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Dielectric response of the charged Bose gas in the random-phase approximation

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A closed form for the dielectric function of a charged Bose gas is found in the random-phase approximation. This dielectric constant is used to explore the quasiparticle energy spectrum (i.e., the allowed modes of oscillation of the gas), the damping of these quasiparticles, and the long-ranged form of the electrostatic potential around a test charge in the gas. We do this around the three temperature regions of interest; namely, $T=0$, $T=\infty$, and $T=T_c$, where T_c is the transition temperature of the gas.

I. INTRODUCTION

The interacting Bose gas is a very complex, and as yet not fully solved, problem. It not only contains the difficulties of the many-body problem, but has the added complication of the critical phenomenon that is a consequence of Bose-Einstein condensation.

Apart from the intrinsic value of solving a particular many-body problem, a solution is desirable as an interacting Bose gas can serve as a model of several real physical systems. In particular, the specialized situation of a gas of charged bosons can serve as a model for a superconductor.[1] There is also astrophysical interest[2] in that the centers of white dwarf stars, and possibly also novas and supernovas, consist mainly of ^{4}He nuclei, which are, of course, charged bosons.

Following the initial work done on the neutral hard-sphere Bose gas,[3,5] many studies on the interacting Bose gas have dealt specifically with the Coulomb interaction because of the above physical applications. In this paper we also deal with the Coulomb interaction. Hopefully, many of the results obtained will be characteristic of more complicated interactions that also contain a long-range part.

The charged Bose gas was first investigated in 1961 by Foldy,[4] who worked at zero temperature ($T=0$) and calculated the energy and quasiparticle energy spectrum of the gas using a method proposed by Bogoliubov.[5] Since then, various approaches have been taken with a view to obtaining information about different aspects of the problem.

Further investigations have been carried out at $T=0$ to look at corrections to Foldy's results for the ground-state energy and quasiparticle energy spectrum.[6–12] The thermodynamic functions near $T=0$ have also been looked at,[13,14] as has the presence of a Meissner effect at $T=0$.[13]

Also, work has been carried out at or near T_c, the transition temperature for the gas. This work has been concerned with the critical exponents,[15,16] the change in the transition temperature from that of the ideal gas,[16,17] and refinements to the ideal single-particle energy spectrum.[16,18]

We now introduce a different approach that, unlike previous works, enables us to consider all temperature regions.

We use a dielectric-constant formalism in the random-phase approximation. This approximation, hereafter called the RPA, means we take the distribution of particles in our charged Bose gas to be that of an ideal Bose gas. We obtain a closed-form expression for the dielectric constant and then go on to investigate the quasiparticle energy spectrum and the long-ranged form of the electrostatic potential around a test charge in the gas. We also look at the damping of the quasiparticles (i.e., the damping of the oscillations of the gas).

Fetter[17] also studied the static screening in the RPA. However, he worked strictly at T_c and above, and we are able to extend his results to below T_c, which is the interesting condensation region.

II. MATHEMATICAL DEVELOPMENT

Consider a gas of N identical spinless bosons with mass m and charge e in a box of volume Ω, together with a background of stationary particles of opposite charge to preserve charge neutrality.

The Fourier-transformed dielectric constant $\epsilon(\vec{q},\omega)$ is well known for this system and can be found in many books. In particular, Harris[19] derives it by looking at a small disturbance on the system in equilibrium, and the result is

$$\epsilon(\vec{q},\omega)=1+\sum_{\vec{p}}\frac{4\pi e^2}{\hbar q^2\Omega}\left(\frac{F_0(\vec{p})-F_0(\vec{p}-\vec{q})}{\omega-(\hbar/m)\vec{p}\cdot\vec{q}+\hbar q^2/2m}\right). \tag{1}$$

Here ω is the frequency of a small oscillation of the gas about equilibrium. Thus $\hbar\omega$ is the energy of the quasiparticle associated with the oscillation; $\vec{q}$ is the Fourier-transform parameter and represents the wave number of the frequency ω; $\vec{p}$ refers to the values of wave number that a free particle in the box would be allowed to have. [If we impose cyclic boundary conditions, then $\vec{p}=(2\pi/\Omega^{1/3})\vec{n}$, where $n_x, n_y, n_z = 0, \pm1, \pm2, \cdots$.] $F_0(\vec{p})$ is the equilibrium distribution of the bosons. We now make the RPA and take for $F_0(\vec{p})$

$$F_0(\vec{p}) = 1/(z^{-1}\, e^{\hbar^2 p^2/2mkT} - 1)\ . \tag{2}$$

This is the ideal Bose gas distribution function at temperature T[20]; k is Boltzmann's constant; z is the fugacity and for bosons $0 \le z \le 1$. Thus

$$\epsilon(\vec{q}, \omega) = 1 + \frac{4\pi e^2}{\hbar\Omega q^2} \sum_{\vec{p}} \left[\left(\frac{1}{z^{-1}\, e^{\hbar^2 p^2/2mkT} - 1} - \frac{1}{z^{-1}\, e^{\hbar^2(\vec{p}-\vec{q})^2/2mkT} - 1}\right) \Big/ \left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} + \frac{\hbar q^2}{2m}\right)\right] . \tag{3}$$

The transition temperature T_c for ideal bosons is given by[20]

$$T_c = \frac{2\pi\hbar^2}{mk}\left(\frac{1}{\rho\zeta(\frac{3}{2})}\right)^{2/3} ,$$

where $\rho = N/\Omega$ and $\zeta(x)$ is the Riemann zeta function.

Bose-Einstein condensation starts to set in at T_c, and for $T \le T_c$ we have $z = 1$. Thus for $T \le T_c$ it is clear from Eq. (3) that because of the singular nature of $1/(z^{-1} - 1)$ for $z = 1$, we must take careful account of the $\vec{p} = \vec{0}$ and $\vec{p} = \vec{q}$ modes. In Appendix A we take careful account of these modes, and we indicate the mathematical steps required to reduce Eq. (3) to the following more manageable result.

$T \le T_c$:

$$\epsilon(\vec{q}, \omega) = 1 - \frac{\omega_p^2}{\omega^2 - \hbar^2 q^4/4m^2}\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] + \frac{\omega_p^2}{\rho}\frac{1}{4\pi^2}\frac{m^2}{\hbar^2 q^3}\frac{2mkT}{\hbar^2}\frac{\pi}{2i}\sum_{j=1}^{\infty}\frac{1}{j}\{[1 + \phi(DC_j)]\, e^{D^2 C_j^2} - [1 + \phi(BC_j)]\, e^{B^2 C_j^2}\} , \tag{4}$$

$T \ge T_c$:

$$\epsilon(\vec{q}, \omega) = 1 + \frac{\omega_p^2}{\rho}\frac{1}{4\pi^2}\frac{m^2}{\hbar^2 q^3}\frac{2mkT}{\hbar^2}\frac{\pi}{2i}\sum_{j=1}^{\infty}\frac{z^j}{j}\{[1 + \phi(DC_j)]\, e^{D^2 C_j^2} - [1 + \phi(BC_j)]\, e^{B^2 C_j^2}\} , \tag{5}$$

where $D = i(m\omega/\hbar q + \frac{1}{2}q)$, $B = i(m\omega/\hbar q - \frac{1}{2}q)$, $C_j = A^{1/2} j^{1/2}$, $A = \hbar^2/2mkT$; $\phi(x)$ is the error function,[21] where

$$\phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$$

and $\omega_p^2 = 4\pi e^2 \rho/m$ is the plasma frequency for the gas.

