# Critical interactions for the triangular spin-$s$ Ising model by a spin-restructuring transformation*

A. Nihat Berker
*Department of Physics, University of Illinois, Urbana, Illinois 61801*
(Received 18 February 1975)

A transformation is constructed which maps the spin-$s$ Ising model onto a spin-1/2 Ising model with further-neighbor and many-site interactions. Renormalization-group methods may thus be applied directly to the spin-$s$ problem by prefacing the usual sequence of position-space rescaling transformations with this spin-restructuring transformation. We demonstrate the method by calculating the critical interactions (i.e., the inverse critical temperatures) of the spin-$s$ Ising model on the triangular lattice, using the recent results of Niemeijer and van Leeuwen for the critical subspace of the corresponding spin-1/2 Ising model. Our evaluations agree very well with recent series data.

## I. INTRODUCTION: THE SPIN-$s$-TO-SPIN-$\frac{1}{2}$ TRANSFORMATION

Uncertainty in the convergence properties of the $\epsilon$ expansion[1] away from space dimensionality $d=4$ motivates direct development of renormalization-group transformations[1] for systems with $d=2$, 3. In particular, Niemeijer and van Leeuwen[2] (NvL) and others[3–6] have produced impressive results for the critical properties of the two-dimensional spin-$\frac{1}{2}$ Ising model. Their transformation systematically replaces each group of spins by a single effective spin, thus inducing a change in the length scale and a renormalization of the interaction constants, as earlier envisaged qualitatively by Kadanoff.[7] Starting, for example, with nearest-neighbor pair interactions only, this position-space rescaling transformation generates further-neighbor and many-site interactions; however, one is capable of keeping track of finite numbers of such additional interactions and results[2] suggest that convergence is rapid as successively larger numbers are retained.

In this paper we show how the results of such spin-$\frac{1}{2}$ calculations[2–6] can easily be extended to arbitrary spin[8] by prefacing the sequence of position-space rescaling transformations with a single "spin-restructuring" transformation, which leaves position space (i.e., lattice structure and length scale) intact but maps the spin-$s$ problem onto a spin-$\frac{1}{2}$ problem with appropriately renormalized interactions. Like the rescaling transformations, the spin-restructuring transformation introduces further-neighbor and many-site interactions; however, convergence is again rapid, so that this appears to be no major difficulty.

In the remainder of this section, we develop the spin-restructuring transformation. The renormalized spin-$\frac{1}{2}$ interaction constants can then be obtained perturbatively in powers of the original spin-$s$ interaction constants to any desired order in a cumulant expansion. In Sec. II this general technique is carried out in detail for the spin-$s$ Ising model on the triangular lattice. By locating in the space of spin-$\frac{1}{2}$ interaction constants the intersection of the subspace onto which the spin-$s$ problem is mapped with the critical subspace as given by NvL,[2] the critical spin-$s$ interactions (i.e., the inverse critical temperatures) are evaluated. Our evaluations for a variety of spins agree very well with recent series data of Van Dyke and Camp.[9] Our procedure confirms the critical-exponent universality ($s$ independence), since the spin-$s$ and spin-$\frac{1}{2}$ critical Hamiltonians iterate to a single fixed point. Section III discusses possible application of the method to other problems.

Consider a spin-$s$ Ising model. At each lattice site $i$, $s_i^z$ can take on the values $-s, -s+1, \ldots, s-1$, and $s$. The general Hamiltonian is[2,3]

$$\mathcal{H}(\{\mu_i\}) = \sum_a K_a \mu_a , \qquad (1.1)$$

where $K_a$ are the interaction constants, $\mu_a \equiv \prod_i^a \mu_i \equiv \prod_i^a (s_i^z/s)$ are the corresponding spin products, and the sum $a$ is over all subsets of lattice sites. Now we define a new variable $\sigma_i$ at each lattice site

$$\sigma_i = \operatorname{sgn}(\mu_i) \quad \text{for } \mu_i \neq 0. \qquad (1.2)$$

The transformation is accomplished by an NvL-like partial trace of the partition function

$$e^{\mathcal{H}'(\{\sigma_i\})} = \sum_{\{\mu_i\}_{\text{fixed }\{\sigma_i\}}} e^{\mathcal{H}(\{\mu_i\})}. \qquad (1.3)$$

