$S = 1/2$ XY model on cubic lattices. II. Longitudinal susceptibilities*

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The longitudinal susceptibility for the $S = 1/2$ XY model on cubic lattices is discussed on the basis of its relationship to the free energy. A series expansion for the susceptibility is developed and its behavior is studied at an asymptotic limit. The asymptotic behavior of the series expansion shows that the longitudinal susceptibility is nondivergent at the critical point T_c, its "singular" part behaving as $(T - T_c)^{-\alpha+1}$, where α is the specific-heat exponent. A related quantity, the partial longitudinal susceptibility, which is important in the study of spin dynamics, is also shown to have a similar nondivergent critical behavior.

I. INTRODUCTION

The conventional order-parameter susceptibility for the Ising model is the longitudinal component χ^{zz} and for the XY model it is the transverse component χ^{xx} (or χ^{yy}). Both quantities in three dimensions are known to diverge strongly near T_c.[1] Fisher[2] showed that the transverse susceptibility χ^{xx} for the Ising model behaves near T_c like the energy. One may therefore expect that the longitudinal susceptibility χ^{zz} for the XY model also behaves as the energy. One can in fact show using a scaling argument that this quantity cannot diverge.[3]

The critical behavior of χ^{zz} for the XY model provides a few interesting points. If it is nondivergent, the hypothesis of "critical-point stability" first put forth by Fisher[4] suggests the following thought experiment: Consider an anisotropic, cubic-lattice nearest-neighbor (nn) Heisenberg model in an external field H_z,

$$\mathcal{H}(\lambda) = -\sum_{(ij)} J(S_i^x S_j^x + S_i^y S_j^y) - \lambda \sum_{(ij)} J S_i^z S_j^z - H_z \sum_i S_i^z , \qquad (1)$$

where the anisotropy parameter λ is non-negative. The stability or smoothness hypothesis would predict that as λ is continuously varied, $\chi^{zz}(\lambda)$ should behave *discontinuously*. In particular if we could maintain the temperature of the system near the critical temperature $T \simeq T_c(\lambda)$, then, as λ changes from Ising-like (i.e., $\lambda = 1+$) to XY-like (i.e., $\lambda = 1-$),[5] the hypothesis predicts that the strongly divergent behavior of $\chi^{zz}(\lambda)$ should vanish.

It is customary to extrapolate the critical behavior of a function such as the susceptibility *numerically* from a finite number of exactly known coefficients of a high-temperature series expansion. The series expansion for $\chi^{zz}(\lambda = 0)$ has been obtained.[6] But owing to its nondivergent (or perhaps very weakly divergent) character, it has been found difficult to establish from the series expansion the critical behavior of $\chi^{zz}(\lambda = 0)$. In an earlier paper[7] the critical behavior of $\chi^{xx}(\lambda = 0)$ was deduced from the asymptotic behavior of its high-temperature series. In this paper we shall study in a similar way the critical behavior of $\chi^{zz}(\lambda = 0)$ by directly examining the asymptotic behavior of its series expansion.

The longitudinal susceptibility is closely related to another physical quantity (we shall call it the *partial* longitudinal susceptibility) which appears as a leading term in the f-sum rule.[8] It is an important quantity for the study of the dynamic behavior of an XY paramagnet.[9] The partial susceptibility represents that portion of the susceptibility which contains only the nn correlations; hence, it behaves approximately as the energy of the system. If the two susceptibilities behave similarly in the critical region, then the longitudinal susceptibility must be nondivergent since short-range correlations alone cannot normally lead to a divergence.

In Sec. II we have defined the longitudinal and partial susceptibilities in terms of the anisotropic Heisenberg model. In Sec. III the high-temperature expansions for both quantities are given and shown to be closely related to the expansion for the free energy. In Sec. IV the asymptotic behavior of the longitudinal-susceptibility series is described in relation to the asymptotic behavior of the free-energy series. Finally, in Appendixes A–D our various assertions are proved.

II. LONGITUDINAL SUSCEPTIBILITIES

The zero-field susceptibility is defined in the usual way[1]:

$$\chi^{zz}(\lambda) = \beta^{-1} \frac{\partial}{\partial H_z} \langle M^z \rangle_{H_z=0} , \qquad (2)$$

where in appropriate units

$$M^z = \sum_R S_R^z , \qquad (3)$$

and the angular brackets denote a thermal average. Since $[H(\lambda), M^z]=0$ for all values of λ, using the general "fluctuation" theorem we can readily write down

$$\chi^{zz}(\lambda)=\sum_{RR'}\langle S_R^z S_{R'}^z\rangle-\left\langle\sum_R S_R^z\right\rangle\left\langle\sum_{R'} S_{R'}^z\right\rangle . \quad (4)$$

The second term in the right-hand side (RHS) of (4) vanishes for $T>T_c$. The susceptibility for the XY model is then obtained by letting $\lambda=0$ in (4).