III. QUASIPARTICLE ENERGY SPECTRUM

We have obtained a closed-form expression for $\epsilon(\vec{q}, \omega)$ and we can now use the properties of the error function to investigate $\epsilon(\vec{q}, \omega)$ in the temperature regions of interest, namely,

$$T \to 0,\quad T \to T_c-,\quad T \to T_c+,\quad T \to \infty .$$

The allowed values of ω are then found by solving the equation,[19]

$$\epsilon(\vec{q}, \omega) = 0 . \tag{6}$$

As an interesting quick result, we first show how Foldy's spectrum can be obtained. Putting $T = 0$ in Eq. (A1) yields

$$\epsilon(\vec{q}, \omega, T = 0) = 1 - \frac{\omega_p^2}{\omega^2 - \hbar^2 q^4/4m^2} . \tag{7}$$

This equation shows how the $\vec{p} = \vec{0}$ and $\vec{p} = \vec{q}$ terms that were singled out in Eq. (A1), determine the ground-state properties of the gas.

Equation (6) now yields

$$\omega^2 = \omega_p^2 + \hbar^2 q^4/4m^2 . \tag{8}$$

This is just Foldy's result for the $T = 0$ quasiparticle energy spectrum with the slight difference that we have no depletion of the ground state. That is, we have $\omega_p^2 = 4\pi e^2\rho/m$, whereas Foldy has $\omega_p^2 = 4\pi e^2 \rho_0/m$, where $\rho_0 = N_0/\Omega$ and $N_0 <$, where N_0 is the depleted ground-state occupation. This difference occurs because of the different approximations made in obtaining the result. We use the RPA, whereas Foldy uses the Bogliubov approximation.

In general, we cannot of course solve Eq. (6) so easily, but we can obtain asymptotic solutions for the $q \to 0$ limit. Physically, this is the interesting region, as here the modes are weakly damped. In this limit we know[4] $\omega \simeq \omega_p$, and so therefore we have $|D^2 A| \simeq |B^2 A| \simeq m\omega_p^2/kTq^2 \gg 1$. This region can be investigated using the results of Appendix B as follows.

Equation (4) was expanded using Eq. (B6) from Appendix B and putting $z = 1$. This gives expansions for $\epsilon(\vec{q}, \omega)$ for $T \to 0$ and $T \to T_c-$. For $T \to T_c+$, $z \to 1-$ and thus an expansion for $\epsilon(\vec{q}, \omega)$ for $T \to T_c+$ was also obtained using Eq. (B6). For $T \to \infty$, $z \to 0$, and therefore, Eq. (B3) was used to obtain an expansion for $\epsilon(\vec{q}, \omega)$ for $T \to \infty$.

For $T \to \infty$ and $T \to T_c+$, we need to eliminate the fugacity to have useful expansions. This was done using the Eqs. (C2) and (C3) from Appendix C.

After some algebra we find the following expansions for $\epsilon(\vec{q},\omega)$:

(i) $T\simeq 0$:

$$\epsilon(\vec{q},\omega)=1-\frac{\omega_p^2}{\omega^2-\hbar^2q^4/4m^2}-\frac{q^2kT}{m}\frac{\omega_p^2}{(\omega^2-\hbar^2q^4/4m^2)^3}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\left(3\omega^2+\frac{\hbar^2q^4}{4m^2}\right)\left(\frac{T}{T_c}\right)^{3/2}+\cdots$$
$$+i\,\frac{\omega_p^2}{\rho}\frac{m^3kT}{2\pi\hbar^4q^3}\left(\sinh\left(\frac{\hbar\omega}{2mkT}\right)\exp\left\{-\left[\left(\frac{m\omega}{\hbar q}\right)^2+\frac{q^2}{4}\right]\frac{\hbar^2}{2mkT}\right\}+\cdots\right),$$

(ii) $T<T_c$, $T\simeq T_c$:

$$\epsilon(\vec{q},\omega)=1-\frac{\omega_p^2}{\omega^2-\hbar^2q^4/4m^2}-\frac{q^2kT}{m}\frac{\omega_p^2}{(\omega^2-\hbar^2q^4/4m^2)^3}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\left(3\omega^2+\frac{\hbar^2q^4}{4m^2}\right)\left(\frac{T}{T_c}\right)^{3/2}+\cdots$$
$$+i\,\frac{\omega_p^2}{\rho}\frac{m^3kT}{2\pi\hbar^4q^3}\left(\sinh\left(\frac{\hbar\omega}{2mkT}\right)\exp\left\{-\left[\left(\frac{m\omega}{\hbar q}\right)^2+\frac{q^2}{4}\right]\frac{\hbar^2}{2mkT}\right\}+\cdots\right),$$

(iii) $T>T_c$, $T\simeq T_c$:

$$\epsilon(\vec{q},\omega)=1-\frac{\omega_p^2}{\omega^2-\hbar^2q^4/4m^2}-\frac{q^2kT}{m}\frac{\omega_p^2}{(\omega^2-\hbar^2q^4/4m^2)^3}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\left(3\omega^2+\frac{\hbar^2q^4}{4m^2}\right)\left(\frac{T}{T_c}\right)^{3/2}+\cdots$$
$$+i\,\frac{\omega_p^2}{\rho}\frac{m^3kT}{2\pi\hbar^4q^3}\left(\sinh\left(\frac{\hbar\omega}{2mkT}\right)\exp\left\{-\left[\left(\frac{m\omega}{\hbar q}\right)^2+\frac{q^2}{4}\right]\frac{\hbar^2}{2mkT}\right\}\left(1-\frac{\zeta^2(\frac{3}{2})}{4\pi}\theta^2+\cdots\right)\right),$$

(iv) $T\simeq\infty$:

$$\epsilon(\vec{q},\omega)=1-\frac{\omega_p^2}{\omega^2-\hbar^2q^4/4m^2}-\frac{q^2kT}{m}\frac{\omega_p^2}{(\omega^2-\hbar^2q^4/4m^2)^3}\left(3\omega^2+\frac{\hbar^2q^4}{4m^2}\right)\left[1-\frac{\zeta(\frac{3}{2})}{2^{5/2}}\left(\frac{T_c}{T}\right)^{3/2}+\cdots\right]+\cdots$$
$$+i\,\frac{\omega_p^2}{\rho}\frac{m^3kT}{2\pi\hbar^4q^3}\left(\sinh\left(\frac{h\omega}{2mkT}\right)\exp\left\{-\left[\left(\frac{m\omega}{\hbar q}\right)^2+\frac{q^2}{4}\right]\frac{\hbar^2}{2mkT}\right\}\zeta(\tfrac{3}{2})\left(\frac{T_c}{T}\right)^{3/2}\left[1-\frac{\zeta(\frac{3}{2})}{2^{5/2}}\left(\frac{T_c}{T}\right)^{3/2}+\cdots\right]+\cdots\right),$$

where $\theta=1-(T_c/T)^{3/2}$. All the above expansions are good for

$$q^2kT/m\omega_p^2\ll 1\ .$$

Equation (6) can now be solved iteratively with the above results to yield $\omega=\alpha+i\gamma$ (α,γ real). Thus,