The sum is over all configurations $\{\mu_i\}$ of the spin-$s$ variables which are in accordance with the specified configuration $\{\sigma_i\}$ of the spin-$\frac{1}{2}$ variables, with the added provision that for integer $s$, $\mu_i = 0$ is to contribute equally to $\sigma_i = +1$ and to $\sigma_i = -1$, i.e., terms having $\mu_i = 0$ are kept in the sum, with a factor of $\frac{1}{2}$, for either specification of $\sigma_i$. Then the Hamiltonian $\mathcal{H}'(\{\sigma_i\})$ will have the general form

$$\mathcal{H}'(\{\sigma_i\}) = \sum_b L_b \sigma_b, \quad \sigma_b \equiv \prod_i^b \sigma_i, \quad \sigma_i = \pm 1, \qquad (1.4)$$

where we assume that the new interaction constants will be nonsingular functions of the old ones

$$L_b = L_b(\{K_a\}). \qquad (1.5)$$

This assumption of analyticity is also made for the recursion relations in other works.[2-6] Thus, a spin-$\frac{1}{2}$ Ising model with the same lattice structure results. The partition function has been conserved

$$Z=\sum_{\{\sigma_i\}} e^{\mathcal{H}'(\{\sigma_i\})}=\sum_{\{\mu_i\}} e^{\mathcal{H}(\{\mu_i\})}. \qquad (1.6)$$

It is, of course, impossible to perform directly the summation in (1.3) in order to obtain explicitly (1.5). We therefore treat $\mathcal{H}$ perturbatively and develop a cumulant expansion. First, we introduce the following notation:

$$\langle A\rangle_{\{\sigma_i\}} \equiv (s+\tfrac{1}{2})^{-N} \sum_{\{\mu_i\}\,\mathrm{fixed}\{\sigma_i\}} A, \qquad (1.7)$$

where $N$ is the number of sites. Then (1.3) gives

$$\mathcal{H}'(\{\sigma_i\})=\ln\langle e^{\mathcal{H}(\{\mu_i\})}\rangle_{\{\sigma_i\}}+N\ln(s+\tfrac{1}{2}). \qquad (1.8)$$

The second term contributes only to the additive constant $L_0$ in $\mathcal{H}'$, i.e., to the spin-$\frac{1}{2}$ interaction "coupling" the empty subset of lattice sites; since $L_0$ does not affect the determination of the critical spin-$s$ interactions $K_a^c$, this term is ignored in our present calculation. Expanding the first term in powers of $\mathcal{H}$,

$$\begin{aligned}\mathcal{H}'(\{\sigma_i\})=&\langle\mathcal{H}\rangle+\frac{1}{2!}\left(\langle\mathcal{H}^2\rangle-\langle\mathcal{H}\rangle^2\right)\\&+\frac{1}{3!}\left(\langle\mathcal{H}^3\rangle-3\langle\mathcal{H}\rangle\langle\mathcal{H}^2\rangle+2\langle\mathcal{H}\rangle^3\right)\\&+\frac{1}{4!}\left(\langle\mathcal{H}^4\rangle-4\langle\mathcal{H}\rangle\langle\mathcal{H}^3\rangle-3\langle\mathcal{H}^2\rangle^2\right.\\&\left.+12\langle\mathcal{H}\rangle^2\langle\mathcal{H}^2\rangle-6\langle\mathcal{H}\rangle^4\right)+\cdots. \qquad (1.9)\end{aligned}$$

For each order in this cumulant expansion, we can now explicitly obtain (1.5). This is demonstrated in the Appendix.

## II. APPLICATION TO THE TRIANGULAR ISING MODEL

We now apply this method to the triangular spin-$s$ Ising model with nearest-neighbor pair interactions only; extension to more diversified short-ranged interactions is straightforward. Our starting Hamiltonian is the following special case of (1.1):

$$\mathcal{H}(\{\mu_i\})=K\sum_{\langle ij\rangle}\mu_i\mu_j, \qquad (2.1)$$

where the sum is over all nearest-neighbor pairs. We proceed to obtain the renormalized spin-$\frac{1}{2}$ interactions $L_b$ as power series in $K$ by using (1.9).

In evaluating the right-hand side of (1.9) contributions to each order can be identified with linear graphs (see Fig. 1 and Appendix): each interaction is associated with a bond, and each spin is associated with a vertex. Graphs contributing to the $n$th order thus have $n$ bonds. Because of cancellations within each cumulant order, disconnected graphs do not occur; this is expected, since otherwise the energy per site would not be intensive. Graphs with no odd vertices contribute only to the additive constant $L_0$ in $\mathcal{H}'$; again, since $L_0$ does not affect the determination of the critical spin-$s$ interactions $K^c$, these graphs are ignored in our present calculation. Within each order $n$, we further label with $m$ the remaining connected odd-vertex graphs. All such graphs are shown for $n\leq 4$ in Fig. 1. The contribution to the cumulant expansion of the graph $(n, m)$ is

$$\sum_{\text{weak embeddings}} K^n g_m^{(n)} \prod_i^{\text{odd}} \sigma_i, \qquad (2.2)$$

where the product is over all odd vertices of $(n, m)$, and the sum is over all distinct weak embeddings of $(n, m)$ in the lattice. "Weak embedding" means[10] that the graph is fitted into the lattice so that all its bonds connect nearest-neighbor sites and no more than one of its vertices is assigned to each site. The graph coefficients $g_m^{(n)}$ depend only on $s$. Their evaluation from (1.9) is straightforward, as shown in the Appendix, although tedious; the resulting expressions are given for $n\leq 4$ in Table I.