The zero-field free energy can be similarly defined,

$$F(\lambda)=\ln \mathrm{Tr}\exp[-\beta\mathcal{H}(H_z=0)] . \quad (5)$$

Now observe that

$$\beta^{-1}\frac{\partial}{\partial\lambda}F(\lambda=0)\equiv\beta^{-1}\dot{F}(\lambda=0)$$

$$=\sum_{(RR')}\langle S_R^z S_{R'}^z\rangle . \quad (6)$$

Thus, $\beta^{-1}\dot{F}(\lambda=0)$, which has all the nn pair correlations, is the susceptibility whose long-range spin correlations are all truncated. We might call $\beta^{-1}\dot{F}(\lambda=0)$ the partial susceptibility, denoted by $\tilde{\chi}^{zz}$. For $T>T_c$ we can write

$$\chi^{zz}(0)=\tfrac{1}{4}+\tilde{\chi}^{zz}(0)+\Delta^{zz}(0) , \quad (7)$$

where Δ^{zz} contains all the long-range correlations. As $T\to T_c+$, one would for $\lambda\geq 1$ expect $|\chi^{zz}|\sim|\Delta^{zz}|$ indicating that the critical behavior is determined essentially by long-range correlations alone. However, for the XY model we will show that in the critical region $|\Delta^{zz}|\ll|\chi^{zz}|$. Thus one finds, instead, $\chi^{zz}\sim\tilde{\chi}^{zz}$ as $T\to T_c+$.

III. HIGH-TEMPERATURE EXPANSIONS

For $T>T_c$, the longitudinal susceptibility may be given a series expansion as [see Eq. (4)][10]

$$\chi^{zz}(\lambda=0)=\tfrac{1}{4}+\sum_n (K^n/n!)\,\mathrm{Tr}\sum_{RR'} S_R^z S_{R'}^z P^n$$

$$\equiv\sum_n a_n(K^n/n!) , \quad (8)$$

where $K=\beta J$; the trace (Tr) is to be taken over terms linear in N only (N is the total number of spins) normalized by 2^{-N}; in the second sum $R\neq R'$, otherwise unrestricted; and, using the raising and lowering operators $S^{\pm}=S^x\pm iS^y$, we have reexpressed the XY Hamiltonian in a reduced form

$$P=\sum_{(\alpha\alpha')} S_\alpha^+ S_{\alpha'}^-\equiv(\alpha\alpha') . \quad (9)$$

The partial susceptibility may be similarly expressed [see Eq. (6)],

$$\tilde{\chi}^{zz}(\lambda=0)=\sum_n (K^n/n!)\,\mathrm{Tr}\sum_{(RR')} S_R^z S_{R'}^z P^n$$

$$\equiv\sum_n \tilde{a}_n(K^n/n!) . \quad (10)$$

The expansion (8) appears rather similar to the high-temperature expansion for the zero-field free energy F:

$$F(\lambda=0)=N\ln 2+\sum_n (K^n/n!)\,\mathrm{Tr}\,P^n$$

$$\equiv\sum_n f_n(K^n/n!) . \quad (11)$$

There exists a rather simple relationship between a_n and f_n (hence, between the susceptibility and the free energy). Consider the nth coefficient of expansion (8):

$$a_n=\sum \mathrm{Tr}\, S_R^z S_{R'}^z (S_\alpha^+ S_{\alpha'}^-)(S_\beta^+ S_{\beta'}^-)\cdots(S_\nu^+ S_{\nu'}^-) , \quad (12)$$

where sums on RR' and on the nn pairs $\alpha\alpha'$, $\beta\beta'$, ..., $\nu\nu'$ are implied. Since S^z is itself a traceless operator, the lattice vectors R and R' must both be degenerate with some nn vectors among the set $\alpha\alpha'$, $\beta\beta'$, ..., $\nu\nu'$. The number of ways the vectors RR' can be degenerate is limited due to the relation for $S=\frac{1}{2}$, which may be written as,

$$\mathrm{Tr}[S_u^z\ldots S_v^{\pm}\ldots]=\pm\tfrac{1}{2}\mathrm{Tr}[\ldots S_v^{\pm}\ldots]\,\delta_{uv} . \quad (13)$$