(i) $T\simeq 0$:

$$\alpha^2=\omega_p^2+\frac{\hbar^2q^4}{4m^2}+3\left(\frac{T}{T_c}\right)^{3/2}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\left\{\left(\frac{q^2kT}{m}\right)\left[1+O\left(\frac{\hbar^2q^4}{m^2\omega_p^2}\right)\right]+O\left(\left(\frac{q^2kT}{m}\right)^2\right)\right\},$$
$$\gamma=-\,\omega_p\left[\left(\frac{\pi}{8}\right)^{1/2}\sinh\left(\frac{\hbar\omega_p}{2kT}\right)\frac{1}{\zeta(\frac{3}{2})}\left(\frac{2m}{\hbar q^2}\right)\left(\frac{m}{q^2kT}\right)^{1/2}\exp\left(-\frac{m\omega_p^2}{q^2kT}\right)+\cdots\right]\ , \tag{9}$$

(ii) $T<T_c$, $T\simeq T_c$:

$$\alpha^2=\omega_p^2+\frac{\hbar^2q^4}{4m^2}+3\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\left\{\frac{q^2kT_c}{m}\left[1+O\left(\frac{\hbar^2q^4}{m^2\omega_p^2}\right)\right]+O\left(\left(\frac{q^2kT}{m}\right)^2\right)\right\}[1+O(\theta)]\ ,$$
$$\gamma=-\,\omega_p\left[\left(\frac{\pi}{8}\right)^{1/2}\sinh\left(\frac{\hbar\omega_p}{2kT_c}\right)\frac{1}{\zeta(\frac{3}{2})}\left(\frac{2m}{\hbar q^2}\right)\left(\frac{m}{q^2kT_c}\right)^{1/2}\exp\left(-\frac{m\omega_p^2}{q^2kT_c}\right)+\cdots\right]\ , \tag{10}$$

(iii) $T>T_c$, $T\simeq T_c$:

$$\alpha^2=\omega_p^2+\frac{\hbar^2q^4}{4m^2}+3\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\left\{\frac{q^2kT_c}{m}\left[1+O\left(\frac{\hbar^2q^4}{\omega_p^2m^2}\right)\right]+O\left(\left(\frac{q^2kT}{m}\right)^2\right)\right\}[1+O(\theta)]\ ,$$
$$\gamma=-\,\omega_p\left[\left(\frac{\pi}{8}\right)^{1/2}\sinh\left(\frac{\hbar\omega_p}{2kT_c}\right)\frac{1}{\zeta(\frac{3}{2})}\left(\frac{2m}{\hbar q^2}\right)\left(\frac{m}{q^2kT_c}\right)^{1/2}\exp\left(-\frac{m\omega_p^2}{q^2kT_c}\right)+\cdots\right]\ , \tag{11}$$

(iv) $T\simeq\infty$:

$$\alpha^2=\omega_p^2+\frac{\hbar^2q^4}{4m^2}+3\frac{q^2kT}{m}\left[1+O\left(\frac{\hbar^2q^4}{m^2\omega_p^2}\right)\right]+O\left(\left(\frac{q^2kT}{m}\right)^2\right), \tag{12}$$

$$\gamma = -\omega_p\left[\left(\frac{\pi}{8}\right)^{1/2}\left(\frac{m\omega_p^2}{q^2kT}\right)^{3/2}\exp\left(-\frac{m\omega_p^2}{q^2kT}\right)+\cdots\right] . \tag{13}$$

These expansions are good for

$$\frac{q^2kT}{m\omega_p^2} \ll 1 \quad\text{and}\quad \frac{\hbar^2q^4}{m^2\omega_p^2} \ll 1 . \tag{14}$$

The leading order term in α^2 is ω_p^2, and we shall now discuss which term is the next largest.

(i) $T \simeq 0$: Let R_1 be the ratio of the second and third terms in Eq. (9). Thus

$$R_1 = \frac{1}{24\pi}\frac{[\zeta(\frac{3}{2})]^{5/3}}{\zeta(\frac{5}{2})}q^2\rho^{2/3}\left(\frac{T_c}{T}\right)^{5/2} .$$

For $q^2 \gg (T/T_c)^{5/2}\rho^{-2/3}$, we have $R_1 \gg 1$, and the q^4 term in Eq. (9) dominates over the q^2 term. Thus, under these conditions of q, ρ, and T, we are reflecting Foldy's result at $T=0$. For $q^2 \ll (T/T_c)^{5/2}\times\rho^{-2/3}$, we have $R_1 \ll 1$, and we flip over to a q^2 dominance in Eq. (9).

(ii) $T \simeq T_c$: Let R_2 be the ratio of the second and third terms in Eqs. (10) and (11). Thus

$$R_2 = \frac{1}{24\pi}\frac{[\zeta(\frac{3}{2})]^{5/3}}{\zeta(\frac{5}{2})}q^2\rho^{2/3} .$$

For $q^2 \gg \rho^{-2/3}$ we have $R_2 \gg 1$, and the q^4 term in Eqs. (10) and (11) dominates. For $q^2 \ll \rho^{-2/3}$ we have $R_2 \ll 1$, and the q^2 term in Eqs. (10) and (11) dominates.

(iii) $T \simeq \infty$: Let R_3 be the ratio of the second and third terms in Eq. (12). Thus

$$R_3 = \frac{1}{24\pi}\frac{[\zeta(\frac{3}{2})]^{5/3}}{\zeta(\frac{5}{2})}q^2\rho^{2/3}\left(\frac{T_c}{T}\right) .$$

For $q^2 \gg \rho^{-2/3}(T/T_c)$ we have $R_3 \gg 1$, and the q^4 term in Eq. (12) dominates. For $q^2 \ll \rho^{-2/3}(T/T_c)$ we have $R_3 \ll 1$, and the q^2 term in Eq. (12) dominates. Thus, under these conditions, we have $\alpha^2 = \omega_p^2 + 3q^2kT/m + \cdots$, which is exactly the classical result[22] for $q \to 0$. [Note that our inequality above for q, together with the conditions in Eq. (14), allows $q \to 0$.]

We shall now briefly discuss the damping of the modes. We note that $|\alpha/\gamma| \ll 1$ for $q^2kT/m\omega_p^2 \ll 1$. Thus the oscillations are hardly damped at all for sufficiently small q. But as q becomes larger such that $q^2kT/m\omega_p^2 \sim 1$, then the damping becomes significant.