Thus, a specific spin-$\frac{1}{2}$ interaction $L_b$ is obtained by counting all distinct weak embeddings in the triangular lattice which result in the said interaction. For example, Fig. 2 shows all third-order weak embeddings which result in a particular next-nearest-neighbor pair interaction $L_2$. From this figure and (2.2) we conclude

$$L_2^{(3)}=6K^3 g_1^{(3)}. \qquad (2.3)$$

The general form for such equations is

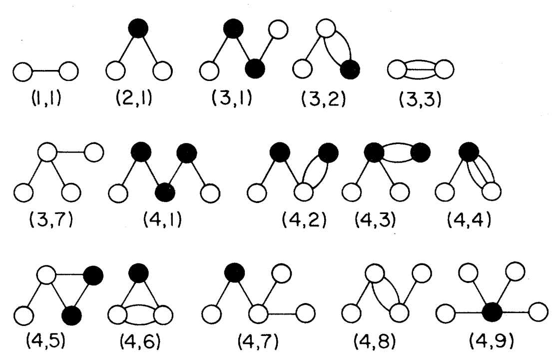


FIG. 1. All odd-vertex linear graphs $(n, m)$ contributing to the fourth-order cumulant expansion for the $s\rightarrow\frac{1}{2}$ transformation. Odd (coupled) vertices are shown with open circles.

TABLE I. Graph coefficients $g_m^{(n)}$ for $n \leq 4$. Each coefficient determines the contribution of the linear graph $(n, m)$ to the cumulant expansion for the $s \to \frac{1}{2}$ transformation, as given by (2.2). $f_l \equiv \langle (u_i)^l \rangle_{\sigma_i=+1}$ as in (1.7) and Appendix, where the explicit evaluation of $g_1^{(2)}$ is presented.

| $n, m$ | $g_m^{(n)}$ |
|---|---|
| 1, 1 | $f_1^2$ |
| 2, 1 | $f_1^2(f_2 - f_1^2)$ |
| 3, 1 | $f_1^2(f_2 - f_1^2)^2$ |
| 3, 2 | $\frac{1}{2}(f_1 f_2 f_3 - f_1^2 f_2^2 - 2 f_1^4 f_2 + 2 f_1^6)$ |
| 3, 3 | $\frac{1}{6}(f_3^2 - 3 f_1^2 f_2^2 + 2 f_1^6)$ |
| 3, 7 | $f_1^3 f_3 - 3 f_1^4 f_2 + 2 f_1^6$ |
| 4, 1 | $f_1^2(f_2 - f_1^2)^3$ |
| 4, 2 | $\frac{1}{2}(f_1 f_2^2 f_3 - f_1^2 f_2^3 - f_1^3 f_2 f_3 - f_1^4 f_2^2 + 4 f_1^6 f_2 - 2 f_1^8$ |
| 4, 3 | $\frac{1}{2}(f_1^2 f_2 f_4 - f_1^2 f_2^3 - 2 f_1^3 f_2 f_3 - 2 f_1^5 f_3 + 10 f_1^6 f_2 - 6 f_1^8$ |
| 4, 4 | $\frac{1}{6}(f_1 f_3 f_4 - f_1^2 f_3^2 - 3 f_1^2 f_2^3 - 3 f_1^3 f_2 f_3 + 6 f_1^4 f_2^2 + 6 f_1^6 f_2 - 6 f_1^8)$ |
| 4, 5 | $f_1 f_2^2 f_3 - f_1^2 f_2^3 - 4 f_1^4 f_2^2 - f_1^5 f_3 + 9 f_1^6 f_2 - 4 f_1^8$ |
| 4, 6 | $\frac{1}{2}(f_2 f_3^2 - 3 f_1^2 f_2^3 - 2 f_1^3 f_2 f_3 + 10 f_1^6 f_2 - 6 f_1^8)$ |
| 4, 7 | $f_1^3 f_2 f_3 - 3 f_1^4 f_2^2 - f_1^5 f_3 + 5 f_1^6 f_2 - 2 f_1^8$ |
| 4, 8 | $\frac{1}{2}(f_1^2 f_3^2 - 2 f_1^3 f_2 f_3 - 3 f_1^4 f_2^2 + 8 f_1^6 f_2 - 4 f_1^8)$ |
| 4, 9 | $f_1^4 f_4 - 3 f_1^4 f_2^2 - 4 f_1^5 f_3 + 12 f_1^6 f_2 - 6 f_1^8$ |

$$L_b = \sum_{n=1}^{\infty} K^n \left( \sum_m A_{bm}^{(n)} g_m^{(n)} \right), \qquad (2.4)$$

so that (2.3) means $A_{2m}^{(3)} = 6\delta_{1m}$. It is through this weak embedding matrix $A_{bm}^{(n)}$ only that the lattice type enters into the problem; its nonzero, $n \leq 4$ elements are given in Table II. Once this matrix and the diagram coefficients $g_m^{(n)}$ are computed, (2.4) provides an explicit realization of the mapping in (1.5) to any desired order in $K$. In our present calculation, we truncate (2.4) after $n = 4$, since subsequent terms generate an ever increasing number of further-neighbor and many-site spin-$\frac{1}{2}$ interactions which are not dealt with in NvL's solution,[2] with which we have to make contact. However, our first four kept orders show good convergence[11] and

