: $\chi_y \sim \text{const} + \text{const}' \times t^{\alpha_G}$, coupled constrained tricritical behavior;

G1: $\chi_y \sim t^{-\alpha_G}$, coupled unconstrained tricritical behavior;

G4: $\chi_y \sim \text{const} + \text{const}' \times t^{1-\alpha_G}$, tricritical behavior with an "irrelevant" coupling of y.

These points and their crossover behavior are given in Fig. 3(b). The crossover between the two sets (S1, G1) → (RI1, I1) is shown in Fig. 2 (see also the Appendix).

The crossover exponent in all cases is found to be equal to the specific-heat exponent which characterizes the corresponding unconstrained system. An amusing crossover is the one from I4 to I1. This is a crossover between two fixed points in both of which the main critical behavior is identical However, I4 is an Ising fixed point with no relevant coupling to the nonordering field. Thus, the singularity of χ_y is extremely weak ($\sim t^{1-\alpha}$). I1 is also an Ising fixed point, but it has a relevant coupling to y, which causes a (weak, $\sim t^{-\alpha}$) divergence of χ_y.

The singular behavior of χ_y as a function of E (note that E is, e.g., the electric field in the case of the polar binary mixture, and the magnetic field H for the antiferromagnet) suggests that a detailed measurement of the (nonlinear) χ_y as a function of E will be extremely valuable. Such a study should obtain both the $t^{-\alpha}$ (or const + t^{α} in the constrained case) and the (const + $t^{1-\alpha}$) behaviors and the crossover between them. The appropriate crossover exponent is also equal to α. Thus in principle a whole $\chi_y(E, t)$ function should be fitted with a single parameter α. The analysis of the dielectric constant in the binary mixture[9] case is currently under study.

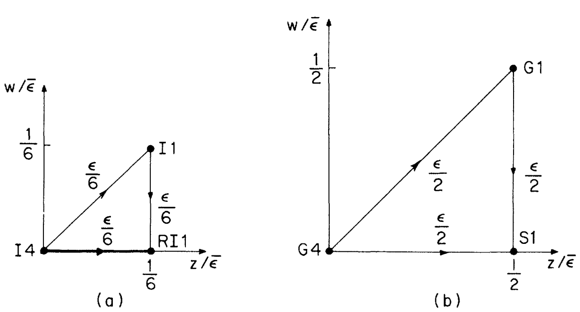

FIG. 3. Hamiltonian flow and the crossover indices between the fixed points of Eq. (4.2) in the z, w plane. Each of the fixed points has a different critical behavior of χ_y. (a) The main critical behavior is Ising or constrained Ising. (b) The main critical behavior is Gaussian or spherical (tricritical).

V. SUMMARY AND CONCLUSIONS

We have applied the RG method to $O(\epsilon)$ for discussing the critical behavior of the coupled Hamiltonian (2.1) with and without the constraint y_0 = const. We have not considered more general constraints[13] but it is possible to see that those can only lead to one of the behaviors found here [plus possible terms which are irrelevant to $O(\epsilon)$]. The critical behavior can stay Ising-like, be renormalized,[12] become first-order, and, for special initial conditions, become tricritical. The tricritical behavior may be Gaussian, spherical, or Ising-like, the former being "doubly tricritical" in a large enough parameter space. An interesting multiplicity of the fixed points was analyzed in detail with the appropriate trajectories in the parameter space.

We find, for the case of s^2y coupling, that the correlation function $\langle y_q y_{-q}\rangle$, as well as $\langle s_q^2 s_{-q}^2\rangle$ and $\langle y_q s_{-q}^2\rangle$, are characterized by a correlation length ξ which is proportional to that of the order parameter ($\langle s_q s_{-q}\rangle$) correlation function. The critical index η_y was found to be $2-2\alpha$, where α is always the specific-heat exponent of the appropriate unconstrained system. The singular part of χ_y is always characterized by the critical exponent $\gamma_y = \alpha$. α here is the specific-heat critical exponent, which is renormalized in the constrained case. Thus, the scaling law $\gamma_y = \nu(2-\eta_y)$ is valid only in the unconstrained system.