Because large q modes appear to be heavily damped, they are not physically interesting and therefore we have not explored them, but dealt only with small q modes [Eq. (14) implies small q]. However, a brief discussion of large q modes in the classical case is given by Jackson.[23]

We note in passing that in Eq. (13), we have recovered the correct form of the Landau damping for the classical case.

IV. ELECTROSTATIC POTENTIAL

We now investigate the electrostatic potential $V(\vec{r})$ about a charge Q immersed in the gas. We are interested in the asymptotic form of $V(\vec{r})$; so we will concern ourselves with the $r \to \infty$ limit of $V(\vec{r})$. Now

$$V(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3q\, e^{i\vec{q}\cdot\vec{r}}\, V(\vec{q}) , \tag{15}$$

and

$$V(\vec{q}) = 4\pi Q/q^2\epsilon(\vec{q}, \omega=0) . \tag{16}$$

Thus we need to investigate $\epsilon(\vec{q}, 0)$, where from Eqs. (4) and (5):

$T \le T_c$:

$$\epsilon(\vec{q},0) = 1 + \frac{4m^2\omega_p^2}{\hbar^2q^2}\left[1-\left(\frac{T}{T_c}\right)^{3/2}\right] + \frac{\omega_p^2}{\rho}\frac{1}{4\pi^2}\frac{m^2}{\hbar^2q^3}\frac{2mkT}{\hbar^2}\frac{\pi}{i}\sum_{j=1}^{\infty}\frac{1}{j}\phi\left(\frac{iqA^{1/2}j^{1/2}}{2}\right)e^{-q^2Aj/4} , \tag{17}$$

$T \ge T_c$:

$$\epsilon(\vec{q},0) = 1 + \frac{\omega_p^2}{\rho}\frac{1}{4\pi^2}\frac{m^2}{\hbar^2q^3}\frac{2mkT}{\hbar^2}\frac{\pi}{i}\sum_{j=1}^{\infty}\frac{z^j}{j}\phi\left(\frac{iqA^{1/2}j^{1/2}}{2}\right)e^{-q^2Aj/4} . \tag{18}$$

We expect that the major contribution to $V(\vec{r})$ for large r comes from the small $\vec{q}$ part of $V(\vec{q})$. Physically, we see this from the fact that to probe large r we need a long-wavelength mode, which means small $\vec{q}$.

We now want small $\vec{q}$ expansions of $\epsilon(\vec{q}, 0)$. These are found from Eqs. (17) and (18) using Eqs. (B2) and (B5) from Appendix B. The fugacity is eliminated in the same fashion as before, using Eqs. (C2) and (C3). We find

(i) $T \simeq 0$:

$$\epsilon(\vec{q},0) = 1 + \frac{1}{q^4}\frac{4m^2\omega_p^2}{\hbar^2}\left[1-\left(\frac{T}{T_c}\right)^{3/2}\right] + \frac{1}{q^3}\frac{\omega_p^2}{\rho}\frac{m^3kT}{2\hbar^4} + O\left(\frac{1}{q^2}\right) ,$$

(ii) $T \leq T_c$, $T \simeq T_c$:

$$\epsilon(\vec{q}, 0) = 1 + \frac{1}{q^4}\frac{4m^2\omega_p^2}{\hbar^2}\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] + \frac{1}{q^3}\frac{\omega_p^2}{\rho}\frac{m^3kT}{2\hbar^4} + O\left(\frac{1}{q^2}\right),$$

(iii) $T = T_c$:

$$\epsilon(\vec{q}, 0) = 1 + \frac{1}{q^3}\frac{\omega_p^2}{\rho}\frac{m^3kT}{2\hbar^4} + O\left(\frac{1}{q^2}\right),$$

(iv) $T > T_c$, $T \simeq T_c$:

$$\epsilon(\vec{q}, 0) = 1 + \frac{1}{q^3}\frac{\omega_p^2 m^3 kT}{\pi\rho\hbar^4}\arctan\left[\left(\frac{\hbar^2q^2}{\theta^2 mkT}\right)^{1/2}\frac{1}{\zeta(\frac{3}{2})}\left(\frac{\pi}{2}\right)^{1/2}\left(1 + \frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{4\pi}\theta + \cdots\right)\right] + O\left(\frac{1}{q^2}\right),$$

(v) $T \simeq \infty$:

$$\epsilon(\vec{q}, 0) = 1 + \frac{1}{q^2}\frac{m\omega_p^2}{kT}\left[1 + \frac{\zeta(\frac{3}{2})}{2^{3/2}}\left(\frac{T_c}{T}\right)^{3/2} + \cdots\right] + O(q^0).$$

Now, our approach is to take $\epsilon(\vec{q}, 0) = 1 +$ (the term that gives greatest contribution as $q \to 0$) and then substitute this into Eqs. (15) and (16), and take the $r \to \infty$ limit to obtain the asymptotic form of $V(\vec{r})$. After doing this and performing the angular integrations, we have

(i) $T \simeq 0$:

$$\lim_{r\to\infty} V(\vec{r}) = \lim_{r\to\infty}\frac{Q}{r}\frac{2}{\pi}\int_0^\infty dq\left(q^3\sin(qr)\bigg/\left\{q^4 + \frac{4m^2\omega_p^2}{\hbar^2}\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]\right\}\right), \tag{19}$$

(ii) $T < T_c$, $T \simeq T_c$:

$$\lim_{r\to\infty} V(\vec{r}) = \lim_{r\to\infty}\frac{Q}{r}\frac{2}{\pi}\int_0^\infty dq\left(q^3\sin(qr)\bigg/\left\{q^4 + \frac{4m^2\omega_p^2}{\hbar^2}\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]\right\}\right), \tag{20}$$

(iii) $T = T_c$:

$$\lim_{r\to\infty} V(\vec{r}) = \lim_{r\to\infty}\frac{Q}{r}\frac{2}{\pi}\int_0^\infty dq\left[q^2\sin(qr)\bigg/\left(q^3 + \frac{\omega_p^2 m^3 kT_c}{2\rho\hbar^4}\right)\right], \tag{21}$$

(iv) $T > T_c$, $T \simeq T_c$:

$$\lim_{r\to\infty} V(\vec{r}) = \lim_{r\to\infty}\frac{Q}{r}\frac{2}{\pi}\int_0^\infty dq\left(q^2\sin(qr)\bigg/\left\{q^3 + \frac{\omega_p^2 m^3 kT}{\rho\pi\hbar^4}\arctan\left[\left(\frac{\hbar^2q^2}{\theta^2 mkT}\right)^{1/2}\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{\zeta(\frac{3}{2})}\left(1 + \frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{4\pi}\theta + \cdots\right)\right]\right\}\right), \tag{22}$$

(v) $T \simeq \infty$:

$$\lim_{r\to\infty} V(\vec{r}) = \lim_{r\to\infty}\frac{Q}{r}\frac{2}{\pi}\int_0^\infty dq\left(q\sin(qr)\bigg/\left\{q^2 + \frac{m\omega_p^2}{kT}\left[1 + \frac{1}{2^{3/2}}\zeta(\tfrac{3}{2})\left(\frac{T_c}{T}\right)^{3/2} + \cdots\right]\right\}\right). \tag{23}$$