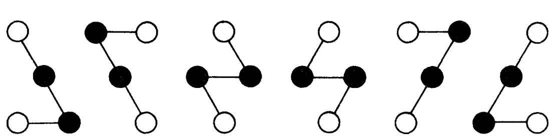

FIG. 2. All third-order weak embeddings in the triangular lattice which result in a particular next-nearest-neighbor pair interaction $L_2$. The two coupled sites are shown with open circles.

yield satisfactory results as will be seen below. All spin-$\frac{1}{2}$ interactions $L_b$ generated by our fourth-order cumulant expansion are shown in Fig. 3. The lowest order in which each interaction first appears and the number of sites it couples are indicated. We note here that the graph coefficients $g_7^{(3)}$, $g_7^{(4)}$, and $g_8^{(4)}$ vanish for half-integer $s$ and $s = \infty$; therefore, the interactions $L_9$, $L_{10}$, $L_{11}$, and $L_{15}$–$L_{25}$ are not generated in these cases.

We have completed the $s \to \frac{1}{2}$ transformation for the triangular Ising model. However, in order to get any use out of it, we now need the solution of the resulting spin-$\frac{1}{2}$ Ising model with many types of interactions. Lacking this, we use the best approximate solution, as provided by NvL.[2] In making contact with their best solution, we introduce two further approximations into our treatment: (i) NvL's best seventh-order cluster expansion is set up so as not to generate (and, therefore, not to deal with) any interaction which cannot be contained within the basic seven-site hexagon of the triangular lattice. We shall therefore ignore[12] interactions $L_4$, $L_5$ first generated by the third-order term in our cumulant expansion, and interactions $L_6$, $L_7$, $L_8$, and $L_{15}$–$L_{25}$ first generated by the fourth-order term. Note that $L_{15}$–$L_{25}$ are already not there for half-integer $s$ and $s = \infty$. In

TABLE II. Nonzero elements of the weak embedding matrix $A_{bm}^{(n)}$ for $n \leq 4$. These relate the interaction constants $L_b$ to the graph coefficients $g_m^{(n)}$, as given in (2.4).

Pair interactions

| $b$ | $n$ = 1, $m$ = 1 | $n$ = 2, $m$ = 1 | $n$ = 3, $m$ = 1 | $n$ = 3, $m$ = 2 | $n$ = 3, $m$ = 3 | $n$ = 4, $m$ = 1 | $n$ = 4, $m$ = 2 | $n$ = 4, $m$ = 3 | $n$ = 4, $m$ = 4 | $n$ = 4, $m$ = 5 | $n$ = 4, $m$ = 6 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 10 | 1 | 10 | 16 | 8 | 4 | 8 | 2 |
| 2 | | 2 | 6 | | | 16 | 20 | 8 | 4 | | |
| 3 | | 1 | 6 | | | 18 | 10 | 4 | 2 | | |
| 4 | | | 3 | | | 16 | | | | | |
| 5 | | | 1 | | | 12 | | | | | |
| 6 | | | | | | 6 | | | | | |
| 7 | | | | | | 4 | | | | | |
| 8 | | | | | | 1 | | | | | |

Four-site interactions

| $b$ | $n$ = 3, $m$ = 7 | $n$ = 4, $m$ = 7 | $n$ = 4, $m$ = 8 | $n$ = 4, $m$ = 9 |
|---|---|---|---|---|
| 9 | 2 | | 4 | |
| 10 | 1 | 5 | 2 | |
| 11 | 1 | 6 | | |
| 12 | | 2 | 1 | 1 |
| 13 | | | | 1 |
| 14 | | 1 | | 1 |
| 15 | | 2 | 1 | |
| 16 | | 1 | 1 | |
| 17 | | | 1 | |
| 18 | | 1 | | |
| 19 | | 2 | | |
| 20 | | 1 | | |
| 21 | | 1 | | |
| 22 | | 2 | | |
| 23 | | 3 | | |
| 24 | | 1 | | |
| 25 | | 2 | | |