We have shown in Appendix A that repeated applications of (13), together with one simple assumption, lead to the desired result

$$a_n=\tilde{a}_n \quad (14)$$

for all $n\geq 1$. Hence,

$$\chi^{zz}(\lambda=0)=\tfrac{1}{4}+\tilde{\chi}^{zz}(\lambda=0) \quad (15)$$

and from (7) it follows that $\Delta(0)=0$. These conclusions are valid if our assumption, that nn vectors in (12) are nondegenerate (e.g., $\alpha\alpha'\neq\beta\beta'$), is valid. As we shall see, the assumption is valid asymptotically (that is, as the order of expansion becomes large).

Because of (13), the diagonal operator S^z contributes to (12) only as numerical prefactor through one of its eigenvalues ($+\frac{1}{2}$ or $-\frac{1}{2}$). Thus, a_n and f_n must have exactly the same set of configurations. The role of the operator may be regarded as merely lattice decorating a given configuration of f_n. Evidently there are a number of ways a given configuration can be decorated depending on the nature of the configuration. This number, a weighting factor, shall be denoted by $\theta_n(i)$, where i refers to the ith diagram of f_n. Then, one can write (12) as

$$a_n=\sum_i \theta_n(i)\,w(n;i)\,f_n(i) \quad (16)$$

with

$$f_n = \sum_i w(n;\, i)\, f_n(i) \ , \tag{17}$$

where $w(n;\, i)$ denotes the weighting factor of the ith diagram $f_n(i)$. One *expects* that asymptotically

$$\lim_{n\to\infty} \theta_n(i) = \theta_n \tag{18}$$

so that in that limit

$$a_n = \theta_n \sum_i w(n;\, i)\, f_n(i)$$

$$= \theta_n f_n \ . \tag{19}$$

In Sec. IV we shall study θ_n in the asymptotic limit $(n\to\infty)$ and show that $\theta \sim O(n)$. Thus, the critical behavior of the susceptibility is related to that of the free energy simply by a temperature derivative.

IV. ASYMPTOTIC BEHAVIOR OF HIGH-TEMPERATURE SERIES

In this section we shall study the asymptotic behavior of the longitudinal-susceptibility series. Owing to the relation (13), it was pointed out that the coefficients of expansion for the susceptibility and free-energy series, respectively, a_n and f_n, both have identical basis graphs and that the two are thus related by (16). As n becomes very large, highly degenerate graphs[11] begin to contribute with greatly diminished weights to the over-all values of the coefficients and they can be neglected from (16). This sort of approximation evidently becomes more accurate as $n\to\infty$. In this spirit we shall examine the asymptotic behavior of $\theta_n(i)$ and in particular extract the n dependence from (16).

Using (9) we can write the nth expansion of the susceptibility as

$$\sum \mathrm{Tr}\, S_R^z S_{R'}^z P^n = \sum \mathrm{Tr}\, S_R^z S_{R'}^z (\alpha\alpha')(\beta\beta')(\gamma\gamma')\cdots(\nu\nu') \ , \tag{20}$$

where sums over RR' and all nn pairs $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, ..., $\nu\nu'$ are implied. By repeated applications of (13) we may replace the S^z in (20) with the following expression:

$$-2\sum \mathrm{Tr}\, S_R^z S_{R'}^z (\alpha\alpha')(\beta\beta')\cdots(\nu\nu') = \left(\sum + \sum{}' + \sum{}'' + \cdots + \sum{}^{(n-1)}\right)\mathrm{Tr}(\alpha\alpha')(\beta\beta')\cdots(\nu\nu') \ , \tag{21}$$

where $\sum$ means a sum on all nn pairs; $\sum'$ means the same sum except with the restriction $\beta\beta' \neq \alpha\alpha'$; $\sum''$ with $\gamma\gamma' \neq \alpha\alpha'$, $\beta\beta'$; etc. Now observe that the first term of the RHS of (21) is just f_n. The remaining $(n-1)$ terms, although more complicated, can nevertheless be similarly related. In Appendixes B and C, it is shown that for $n\to\infty$

$$\left(\sum + \sum{}' + \cdots + \sum{}^{(n-1)}\right)\mathrm{Tr}(\alpha\alpha')(\beta\beta')\cdots(\nu\nu') = \tfrac{1}{3} n\, f_n + O(f_n) \ . \tag{22}$$