Now the integrals for (i), (ii), and (v) are standard integrals.[21] The integral in (iii) is not standard but its asymptotic expansion (large r) has been worked out in Appendix D. We find

(i) $T \simeq 0$:

$$\lim_{r\to\infty} V(\vec{r}) = \frac{Q}{r}\cos(Kr)\,e^{-Kr}, \quad K = \left\{\frac{m^2\omega_p^2}{\hbar^2}\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]\right\}^{1/4},$$

(ii) $T < T_c$, $T \simeq T_c$:

$$\lim_{r\to\infty} V(\vec{r}) = \frac{Q}{r}\cos(Kr)\,e^{-Kr}, \quad K = \left\{\frac{m^2\omega_p^2}{\hbar^2}\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]\right\}^{1/4},$$

(iii) $T = T_c$:

$$\lim_{r\to\infty} V(\vec{r}) = \frac{Q}{r^4}\frac{2\hbar^4\rho}{\pi\omega_p^2 m^3 kT_c},$$

(iv) $T \simeq \infty$:

$$\lim_{r\to\infty} V(\vec{r}) = \frac{Q}{r}e^{-K'r}, \qquad K' = \left\{\frac{m\omega_p^2}{kT}\left[1 + \frac{\zeta(\frac{3}{2})}{2^{3/2}}\left(\frac{T_c}{T}\right)^{3/2} + \cdots\right]\right\}^{1/2}.$$

Consider now the integral for $T > T_c$ and $T \simeq T_c$. Our prescription for finding $\lim_{r\to\infty} V(\vec{r})$ says that we take the largest contributing term to $\epsilon(\vec{q}, 0)$ in the limit $\vec{q} \to 0$, and then integrate. So we might be tempted to expand the arctan for small q and take the first term. Thus we would say

$$\arctan\left[\left(\frac{\hbar^2 q^2}{\theta^2 mkT}\right)^{1/2} \frac{1}{\zeta(\frac{3}{2})}\left(\frac{\pi}{2}\right)^{1/2}\left(1+\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{4\pi}\theta+\cdots\right)\right] \sim \left(\frac{\hbar^2 q^2}{\theta^2 mkT}\right)^{1/2} \frac{1}{\zeta(\frac{3}{2})}\left(\frac{\pi}{2}\right)^{1/2}\left(1+\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{4\pi}\theta+\cdots\right)$$

and then

$$\lim_{r\to\infty} V(\vec{r}) = \lim_{r\to\infty}\frac{Q}{r}\frac{2}{\pi}\int_0^\infty dq\left\{q\sin(qr)\Big/\left[q^2+\frac{\omega_p^2}{\rho}\frac{m^3kT}{\pi\hbar^4}\frac{1}{\theta}\left(\frac{\hbar^2}{mkT}\right)^{1/2}\frac{1}{\zeta(\frac{3}{2})}\left(\frac{\pi}{2}\right)^{1/2}\left(1+\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{4\pi}\theta+\cdots\right)\right]\right\} .$$

Therefore

$$\lim_{r\to\infty} V(\vec{r}) = \frac{Q}{r}e^{-K'' r}, \quad \text{where } K'' = \left[\frac{m^2\omega_p^2}{\theta\rho\hbar^2}\left(\frac{mkT}{\hbar^2}\right)^{1/2}\frac{1}{\sqrt{2\pi}\,\zeta(\frac{3}{2})}\left(1+\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{4\pi}\theta+\cdots\right)\right]^{1/2} .$$

However, for the expansion of the arctan to be valid, we require that θ be greater than zero. So our procedure does not allow us to go down to T_c from above. Thus we are fixed in some temperature range above T_c.

Fetter[17] has given a careful analysis in the high-density limit, of the asymptotic (large r) behavior of the integral in Eq. (22) by considering the poles of the integrand. He finds, for $T > T_c$, $T \simeq T_c$:

$$\lim_{r\to\infty} V(\vec{r}) = \frac{Q}{r}e^{-K''' r},$$

where

(i) for $\theta^3 \gg (m\omega_p^2/\rho)(m/\hbar^2 kT_c)^{1/2}$:

$$K''' = \left[\frac{m^2\omega_p^2}{\theta\rho\hbar^2}\left(\frac{mkT}{\hbar^2}\right)^{1/2}\frac{1}{\sqrt{2\pi}\,\zeta(\frac{3}{2})}\times\left(1+\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{4\pi}\theta+\cdots\right)\right]^{1/2};$$

(ii) for $\theta^3 = 0.022(m\omega_p^2/\rho)(m/\hbar^2 kT_c)^{1/2}$:

$$K''' = \left[3.4\theta^2\left(\frac{mkT}{\hbar^2}\right)\left(1-\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\pi}\theta+\cdots\right)\right]^{1/2};$$

(iii) for $\theta^3 < 0.014(m\omega_p^2/\rho)(m/\hbar^2 kT_c)^{1/2}$:

$$K''' = \left[4.3\theta^2\left(\frac{mkT}{\hbar^2}\right)\left(1-\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\pi}\theta+\cdots\right)\right]^{1/2}.$$

[Note that for large density, $(m\omega_p^2/\rho)(m/\hbar^2 kT_c)^{1/2} \ll 1$.]

In the limit $T = T_c$, Fetter obtains a result which agrees exactly with our result at $T = T_c$. We see that Fetter's result (i) above is what we obtained by expanding the arctan, and Fetter finds it holds in a region where θ is not allowed to go to zero, which agrees with what we argued above.

We thus have a description, for all temperatures, of the asymptotic form of the potential around a test charge. Around $T = \infty$ we have a simple screened potential, i.e., an e^{-Kr}/r form. As we approach T_c from above we still have the simple screened form, but the screening length $1/K$ changes its form near T_c until, in the limit as $\theta \to 0+$, we find that $1/K \to \infty$.

At T_c we have the remarkable change to simple power-law potential so that we no longer have the Yukawa-type screened potential. ($1/K \to \infty$ near T_c is preemptive of the dramatic change in the form of the potential since a power-law potential has infinite screening length.)

As we pass over to $T < T_c$, we now find we again have the simple screened potential form, but together with an oscillatory factor, i.e., a $(\cos Kr) \times (e^{-Kr}/r)$ form. For T very close to T_c and below it, $1/K \to \infty$, as we would expect. As T drops to around zero the form of the potential is the same, but $1/K$ drops to $(\hbar^2/4m^2\omega_p^2)^{1/4}$.

Thus the asymptotic form of $V(\vec{r})$ displays quite dramatic behavior around T_c, and this is a direct example of the effect Bose-Einstein condensation has on the properties of the gas.