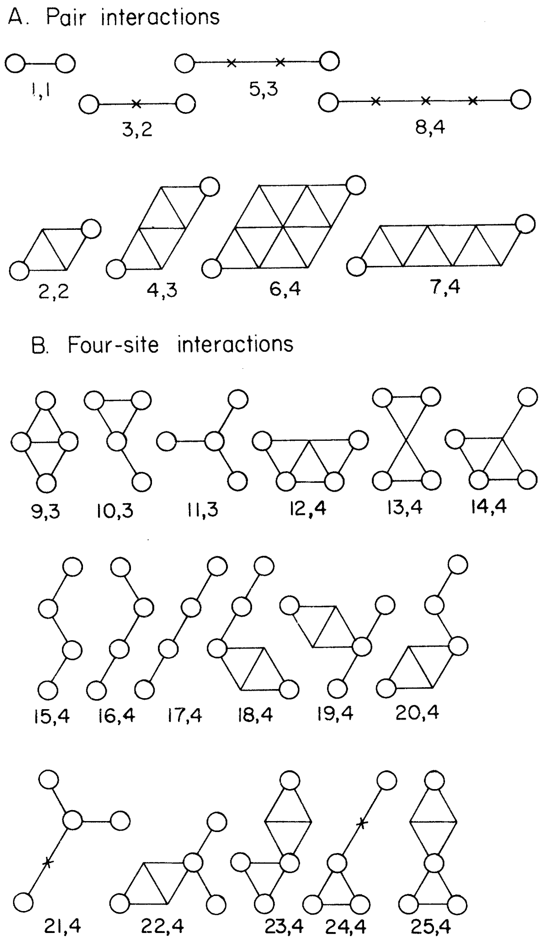


FIG. 3. All spin-$\frac{1}{2}$ interactions $L_b$ generated by the fourth-order cumulant expansion for the $s \to \frac{1}{2}$ transformation. Under each interaction, its label $b$ and the lowest order in which it first appears are given. The sites which it couples are shown with open circles.

view of the accuracy achieved by NvL, who keep these interactions out of their treatment, we feel that our results will not be much affected by this omission. (ii) We follow NvL in the evaluation of the critical interactions by using, not the true critical surface, but rather the plane tangent to it at the fixed point. These authors report that there is, in fact, very little curvature around the fixed point. Since our transformation maps the critical spin-$s$ systems onto the same vicinity, as shown in Table III, we again feel justified.

The locus of critical interactions $L_b^c$ is given by NvL's tangent plane

$$\sum_b r_b L_b^c - L_I^c = 0, \tag{2.5}$$

where the eigenvector $r_b$ is

$$\{r_1, r_2, r_3, r_9 - r_{14}\} = \{1, 1.607, 1.811, 1.248, 5.782, 1.083, 2.808, 1.372, 3.081\}, \tag{2.6}$$

and $L_I^c = 0.27416$ is NvL's best value for the nearest-neighbor-only critical interaction. Our $s \to \frac{1}{2}$ transformation maps critical spin-$s$ systems onto critical spin-$\frac{1}{2}$ systems. Thus, we substitute the truncated (2.4) with $K = K^c$ into (2.5) to obtain

$$\sum_{n=1}^{4} (K^c)^n \left( \sum_{b,m} r_b A_{bm}^{(n)} g_m^{(n)} \right) - L_I^c = 0. \tag{2.7}$$

Equation (2.7) is a polynomial equation for the critical spin-$s$ interactions $K^c$. Since it was derived from the tangent plane, it is, of course, valid only for solutions which map onto $L_b^c = L_b(K^c)$ near NvL's fixed point $L_b^*$.

The critical spin-$s$ interactions $K^c$ obtained from (2.7) are shown in Fig. 4 and Table IV. Also shown are values obtained by using first-, second-, or third-order cumulant expansions instead of fourth for the $s \to \frac{1}{2}$ transformation. Series data of Van Dyke and Camp[9] is shown for comparison. Our deviations from this series data are shown in Fig. 5. Within each order cumulant expansion, we note two separate trends: (i) the lower spins deviate less, as might be expected; (ii) the half-integer spins deviate less, again as expected, since the $s \to \frac{1}{2}$ transformation generates fewer contributions because of vanishing graph coefficients. Our fourth-order values deviate by 0.8 to 10 parts in $10^3$ from the series data (which has an uncertainty of about 1 part in $10^3$). Finally, Table III justifies our use of the tangent plane by giving the NvL fixed point $L_b^*$, NvL's tangent plane extrapolation point $L_I^c$, and typical examples of our tangent plane extrapolation points $L_b(K^c)$.

## III. DISCUSSION

We have illustrated how a transformation based on the usual partial tracing of the partition function can be used to map a given system onto a more manageable one with different structure. By prefacing the sequence of rescaling transformations with one such "restructuring" transformation, the results of a specific renormalization-group calculation can be extended. In this paper, we have presented a *spin*-restructuring transformation. *Lattice*-restructuring transformations can be developed along the lines of the well-known dual, decoration-iteration and star-triangle transformations.[13] This brings us to the limitation mentioned in Sec. II: in order to get any use out of a restructuring transformation, one needs the solution of the resulting system with many types of interactions. Furthermore, the manner in which contact is made with such a (doubtless approximate) solution limits one's

TABLE III. Tangent plane extrapolations.