Hence, from (8), (20), and (21) we obtain for $n\to\infty$

$$a_n \sim n\, f_n + O(f_n) \tag{23}$$

and

$$\chi^{zz} \sim \frac{\partial}{\partial K} F \ . \tag{24}$$

Thus, if the critical behavior of the free energy F is defined as $F \sim \Delta T^{-\alpha+2}$, where $\Delta T = T - T_c$ and α is the specific-heat exponent, then[12] it follows from (24) that $\chi^{zz} \sim \Delta T^{-\alpha+1}$.

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APPENDIX A: PROOF OF EQ. (14)

Let, in (12), R be fixed, say $R = \alpha$, and consider the sum over R' first. Then, with (13),

$$\frac{1}{2}\sum_{R'} \mathrm{Tr}\, S_{R'}^z (S_\alpha^+ S_{\alpha'}^-)(S_\beta^+ S_{\beta'}^-)\cdots(S_\nu^+ S_{\nu'}^-)\,\delta_{R\alpha}$$

$$= \frac{1}{2}\sum_{R'} \mathrm{Tr}\, S_{R'}^z (S_\alpha^+ S_{\alpha'}^-)(S_\beta^+ S_{\beta'}^-)\cdots(S_\nu^+ S_{\nu'}^-)\,\delta_{R\alpha}$$

$$\times[\delta_{R'\alpha'} + (\delta_{R'\beta} + \delta_{R'\beta'}) + \cdots + (\delta_{R'\nu} + \delta_{R'\nu'})] \ , \tag{A1}$$

where we have assumed that the nn vectors are nondegenerate (e.g., $\alpha\alpha' \neq \beta\beta'$). If they are degenerate, there is an overcounting which must be handled separately (see Appendix C). Now observe that except for the first Kronecker δ, $\delta_{R'\alpha'}$, the others are all exactly paired, each pair corresponding to a pair of nn lattice vectors. Hence, the application of (13) leads to an exact cancellation of all these paired terms, leaving the one unpaired (leading) term. Now carrying out the sum over R, we obtain

$$-(\tfrac{1}{2})^2\, \mathrm{Tr}\,(S_\alpha^+ S_{\alpha'}^-)(S_\beta^+ S_{\beta'}^-)\cdots(S_\nu^+ S_{\nu'}^-)$$

$$\times(\delta_{R\alpha}\delta_{R'\alpha'} + \delta_{R\beta}\delta_{R'\beta'} + \cdots + \delta_{R\nu}\delta_{R'\nu'})$$

$$= \mathrm{Tr}\,(S_R^z S_{R'}^z)\,(S_\alpha^+ S_{\alpha'}^-)\,(S_\beta^+ S_{\beta'}^-)\cdots(S_\nu^+ S_{\nu'}^-)\ . \qquad \text{(A2)}$$

That is, the lattice vectors $\vec{R}$ and $\vec{R}'$ are now themselves nn vectors $|\vec{R}-\vec{R}'| = R_0$, where R_0 is the nn distance. This is precisely the restriction on the lattice sum for the partial susceptibility $\tilde{\chi}^{zz}$ [see Eq. (10)].

APPENDIX B: ASYMPTOTIC BEHAVIOR

Consider the second term of (21). The restriction on the sum $\beta\beta' \neq \alpha\alpha'$ may be removed in the following way: with the unaffected nn pairs denoted as Q,

$$\sum{}' \mathrm{Tr}(\alpha\alpha')(\beta\beta')Q$$
$$= \sum \mathrm{Tr}[(\alpha\alpha') - (\alpha\beta) - (\beta'\alpha') + (\beta'\beta)](\beta\beta')Q\ . \qquad \text{(B1)}$$

Observe that the first term of the RHS of (B1) again reproduces f_n. The second term can also be related to f_n. For $n>1$, the following relation holds[13]:

$$\sum \mathrm{Tr}(\alpha\beta)(\beta\beta')Q$$
$$= \frac{1}{n-1}\sum \mathrm{Tr}(\alpha\alpha')(\beta\beta')\cdots(\nu\nu')\ . \qquad \text{(B2)}$$