V. SUMMARY OF RESULTS

We have given a description of the charged Bose gas, in the random-phase approximation, that covers all temperature regions. The RPA result is valid in the high-density limit.[17,26] Many authors[4,6-13] have studied the charged Bose gas in the high-density limit, and particular attention has been paid to the calculation of the ground-state energy and the $T = 0$ low-lying excitation spectrum of the gas. This has been done from the point of view of the Bogliubov approximation, as well as many other perturbation treatments. While we have not calculated the ground-state energy, we can compare our result for the $T = 0$ excitation spectrum. We did this in Sec. III and as noted there we obtained, except for the depletion of the ground state, the same result as Foldy[4] did in his high-density treatment. Also, as noted in Sec. IV, we agree with Fetter's[17] high-density result for the form of the screened potential. All this is further evidence of the fact that our RPA treatment is valid in the high-density limit. We note that for the electron gas the RPA is also valid in the high-density limit.

In the limit $T \to \infty$ we recover, as expected, the classical results for ω, the Landau damping, and the asymptotic form of the electrostatic potential around an impurity. We have obtained Foldy's expression for ω at $T=0$ and extended it for $T>0$. We have given the form of ω around T_c and found that it shows characteristics of the classical result. We recover Fetter's results for the form of the potential both above T_c and at T_c, and we are able to extend his results to below the transition temperature where we show the existence of an oscillatory factor.

The above theory could now be improved by making the whole calculation self-consistent. This is done by using the form of the static screening already obtained to determine the changes in the chemical potential and the single-particle energy spectrum, due to the Coulomb interaction, from that of the ideal gas. This would then be substituted back into Eq. (1) and the whole calculation worked through again. Fetter[17] and Bishop[16] have used the self-consistent approach to work out the change in the transition temperature for the interacting gas from that of the ideal gas.

Further extensions of the work within the RPA would be to study the interacting charged Bose gas in an applied magnetic field. This problem has already been studied for the case of an ideal charged Bose gas by Schafroth,[1] where he displays the Meissner effect characteristic of superconductors, and by Fetter[13] who did the same for the interacting charged Bose gas using the Bogliubov approximation.

It would be most interesting to see the magnetic response of our interacting charged Bose gas displayed and the natural starting point would be the RPA. We hope to discuss this in a future publication.

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APPENDIX A

The value of $F_0(\vec{0})$, which is the ground-state term, can be shown in the thermodynamic limit to be[20]

$$F_0(\vec{0}) = 1/(z^{-1}-1) = N[1-(T/T_c)^{3/2}] \quad \text{for } T \le T_c \, .$$

Using this, together with the identity

$$\frac{1}{x^{-1}-1} = \sum_{j=1}^{\infty} x^j \quad \text{for } |x|<1 \, ,$$

we obtain the following from Eq. (3):

$T \le T_c$:

$$\epsilon(\vec{q},\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \hbar^2 q^4/4m^2}\left[1-\left(\frac{T}{T_c}\right)^{3/2}\right] + \frac{4\pi e^2}{q^2\Omega\hbar}\sum_{j=1}^{\infty}\left\{\sum_{\vec{p}\ne\vec{0}}\left[\exp\left(-\frac{\hbar^2 p^2}{2mkT}j\right)\Big/\left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} + \frac{\hbar q^2}{2m}\right)\right]\right.$$
$$\left. - \sum_{\vec{p}\ne\vec{q}}\left[\exp\left(-\frac{\hbar^2(\vec{p}-\vec{q})^2}{2mkT}j\right)\Big/\left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} + \frac{\hbar q^2}{2m}\right)\right]\right\} , \tag{A1}$$

$T \ge T_c$:

$$\epsilon(\vec{q},\omega) = 1 + \frac{4\pi e^2}{q^2\Omega\hbar}\sum_{j=1}^{\infty} z^j\left\{\sum_{\vec{p}}\left[\exp\left(-\frac{\hbar^2 p^2}{2mkT}j\right)\Big/\left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} + \frac{\hbar q^2}{2m}\right)\right]\right.$$
$$\left. - \sum_{\vec{p}}\left[\exp\left(-\frac{\hbar^2(\vec{p}-\vec{q})^2}{2mkT}j\right)\Big/\left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} + \frac{\hbar q^2}{2m}\right)\right]\right\} , \tag{A2}$$

where $\omega_p^2 = 4\pi e^2\rho/m$ is the plasma frequency for the gas. Note that the second term in Eq. (4) represents the $\vec{p}=\vec{0}$ and $\vec{p}=\vec{q}$ contributions, which are the ground-state terms.

Using the prescription, valid in the limit $\Omega \to \infty$,

$$\sum_{\vec{p}} \to \frac{\Omega}{(2\pi)^3}\int d^3p \, .$$

We obtain:

$T \le T_c$:

$$\epsilon(\vec{q},\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \hbar^2 q^4/4m^2}\left[1-\left(\frac{T}{T_c}\right)^{3/2}\right] + \frac{4\pi e^2}{\hbar q^2}\frac{1}{(2\pi)^3}\sum_{j=1}^{\infty}\left\{\int d^3p\left[\exp\left(-\frac{\hbar^2 p^2}{2mkT}j\right)\Big/\left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} + \frac{\hbar q^2}{2m}\right)\right]\right.$$
$$\left. - \int d^3p\left[\exp\left(-\frac{\hbar^2(\vec{p}-\vec{q})^2}{2mkT}j\right)\Big/\left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} + \frac{\hbar q^2}{2m}\right)\right]\right\} ,$$

$T \geq T_c$:

$$\epsilon(\vec{q}, \omega) = 1 + \frac{4\pi e^2}{\hbar q^2}\frac{1}{(2\pi)^3}\sum_{j=1}^{\infty} z^j \left\{ \int d^3p \left[\exp\left(-\frac{\hbar^2 p^2}{2mkT}j\right) \Big/ \left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} + \frac{\hbar q^2}{2m}\right)\right] - \int d^3p \left[\exp\left(-\frac{\hbar^2(\vec{p}-\vec{q})^2}{2mkT}j\right) \Big/ \left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} + \frac{\hbar q^2}{2m}\right)\right]\right\} .$$

Using the transformation of variable $\vec{p} \to \vec{p} - \vec{q}$, we can show

$$\int d^3p \left[\exp\left(-\frac{\hbar^2(\vec{p}-\vec{q})^2}{2mkT}j\right) \Big/ \left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} + \frac{\hbar q^2}{2m}\right)\right] = \int d^3p \left[\exp\left(-\frac{\hbar^2 p^2}{2mkT}j\right) \Big/ \left(\omega - \frac{\hbar}{m}\vec{p}\cdot\vec{q} - \frac{\hbar q^2}{2m}\right)\right] .$$

Using this, we now take $\vec{q}$ as the polar direction, perform the angular integrations and integrate once by parts to obtain:

$T \leq T_c$:

$$\epsilon(\vec{q}, \omega) = 1 - \frac{\omega_p^2}{\omega^2 - \hbar^2 q^4/4m^2}\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] - \frac{\omega_p^2}{\rho}\frac{m^2}{\hbar^2 q^2}\frac{2mkT}{4\pi^2\hbar^2}\sum_{j=1}^{\infty}\frac{1}{j}\int_0^{\infty} dp\, p \exp\left(-\frac{\hbar^2 p^2}{2mkT}j\right) \times\left\{\left(\frac{m\omega}{\hbar q} + \frac{q}{2}\right)\Big/\left[p^2 - \left(\frac{m\omega}{\hbar q}+\frac{q}{2}\right)^2\right] - \left(\frac{m\omega}{\hbar q} - \frac{q}{2}\right)\Big/\left[p^2 - \left(\frac{m\omega}{\hbar q}-\frac{q}{2}\right)^2\right]\right\} , \tag{A3}$$

$T \geq T_c$:

$$\epsilon(\vec{q}, \omega) = 1 - \frac{\omega_p^2}{\rho}\frac{m^2}{\hbar^2 q^2}\frac{2mkT}{4\pi^2\hbar^2}\sum_{j=1}^{\infty}\frac{z^j}{j}\int_0^{\infty} dp\, p \exp\left(-\frac{\hbar^2 p^2}{2mkT}j\right)\left\{\left(\frac{m\omega}{\hbar q} + \frac{q}{2}\right)\Big/\left[p^2 - \left(\frac{m\omega}{\hbar q}+\frac{q}{2}\right)^2\right] - \left(\frac{m\omega}{\hbar q} - \frac{q}{2}\right)\Big/\left[p^2 - \left(\frac{m\omega}{\hbar q}-\frac{q}{2}\right)^2\right]\right\} . \tag{A4}$$

We see that the integral in the above two expressions is singular for real ω. As Landau[22] has pointed out, a correct treatment of the initial value problem would not yield this difficulty. Carrying out the correct procedure is equivalent to taking the above results and putting $\omega = \omega + i\gamma$ where $\gamma \to 0+$. We can now do the integrals in Eqs. (A3) and (A4) and we find the results as given in Eqs. (4) and (5) of Sec. II.

APPENDIX B

We are interested in expansions of

$$\sum_{j=1}^{\infty}\frac{z^j}{j}\phi(ix^{1/2}j^{1/2})\, e^{-xj}$$

for both $x \gg 1$ and $x \ll 1$ around the regions $z = 0$ ($T = \infty$) and $z = 1$ ($T \leq T_c$). Now[21]

$$e^{-xj}\phi(ix^{1/2}j^{1/2}) = \frac{2}{\sqrt{\pi}}e^{-jx}\int_0^{ix^{1/2}j^{1/2}} e^{-t^2}dt$$

$$= \frac{2}{\sqrt{\pi}}e^{-xj}\, ix^{1/2}j^{1/2}\int_0^1 e^{xjy^2}dy$$

$$= \frac{2}{\sqrt{\pi}}ix^{1/2}j^{1/2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\,\Gamma(s)x^{-s}j^{-s} \times\left(\int_0^1 (1-y^2)^{-s}dy\right), \quad c>0$$

where we have used[24]

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)x^{-s}ds , \quad c>0 \tag{B1}$$

which is the Mellin integral representation of e^{-x} (for $x \geq 0$). $\Gamma(x)$ is the γ function. Now[21]

$$\int_0^1 (1-y^2)^{-s}dy = \frac{\sqrt{\pi}}{2}\frac{\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)} , \quad \operatorname{Re} s < 1 .$$

Therefore,

$$e^{-xj}\phi(ix^{1/2}j^{1/2}) = ix^{1/2}j^{1/2}\frac{1}{2\pi i} \times\int_{c-i\infty}^{c+i\infty} ds\, x^{-s}j^{-s}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)} , \quad 0<c<1 .$$

(i) $z \simeq 0$.

$$\sum_{j=1}^{\infty}\frac{z^j}{j}\phi(ix^{1/2}j^{1/2})\, e^{-xj} = ix^{1/2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, x^{-s}g_{1/2+s}(z)\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)} , \quad 0<c<1$$

where $g_\alpha(z) = \sum_{j=1}^{\infty} z^j/j^\alpha$ and this function has no poles, as a function of α, around $z = 0$.

Closing the contour in the left half-plane and using Cauchy's theorem, we obtain an expression for small x. Closing in the right half-plane we obtain an expansion for large x. We find,

$x \ll 1$:

$$\sum_{j=1}^{\infty}\frac{z^j}{j}\phi(ix^{1/2}j^{1/2})\, e^{-xj} = i\sum_{p=0}^{\infty}\frac{(-1)^p x^{p+1/2}}{\Gamma(p+\frac{1}{2})}g_{1/2-p}(z) , \tag{B2}$$

$x \gg 1$:

$$\sum_{j=1}^{\infty} \frac{z^j}{j} \phi(ix^{1/2} j^{1/2})\, e^{-xj} = i \sum_{p=0}^{\infty} \frac{(-1)^p}{\Gamma(\frac{1}{2}-p) x^{p+1/2}} g_{3/2+p}(z) \ . \tag{B3}$$

(ii) $z \simeq 1$. We must now Mellin transform z^j as well. Now $z = e^{\ln z}$ and since $z \leq 1$, $\ln z \leq 0$, so we use Eq. (B1) to write

$$z^j = e^{-(-\ln z) j} = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} dt\, \Gamma(t) (-\ln z)^{-t} j^{-t}\ , \quad d > 0 \ . \tag{B4}$$

Therefore

$$\sum_{j=1}^{\infty} \frac{z^j}{j} \phi(ix^{1/2} j^{1/2})\, e^{-xj} = \left(\frac{1}{2\pi i}\right)^2 \int_{d-i\infty}^{d+i\infty} dt \int_{c-i\infty}^{c+i\infty} ds\, i x^{1/2} \Gamma(s) \Gamma(t) \frac{\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)} (-\ln z)^{-t} x^{-s} \sum_{j=1}^{\infty} \frac{1}{j^{1/2+s+t}}\ ; \quad 0 < c < 1,\ d > 0 \ .$$

Therefore

$$\sum_{j=1}^{\infty} \frac{z^j}{j} \phi(ix^{1/2} j^{1/2})\, e^{-xj} = \left(\frac{1}{2\pi i}\right)^2 \int_{d-i\infty}^{d+i\infty} dt \int_{c-i\infty}^{c+i\infty} ds\, i x^{1/2} \Gamma(s) \Gamma(t) \frac{\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)} (-\ln z)^{-t} x^{-s} \zeta(\tfrac{1}{2}+s+t)\ ;$$

$$0 < c < 1\ , \ d > 0\ , \ c + d > \tfrac{1}{2} \ .$$

If we close both contours in the left half-plane, we obtain the expansion appropriate to $\ln z \simeq 0$ ($z \simeq 1$) and x small. If we close the s contour in the right half-plane and the t contour in the left half-plane, we get the expansion appropriate to $\ln z \simeq 0$ ($z \simeq 1$) and x large. We find,

for $x \ll 1$,

$$\sum_{j=1}^{\infty} \frac{z^j}{j} \phi(ix^{1/2} j^{1/2})\, e^{-xj} = i \left\{ 2\arctan\left[\left(\frac{x}{-\ln z}\right)^{1/2}\right] + \sum_{\alpha=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{\alpha+p} (-\ln z)^{\alpha} x^{p+1/2}}{\alpha!\, \Gamma(\frac{3}{2}+p)} \zeta(\tfrac{1}{2} - \alpha - p) \right\} \ ; \tag{B5}$$

for $x \gg 1$,

$$\sum_{j=1}^{\infty} \frac{z^j}{j} \phi(ix^{1/2} j^{1/2})\, e^{-xj} = i \left\{ -2\arctan\left[\left(\frac{-\ln z}{x}\right)^{1/2}\right] + \sum_{\alpha=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{\alpha+p} \zeta(\frac{3}{2}+p-\alpha)(-\ln z)^{\alpha}}{\alpha!\, \Gamma(\frac{1}{2}-p) x^{p+1/2}} \right\} , \tag{B6}$$

where we have used[21]