| $b$ | NvL fixed point $L_b^*$ | NvL's tangent plane extrapolation point $L_I^c$ | Typical examples of our tangent plane extrapolation points $L_b(K^c)$ (fourth-order cumulant expansion) $s=1$ | $s=4.5$ | $s=\infty$ |
|---|---|---|---|---|---|
| 1 | 0.3069 | 0.2742 | 0.2003 | 0.2093 | 0.2076 |
| 2 | −0.0183 | 0 | 0.0343 | 0.0247 | 0.0253 |
| 3 | −0.0214 | 0 | 0.0216 | 0.0144 | 0.0148 |
| 9 | 0.0034 | 0 | −0.0030 | 0 | 0 |
| 10 | 0.0066 | 0 | −0.0020 | 0 | 0 |
| 11 | 0.0036 | 0 | −0.0017 | 0 | 0 |
| 12 | −0.0022 | 0 | −0.0006 | −0.0001 | −0.0001 |
| 13 | −0.0016 | 0 | −0.0003 | −0.0001 | −0.0001 |
| 14 | −0.0009 | 0 | −0.0004 | −0.0001 | −0.0001 |
| six-site $L_{26}$ | 0.0003 | 0 | 0 | 0 | 0 |
| six-site $L_{27}$ | 0.00004 | 0 | 0 | 0 | 0 |

accuracy.

It is also possible at least formally to apply the transformation developed in Sec. I to other, non-Ising systems; however, results must be interpreted with care. For example, the classical $XY$ model can be mapped onto the space of spin-$\frac{1}{2}$ Ising interaction constants by writing in place of (1.2),

$$\sigma_i = \begin{cases} +1 & \text{for} \quad 0 \le \theta_i < \pi, \\ -1 & \text{for} \quad -\pi \le \theta_i < 0. \end{cases} \tag{3.1}$$

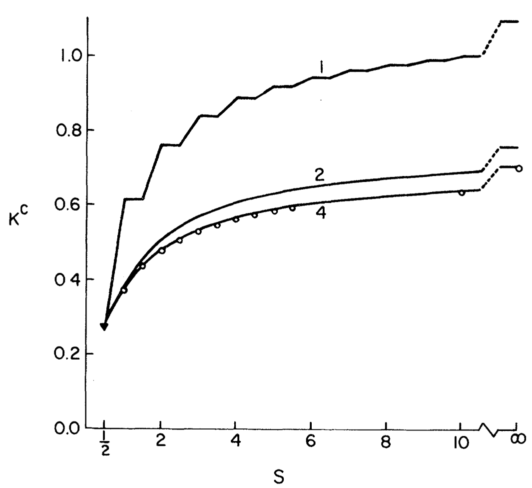


FIG. 4. Critical spin-$s$ interactions $K^c$. Curves 1, 2, and 4 represent results from the first-, second-, and fourth-order cumulant expansions for the $s \to \frac{1}{2}$ transformation. Results from the third-order cumulant expansion would barely be distinguishable from curve 4 in this figure. Open circles show Van Dyke and Camp's series data (Ref. 9) (the size of this symbol is not related to their uncertainty, which would not show in this figure; see Table IV and Fig. 5); ▼ shows NvL's result (Ref. 2), which in this figure is indistinguishable from the exact solution.

The cumulant expansion (1.9) follows as before and produces effective spin-$\frac{1}{2}$ Ising interactions. Analogous transformations are possible for the Heisenberg model and for models with higher spin dimensionalities. These systems do not have Ising critical exponents, so either (i) the restructuring transformation is itself singular for such a critical system at least, or (ii) the restructuring transformation maps such a critical system onto another critical subspace flowing into an "unusual" fixed point characteristic of the original spin dimensionality, possibly dominated by long-ranged interactions, which are known to affect critical behavior.[14] (For the classical $XY$ model in $d=2$, the appearance of long-ranged interactions is suggested by the work of Kosterlitz and Thouless,[15] who approximately transformed it into what can be viewed as an $s=1$ Ising model with Coulomb interactions, thus giving a possible ex-

TABLE IV. Critical spin-$s$ interactions $K^c$. Results from first-, second-, third-, and fourth-order cumulant expansions for the $s \to 1/2$ transformation are given together with Van Dyke and Camp's series data.

| $s$ | First cum | Second cum | Third cum | Fourth cum | Series[a] |
|---|---|---|---|---|---|
| 1 | 0.617 | 0.385 | 0.365 | 0.3746 | 0.3737–0.3740 |
| 1.5 | 0.617 | 0.455 | 0.434 | 0.4348 | 0.4348–0.4355 |
| 2 | 0.762 | 0.508 | 0.479 | 0.4798 | 0.4769–0.4776 |
| 2.5 | 0.762 | 0.542 | 0.511 | 0.5096 | 0.5068–0.5079 |
| 3 | 0.840 | 0.570 | 0.537 | 0.5344 | 0.5302–0.5311 |
| 3.5 | 0.840 | 0.590 | 0.555 | 0.5522 | 0.5479–0.5491 |
| 4 | 0.888 | 0.608 | 0.571 | 0.5679 | 0.5624–0.5637 |
| 4.5 | 0.888 | 0.621 | 0.583 | 0.5796 | 0.5744–0.5757 |
| 5 | 0.921 | 0.633 | 0.594 | 0.5904 | 0.5845–0.5858 |
| 10 | 0.999 | 0.690 | 0.647 | 0.6422 | 0.6353–0.6369 |
| 50 | 1.075 | 0.744 | 0.698 | 0.6919 | 0.6840–0.6859 |
| ∞ | 1.097 | 0.759 | 0.711 | 0.7057 | 0.6988–0.6997 |

[a]Ranges of values given by Van Dyke and Camp (Ref. 9) are exhibited.