The third term leads to the same results as (B2). The last remaining term, denoted as A, will be treated later. Hence, together, for $n>1$,

$$\sum{}' \mathrm{Tr}(\alpha\alpha')(\beta\beta')Q = \left(1 - \frac{2}{n-1}\right) f_n + A_1^{(n)}\ , \qquad \text{(B3)}$$

where

$$A_1^{(n)} = \sum \mathrm{Tr}(\beta'\beta)(\beta\beta')Q\ . \qquad \text{(B4)}$$

Now consider the third term of the RHS of (21), where the restrictions on the sums are $\gamma\gamma' \neq \alpha\alpha'$, $\beta\beta'$. These restrictions can be more readily seen on Fig. 1. Proceeding as in the previous case, we obtain for $n>1$,

$$\sum{}'' \mathrm{Tr}(\alpha\alpha')(\beta\beta')(\gamma\gamma')Q$$
$$= \left(1 - \frac{4}{n-1}\right) f_n + A_2^{(n)} + B_2^{(n)}\ , \qquad \text{(B5)}$$

where

$$A_2^{(n)} = \sum \mathrm{Tr}[(\alpha\alpha')(\gamma'\gamma) + (\gamma'\gamma)(\beta\beta')](\gamma\gamma')Q\ , \qquad \text{(B6)}$$

$$B_2^{(n)} = \sum \mathrm{Tr}[(\alpha\gamma)(\gamma'\beta') + (\gamma'\alpha')(\beta\gamma)](\gamma\gamma')Q\ . \qquad \text{(B7)}$$

As may be apparent from (B6) and (B7), the remainders $A_2^{(n)}$ and $B_2^{(n)}$ are of different types. Whereas the former has unmixed nn lattice vectors, the latter has mixed nn lattice vectors. These are, as turned out, the only two types of remainders that the other terms of (21) can have.

The fourth term of the RHS of (21) can thus be immediately written down. For $n>1$,

$$\sum{}''' \mathrm{Tr}(\alpha\alpha')(\beta\beta')(\gamma\gamma')(\delta\delta')Q$$
$$= \left(1 - \frac{6}{n-1}\right) f_n + A_3^{(n)} + B_3^{(n)}\ , \qquad \text{(B8)}$$

where the remainders $A_3^{(n)}$ and $B_3^{(n)}$ are defined as (B6) and (B7), respectively, but with one added degree of freedom through a new pair of nn lattice vectors $\delta\delta'$. That is,

$$A_3^{(n)} = \sum \mathrm{Tr}[(\alpha\alpha')(\beta\beta')(\delta'\delta) + (\alpha\alpha')(\delta'\delta)(\gamma\gamma')$$
$$+ (\delta'\delta)(\beta\beta')(\gamma\gamma')](\delta\delta')Q\ . \qquad \text{(B9)}$$

$$B_3^{(n)} = \sum \mathrm{Tr}[(\alpha\delta)(\delta'\beta')(\gamma\gamma') + (\alpha\delta)(\beta\beta')(\delta'\gamma')$$
$$+ (\delta'\alpha')(\beta\delta)(\gamma\gamma') + (\alpha\alpha')(\beta\delta)(\delta'\gamma') + (\delta'\alpha')(\beta\beta')(\gamma\delta)$$
$$+ (\alpha\alpha')(\delta'\beta')(\gamma\delta)](\delta\delta')Q\ . \qquad \text{(B10)}$$

By induction it is possible to write a similar expression for any other term in (21). Thus, we obtain

$$\left(\sum + \sum{}' + \cdots + \sum{}^{(n-1)}\right) \mathrm{Tr}(\alpha\alpha')(\beta\beta')\cdots(\nu\nu')$$
$$= A^{(n)} + B^{(n)}\ , \qquad \text{(B11)}$$

where[14]

$$A^{(n)} = A_1^{(n)} + A_2^{(n)} + \cdots + A_{n-1}^{(n)}\ , \qquad \text{(B12)}$$

$$B^{(n)} = B_2^{(n)} + B_3^{(n)} + \cdots + B_{n-1}^{(n)}\ . \qquad \text{(B13)}$$