Adding a constant field E conjugate to y, results in an ideal critical behavior, with a T_c shift linear in E, in agreement with Fisher's general assumption.[7,12,14] By taking the derivative of the free energy with respect to E, one immediately finds that the singular part of $\langle y\rangle$ at a constant E behaves as $|t|^{\beta_y}$ when $|t|\to 0$, where

$$\beta_y = 1-\alpha\ . \tag{5.1}$$

This relation is derived above T_c. Below T_c, there should be a singular contribution to $\langle y\rangle$ characterized by this value of β_y.

The singular part of χ_y was obtained for s^2y^2 coupling. When the field E conjugate to y vanishes, χ_y has a singular part which behaves like $t^{1-\alpha}$. For $E\neq 0$ or the constrained case, the critical behavior changes, with a crossover exponent equal to the corresponding unconstrained α, to that of the s^2y case, i.e., a divergence $t^{-\alpha}$ in the unconstrained case, and a singular part $t^{-\alpha_R}$ ($\alpha_R<0$) in the constrained case. We believe that these results will be relevant for the interpretation of magnetic susceptibility measurements in antiferromagnets,[6] dielectric measurements in polar liquids and liquid-mixture critical points,[9] and possibly many additional cases.

Thus, the coupling to a nonordering parameter can lead to a variety of types of phase transitions.[23] Experiments that couple directly to the nonordering parameter may yield interesting information on the phase transition. Some examples[9,17] for such experimental probes were given in Sec. I.

Our treatment was restricted to the case where the number of components n of the order parameter is 1, and the unconstrained specific-heat index, $\alpha_{\rm I}$ or $\alpha_{\rm G}$, is positive. We hope to analyze the more general problem, as well as the $O(\epsilon^2)$ corrections, in future work. The latter corrections are important in the case of biquadratic (s^2y^2) coupling, when the field conjugate to y is zero.

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APPENDIX: DETAILS OF RG RESULTS IN GENERAL PARAMETER SPACE

The RG transformation (2.6) has 16 different fixed points. These fixed points are divided into four groups, each with four distinct fixed points. All the fixed points within a group correspond to the same critical behavior of the order parameter. The four critical behaviors are Ising (I), renormalized Ising (RI), Gaussian (G), and spherical (renormalized Gaussian) (S). The fixed points, their critical indices ν, the eigenvalues $\lambda_i = b^{\phi_i}$, and the corresponding eigenvectors of the linearized RG, R_L^b, defined by

$$\begin{pmatrix}\delta u'\\ \delta z'\\ \delta v'\\ \delta w'\end{pmatrix} = R_L^b \begin{pmatrix}\delta u\\ \delta z\\ \delta v\\ \delta w\end{pmatrix},$$

are given in Table III.

The multiplicity of the fixed points for each group results from the large parameter space that we have chosen. In fact, one can generate different forms of $\mathcal{H}$ which obviously have the same critical behavior by transformations within the variables space. By such transformations, one can span different subspaces of the parameter space to which the different fixed points describing the same criticial behavior belong. The main reasons that we chose this large parameter space were the convenience of calculating $\langle y_q y_{-q}\rangle$, the possibility of obtaining the crossover between different critical behaviors of y (Sec. IV), and the fact that higher-order terms could be dealt with.

The unconstrained Hamiltonian (2.7) corresponds to the initial condition $z=w$ and $v=0$. This is the most stable fixed point in group (I): I1. By integrating either $y_{q=0}$ or $y_{q\neq 0}$, this Hamiltonian goes to a subspace of the parameter space in which I2 or I3, respectively, are the stable fixed points. This integration reduces by one the number of parameters, and thus reduces by one the stability of the Hamiltonian in the total space. By integrating both $y_{q\neq 0}$ and y_0, the Hamiltonian goes to a more restricted subspace in which I4 is stable.