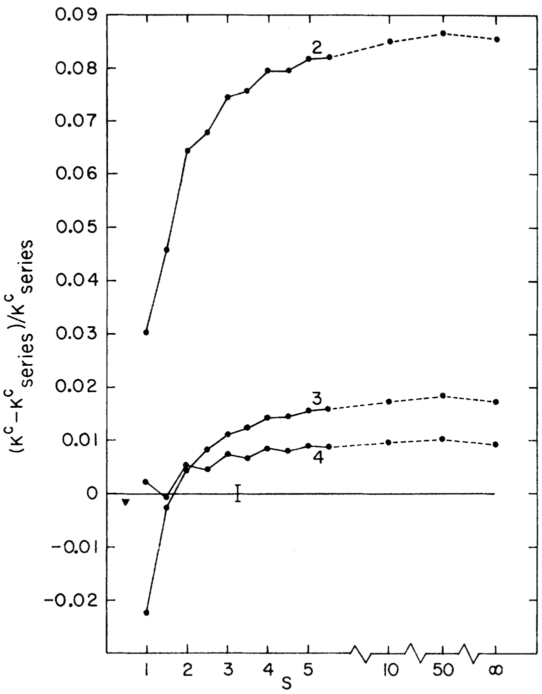


FIG. 5. Deviations $(K^c - K^c_{series})/K^c_{series}$ of our critical spin-$s$ interactions $K^c$ from Van Dyke and Camp's (Ref. 9) series data $K^c_{series}$. Curves 2, 3, and 4 represent the deviations of the second-, third-, and fourth-order cumulant expansions for the $s \to \frac{1}{2}$ transformation. The error bar shows series data uncertainty. ▼ shows NvL's deviation (Ref. 2) from the exact solution.

planation for the Stanley–Kaplan phase transition.[16]) Once the mechanism of the exponent change is understood, the method may prove of practical value. We plan to investigate this point further.

## ACKNOWLEDGMENTS

Many thanks are due to Professor M. Wortis for extensive guidance and encouragement. I thank Dr. G. Grinstein for a critical reading of the manuscript and helpful discussions. I thank Dr. W. J. Camp and Dr. J. P. Van Dyke for providing the series results given in Table IV.

## APPENDIX: CUMULANT EXPANSION

We demonstrate the cumulant expansion (1.9) by evaluating its second-order term, starting with the spin-$s$ Hamiltonian (2.1),

$$\frac{1}{2!}\left[\langle \mathcal{H}(\{\mu_i\})^2\rangle - \langle \mathcal{H}(\{\mu_i\})\rangle^2\right] = K^2 \frac{1}{2!} \sum_{\langle ij\rangle \langle kl\rangle} \left[\langle \mu_i\mu_j\mu_k\mu_l\rangle - \langle \mu_i\mu_j\rangle\langle \mu_k\mu_l\rangle\right] . \quad \text{(A1)}$$

The terms in which $\langle ij\rangle$ and $\langle kl\rangle$ have no spin in common are zero, since, in general,

$$\left\langle \prod_i (\mu_i)^{l_i}\right\rangle = \prod_i \langle(\mu_i)^{l_i}\rangle = \prod_i f_{l_i}(\sigma_i)^{l_i} , \quad \text{(A2)}$$

where

$$f_{l_i} \equiv \langle(\mu_i)^{l_i}\rangle_{\sigma_i=+1} = (s+\tfrac{1}{2})^{-1} \sum_{\mu_i>0} (\mu_i)^{l_i} \quad \text{(A3)}$$

are single-spin sums; for example,

$$f_1 = \begin{cases} (s+1)/(2s+1) & \text{for integer } s, \\ (2s+1)/4s & \text{for half-integer } s, \end{cases} \quad \text{(A4)}$$
$$f_2 = (s+1)/3s, \quad \text{etc.}$$

Then the right-hand side of (A1) equals

$$K^2 \sum_{\langle ij\rangle} \left[\langle(\mu_i)^2(\mu_j)^2\rangle - \langle \mu_i\mu_j\rangle^2\right] + K^2 \sum_{\substack{\langle ij\rangle\langle jk\rangle \\ i\neq k}} \left[\langle \mu_i(\mu_j)^2\mu_k\rangle - \langle\mu_i\mu_j\rangle\langle\mu_j\mu_k\rangle\right]$$
$$= K^2 \sum_{\langle ij\rangle} (f_2^2 - f_1^4) + K^2 \sum_{\substack{\langle ij\rangle\langle jk\rangle \\ i\neq k}} (f_1^2 f_2 - f_1^4)\sigma_i\sigma_k$$
$$\equiv K^2 \sum_{\langle ij\rangle} g_0^{(2)} + K^2 \sum_{\substack{\langle ij\rangle\langle jk\rangle \\ i\neq k}} g_1^{(2)}\sigma_i\sigma_k . \quad \text{(A5)}$$

where (A2) was used.