Now the lattice vectors which appear in (B1) and (B10), also in (B6) and (B7), are arbitrary. This implies that the three separate components of $A_3^{(n)}$, for example, are each identical. Hence, for $1 \le l \le n-1$,

$$A_l^{(n)} = l A_1^{(n)} \qquad \text{(B14)}$$

and

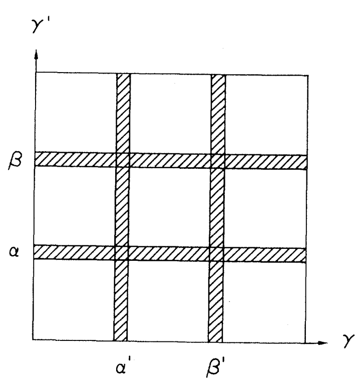

FIG. 1. $\sum{}'' \mathrm{Tr}(\alpha\alpha')(\beta\beta')(\gamma\gamma')Q$. The restrictions imposed on the lattice sums [see Eq. (B5)] are $\gamma\gamma' \neq \alpha\alpha'$, $\beta\beta'$ only. The allowed values of the nn vectors $\gamma\gamma'$ are schematically shown here. For fixed values of $\alpha\alpha'$ and $\beta\beta'$ (shaded strips), the allowed values of $\gamma\gamma'$ correspond to the unshaded areas. All other nn vectors contained in Q [see Eq. (B1)] have unrestricted lattice sums and are thus not shown here.

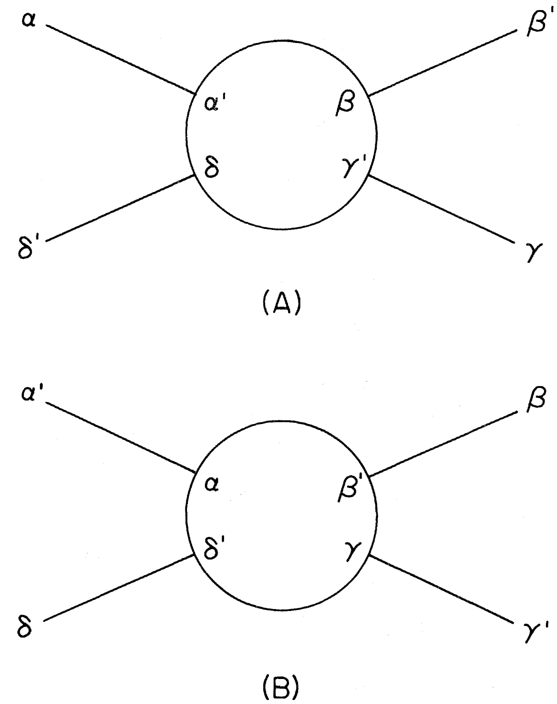

FIG. 2. Degenerate lattice vectors in Eq. (C1) are schematically illustrated. Lattice vectors which appear on the circles are degenerate, e.g., $\alpha'=\beta=\gamma'=\delta$ in the upper figure. (Note that the four inner vectors in the upper or lower circle become coincident if the circle shrinks to a point). Pairs of vectors connected by straight lines, e.g., $\alpha\alpha'$, are nearest neighbors with lines denoting nn distances. The four outer vectors, e.g., $\alpha\beta'\gamma\delta'$ in the upper figure, are thus interlinked via one common nn point.

$$B_l^{(n)}=\tfrac{1}{2}l(l-1)B_2^{(n)}\,. \tag{B15}$$

With (B14) and (B15), $A^{(n)}$ and $B^{(n)}$ can be directly given by

$$A^{(n)}=\tfrac{1}{2}n(n-1)A_1^{(n)} \tag{B16}$$

and

$$B^{(n)}=\tfrac{1}{3}n(n-1)(n-2)B_2^{(n)}\,. \tag{B17}$$

Hence, the asymptotic behavior of the susceptibility series is contained in the remainders A_1 and B_2. It is shown in Appendix D that for $n\to\infty$

$$|A_1^{(n)}|<|B_2^{(n)}|\,. \tag{B18}$$

Also, it follows from (B2) that for $n\gg 1$

$$B_2^{(n)}=\frac{2}{(n-1)^2}f_n\,. \tag{B19}$$

The above results (B18) and (B19), together with (B11), (B16), and (B17), give for $n\to\infty$

$$A^{(n)}+B^{(n)}=\tfrac{1}{3}nf_n+O(f_n)\,. \tag{B20}$$

APPENDIX C: DEGENERATE EXPANSIONS

The expansions (A1), (A2), and (22) are valid provided that nn vectors are all nondegenerate. When nn vectors are multiply degenerate, the cancellations described in obtaining (A2) are not exactly satisfied. This sort of multiple degeneracy in nn vectors occurs at higher-order expansions. We shall show below the simplest cases of multiple degeneracy which can lead to imperfect cancellations. Consider the following:

$$D(1)=-2\sum \mathrm{Tr}\,S_R^zS_{R'}^z(\alpha\alpha')(\beta\beta')(\gamma\gamma')(\delta\delta')\cdots(\nu\nu')\times\delta_{\alpha'\beta}\delta_{\beta\gamma'}\delta_{\gamma'\delta}\,. \tag{C1}$$