By a further restriction of the subspaces of points I, by demanding that $u=z/2$, we obtain spaces in which the G's are the stable fixed points. We have chosen our notation in such a way that the same transformation that carries I1 to I2 also carries G1 into G2, etc.

We now go on to discuss the fixed points RI and S of constrained systems. A simple way to add a constraint to the Hamiltonian is to fix the value of $y_0=\theta$. The Hamiltonian will be then

$$-\mathcal{H} = -H\left(r+\frac{\mu_0\theta}{\Omega}\ u;\ s\right) + \mu\int_q'\int_{q'} y_q s_{q'} s_{-q-q'} + \tfrac{1}{2}\beta\int_q' y_q y_{-q}\ . \tag{A1}$$

This has the stable fixed point RI1. It has the renormalized Ising index $\nu_{\rm RI}=\nu_{\rm I}/(1-\alpha_{\rm I})$, as all other Hamiltonians in the group RI. RI2 is obtained by integrating $y_{q\neq 0}$ from Eq. (3.3), and thus is "once unstable."

The constraint can be imposed on the system with the Hamiltonian (2.7) employing a delta function $\delta(\int y(x)-\theta) = (1/2\pi i)\int_{-i\infty}^{i\infty} d\kappa \exp[\kappa(y-\theta)]$, changing of the order of integration $\int dy_q \int d\kappa \to \int d\kappa \int dy_q$, and the shifting of y_0:

$$y_0 \to y_0 + \frac{1}{\beta}\left[\kappa\Omega + (1-i)\mu \int s^2\right]$$

and (A2)

$$\kappa \to \kappa + \frac{\mu}{\Omega}\int s^2 + \frac{\theta}{\Omega}$$

(which do not spoil the convergence of the integrals in the partition function), taking the Hamiltonian into

$$-\mathcal{H} = -H(r, u; s) + \tfrac{1}{2}\beta \int y_q y_{-q} - \frac{\mu^2}{2\Omega\beta}\left(\int_x s^2(x)\right)^2 + \frac{i\mu}{\Omega} y_0 \int s^2(x) + \mu \int_q' \int_{q'} y_q s_{q'} s_{-q-q'} - \frac{\kappa^2\Omega}{2\beta} . \quad \text{(A3)}$$

This Hamiltonian corresponds to the fixed point RI3. Integrating $y_{q\neq 0}$ in this Hamiltonian yields RI4. The above method can also be used for dealing with more general constraints, which can be shown to lead to Hamiltonians whose forms differ from the above one by irrelevant terms only.

The fixed points S1, S2, S3, S4 correspond to RI1, RI2, RI3, RI4, respectively, in the subspace $2u = z$.

In order to better understand the Gaussian and the spherical cases, let us examine again the Hamiltonian (2.7). We have seen that by integrating y_q we obtain an effective u

$$u_{\text{eff}} = u - z/2 , \quad \text{(A4)}$$

when $\frac{1}{2}z < u$, the phase transition is second-order, and the renormalized Hamiltonian will flow to one of the I or RI fixed points. When we have a positive $S^6(x)$ term, the Hamiltonian would have become the usual one used to describe first-order transitions for $u_{\text{eff}} < 0$ and tricritical point ($u_{\text{eff}} = 0$, or $u = z/2$).[22,7,24] Thus G is the tricritical point in the unconstrained subspace, and S is the same in the constrained (renormalized) subspace.

One should also note that the Ising fixed point may also become a tricritical point, in a large parameter space obtainable, e.g., when the initial $\beta_0 \neq \beta$ and/or $\mu_0 \neq \mu$. This is the situation, for example, in the magnetoelastic case.[2,3,13] In this picture the Gaussian point becomes a higher-order critical point, corresponding to the case where two tricritical lines meet.

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