These two terms are of the form given in (2.2). The first term contributes to the additive constant $L_0$ in $\mathcal{H}'$,

$$L_0^{(2)} = (q/2) N K^2 g_0^{(2)}, \quad \text{(A6)}$$

where $q$ is the lattice coordination number. The second term corresponds to the linear graph (2, 1) in Fig. 1 and contributes to the nearest-, next-nearest-, and third-neighbor pair interactions $L_1$, $L_2$, and $L_3$. Counting its weak embeddings in the triangular lattice as illustrated in Fig. 2 and Eq. (2.3),

$$L_1^{(2)} = 2K^2 g_1^{(2)}, \quad L_2^{(2)} = 2K^2 g_1^{(2)}, \quad L_3^{(2)} = K^2 g_1^{(2)}, \quad \text{(A7)}$$

which are the $n=2$ contributions in (2.4).

---

*Research supported in part by the National Science Foundation under Grant No. NSF-GP-40395.

[1]K. G. Wilson, Phys. Rev. B 4, 3174 (1971); 4, 3184 (1971); K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972). For a general review, see K. G. Wilson and J. Kogut, Phys. Repts. 12, 75 (1974).

[2](a) Th. Niemeijer and J. M. J. van Leeuwen, in *Renormalization Group in Critical Phenomena and Quantum Field Theory: Proceedings of a Conference*, edited by J. D. Gunton and M. S. Green (Temple University, Philadelphia, Pa., 1974), p. 42; (b) Phys. Rev. Lett. 31, 1411 (1973); (c) Physica (Utr.) 71, 17 (1974); (d) S.-C. Hsu, Th. Niemeijer, and J. D. Gunton, Phys. Ref. B 11, 2699 (1975).

[3]L. P. Kadanoff, in Ref. 2(a), p. 21; L. P. Kadanoff and A. Houghton, Phys. Rev. B 11, 377 (1975).

[4]C.-S. Hsu, Ph.D. thesis (Columbia University, 1974) (unpublished).

[5]K. Subbarao, Phys. Rev. B 11, 1165 (1975).

[6]M. Nauenberg and B. Nienhuis, Phys. Rev. Lett. 33, 944 (1974).

[7]L. P. Kadanoff, Physics 2, 263 (1966).

[8]Higher-spin Ising models have previously been studied by M. E. Fisher [Phys. Rev. 113, 969 (1959)] within the context of decoration-iteration and star-triangle transformations: the decorating and central spins, respectively, are considered for spin-$s$ while the remaining spins are kept as spin-$\frac{1}{2}$.

[9]J. P. Van Dyke and W. J. Camp, Phys. Rev. B (to be published); W. J. Camp and J. P. Van Dyke, Phys. Rev. B 11, 2579 (1975), Tables XVI and XVIII.

[10]M. F. Sykes, J. W. Essam, B. R. Heap, and B. J. Hiley, J. Math. Phys. 7, 1557 (1966).

[11]This convergence can be seen in Figs. 4 and 5, and Table IV by comparing the critical spin-$s$ interactions $K^c$ obtained from first-, second-, third-, or fourth-order cumulant expansions for the $s \rightarrow \frac{1}{2}$ transformation. We have also verified that in the fourth-order cumulant expansion, contributions to $L_b(K^c)$ from successive orders decrease in magnitude.

[12]A somewhat more consistent procedure might be to include all the interactions generated by the $s \rightarrow \frac{1}{2}$ transformation as input in an initial NvL seventh-order cluster expansion rescaling transformation. The unwanted interactions would, of course, not be regenerated as output; the other output interactions could then be processed as below.

[13]For a general review of these transformations, see I. Syozi, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 1, p. 269.

[14]M. E. Fisher, S. Ma, and B. G. Nickel, Phys. Rev. Lett. 29, 917 (1972).

[15]J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973); J. M. Kosterlitz, *ibid.* 7, 1046 (1974); for a direct application of NvL theory to the triangular classical $XY$ model, see D. M. Lublin, Phys. Rev. Lett. 34, 568 (1975).

[16]H. E. Stanley and T. A. Kaplan, Phys. Rev. Lett. 17, 913 (1966); H. E. Stanley, *ibid.* 20, 589 (1968); M. A. Moore, *ibid.* 23, 861 (1969).