The degenerate vectors are also illustrated in Fig. 2(a). Those vectors which lie on the circle are degenerate (note that the four vectors become coincident if the circle shrinks to a point). The straight lines connecting pairs of vectors (e.g., $\alpha\alpha'$) indicate nn distances. Assume that the other vectors $\alpha\beta'\gamma\delta'$ are nondegenerate. The operators S^z may now be replaced by their eigenvalues. As was pointed out, the diagonalization processes can be viewed as decorations of vertex points.

The decorations may be divided into two types: (i) those connecting nn pairs (e.g., $\alpha\alpha'$ or $\alpha\beta$) and (ii) those connecting non-nn pairs (e.g., $\alpha\beta'$ or $\beta'\delta'$). The difference between the expansions for the susceptibility and the partial susceptibility, a_n and $\tilde{a}_n$, respectively, is in the type-(ii) decorations. The decorations for a_n leave these non-nn distances unchanged, but the decorations for $\tilde{a}_n$ can convert the non-nn distances into nn distances (thereby reducing the sizes of the configurations). Applications of (13) show that the type-(i) decorations are still exactly canceled, but the type-(ii) decorations are not.

There is a second choice of multiple degeneracy, which is entirely equivalent to (C1), and is given below:

$$D(2)=-2\sum \mathrm{Tr}\,S_R^zS_{R'}^z(\alpha\alpha')(\beta\beta')(\gamma\gamma')\cdots(\nu\nu')\times\delta_{\alpha\beta'}\delta_{\beta'\gamma}\delta_{\gamma\delta'}\,. \tag{C2}$$

In Fig. 2(b), the degenerate vectors are illustrated. The operators S^z may be similarly replaced by their eigenvalues. As in (C1), the type-(ii) decorations are not all exactly canceled. Thus, these remainders as well as those from (C1) together contribute to the expansions for the susceptibility and the partial susceptibility. This conclusion is still valid even if the four vectors, e.g., $\alpha\beta'\gamma\delta'$ for (C1), are not all distinct. If, for example, $\gamma=\delta'$ in (C1), one still obtains the same results.

Using the arguments such as (B2), one can show that for large n,

$$|D(1)+D(2)| < \frac{2}{(n-1)(n-2)(n-3)} \left| \sum \mathrm{Tr}(\alpha\alpha')(\beta\beta')\cdots(\nu\nu') \right| . \tag{C3}$$

That is, as $n\to\infty$, the remainders behave as $O(n^{-3}f_n)$. Thus, compared with $B_2^{(n)}$ [see (B19)], which behaves as $O(n^{-2}f_n)$, the remainders due to the fourfold degeneracy do not contribute asymptotically. The next permissible multiple degeneracy is a sixfold degeneracy. A similar argument may be applied to show that the remainders due to the sixfold degeneracy behave as $O(n^{-5}f_n)$ and are therefore asymptotically of no significance.

Multiple degenerate terms such as (C1) can occur at higher-order expansions. The first such term is found in fifth order ($n=5$). Hence, up to fourth order, the susceptibility and the partial susceptibility are expected to be term-by-term identical. There are several fourfold degenerate terms in sixth order. Since those factors, which distinguish $\tilde{a}_n$ from a_n, are asymptotically not significant, the two expansions must have the same asymptotic behavior. Finally, the expression (21) is also valid asymptotically since multiply degenerate terms are not included in the expansion.

APPENDIX D: $A_1^{(n)} < B_2^{(n)}$

The two restrictively summed expansion terms are again shown below:

$$A_1 = \sum \mathrm{Tr}(\alpha\alpha')(\beta\beta')(\gamma\gamma')\cdots(\nu\nu')\,\delta_{\alpha\beta'}\,\delta_{\alpha'\beta}\,, \tag{D1}$$

$$B_2 = \sum \mathrm{Tr}(\alpha\alpha')(\beta\beta')(\gamma\gamma')\cdots(\nu\nu')\,\delta_{\alpha\gamma'}\,\delta_{\beta'\gamma}\,. \tag{D2}$$

For small n, it is relatively easy to show numerically that these quantities are positive. Although it is not essential, we shall thus assume (by induction) that both $A_1^{(n)}$ and $B_2^{(n)}$ remain non-negative as $n\to\infty$.