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}\ , \quad x^2 \leq 1 \ .$$

APPENDIX C

We want to obtain expressions for $z = z(\Omega, T)$ for $z \to 0$ ($T \to \infty$) and $z \to 1-$ ($T \to T_c +$).

Now, the number equation for ideal bosons yields[20]

$$\rho\left(\frac{2\pi\hbar^2}{mkT}\right) = g_{3/2}(z)\ , \quad T \geq T_c$$

or

$$\zeta(\tfrac{3}{2})\left(\frac{T_c}{T}\right)^{3/2} = g_{3/2}(z)\ , \quad T \geq T_c \ . \tag{C1}$$

(i) for $z \simeq 0$.

$$g_{3/2}(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^{3/2}} = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \cdots \ .$$

Thus $g_{3/2}(z)$ is already in the form of a rapidly converging series and we simply have to invert Eq. (C1). This yields

$$z = \zeta(\tfrac{3}{2})\left(\frac{T_c}{T}\right)^{3/2} - \frac{1}{2^{3/2}} \zeta^2(\tfrac{3}{2})\left(\frac{T_c}{T}\right)^3 + \cdots \ . \tag{C2}$$

This expansion is good for $T \to \infty$.

(ii) for $z \lesssim 1$.

$$g_{3/2}(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^{3/2}}$$

$$= \sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \Gamma(s)(-\ln z)^{-s} j^{-s}\ , \quad c > 0$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \Gamma(s)(-\ln z)^{-s} \zeta(\tfrac{3}{2}+s)\ , \quad c > 0$$

where we have used Eq. (B4).

Closing the contour in the left half-plane, we obtain the following expansion for $\ln z \simeq 0$ ($z \simeq 1$):

$$g_{3/2}(z) = -2\sqrt{\pi}\,(-\ln z)^{1/2} + \sum_{p=0}^{\infty} \frac{\zeta(\frac{3}{2}-p)}{p!} (\ln z)^p \ .$$

Thus

$$\zeta(\tfrac{3}{2})\left(\frac{T_c}{T}\right)^{3/2} = -2\sqrt{\pi}\,(-\ln z)^{1/2} + \sum_{p=0}^{\infty} \frac{\zeta(\frac{3}{2}-p)}{p!} (\ln z)^p\ , \quad T \geq T_c \ .$$

Inverting this equation yields

$$\ln z = -\frac{\zeta^2(\frac{3}{2})}{4\pi} \theta^2 - \frac{\zeta(\frac{1}{2})\zeta^3(\frac{3}{2})}{8\pi^2} \theta^3 - \cdots\ , \quad T \geq T_c \ . \tag{C3}$$

This expansion is good for $T \gtrsim T_c$.

APPENDIX D

We want an expansion for

$$\int_0^\infty \frac{q^2 \sin(qr)\, dq}{q^3+q_0^3} \quad \text{for } r\to\infty .$$

$$I=\int_0^\infty \frac{q^2\sin(qr)}{q^3+q_0^3}dq=\int_0^\infty \frac{x^2\sin(q_0 rx)}{x^3+1}dx .$$

If we convert $x^2/(x^3+1)$ to partial fractions and add and subtract a term, we have

$$I=\frac{1}{3}\int_0^\infty \left(\frac{1}{x+1}-\frac{2x+1}{x^2+x+1}\right)\sin(q_0 rx)\,dx + \frac{1}{3}\int_{-\infty}^\infty \left(\frac{2x-1}{x^2-x+1}\right)\sin(q_0 rx)\,dx .$$

The second integral is a standard[22] one, and we have

$$I=\frac{1}{3}\int_0^\infty \left(\frac{1}{x+1}-\frac{2x+1}{x^2+x+1}\right)\sin(q_0 rx)\,dx + \frac{2\pi}{3}\cos\left(\frac{q_0 r}{2}\right)e^{-q_0 r\sqrt{3}/2} .$$

Splitting $(2x+1)/(x^2+x+1)$ into partial fractions and using the result[25]

$$\int_0^\infty \frac{\sin t}{t+z}dt=\int_0^\infty \frac{e^{-zt}}{1+t^2}dt, \quad \mathrm{Re}\, z>0$$

we obtain

$$I'=\int_0^\infty \left(\frac{1}{x+1}-\frac{2x+1}{x^2+x+1}\right)\sin(q_0 rx)\,dx =\int_0^\infty \frac{dx}{x^2+1}\left[e^{-q_0 rx}-2\cos\left(\frac{q_0 r\sqrt{3}\,x}{2}\right)e^{-q_0 rx/2}\right] .$$

Using Eq. (B1) and also[24]

$$e^{-\alpha x}\sin\beta x=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\,\Gamma(s)(\alpha^2+\beta^2)^{-s/2} \times\sin[s\arctan(\beta/\alpha)]; \quad c>-1, \quad \mathrm{Re}\,\alpha>|\mathrm{Im}\,\beta|$$

we have

$$I'=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, \frac{\Gamma(s)}{(q_0 r)^s}\left[1-2\cos\left(\frac{\pi s}{3}\right)\right] \times\int_0^\infty \frac{dt\, t^{-s}}{1+t^2}, \quad c>0 .$$

Now[21]

$$\int_0^\infty \frac{dt\,t^{-s}}{1+t^2}=-\frac{\pi}{2i^{s+1}}[1-(-1)^s]\frac{1}{\sin[\pi(1-s)]}, \quad -1<c<1 .$$

Substituting this into the expression for I' and closing the contour in the right half-plane (to obtain an expansion for large r), we find

$$I'=3\sum_{k=0}^\infty \frac{(-1)^k\Gamma(3+6k)}{(q_0 r)^{3+6k}} .$$

Therefore,

$$\int_0^\infty \frac{q^2\sin(qr)}{q^3+q_0^3}dq =\frac{2\pi}{3}\cos\left(\frac{q_0 r}{2}\right)e^{-q_0 r\sqrt{3}/2}+\sum_{k=0}^\infty \frac{(-1)^k\Gamma(3+6k)}{(q_0 r)^{3+6k}} .$$

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