Consideration of the restrictions in (D1) and (D2) shows that, for n large, $B_2^{(n)}$ contains the complete set of diagrams for f_n, whereas $A_1^{(n)}$ contains only a partial set of diagrams. Among those missing are the *leading* term of f_n,

$$\sum \mathrm{Tr}(\alpha\alpha')(\beta\beta')(\gamma\gamma')\cdots(\mu\mu')(\nu\nu') \times \delta_{\alpha'\beta}\delta_{\beta'\gamma}\cdots\delta_{\mu'\nu}\delta_{\nu'\alpha}$$

and its derivative terms resulting from further degeneracy among nn vectors. The removal of these terms from f_n thus establishes the inequality between $A_1^{(n)}$ and $B_2^{(n)}$.

One can also use arguments similar to (B2) to show that $A_1^{(n)} = O(n^{-3}f_n)$. Hence, the inequality is again satisfied asymptotically.

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[1]For a recent review on the critical behavior of the XY model, see D. D. Betts, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1974), Vol. III.

[2]M. E. Fisher, Physica 26, 618 (1960).

[3]M. E. Fisher (private communication, 1972). The author wishes to thank Dr. Fisher for showing him this argument.

[4]M. E. Fisher, Phys. Rev. 176, 257 (1968). The hypothesis states that "mild" perturbations merely produce a λ line along which critical exponents do not change (although amplitudes and critical temperatures may vary smoothly). Also, on universality or smoothness, see R. B. Griffiths, Phys. Rev. Lett. 24, 1479 (1970); R. Abe, Prog. Theor. Phys. 44, 339 (1970); and L. P. Kadanoff, in *Proceedings of the Enrico Fermi Summer School, Varenna*, 1970, edited by M. S. Green (Academic, New York, 1971). D. Jasnow and M. Wortis [Phys. Rev. 176, 739 (1968)] have also studied the changes of critical exponents caused by changes of symmetry of the ground state. Based on numerical evidence they were led to a related hypothesis which reinforces Fisher's original idea.

[5]Note that the two regions are separated by the isotropic Heisenberg limit ($\lambda=1$). Thus, there are two stages in the discontinuity: from the Ising-like region to the Heisenberg limit and from the XY-like region to the Heisenberg limit.

[6]T. Obokata, I. Ono, and T. Oguchi, J. Phys. Soc. Jpn. 23, 516 (1967); K. Pirnie (private communication to D. D. Betts, 1968). The author wishes to thank Dr. Pirnie and Dr. Betts for this unpublished information. M. H. Lee, J. Math. Phys. 12, 61 (1971); J. Rogiers, and R. Dekeyser, Phys. Lett. A 46, 206 (1973). Also see D. W. Wood and N. W. Dalton [J. Phys. C 5, 1675 (1972)] who have obtained $\chi^{**}(\lambda)$ for general spin through sixth order.

[7]M. H. Lee, Phys. Rev. B 8, 1203 (1973).

[8]M. H. Lee, Phys. Rev. B 8, 3290 (1973).

[9]R. V. Ditzian and D. D. Betts, Can. J. Phys. 50, 129 (1972); S. R. Mattingly and D. D. Betts, *ibid.* 50, 2415 (1972).

[10]M. H. Lee, see Ref. 6.

[11]A highly degenerate graph satisfies the inequality $N_l \gg N_v$, where N_l and N_v are, respectively, the numbers of directed lines and vertices of the given graph. See also D. D. Betts, C. J. Elliott, and M. H. Lee, Can. J. Phys. 48, 1566 (1970).

[12]The best numerical evidence shows that in three dimensions $\alpha\approx 0$ [D. D. Betts and J. R. Lothian, Can. J. Phys. 51, 2249 (1973)] and in two dimensions the specific heat is nonsingular [D. D. Betts, J. T. Tsai, and C. J. Elliott, in Proceedings of the International Conference on Magnetism, Moscow, 1973 (unpublished).]

[13]For (B2) to be nonvanishing, all nn vectors must be appropriately degenerate at least once (see Ref. 10). The choice $\alpha'=\beta$ represents only one of $(n-1)$ possible choices for α'.

[14]The leading terms (those proportional to f_n) are exactly canceled and only remainders $A^{(n)}$ and $B^{(n)}$ survive.